Model theory

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1. Semantic and syntax

1.1. Structures

1.2. Formulas

1.3. Theories

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Theorem 1.1 (Downwards Löwenheim-Skolem)**:**

Let M be a structure and $A \subseteq M$. Then, there exists $M_0 \preccurlyeq M$ containing A such that $|M_0| \leqslant |A| + |\mathcal{L}|$.

Proof. Let $\kappa = |A| + |\mathcal{L}|$. We define $(A_j)_{j\in\mathbb{Z}_{\geqslant 0}}$ by induction. Let $A_0 := A$. Let us now assume A_j has been built. Let $(\varphi_i(x_i, \overline{y}_i), \overline{a}_i) : i \in \kappa)$ be an enumeration of all formulas $\varphi(x, \overline{y})$ and tuples $\overline{a} \in A_j^{\overline{y}}$. Let $I_0 := \{i \in \kappa : M \vDash (\exists x_i \varphi_i)[\overline{a}_i]\}$. For all $i \in I_0$, pick any $c_i \in M^x$ such that $M \vDash \varphi_i[c_i, \overline{a}_i]$. Let $A_{j+1} := A_j \cup \{c_i : i \in I_0\}$.
Let $M_0 := \bigcup_j A_j$. By construction, $|A_j| \leqslant \kappa$, so $|M_0| \leqslant \kappa$ — recall that, by definition, $|\mathcal{L}| \geqslant \aleph_0$.

Claim 1.2: *For all formula $\varphi(x, \overline{y})$ and M_0-assignment δ, if $M \vDash (\exists x\ \varphi)[\delta]$ then there exists $c \in M_0$ such that $M \vDash \varphi[\delta, x \to c]$.*

Proof. Since $\delta(\overline{y})$ is finite, there exists j such that $\overline{a} := \delta(\overline{y}) \in A_j$. Since $M \vDash (\exists x\varphi)[\overline{a}]$, by construction, there exists $c \in A_{j+1} \subseteq M_0$ such that $M \vDash \varphi[c, \overline{a}]$. ⧫

By Tarski-Vaught, $M_0 \preccurlyeq M$. ■

Definition 1.3:
Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two languages and let M be an $\mathcal{L}$-structure. We define $M|_{\mathcal{L}_0}$ to be the $\mathcal{L}_0$-structure whose sorts are the same as M and where the $\mathcal{L}_0$-symbols are interpreted as in M.
If M_0 is an $\mathcal{L}_0$-structure, an $\mathcal{L}$-enrichment of M_0 is an $\mathcal{L}$-structure M such that $M|_{\mathcal{L}_0} = M_0$.

Let M be an structure and let $A \subseteq M$. We define $\mathcal{L}(A) := \mathcal{L} \cup \{c_a : a \in A\}$, where c_a is a new constant sorted as a. We define M_A to be the natural $\mathcal{L}(A)$-enrichment of M defined by $c_a^M = a$, for all $a \in A$.

Definition 1.4 (Diagrams)**:**
Let M be an structure and let $A \subseteq M$. We define:

(i) the quantifier free diagram *of A in M to be*

$$\Delta_M(A) := \{\varphi \text{ quantifier free } \mathcal{L}(A)\text{-sentence} : M_A \vDash \varphi\};$$

(ii) the elementary diagram *of A in M to be*

$$\mathcal{D}_M^{\mathrm{el}}(A) := \mathrm{Th}(M_A) = \{\varphi\ \mathcal{L}(A)\text{-sentence} : M_A \vDash \varphi\}.$$

Proposition 1.5:
Let M and N be structures and let $A \leqslant M$.

(i) There exists an $\mathcal{L}(A)$-enrichment N_A of N such that $N_A \vDash \Delta_M(A)$ if and only if there exists an embedding $\theta : A \to N$.

(ii) There exists an $\mathcal{L}(M)$-enrichment N_M of N such that $N_M \vDash \mathcal{D}_M^{\mathrm{el}}(M)$ if and only if there exists an elementary embedding $\theta : M \to N$.

Proof.

(i) Let us first assume that there exists an $\mathcal{L}(A)$-enrichment N_A of N such that $N_A \vDash \Delta_M(A)$. Let $\theta(a) := c_a^{N_A}$. Let us show that θ is an embedding.

