Silvain Rideau silvain.rideau@berkeley.edu
1091 Evans `www.normalesup.org/~srideau/en/teaching`

Homework 5

Problem 1 :
Let $\mathcal{L}_< := \{X; <: X^2\}$. We say that a total order $X, <$ is discrete if for all $x \in X$, $\{y \in X : y < x\}$ has a maximal element and $\{y \in X : x < y\}$ has a minimal element. Note that this definition of discrete order implies that there are no end points.

1. Give a theory T whose models are exactly the non empty discrete total orders.
2. Let $(X, <)$ be an order, show that the following are equivalent:
   a) $X \vDash T$;
   b) there exists a (non empty) total order $(Y, <)$ such that X is $\mathcal{L}_<$-isomorphic to $\mathbb{Z} \times Y$ with the right to left lexicographic order (i.e. $(n, x) < (m, y)$ if $x < y$ or $x = y$ and $n < m$).
3. Show that T does not eliminate quantifiers.
4. Let $\mathcal{L}_s := \mathcal{L}_> \cup \{s : X \to X\}$. Give a theory T_s whose models are exactly the models of T where s is interpreted as the successor function, i.e. $s(x) = \min\{y \in X : y > x\}$.
5. Show that T_s eliminates quantifiers and is complete.
6. Show that T is complete.

Problem 2 :
Let M be an $\mathcal{L}$-structure and λ be a cardinal. We say that M is a λ-model if $|M| = \lambda$ and for every $\mathcal{L}$-formula φ, either $\varphi(M)$ is finite or $|\varphi(M)| = \lambda$.

1. Let M be an $\mathcal{L}$ structure and λ be a cardinal greater or equal to $|M|$ and $|\mathcal{L}|$. Show that there exists $N \succcurlyeq M$ which is a λ-model.
2. Let T be an $\mathcal{L}$-theory and λ be a cardinal greater than $|\mathcal{L}|$. Assume that all the λ-models of T are isomorphic. Show that T is complete.
3. Assume that T is λ-categorical for some $\lambda \geqslant |\mathcal{L}|$. Show that every model of T of cardinality λ is a λ-model.

Problem 3 :
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. Let $\kappa = \aleph_0 + |A|$. Assume M is strongly κ^+-homogeneous and κ^+-saturated. Let $X \subseteq M^x$ be $\mathcal{L}(M)$-definable and assume that for all $\sigma \in \mathrm{Aut}(M/A)$, $\sigma(X) = X$. Show that X is $\mathcal{L}(A)$-definable.