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Homework 6

Problem 1 :
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $a \in M$ is algebraic over A if there exists an $\mathcal{L}(A)$-formula φ such that $M \vDash \varphi(a)$ and $|\varphi(M)|$ is finite. We define $\operatorname{acl}(A) = \{a \in M : a \text{ is algebraic over } A\}$.

1. Show that $|A| \leqslant |\operatorname{acl}(A)| \leqslant |A| + |\mathcal{L}|$.

   Solution: To each element of $\operatorname{acl}(A)$ we can associate a formula φ_a such that $M \vDash \varphi_a(a)$ and $|\varphi_a(M)|$ infinite. The map $a \mapsto \varphi_a$ is not injective but it has finite fibers: any element such that $\varphi_a = \varphi_b$ must be in $\varphi_a(M) = \varphi_b(M)$ which is finite. So $|A| \leqslant \sum_{\varphi_a} |\varphi_a(M)| \leqslant |\mathcal{L}(A)|\aleph_0 = |A| + |\mathcal{L}|$.

2. Show that $\operatorname{acl}(\operatorname{acl}(A)) = \operatorname{acl}(A)$.

   Solution: Note that a is the only realisation of the formula $x = a$ so any $a \in A$ is algebraic over A. It follows that $\operatorname{acl}(A) \subseteq \operatorname{acl}(\operatorname{acl}(A))$. Now pick $a \in \operatorname{acl}(\operatorname{acl}(A))$. There exists a tuple $c \in \operatorname{acl}(A)^x$ and an $\mathcal{L}(A)$-formula$\varphi(y,x)$ such that $M \vDash \varphi(a,c)$ and $|\varphi(M)| = n < \aleph_0$. Let c_i denote the components of c. Since $c_i \in \operatorname{acl}(A)$, there exists $\mathcal{L}(A)$-formulas $\psi_i(x_i)$ such that $M \vDash \psi_i(c_i)$ and $|\psi_i(M)| = m_i < \aleph_0$.

   Let $\theta(y) := \exists x_1 \ldots \exists x_{|c|} \bigwedge_i \psi_i(x_i) \wedge \varphi(y,x) \wedge (\forall y_0 \ldots \forall y_n (\bigwedge_j \varphi(y_j,x)) \to (\bigvee_{j_1 \neq j_2} y_{j_1} = y_{j_2}))$. Then $M \vDash \theta(a)$ and $|\theta(M)| \leqslant n \prod_i n_i < \aleph_0$. Indeed there are at most n_i choices for each x_i and for each of this choices at most n choices for y. So $a \in \operatorname{acl}(A)$.

3. Let $a \in \operatorname{acl}(A)$, show that $\operatorname{tp}(a/A) \in \mathcal{S}^M(A)$ is isolated.

   Solution: Let $\varphi(x)$ be an $\mathcal{L}(A)$-formula such that $M \vDash \varphi(a)$ and $|\varphi(M)| = \{a_1, \ldots, a_k\}$. Note that $M \vDash \forall y \varphi(y) \to \bigvee_i y = a_i$. In particular, the only realisations of φ in elementary extensions of M are the a_i. It follows that $\langle\varphi\rangle = \{\operatorname{tp}(a_i/A) : 0 < i \leqslant k\}$ is finite. Because $\mathcal{S}^M_y(A)$ is Hausdorff, we can find an open U such that $U \cap \langle\varphi\rangle = \{\operatorname{tp}(a/A)\}$ which is therefore open, i.e. $\operatorname{tp}(a/A)$ is isolated.

4. Show that $a \in \operatorname{acl}(A)$ if and only if for all $N \succcurlyeq M$, $\operatorname{tp}(a/A)$ only has finitely many realisations in N.

   Hint: Use compactness to prove that $a \notin \operatorname{acl}(A)$ implies that there is some $N \succcurlyeq M$ in which $\operatorname{tp}(a/A)$ has infinitely many realisations.

   Solution: We have already proved in the previous question that for all $N \succcurlyeq M$, if $\varphi(M)$ is finite, $\varphi(N) = \varphi(M)$ is finite. Since $\varphi \in \operatorname{tp}(a/A)$, it follows that all realisations of $\operatorname{tp}(a/A)$ in any $N \succcurlyeq M$ are in the finite set $\varphi(M)$.

