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Homework 3

Problem 1 (Vaught's test) **:**
We say that a theory T is κ-categorical if all models of T of cardinality κ are isomorphic. Show that if T only has infinite models and is κ-categorical for some $\kappa \geqslant |\mathcal{L}|$, then T is complete.

Problem 2 (Vector spaces) **:**
Let K be an field (if you prefer, take $K = \mathbb{R}$). Let $\mathcal{L}$ be the language with one sort $\mathbf{V}$ a constant 0, a function $- : \mathbf{V} \to \mathbf{V}$, a function $+ : \mathbf{V}^2 \to \mathbf{V}$ and for all $x \in K$, a function $\lambda_x : \mathbf{V} \to \mathbf{V}$. Any K-vector space can naturally be made into an $\mathcal{L}$-structure by interpreting + as the addition, 0 the additive identity, − as the additive inverse and λ_x as scalar multiplication be x

1. Show there exist a theory T whose models are all the infinite K-vector spaces.
2. Let $V_i \leqslant U_i$, for $i = 1, 2$ be K-vector spaces such that $\dim(U_1) = \dim(U_2) > \dim(V_1) = \dim(V_2)$. Show that there exists an isomorphism $f : U_1 \to U_2$ such that $f(V_1) = f(V_2)$.

   This is just linear algebra and is the only linear algebra fact you'll need to do the rest.
3. Show that T is κ-categorical for all $\kappa > |\mathcal{L}|$. Conclude that T is complete.
4. Let $V \leqslant U$ be two infinite K-vector spaces such that $\dim(V) < \dim(U)$ and $\dim(U) > |\mathcal{L}|$, then $V \preccurlyeq U$.
5. Show that this remains true without the dimension hypothesis.
6. Let $U \vDash T$, $V \leqslant U$ and $\pi(x) = \{\neg x = c : c \in V\}$. Let $a, b \in U$ be two realizations of π. Show that $\mathrm{tp}^U(a/V) = \mathrm{tp}^U(b/V)$.
7. Let x be a single variable, show that $\mathcal{S}_x^U(V) = \{\mathrm{tp}^U(a/V) : a \in V\} \cup \{p_\infty\}$ where $p_\infty \supseteq \pi$ (in particular, you have to show that there is a unique complete type over V extending π).
8. Pick $p \in \mathcal{S}_x^U(V) \smallsetminus \{p_\infty\}$. Show that $\{p\}$ is open.
9. Show that $X \subseteq \mathcal{S}_x^U(V)$ containing p_∞ is open if and only if X is cofinite.
10. Let $\varphi(x)$ be an $\mathcal{L}$-formula, show that $\varphi(V)$ is finite or cofinite.

If you don't like linear algebra, you can replace T by the theory of infinite sets in the language with one sort and no symbol and dimension by cardinal. All of the (obvious adaptations of the) questions above remain true