- Pick any $a_1 \neq a_2 \in A$. Then $M_A \vDash c_{a_1} \neq c_{a_2}$ which is a quantifier free $\mathcal{L}(A)$-formula and hence $N_A \vDash c_{a_1} \neq c_{a_2}$, i.e. $\theta(a_1) = c_{a_1}^{N_A} \neq c_{a_2}^{N_A} = \theta(a_2)$. So θ is injective.
- If c is a constant and $A \leqslant M$, $c^M = c^A$ and $M_A \vDash c = c_{c^A}$, a quantifier free $\mathcal{L}(A)$-formula. It follows that $N_A \vDash c = c_{c^A}$, i.e. $c^N = c^{N_A} = c_{c^A}^{N_A} = \theta(c^A)$.
- If $f \in \mathfrak{F}_{\overline{X}}$ and $a_i \in X_i(A)$, since $A \leqslant M$, $M_A \vDash c_{f^A(\overline{a})} = f(c_{\overline{a}})$, where $c_{\overline{a}}$ is the tuple $(c_{a_i})_i$. We therefore have $N_A \vDash c_{f^A(\overline{a})} = f(c_{\overline{a}})$, that is, $\theta(f^A(\overline{a})) = c_{f^A(\overline{a})}^{N_A} = f^N(c_{\overline{a}}^{N_A}) = f^n(\theta(\overline{a}))$.
- If $P \in \mathfrak{R}_{\overline{X}}$ and $a_i \in X_i(A)$, since $A \leqslant M$, $\overline{a} \in R^A$ if and only if $M_A \vDash R(c_{\overline{a}})$, which is equivalent to $N_A \vDash R(c_{\overline{a}})$, i.e. $c_{\overline{a}}^{N_A} = \theta(\overline{a}) \in R^N$.

Conversely, assume that $\theta : A \to N$ is a embedding. We define an $\mathcal{L}(A)$-enrichment of N by $c_a^{N_A} := \theta(A)$. For all quantifier free $\mathcal{L}$-formula $\varphi(\overline{x})$ and $\overline{a} \in A^{\overline{x}}$, we have:

$$\begin{aligned} M_A \vDash \varphi[c_{\overline{a}}] &\iff M \vDash \varphi[\overline{a}] \\ &\iff A \vDash \varphi[\overline{a}] \\ &\iff N \vDash \varphi[\theta(\overline{a})] \\ &\iff N_A \vDash \varphi[c_{\overline{a}}]. \end{aligned}$$

So $N_A \vDash \Delta_M(A)$.

(ii) Let us first assume that there exists an $\mathcal{L}(M)$-enrichment of N of N such that $N_M \vDash \mathcal{D}^{\mathrm{el}}_M(M)$. Let $\theta : M \to N$ be defined by $\theta(a) := c_a^{N_M}$. In (i), we proved that θ was an embedding, let us prove that it is elementary. Pick any $\mathcal{L}(M)$-formula $\varphi(\overline{x})$ and $\overline{a} \in M^{\overline{x}}$. We have:

$$\begin{aligned} M \vDash \varphi[\overline{a}] &\iff M_M \vDash \varphi[c_{\overline{a}}] \\ &\iff N_M \vDash \varphi[c_{\overline{a}}] \\ &\iff N \vDash \varphi[\theta(\overline{a})]. \end{aligned}$$

So θ is elementary.

Conversely assume that there exists an elementary embedding $\theta : M \to N$. Let us define an $\mathcal{L}(M)$-enrichment N_M of N by $c_a^{N_M} := \theta(a)$, for all $a \in M$. For all $\mathcal{L}$-formula $\varphi(\overline{x})$ and $\overline{a} \in M^{\overline{x}}$, we have:

$$\begin{aligned} M_M \vDash \varphi[c_{\overline{a}}] &\iff M \vDash \varphi[\overline{a}] \\ &\iff N \vDash \varphi[\theta(\overline{a})] \\ &\iff N_M \vDash \varphi[c_{\overline{a}}]. \end{aligned}$$

So $N_M \vDash \mathcal{D}^{\mathrm{el}}_M(M)$.

■

If $N \vDash \Delta_M(A)$, we may, at the cost of renaming points of N, assume that $A \subseteq N$ and that, for all $a \in A$, $c_a^N = a$. In particular, if $N \vDash \mathcal{D}^{\mathrm{el}}_M(M)$, by renaming points in N, we may assume that $M \preccurlyeq N$.

Theorem 1.6 (Upwards Löwenheim-Skolem)**:**

Let M be an infinite structure and $\kappa \geqslant |M| + |\mathcal{L}|$. Then there exists $N \succcurlyeq M$ with $|N| = \kappa$.

