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Solutions to homework 8

Problem 1 :
Let M an $\mathcal{L}$-structure. We say that an $\mathcal{L}(M)$-formula $\varphi(x,y)$, where x and y are sorted in the same way, has the order property in M if there exists $A = (a_i)_{i\in\mathbb{Z}_{>0}}$ tuples in M such that $M \vDash \varphi(a_i, a_j)$ if and only if $i < j$.

1. Assume that there exists an $\mathcal{L}$-formula $\varphi(x,y)$ with the order property in M, show that there exists an indiscernible sequence $(a_i)_{i\in\mathbb{Z}_{>0}}$ in some $N \succcurlyeq M$ such that $N \vDash \varphi(a_i, a_j)$ if and only if $i < j$.

   Solution: Let $\{c_i : i \in \mathbb{Z}_{>0}\}$ be new constants and $\Sigma = \mathcal{D}^{\mathrm{el}}(M) \cup \{\varphi(c_i, c_j) : i < j \in \mathbb{Z}_{>0}\} \cup \{\neg\varphi(c_i, c_j) : j \leqslant i \in \mathbb{Z}_{>0}\} \cup \{\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) : i_1 < \ldots i_n \in \mathbb{Z}_{>0}$ and $j_1 < \ldots j_n \in \mathbb{Z}_{>0}\}$. It suffices to show that Σ is finitely satisfiable. Let $\Sigma_0 \subseteq \Sigma$ be finite. Then there exists $\mathcal{L}$-formulas $(\psi_l)_{l<k}$ such that $\Sigma_0 \subseteq \mathcal{D}^{\mathrm{el}}(M) \cup \{\varphi(c_i, c_j) : i < j \in \mathbb{Z}_{>0}\} \cup \{\neg\varphi(c_i, c_j) : j \leqslant i \in \mathbb{Z}_{>0}\} \cup \{\psi_l(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi_l(c_{j_1}, \ldots, c_{j_n}) : i_1 < \ldots i_n \in \mathbb{Z}_{>0}, j_1 < \ldots j_n \in \mathbb{Z}_{>0}$ and $l < k\}$ — note that provided we add useless variables to the ψ_l we can assume that the ψ_l all have the same variables. Define $f : [A]^n \to 2^k$ by $f(\{a_{i_1}, \ldots, a_{i_n})(l) = 1$ if and only if $M \vDash \psi_l(a_{i_1}, \ldots, a_{i_n})$, where $i_1 < \ldots i_n \in \mathbb{Z}_{>0}$. By Ramsey's theorem, there exists $X \subseteq A$ infinite monochromatic. Let $g : \mathbb{Z}_{>0} \to X$ be an infinite strictly increasing function. Then, interpreting c_i as $a_{g(i)}$ makes M into a model of Σ_0.

2. Let I be a totally ordered set and $A := (a_i)_{i\in I}$ be an indiscernible sequence in some $\mathcal{L}$-structure M. Assume that A is not an indiscernible set. Show that there exists a formula $\varphi(x_1, \ldots, x_n)$ and a transposition $\tau = (i\ i+1)$ such that $M \vDash \varphi(a_1, \ldots, a_n)$ and $M \vDash \neg\varphi(a_{\tau(1)}, \ldots, a_{\tau(n)})$.

   Solution: If A is not indiscernible, then exists $i_1 < \ldots < i_n \in I$, $j_1, \ldots, j_n \in I$ and an LL-formula $phi(x_1, \ldots, x_n)$ such that $M \vDash \varphi(a_{i_1}, \ldots, a_{i_n})$ and $M \vDash \neg\varphi(a_{j_1}, ldots, a_{j_n})$. There exists a permutation $\sigma \in \mathfrak{S}_n$ such that $j_{\sigma^{-1}(1)} < \ldots < j_{\sigma^{-1}(n)}$. Since A is an indiscernible sequence, we also have $M \vDash \neg\varphi(a_{i_{\sigma(1)}}, \ldots, a_{i_{\sigma(n)}})$. Since $\mathfrak{S}_n$ is generated by the permutations $(i\ i+1)$, there exists finitely many such transpositionsτ_k such that $\sigma = \prod_k \tau_k$. Let $\sigma_l = \prod_{k<l} \tau_k$ and l_0 be maximal such that $M \vDash \varphi(a_{i_{\sigma_l(1)}}, \ldots, a_{i_{\sigma_l(n)}})$. Then $M \vDash \neg\varphi(a_{\tau_l\circ\sigma_l(1)}, \ldots, a_{\tau_l\circ\sigma_l(n)})$. Reordering the variables, we get that $M \vDash \varphi(a_{i_1}, \ldots, a_{i_n})$ and $M \vDash \neg\varphi(a_{i_1}, \ldots, a_{i+1}, a_i, \ldots, a_{i_n})$.

3. Let A be as above, show that there is a $\mathcal{L}(N)$-formula with the order property in $N \succcurlyeq M$.

   Solution: Let A be an indiscernible sequence of order $\omega\cdot 3$ in some $N \succcurlyeq M$ with the same type as A. Let $psi(x,y) := \varphi(a_1, \ldots, a_{i-1}, x, y, a_{\omega 2}, \ldots, a_{\omega 2+n-i-2}) \wedge x \neq y$. Since A and A have the same type, $M \vDash \varphi(a_1, \ldots, a_{i-1}, a_{\omega+k}, a_{\omega+l}, a_{\omega 2}, a_{\omega 2+n-i-2})$ if and only if $k < l$.