   Let us now prove the converse and assume that $a \notin \operatorname{acl}(A)$. Let $C = \{c_i : i \in \omega\}$ be a set of new constants and $\Sigma := \mathcal{D}^{\mathrm{el}}(M) \cup \{c_i \neq c_j : i < j\} \cup \{\varphi(c_i) : i \in \omega \text{ and } \varphi \in \operatorname{tp}(a/A)\}$. Let us prove that Σ is finitely satisfiable. Let $\Sigma_0 \subseteq \Sigma$ be finite, we have $\Sigma_0 \subseteq \mathcal{D}^{\mathrm{el}}(M) \cup \{c_i \neq c_j : 0 \leqslant i < j < n\} \cup \{\varphi_j(c_i) : 0 \leqslant i < n \text{ and } 0 \leqslant j < k\}$ for some $\varphi_j \in \operatorname{tp}(a/A)$. Let $\psi := \bigwedge_j \varphi_j$. Since $M \vDash \psi(a)$, ψ is an $\mathcal{L}(A)$-formula and

$a \notin \mathrm{acl}(A)$, $|\psi(M)| = \aleph_0$. In particular we can find n distinct realisation of ψ in M to interpret the c_i.

Let $N^\star \models \Sigma$. We have $N := N^\star|_{\mathcal{L}} \succcurlyeq M$ and every $c^{N^\star}$ is a distinct realisation of $\mathrm{tp}(a/A)$.

5. Let $\kappa = |\mathcal{L}| + |A|$. Assume that M is κ^+-saturated. Show that:

$$\mathrm{acl}(A) = \bigcap_{A \subseteq N \preccurlyeq M} N.$$

Solution: Pick $N \preccurlyeq M$ containing A, $a \in \mathrm{acl}(A)$ and φ an $\mathcal{L}(A)$-formula such that $|\varphi(M)| = n < \aleph_0$ and $M \models \varphi(a)$. Let $\theta = \exists y_1 \ldots \forall y_n \bigwedge_i \varphi(y_i) \wedge \bigwedge_{j \neq i} y_j \neq y_i$. We have $M \models \theta$ so $N \models \theta$ and $|\varphi(N)| \geqslant n = |\varphi(M)|$. Since $N \preccurlyeq M$, $\varphi(N) \subseteq \varphi(M)$ and hence they are equal. It follows that $a \in \varphi(N) \subseteq N$.

Conversely, assume $a \notin \mathrm{acl}(A)$ and let $M_0 \preccurlyeq M$ contain $A \cup \{a\}$ and have cardinality $\kappa = |A| + \mathcal{L}$. Let C_0 be the set of all realisations of $\mathrm{tp}(a/A)$ in M_0 (there are at most $|M_0| = \kappa$ of them). Let $\Sigma(x) := \mathrm{tp}(a/A) \cup \{x \neq c \mod c \in M_0\}$. Let $\Sigma_0 \subseteq \Sigma$ be finite, then we may assume that $\Sigma_0 = \{\varphi(x)\} \cup \{x \neq c_i : 0 \leqslant i < n\}$ for some $\varphi \in \mathrm{tp}(a/A)$ and $c_i \in C$. Since $\varphi(M)$ is infinite, we can find a realisation that avoids the c_i. So Σ is a partial type over at most κ parameters so it is realized in M be some c_0. Note that $c_0 \notin M_0$.

We have $\mathrm{tp}(a/A) = \mathrm{tp}(c_0/A)$ the map sending c_0 to a and fixing A is a partial elementary embedding. By weak κ^+ homogeneity (and induction), we can extend f to a partial elementary embedding $g : M_0 c_0 \to M$. Let $N = g(M_0)$. Since g is partial elementary and $M_0 \preccurlyeq M$, we also have $N \preccurlyeq M$. Since g fixes A, we also have $A \subseteq N$. Finally, since $c_0 \notin M_0$, we have $a \notin N$.

6. Let $\mathrm{Aut}(M/A)$ be the set of $\mathcal{L}$-automorphisms of M that fix A pointwise. We say that $a \in M$ has a finite orbit under the action of $\mathrm{Aut}(M/A)$ if the set $\{\sigma(a) : \sigma \in \mathrm{Aut}(M/A)\}$ is finite.

Assume that M is strongly κ^+-homogeneous and κ^+-saturated. Show that:

$$\mathrm{acl}(A) = \{a \in M : a \text{ has a finite orbit under the action of } \mathrm{Aut}(M/A)\}.$$

Solution: Let us assume that $a \in \mathrm{acl}(A)$, then there is an $\mathcal{L}(A)$-formula $\varphi(x)$ such that $M \models \varphi(a)$ and $|\varphi(M)|$ is finite. Every automotphism in $\mathrm{Aut}(M/A)$ preserves $\varphi(M)$ so the orbit of a under the action of $\mathrm{Aut}(M/A)$ is contained in $\varphi(M)$ and hence is finite.