Proof. Let X be a sort such that $X(M)$ is infinite and let T be the $\mathcal{L} \cup \{c_i : i \in \kappa\}$-theory $\mathcal{D}^{\mathrm{el}}_M(M) \cup \{c_i \neq c_j : i \neq j \in \kappa\}$, where the c_i are new X-constants. Then T is finitely satisfiable. Indeed, let $T_0 \subseteq T$ be finite. Then there exists $I_0 \subseteq \kappa$ finite such that $T_0 \subseteq \mathcal{D}^{\mathrm{el}}_M(M) \cup \{c_i \neq c_j : i \neq j \in I_0\}$. Since $X(M)$ is infinite, we can find distinct $(a_i)_{i \in I_0} \in X(M)$ and interpreting c_i as a_i we get an enrichment of M which is a model of T_0.
Let $N \vDash T$. We may assume that $M \preccurlyeq N|_{\mathcal{L}}$. By downwards Löwenheim-Skolem (cf. Theorem **(1.1)**), applied to $M \subseteq N$, we may assume that $|N| \leqslant |\mathcal{L} \cup \{c_i : i \in \kappa\}| + |M| = \kappa$. Note that since all the c_i^N are distinct, $|N| = \kappa$. ■

Corollary 1.7:
Let T be a theory with infinite models, then T has a model of every cardinality $\kappa \geqslant |L|$.

Proof. Let $M \vDash T$ be finite. If $\kappa \geqslant |M|$, applying upwards Löwenheim-Skolem (cf. Theorem **(1.6)**), we find $N \succcurlyeq M$ such that $|N| = \kappa$. In particular, $N \vDash T$. If $\kappa \leqslant |M|$, pick any $A_0 \subseteq M$ with $|A_0| = \kappa$. By downwards Löwenheim-Skolem (cf. Theorem **(1.1)**), we find $N \preccurlyeq M$ containing A with $|N| \leqslant |A| + |\mathcal{L}| = \kappa$. But since $\kappa = |A| \leqslant |N|$, $|N| = \kappa$. ■ 09/05

2. Types

2.1. Definitions

Definition 2.1:
Let M be a structure. A set $X \subseteq M^{\overline{x}}$ is said to be definable if there exists a formula $\varphi(\overline{x})$ such that $X = \varphi(M) := \{\overline{a} \in M^{\overline{x}} : M \vDash \varphi[\overline{a}]\}$.
A function is said to be definable if its graph is definable.

If $\theta_i : M \to N$ is an elementary embedding and φ is a formula, then $\varphi(M) = \theta^{-1}(\varphi(N))$. It follows that definable sets should be considered as functors from the category of models of $\mathrm{Th}(M)$, with elementary embeddings, to the category of sets. Equivalently, they can be considered as an equivalence class of formulas (under equivalence in $\mathrm{Th}(M)$).
If $A \subseteq M$, we say that $X \subseteq M^{\overline{x}}$ is A-definable if it is definable in the $\mathcal{L}(A)$-structure M_A, i.e. there exists a formula $\varphi(\overline{x}, \overline{y})$ and $\overline{a} \in A^{\overline{y}}$ such that $X = \varphi(M, \overline{a})$.

Definition 2.2:
Let T be a theory and $\pi(\overline{x})$ a set of formulas. We say

Definition 2.3:
Let $\pi(\overline{x})$ a set of formulas.

(i) let M be a structure. We say that π is satisfiable in M, if there exists $\overline{a} \in M^{\overline{x}}$ such that for all $\varphi \in \pi$, $M \vDash \varphi[\overline{a}]$. Dually, we say that $\overline{a}$ realizes π in M. We write $M \vDash \pi[\overline{a}]$.

(ii) Let T be a theory. The set π is said to be a partial $\overline{x}$-type *of T if for all finite $\pi_0 \subseteq \pi$, π_0 is satisfiable in some model T. We say that π is finitely satisfiable in models of T.*

(iii) An $\overline{x}$-type *p is a partial $\overline{x}$-type that is maximal for inclusion.*

We sometimes refer to types as *complete* types to distinguish them from partial types. Partial types are filters on the Boolean algebra of definable sets and types are ultrafilters on that algebra.

Remark 2.4:
Let T be a complete theory. A set of formulas $\pi(\overline{x})$ is a partial type of T if and only if for all $M \vDash T$, π is satisfiable in M.

Proof. Assume that π is a partial type and fix $M \vDash T$ and $\pi_0 \subseteq \pi$ finite. By definition, there exits $N \vDash T$ such that $N \vDash \exists\overline{x} \bigwedge_{\varphi\in\pi_0} \varphi(\overline{x})$. Since T is complete, it follows that we also have $M \vDash \exists\overline{x} \bigwedge_{\varphi\in\pi_0} \varphi(\overline{x})$. The converse is obvious. ■

Notation 2.5:

(i) If $A \subseteq M$, a (partial) type of M over A is a (partial) type of $\mathcal{D}^{\mathrm{el}}_M(A)$.