4. T be an $\mathcal{L}$-theory, show that the following are equivalent:

   a) There exists $M \vDash T$ and and indiscernible sequence in M which is not an indiscernible set;

b) There exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}(M)$-formula with the order property in M;

c) There exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}$-formula with the order property in M;

Solution: It is proved in Question 3 that a) implies b). It is clear that c) is a consequence of b). So there remains to prove that c) implies a). Assume c). By Question 1, we find an indiscernible sequence $A = (a_i)_{i\in\mathbb{Z}_{>0}}\} \subseteq N \vDash T$ such that $N \vDash \varphi(a_i, a_j)$ if and only of $i < j$. So A is not an indiscernible set.

5. Assume that there exists an $\mathcal{L}$-formula $\varphi(x,y)$ with the order property in M, show that for any total order $(I,<)$, there exists $(a_i)_{i\in I}$ tuples in $N \succcurlyeq M$ such that $N \vDash \varphi(a_i, b_j)$ if and only if $i < j$.

   Solution: By the first question, there exists $A_0 = \{a_i \in i \in \mathbb{Z}_{>0}\} \subseteq N_0 \succcurlyeq M$ an indiscernible sequence such that $N_0 \vDash \varphi(a_i, a_j)$ if and only if $i < j$. Then there exists $A \subseteq N \succcurlyeq N_0$ an indiscernible sequence indexed by I such that $\mathrm{tp}(A) = \mathrm{tp}(A_0)$. In particular, for all $i, j \in I$, if $i < j$, $N \vDash \varphi(a_i, a_j)$ and if $j \leqslant i$, $N \vDash \neg\varphi(a_i, a_j)$.

   This is slightly overkill since we obtain an indiscernible A in the end. We can obviously also do it using a straighforward compactness argument.

6. Let φ and $(a_i)_{i\in I}$ be as in the previous question. Show that there is an injective map from the set of proper cuts of I (i.e. downwards closed strict subsets of I) into $\mathcal{S}_x(\bigcup_{i\in I} a_i)$.

   Solution: Let D be a cut of I and let $\pi_D(x) := \{\varphi(a_i, x) : i \in D\} \cup \{\neg\varphi(a_i, x) : i \notin D\}$. Then π_D is finitely satisfaible. Indeed, if $\pi_0 \subseteq \pi$ is finite and I_0 is the set of $i \in I$ such that a_i appears in π_0; if $I_0 \smallsetminus D$ is non empty then a_{i_0} for i_0 minimal in $I_0 \smallsetminus D$ realizes π_0 and if it is empty, any a_i for $i \in I \smallsetminus D$ realizes π_0. Let $p_D \supseteq \pi_D$ be any complete type over $\bigcup_{i\in I} a_i$. Note that $\varphi(a_i, x) \in p_D$ if and only if $i \in D$. It follows that the map $D \mapsto p_D$ is injective.

7. (This is really much more set theoretic) Let κ be a cardinal and μ be the smallest cardinal such that $\kappa < \kappa^\mu$. Let $\kappa^{<\mu}$ be the set of all function from some $\alpha < \mu$ into κ. Order $\kappa^{<\mu}$ lexicographically (i.e. $f < g$ if there exists α that for all $\beta < \alpha$, $f|_\beta = g|_\beta$ and either $f(\alpha)$ is not defined or $f(\alpha) < g(\alpha)$). Show that $\kappa^{<\mu}$ is a total order of size $\leqslant \kappa$ with $> \kappa$ many cuts.

   Solution: Pick some $f \in \kappa^\mu$ and define $D_f := \{g \in \kappa^{<\mu} : \exists \alpha\mu,\, g \leqslant f|_\alpha\}$, which is a cut of $\kappa^{<\mu}$. Note that the map $f \mapsto D_f$ is injective. Indeed, if $f \neq g$, there exists a minimal α such that $f(\alpha) \neq g(\alpha)$. We may assume that $f(\alpha) < g(\alpha)$. Then $g|_\alpha \in D_g \smallsetminus D_f$.

   It follows that there are least $\kappa^\mu > \kappa$ cuts in $\kappa^{<\mu}$. Moreover, $\kappa^{<mu} = \bigcup_{\alpha<\mu} \kappa^\alpha$. Since $\alpha < \mu$, $|\alpha| < \mu$ and hence, by construction, $|\kappa^\alpha| \leqslant \kappa$. Since $\kappa^\kappa > \kappa$, we have $\mu \leqslant \kappa$ and hence $|\kappa^{<\mu}| \leqslant \sum_{\alpha<\mu} |\kappa^\alpha| \leqslant \kappa^2 = \kappa$.

8. Let T be an $\mathcal{L}$-theory and assume that there exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}$-formula with the order property in M, show that T is not κ-stable for any cardinal κ.

   Solution: By Question 5, we can find $N \vDash T$ and a_i, $b_i \in N$ for $i \in \kappa^{<\mu}$ where μ is as in Question 7. By Question 6, $|\mathcal{S}_x(\bigcup_{i\in\kappa^{<mu}})| \geqslant \kappa^\mu > \kappa$, contradicting κ-stability of T.

The converse is also true (this is a theorem of Shelah), but it is much harder.