Conversely, let us assume that $a \notin \mathrm{acl}(A)$. By the same compactness as before, because M is κ^+-saturated, we can find infinitely many realisations of $\mathrm{tp}(a/A)$ in M. Note that since M is κ^+-homogeneous, two points in M are in the same orbit under the action of $\mathrm{Aut}(M/A)$ if and only if they have the same type over A. So the orbit of a under the action of $\mathrm{Aut}(M/A)$ is infinite.

7. Let $\mathcal{R} := (\mathbb{R}, +, \cdot)$. Show that the only automorphism of $\mathcal{R}$ is the identity, but that $\mathrm{acl}(\mathbb{Q})$ is a strict subset of $\mathbb{R}$.

Solution: The element $0 \in \mathbb{R}$ is the unique realisation of $\varphi(x) := \forall y\, x + y = y$ and therefore every $\sigma \in \mathrm{Aut}(\mathcal{R})$ fixes 0. Similarly 1 is the only realisation of $\psi(x) := \forall y\, y \cdot x = y$ and therefore 1 is also fixed by $\mathrm{Aut}(\mathcal{R})$. Since $n = \sum_{i=1}^{n} 1$ and $+$ is preserved by $\mathrm{Aut}(\mathcal{R})$, it follows that $\mathbb{Z}$ is preserved by $\mathrm{Aut}(\mathcal{R})$. Finally, since there is a unique way of extending a ring morphism to the field of fractions, $\mathbb{Q}$ is also fixed by $\mathrm{Aut}(\mathcal{R})$.

Note that $x < y$ on $\mathbb{R}$ si defined by $\theta(x,y) := \exists z\, y - x = z^2$ and any $\sigma \in \mathrm{Aut}(\mathcal{R})$ has to be strictly increasing. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ for the order topology and $\mathbb{Q}$ is fixed by $\mathrm{Aut}(\mathcal{R})$, it follows that all of $\mathbb{R}$ is fixed by $\mathrm{Aut}(\mathcal{R})$, i.e. the only automorphism of $\mathcal{R}$ is the identity.

But $|\mathbb{R}| = 2^{\aleph_0} > \aleph_0 = |\mathrm{acl}(\mathbb{Q})|$ so $\mathrm{acl}(\mathbb{Q}) \subset \mathbb{R}$.

Problem 2 :
Let $\mathcal{L}$ be a countable language and T be a complete $\mathcal{L}$-theory.

1. Pick $p \in \mathcal{S}_x(T)$ and c a new tuple of constants. Let $T_p := T \cup p(c)$. Show that if T_p is $\aleph_0$-categorical, then so is T.

   Hint: Show that the restriction map $\mathcal{S}_y(T_p) \to \mathcal{S}_y(T)$ is unto.

   Solution: Pick $q \in \mathcal{S}_y(T)$. Let $N \vDash T$ realize both p and q. Let $c \in N^x$ realize p and $a \in N^y$ realize q. Let N_c be the obvious enrichment of N to a model of T_p. Let $r := \mathrm{tp}^{N_c}(a) \in \mathcal{S}_y(T_p)$, then $r|_{\mathcal{L}} = \mathrm{tp}N(a) = p$. So the restriction map $\mathcal{S}_y(T_p) \to \mathcal{S}_y(T)$ is unto.

   If T_p is $\aleph_0$-categorical, then for all y, $\mathcal{S}_y(T_p)$ is finite. It follows that $\mathcal{S}_y(T)$ is also finite and by the previous question T is $\aleph_0$-categorical.

2. Assume that T is not $\aleph_0$-categorical, show that it has a least three countable models up to isomorphism.

   Solution: Since T is not $\aleph_0$-categorical, we can find a type $p \in \mathcal{S}_x(T)$ such that p is not isolated. Let M be a countable model omitting p and N a countable model realizing p (and fix $c \in N^x$ a realization of p). Let T_p be as above (for the p we just found). By the previous question we can find $q \in \mathcal{S}_y(T_p)$ which is not isolated. By omitting types we may assume that $N_c \vDash T_p$ omits q.

   By compactness, we can find $N^1 \succcurlyeq N$ countable such that for all c' in N realizing p, $N^1_{c'}$ realizes q (where the constants c are now interpreted as c'). Iterating this construction, we find $N^{i+1} \succcurlyeq N^i$ such that for all c' in N^i realizing p, $N^{i+1}_{c'}$ realizes q. Let $R = \bigcup_{i\in\omega} N^i$. By the chain theorem, $R \vDash T$ (and it is countable). Moreover, for all c' in R realizing p, $R_{c'}$ realizes q.

   Because R and N both realize p they are not isomorphic to M which omits it. Moreover, if $f : N \to R$ is an isomorphism, then $f(c)$ realizes p in R so we can find d realizing q in $R_{f(c)}$. But it would follow that $f^{-1}(d)$ realizes q in N_c, a contradiction.