(ii) We denote by $\mathrm{S}_{\overline{x}}(T)$ the set of $\overline{x}$-types of T and by $\mathrm{S}^M_{\overline{x}}(A)$, the set of $\overline{x}$-types of M over A.

If $A \subseteq M \preccurlyeq N$, then, for all $\overline{x}$, $\mathrm{S}^M_{\overline{x}}(A) = \mathrm{S}^M_{\overline{x}}(A)$ — in fact, it suffice that $\vDash \mathcal{D}^{\mathrm{el}}_M(A)$.

Example 2.6:

1. Let T be a theory, $M \vDash T$ and $\overline{a} \in M^{\overline{x}}$. Then $\mathrm{tp}^M(\overline{a}) := \{\varphi(\overline{x}) \text{ formula} : M \vDash \varphi[\overline{a}]\} \in \mathrm{S}_{\overline{x}}(T)$.
   If $A \subseteq M$, $\mathrm{tp}^M(\overline{a}/A) := \{\varphi(\overline{x})\ \mathcal{L}(A)-\text{formula} : M \vDash \varphi[\overline{a}]\} \in \mathrm{S}^M_{\overline{x}}(A)$.
2. If you consider the empty tuple of variables: $\mathrm{S}_{\varnothing}(T)$ is the set of completions of T and $\mathrm{S}^M_{\varnothing}(A) = \{\mathcal{D}^{\mathrm{el}}_M(A)\}$.
3. Let $\mathcal{L}_<$ be the one sorted language with a binary symbol $<$. Let $M = (\mathbb{Q}; <)$ and $A = \mathbb{Z}$. Then $\{x > n : n \in \mathbb{Z}\}$ is a partial type. We will see later that it is in fact a type.

Remark 2.7:
A partial type $\pi(\overline{x})$ of some satisfiable theory T is complete if and only if for every formula $\varphi(\overline{x})$, $\varphi \in \pi$ or $\neg\varphi \in \pi$.

Proof. Let us assume π is a type and $\varphi \notin \pi$. Since π is maximal for inclusion, $\pi \cup \{\varphi\}$ is not finitely satisfiable, so we can find $\pi_0 \subseteq \pi$ finite such that for all $M \vDash T$ and $\overline{a} \in M^{\overline{x}}$, if $M \vDash \pi_0[\overline{a}]$ then $M \nvDash \varphi[\overline{a}]$, i.e. $M \vDash \neg\varphi[\overline{a}]$. Now pick any $\pi_1 \subseteq \pi$ finite. By the above, $\pi_1 \cup \{\neg\varphi\}$ is satisfiable in any model of T by a realization of $\pi_1 \cup \pi_0$. It follows that $\pi \cup \{\neg\varphi\}$ is finitely satisfiable and hence, by maximality, $\neg\varphi \in \pi$.
Conversely, if for every formula $\varphi(\overline{x})$, $\varphi \in \pi$ or $\neg\varphi \in \pi$, then π is maximal since the finite set $\{\varphi, \neg\varphi\}$ is not satisfiable in any model of T. ∎

Proposition 2.8:

(i) Let $\pi(\overline{x})$ be a partial type of T. Then there exists $M \vDash T$ realizing π.

(ii) Let $A \subseteq M$ some structure, and π a partial type of M over A. Then there exist $N \succcurlyeq M$ realizing π.

Proof.

(i) Let $\overline{c}$ be a new tuple of constants sorted like $\overline{x}$. Then the $\mathcal{L} \cup \{\overline{c}\}$-theory $T \cup \pi[\overline{c}]$ is finitely satisfiable by definition of partial types. So, by compactness, we find an $\mathcal{L} \cup \{\overline{c}\}$-structure $M_{\overline{c}} \vDash T \cup \pi[\overline{c}]$, i.e. $M|_{\mathcal{L}} \vDash T$ and $M \vDash \pi[\overline{c}^M]$.

(ii) Note that π is finitely satisfiable in $M \vDash \mathcal{D}^{\mathrm{el}}_M(A)$, so π is also a partial type in $\mathcal{D}^{\mathrm{el}}_M(M)$. We can now apply (i) to conclude.

■

Corollary 2.9:
Every partial type $\pi(\overline{x})$ is included in a complete type.

We can prove this directly using Zorn, but let us use what we have done so far:

Proof. Since π is a partial type, then by Proposition **(2.8)**.(i), we find $M \vDash T$ and $\overline{a} \in M^{\overline{x}}$ realizing π. Then $\mathrm{tp}^M(a)$ is a complete type containing π. ■

Definition 2.10:
Let T be a theory and $\overline{x}$ a tuple of variables. We define a topology on $\mathrm{S}_{\overline{x}}(T)$ whose basis of open sets is given by sets of the form $\langle\varphi\rangle := \{p \in \mathrm{S}_{\overline{x}}(T) : \varphi \in p\}$ for all formula $\varphi(\overline{x})$.

