Silvain Rideau silvain.rideau@berkeley.edu
1091 Evans www.normalesup.org/~srideau/en/teaching

Homework 4

Problem 1 :
Let $\mathcal{L}_n$ be the language with one sort and n binary predicates $(E_i)_{0 \leqslant i < n}$.

1. Give an $\mathcal{L}_n$-theory T_n such that in models of T_n, for all i, the E_i are equivalence relations, E_{i+1} is finer than E_i (i.e. every E_{i+1}-class is included in an E_i-class), E_0 has infinitely many classes, every E_i class is covered by infinitely many E_{i+1}-classes and the classes of E_{n-1} are infinite.

2. How many countable models does T_n have, up to isomorphism?

3. Let M, $N \vDash T_n$, $A \subseteq M$ finite, $f : A \to N$ be an embedding and $a \in M$. Show that f can be extended to a.

4. Show that any f as in the previous question is a partial elementary embedding from M into N.

5. Show that f is also elementary even when A is not finite.

6. Let $M \vDash T_n$, $A \subseteq M$ non empty and $p \in \mathcal{S}_x^M(A)$ where $|x| = 1$. To simplify notations, let E_n be the equality and E_{-1} be the trivial equivalence relation with just one equivalence class. Show that one (and only one) of the following holds:
   - there exists $a \in A$ such that p is the unique type containing $x = a$;
   - there exists $i \in \{-1, 0, \ldots, n\}$ and $a \in A$ such that p is the unique type containing $x E_i a$ and, for all $c \in A$, $\neg x E_{i+1} c$.

7. Show that T_n is κ-stable for all $\kappa \geqslant \aleph_0$.

8. Show that T_n has a saturated model of cardinality κ for all $\kappa \geqslant \aleph_0$.

9. Let $\mathcal{L}_\infty := \bigcup_n \mathcal{L}_n$ and $T_\infty := \bigcup_n T_n$. Show that T_∞ is a satisfiable $\mathcal{L}_\infty$-theory.

10. Let $M \vDash T_\infty$ and $p \in \mathcal{S}_x^M(M)$ where $|x| = 1$. Let E_{-1} be the trivial equivalence relation with just one equivalence class. Show that one (and only one) of the following holds:
    - there exists $a \in M$ such that p is the unique type containing $x = a$;
    - there exists $a \in M$ such that p is the unique type containing $\neg x = a$ and $x E_i a$ for all $i \in \mathbb{Z}_{\geqslant 0}$;
    - there exists $i \in \mathbb{Z}_{\geqslant 0}$ and $a \in M$ such that p is the unique type containing $x E_i a$ and, for all $c \in M$, $\neg y E_{i+1} c$;
    - there is no $a \in M$ such that p contains $x E_i a$, for all $i \in \mathbb{Z}_{\geqslant 0}$, and there exists $a_i \in M$ such that for all $i \in \mathbb{Z}_{\geqslant 0}$, p is the unique type containing $x E_i a_i$ and $\neg x E_{i+1} a_i$, for all $i \in \mathbb{Z}_{\geqslant 0}$.

11. Let $M \vDash T_\infty$, $A \subseteq M$ be infinite and x a tuple of variables. Show that $|\mathcal{S}_x^M(A)| \leqslant |A|^{\aleph_0}$ and that this bound is sharp.