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Homework 8

Problem 1 :
Let M an $\mathcal{L}$-structure. We say that an $\mathcal{L}(M)$-formula $\varphi(x,y)$, where x and y are sorted in the same way, has the order property in M if there exists $X = (a_i)_{i\in\mathbb{Z}_{>0}}$ tuples in M such that $M \vDash \varphi(a_i, a_j)$ if and only if $i < j$.

1. Assume that there exists an $\mathcal{L}$-formula $\varphi(x,y)$ with the order property in M, show that there exists an indiscernible sequence $(a_i)_{i\in\mathbb{Z}_{>0}}$ in some $N \succcurlyeq M$ such that $N \vDash \varphi(a_i, a_j)$ if and only if $i < j$.

2. Let I be a totally ordered set and $A := (a_i)_{i\in I}$ be an indiscernible sequence in some $\mathcal{L}$-structure M. Assume that A is not an indiscernible set. Show that there exists a formula $\varphi(x_1, \dots, x_n)$ and a transposition $\tau = (i\ i+1)$ such that $M \vDash \varphi(a_1, \dots, a_n)$ and $M \vDash \neg\varphi(a_{\tau(1)}, \dots, a_{\tau(n)})$.

3. Let A be as above, show that there is a $\mathcal{L}(N)$-formula with the order property in $N \succcurlyeq M$.

4. T be an $\mathcal{L}$-theory, show that the following are equivalent:
   a) There exists $M \vDash T$ and and indiscernible sequence in M which is not an indiscernible set;
   b) There exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}(M)$-formula with the order property in M;
   c) There exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}$-formula with the order property in M;

5. Assume that there exists an $\mathcal{L}$-formula $\varphi(x,y)$ with the order property in M, show that for any total order $(I,<)$, there exists $(a_i)_{i\in I}$ tuples in $N \succcurlyeq M$ such that $N \vDash \varphi(a_i, b_j)$ if and only if $i < j$.

6. Let φ and $(a_i)_{i\in I}$ be as in the previous question. Show that there is an injective map from the set of proper cuts of I (i.e. downwards closed strict subsets of I) into $\mathcal{S}_x(\bigcup_{i\in I} a_i)$.

7. (This is really much more set theoretic) Let κ be a cardinal and μ be the smallest cardinal such that $\kappa < \kappa^\mu$. Let $\kappa^{<\mu}$ be the set of all function from some $\alpha < \mu$ into κ. Order $\kappa^{<\mu}$ lexicographically (i.e. $f < g$ if there exists α that for all $\beta < \alpha$, $f|_\beta = g|_\beta$ and either $f(\alpha)$ is not defined or $f(\alpha) < g(\alpha)$). Show that $\kappa^{<\mu}$ is a total order of size $\leqslant \kappa$ with $> \kappa$ many cuts.

8. Let T be an $\mathcal{L}$-theory and assume that there exists $M \vDash T$ and $\varphi(x,y)$ an $\mathcal{L}$-formula with the order property in M, show that T is not κ-stable for any cardinal κ.

The converse is also true (this is a theorem of Shelah).