The $\langle\varphi\rangle$ do form a basis of open sets since $\langle\varphi\rangle \cap \langle\psi\rangle = \langle\varphi \wedge \psi\rangle$ and $\langle\varphi\rangle \cup \langle\psi\rangle = \langle\varphi \vee \psi\rangle$. A general closed set of $\mathrm{S}_{\overline{x}}(T)$ is of the form $\langle\pi\rangle := \bigcap_{\varphi\in\pi}\langle\varphi\rangle = \{p \in \mathrm{S}_{\overline{x}}(T)\} : \pi \subseteq p\}$ where π is a partial $\overline{x}$-type of T. Note also that the complement of $\langle\varphi\rangle$ is $\langle\neg\varphi\rangle$ which is also open. So $\langle\varphi\rangle$ is both open and closed — i.e. clopen — and $\mathrm{S}_{\overline{x}}(T)$ is totally disconnected. In particular, it is Hausdorff.

Theorem A:

For all theory T and tuple of variables $\overline{x}$, the space $\mathrm{S}_{\overline{x}}(T)$ is compact.

Proof. Assume $\mathrm{S}_{\overline{x}}(T) = \bigcup_{i\in I} V_i$ where the V_i are open. Without loss of generality, we may assume that $V_i = \langle\varphi_i\rangle$ for some formula $\varphi_i(\overline{x})$. Let $\pi := \{\neg\varphi_i : i \in I\}$. If π is finitely satisfiable in models of T, then, by Corollary **(2.9)** we can find a complete type $p \supseteq \pi$. Then $p \notin \langle\varphi_i\rangle$ for all i, contradicting the fact that $\mathrm{S}_{\overline{x}}(T) = \bigcup_{i\in I}\langle\varphi_i\rangle$.
It follows that there exists $I_0 \subseteq I$ finite such that for all $M \vDash T$, $M \vDash \forall\overline{x} \bigvee_{i\in I_0} \neg\varphi_i$ and thus $\mathrm{S}_{\overline{x}}(T) = \bigcup_{i\in I_0}\langle\varphi_i\rangle$. ■

Corollary 2.11:
If $U \subseteq \mathrm{S}_{\overline{x}}(T)$ is clopen, then there exists a formula φ such that $U = \langle\varphi\rangle$.

Proof. Since U is open, $U = \bigcup_{i\in I}\langle\varphi_i\rangle$ for some formulas $\varphi_i(\overline{x})$. Since U is closed it is compact and there exists $I_0 \subseteq I$ finite such that $U = \bigcup_{i\in I_0}\langle\varphi_i\rangle = \langle\bigvee_{i\in I_0}\varphi_i\rangle$. ■

2.2. Saturation

Definition 2.12:
Let M be a structure and κ be an infinite cardinal.

(i) We say that M is κ-saturated if for every $A \subseteq M$ such that $|A| < \kappa$ and $p \in \mathrm{S}_{\overline{x}}(A)$, p is realized in M.

(ii) We say that M is saturated if it $|M|$-saturated.

Remark 2.13:
If M is κ-saturated, then $|M| \geqslant \kappa$. Indeed, for any sort X, the partial type $\{x \neq a : a \in X(M)\}$ cannot be realized in M so $|X(M)| \geqslant \kappa$.

Remark 2.14:
The structure M is κ-saturated if and only if for all $A \subseteq M$ such that $|A| < \kappa$ and $p \in \mathrm{S}_x(A)$ where x is a single variable, p is realized in M

Proof. Let us assume that the second statement holds — the converse is obvious. We show by induction on $|x|$ that, for all $A \subseteq M$ with $|A| < \kappa$, any $p \in \mathrm{S}_{\overline{x}}(A)$ is realized in A. If $|\overline{x}| = 1$, that is our hypothesis. Let us now assume $\overline{x} = (\overline{y}, z)$ where z is a single variable. Let $q := \{\varphi(\overline{y}) : \varphi \in p\} \in \mathrm{S}_{\overline{y}}(A)$. By induction we find $\overline{c} \in M^{\overline{x}}$ such that $M \vDash q[\overline{c}]$. Let $p(\overline{c}, z) = \{\varphi(\overline{c}, z) : \varphi \in p\} \in \mathrm{S}_z(A \cup \overline{c})$. By hypothesis we can find $d \in M$ such that $M \vDash p[\overline{c}, d]$, i.e. p is realized in M. ∎

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Proposition 2.15:
Let M be an $\mathcal{L}$-structure, $|M| \geqslant |\mathcal{L}|$. We can find $N \succcurlyeq M$ which is $|M|^+$-saturated and such that $|N| \leqslant 2^{|M|}$.