3. Let $\mathcal{L}_C = \mathcal{L}_< \cup \{c_i : i \in \omega\}$ and $\mathrm{DLO}_C := \mathrm{DLO} \cup \{c_i < c_j : i < j\}$. Show that DLO_C is complete and has three countable models up to isomorphism.

   Solution: Since DLO eliminates quantifiers, for any $M \vDash \mathrm{DLO}$ and $C \subset M$, $\mathrm{DLO} \cup \Delta_M(C)$ is complete. Note that DLO_C is exactly an instance of such a theory when C is taken to be an increasing chain. So DLO_C is complete.

   Let M, N and R be $\mathbb{Q}$ with the usual order. Let $c_i^M = i$, $c_i^N = -\frac{1}{i}$ and $c_i^R = \sum_{j=0}^{i} \frac{1}{j!}$. In M, the sequence c_i is unbounded. In N the sequence has a lower upper bound (which is 0). In R, the sequence is bounded but has no lower upper bound (because e is not rational). So these three models cannot be isomorphic. Indeed any isomorphism would send a upper bound to an upper bound and a lower upper bound to a lower upper bound.

Let us now show that any countable $S \vDash \mathrm{DLO}_C$ is isomorphic to one of the above. Assume that the sequence c_i^S has no upper bound in S and let I be the set of partial $\mathcal{L}(C)$-embeddings with finite domains from S to M. Since DLO_C is complete, I contains $\varnothing \to M$. Let us now assume that we have a partial embedding $f : A \to M$ for some $A \subseteq S$ finite and $a \in S$. If $a \in A \cup C$, then the extension is quite obvious. Otherwise let $D := \{d \in \{-\infty\} \cup A \cup C : d < a\}$. The set D must be finite, otherwise $C \subseteq D$ and a is an upper bound of C in S, a contradiction. Let d_0 be the maximal element of D and c_{i_0} the maximal element of $D \cap C$. Let d_1 be the minimal element of $\{c_{i_0+1}\} \cup (A \smallsetminus D)$, then $\mathrm{tp}(a/A \cup C)$ is isolated by $d_0 < x < d_1$. By density and absence of endpoints, this formula is realized in M by some b (where d_i is interpreted as $f(d_i)$ if $d_i \in A$, as c^M if $d_i \in C$ and as $-\infty$ if $d_i = -\infty$). It is easy to check that we can extend f by sending a to b. The last part of the back-and-forth is symmetric. Since M and S are countable and I has the back-and-forth, they are isomorphic.

Let us now assume that C has a lower upper bound in S and let us call that bound e. As above, we show that the set I of partial $\mathcal{L}_C$-embeddings from S to N with finite domain containing e and sending e to 0 has the back-and-forth. The set I is non empty because the map $e \mapsto 0$ can easily seen to be a partial $\mathcal{L}_C$-embedding. Let $f : A \to N$ be a partial embedding with finite domain sending e to 0. Let $D := \{d \in A \cup C \cup \{e, -\infty\} : d < a\}$. If $a \in A \cup C \cup \{e\}$ or D is finite, we proceed as before. If D is infinite, then we must have $C < a$, but because e is a lower upper bound, we also have $e \in D$. Let $d_0 := \max D \cap A$ and $d_1 := \min\{+\infty\} \cup A \smallsetminus D$. Then $C < e \leqslant d_0$ and $\mathrm{tp}(a/A \cup C \cup \{e\})$ is isolated by $d_0 < x < d_1$. By density, we can also b in N such that $f(d_0) < b < f(d_1)$ (with the obvious convention that $f(+\infty) = +\infty$) and we extend f by sending a to b. It follows that S is isomorphic to N.

Finally, let us assume that C has an upper bound, but no lower upper bound. As above, we show that the set I of partial $\mathcal{L}_C$-embeddings from S to R with finite domain has the back-and-forth. The set I is non empty because the map $\varnothing \to N$ is in I by completeness. Let $f : A \to N$ be a partial embedding with finite domain. Let $D := \{d \in A \cup C \cup \{-\infty\} : d < a\}$. If $a \in A \cup C$ or D is finite, we proceed as before. If D is infinite, then we must have $C < a$. There are now two cases. If there exists an $d \in D$ such that $d > C$, then we define $d_0 := \max D \cap A$ and we proceed as before. If there is no $d \in D$ such that $d > C$, let $d_1 := \min\{+\infty\} \cup A \smallsetminus D$. Because C has no lower upper bound in R, we can find $b \in R$ such that $C < b < d_1$. We can now extend f by sending a to b. It follows that S is isomorphic to R.