2.3. Homogeneity

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Proposition 2.16:
Let M, N be homogeneous with $|M| = |N|$. Assume that:

- ($\star$) *for every $A \subseteq M$, $|A| < |M|$ and partial elementary embedding $f : M \to N$ with domain A, any $p \in \mathrm{S}_{\overline{x}}(A)$ is realized in M if and only if $f_\star p$ is realized in N.*

Then $M \cong N$.

Remark 2.17:
Assumption ($\star$) is equivalent to:

- ($\star'$) for all potentially infinite tuple of variables $\overline{x}$ with $|\overline{x}| < \kappa$, M and N realize the same types in $\mathrm{S}_{\overline{x}}(T)$, where $T = \mathrm{Th}(M) = \mathrm{Th}(N)$.

Proposition 2.18:
Let M be a structure and $\kappa \geqslant |\mathcal{L}|$ a cardinal. The following are equivalent:

- *(i) M is κ-saturated.*
- *(ii) M is weakly κ-homogeneous and κ^+-universal.*

Proof.

(ii)⇒(i) Pick any $A \subseteq M$ with $|A| < \kappa$ and any $p \in \mathrm{S}_x^M(A)$. We know that there exists $N \vDash \mathcal{D}_M^{\mathrm{el}}(A)$ containing $c \vDash p$. By downwards Löwenheim-Skolem, we may assume that $|N| = |A| + |\mathcal{L}| \leqslant \kappa$. By κ^+-universality, there exists an elementary embedding $f : N \to M$. The map $g : f(A^N) \to A$ defined by $g(f(a^N)) = a$ is a partial elementary embedding so, by κ-homogeneity, it extends to $f(c)$. For all formula $\varphi(x, a) \in p$, we have that $N \vDash \varphi(c, a^N)$. It follows that $M \vDash \varphi(f(c), f(a^N))$ and hence $M \vDash \varphi(g(f(c)), g(f(a^N)))$, i.e. $M \vDash \varphi(g(f(c)), a)$. So $g(f(c)) \vDash p$. 09/19

(i)⇒(ii) Let $f : M \to N$ be a partial elementary embedding with domain A, $|A| < \kappa$, and $a \in M$. Let $p := \mathrm{tp}^M(a/A)$. By κ-saturation, $f_\star p$ is realized by some $c \in M$. Extending f by $a \mapsto c$, we get an elementary embedding. Now consider $N \equiv M$ with $\lambda := |N| \leqslant \kappa$. Enumerate N as $\{a_i : i \in \lambda\}$. We build, by induction on $i \in \kappa$, a partial elementary embedding $f_i : N \to M$ with domain $\{a_j : j \leqslant i\}$ and such that f_i extends f_j whenever $i \leqslant j$. Assuming that f_j is build for every $j < i$, let $g = \bigcup_{j<i} f_j$. It is an partial elementary embedding with domain $A_i := \{a_i : i < j\}$. Let $p = \mathrm{tp}(a_i/A_i)$. By κ-saturation, $f_\star p$ is realized by some $c \in M$. Setting $f_i(a_i) = c$ will provide us with the required extension of g. We may now define $f = \bigcup_{j\leqslant\lambda}$ which is indeed an elementary embedding from N to M.

■

Corollary 2.19:
Let $M \equiv N$ be saturated with $|M| = |N|$. Then $M \simeq N$.

Proof. By Proposition **(2.18)**, M and N are homogeneous. Moreover, for every $A \subseteq M$ with $|A| < |M|$ and partial elementary $f : M \to N$ with domain A, any $p \in \mathrm{S}_{\overline{x}}(A)$ is realized in M by saturation and so is $f_\star p$. So, by Proposition **(2.16)**, $M \simeq N$. ■

Proposition 2.20:
Let M be a structure and κ be an infinite cardinal. Then there exists $N \succcurlyeq M$ which is κ^+-saturated and strongly κ^+-homogeneous.

Proof. By upwards Löwenheim-Skolem, we may assume $|M| = \kappa \geqslant \mathcal{L}$. We build by induction, using Proposition **(2.15)**, structures $(M_i)_{i<\kappa^+}$ such that $M_0 = M$, M_{i+1} is $|M_i|^+$-saturated and $M_i \preccurlyeq M_j$ whenever $i < j$. Let $N = \bigcup_{i<\kappa^+}$. Then, by Tarski's union theorem, $N \succcurlyeq M$.
Let $A \subseteq M$ have cardinality strictly smaller than κ^+. By regularity of κ^+, there exists $i < \kappa^+$ such that $A \subseteq M_i$. By construction, any type over A is realized in $M_{i+1} \preccurlyeq N$.
Now let $f : N \to N$ be a partial elementary embedding on some domain $A \subseteq N$ with $|A| < \kappa^+$. Then, by regularity, there exists an $i < \kappa^+$ such that f is a partial elementary embedding from $M_i \to M_i$. We build, by induction on $j < \omega$, elementary embeddings $f_j : M_{i+2j} \to M_{i+2j+1}$ and $g_j : M_{i+2j+1} \to M_{i+2j+2}$ such that f_{i+1} extends g_i^{-1} and g_i extends f_i^{-1}. By convention, we set $g_{-1} = f^{-1}$.
Assume g_j is build, note that g_j^{-1} is a partial elementary embedding from M_{i+2j+2} into $M_{i+2j+1} \preccurlyeq M_{i+2j+3}$. By the same proof that κ-saturation implies κ^+-universality, we extend g_j^{-1} to a map from M_{i+2j+2} to M_{i+2j+3}. The inverse of this extension is f_{j+1}. We

build g_j out of f_j is a similar way. Then $h = \bigcup_{j<\omega} f_j = \bigcup_{j<\omega} g_j$ is an automorphism of $M_{i+\omega}$.
The set of automorphisms of some M_i extending f is therefore non empty—where M_{κ^+} denote N. It is not hard to see that chains of such automorphisms have an upper bound (namely the union of these automorphisms which is an automorphism of the union of the domains). By Zorn, there is a maximal $g : M_i \to M_i$ extending f. But we have just shown above that if $i < \kappa$, we can extend g to an automorphism of $M_{i+\omega}$. So g must be an automorphism of N. ■

3. Quantifier elimination

3.1. Definition and consequences

Definition 3.1 (Elimination of quantifiers)**:**
A theory T is said to eliminate quantifiers *if for every formula $\varphi(\overline{x})$, there exists a quantifier free formula $\psi(\overline{x})$ such that $T \vDash \forall \overline{x}\ (\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.*

Remark 3.2:

1. Due to the fact that the empty structure is not always considered a structure and issues with sentences when there are no constants, the definition of quantifier elimination might vary slightly.

2. If $\mathcal{L}$ has no constants, there are no quantifier free sentences, so we add a new form of atomic formula $\perp$ which never holds. With that convention, elimination of quantifiers in a language without constants implies completeness.

Example 3.3:

1. k-vector spaces for a fixed k in the language with one sort V, constant $0 : V$, function symbols $+ : V^2 \to V$, $- : V \to V$ and $\lambda_x : V \to V$ for all $x \in k$.

2. Infinite sets without structure in the language with one sort and no symbols.

3. Algebraically closed fields in the ring language.

4. Dense linear orders with no endpoints: $\mathbb{Q}, <$, $\mathbb{R}, <$.

5. Divisible Abelian groups: $\mathbb{Q}, +$, $\mathbb{R}, +$.

6. Divisible ordered Abelian group: $\mathbb{Q}, +, <$, $\mathbb{R}, +, <$.

7. The random graph in the language with one sort G and a predicate $E : G^2$ for the edge relation.

8. The theory of $\mathbb{Z}$ in the ring language does not eliminate quantifiers.

We will prove most of these quantifier elimination results later, when we have a more tools at our disposal.

Proposition 3.4:
Let T be a theory . We define $\mathcal{L}^\star := \langle\cup\rangle\{R_\varphi \subseteq \prod_i S_{x_i} : \varphi(\overline{x})$ $\mathcal{L}$-formula$\}$ and $T^\star := T \cup \{\forall \overline{x}\ R_\varphi(\overline{x}) \leftrightarrow \varphi(\overline{x})\}$. Then any model of T has a unique enrichment to a model of $T^\star$, every $\mathcal{L}^\star$-formula is equivalent, modulo $T^\star$, to an $\mathcal{L}$-formula and $T^\star$ eliminates quantifiers.

Proof. The first assertion is obvious since $T^\star$ specifies how R_φ should be interpreted. The second assertion is proved by induction on formulas. The atomic case is the definition of $T^\star$ and the rest of the induction is obvious. The third fact is then a easy consequence of the second one. ∎

We have just seen that without changing the definable sets, we can make T have quantifier elimination. So one might wonder what use is quantifier elimination. But in the previous construction we changed the quantifier free definable sets, which are the only ones we really understand. This is therefore a very useful abstract construction, but it serves no purpose as far as understanding what definable sets in a given structure look like.

Definition 3.5:
Let M, N be structures. A partial embedding from M to N is a map $f : A \to M$, where $A \subseteq M$ such that for all quantifier free formula $\varphi(\overline{x})$ and all $\overline{a} \in A^{\overline{x}}$, $M \vDash \varphi(\overline{a})$ if and only if $N \vDash \varphi(f(\overline{a}))$.

Remark 3.6:
If $A \leqslant M$, a partial embedding from M to N with domain A is exactly an embedding $A \to N$. Moreover, any partial embedding extends uniquely to the structure generated by its domain.

Lemma 3.7:
Let M, N be structure and $A \subseteq M$. The following are equivalent:

- *there is a partial embedding $f : M \to N$ with domain A;*
- *there exists an $\mathcal{L}(A)$-enrichment N_A of N with $N_A \vDash \Delta_M(A)$.*

Proposition 3.8:
Let T eliminate quantifiers, $M, N \vDash T$ and $f : M \to N$ be a partial embedding. Then f is elementary.

Proof. Pick any formula $\varphi(x)$ and $a \in M^x$. By quantifier elimination, there exists a quantifier free formula $\psi(x)$ such that $T \vDash \forall x\ (\varphi(x) \leftrightarrow \psi(x))$. Then, since $M \vDash T$, $M \vDash \varphi(a)$ if and only if $M \vDash \psi(a)$. Because f is a partial embedding, this equivalent to $N \vDash \varphi(f(a))$. Since $N \vDash T$, this last statement is equivalent to $N \vDash \varphi(f(a))$. So we do have $M \vDash \varphi(a)$ if and only if $N \vDash \varphi(f(a))$ ∎

Quantifier elimination therefore allows us to prove that certain embeddings are elementary by just checking that they are embeddings, which is much easier. This can be used to determine completions of theories, describe types...

3.2. Criterion's for elimination of quantifiers

Definition 3.9:
Let M, N be structures, A set I of partial embedding from M to N is said to have the back and forth if:

(i) $I \neq \varnothing$;

(ii) for all $f \in I$ and $a \in M$, there exists $g \in I$ extending f which i defined at a;

(iii) for all $f \in I$ and $b \in N$, there exists $g \in I$ extending f whose image contains b.

Proposition 3.10:
Let M, N be structure and I a set of partial embeddings from M to N with the back and forth. Then $M \equiv N$. Moreover, any $f \in I$ is elementary.

Proof. We first prove by induction on $\varphi(x)$, that for any $f \in I$ and tuple a in the domain of f, $M \models \varphi(a)$ if and only if $N \models \varphi(f(a))$. The case of atomic formulas follows from the fact that f is a partial embedding. The Boolean combinations are easy to deal with. There remains to consider existential quantification. Assume $M \models \exists x\ \varphi(x, a)$, then we can find some $c \in M$ such that $M \models \varphi(c, a)$. By the back and forth, we can extend f to some $g \in I$ defined on c. Then $N \models \varphi(g(c), g(a))$ and hence $N \models \exists x\ \varphi(x, f(a))$. The converse is proved similarly. ∎

Proposition 3.11:
Let M, N be $\aleph_0$-saturated. The following are equivalent:

(i) $M \equiv N$;

(ii) the set I of partial elementary embeddings with finite domain from M to N has the back and forth.

Proof. It is proved in Proposition **(3.10)** that (ii) always implies (i). There remains to prove that (i) implies (ii). Let us assume that $M \equiv N$ holds. Then the map $\varnothing \to N$ is a partial elementary embedding (since the domain is empty, we only have to check sentences). Now, pick any partial elementary embedding $f : M \to N$ with finite domain A and any $a \in M$. Let $p := \mathrm{tp}(a/A)$. Then $f_\star p \in \mathrm{S}^N(f(A))$ is realized by some $c \in N$, by $\aleph_0$-saturation, and f can be extended to a partial elementary embedding by sending a to c. The reverse direction is proved similarly. ∎

Remark 3.12:

1. If M, N are structure, $|M| = |N|$ and the set I of all partial elementary embeddings from M to N, with domain strictly smaller than $|M|$, has the back and forth, then any $f \in I$ extends to an isomorphism from M to N.

2. If M and N are countable, they are isomorphic if and only if there exists a family of partial embeddings with the back and forth.

3.3. Examples

4. Omission of types

09/29

4.1. Isolated types

Theorem 4.1 (Omitting types):

Assume $|\mathcal{L}| = \aleph_0$. Let T be a theory and $(\pi_i(x_i))_{i\in\omega}$ a collection of non isolated (partial) types. Then, there exists $M \vDash T$ countable such that none of the π_i are realized in M.

4.2. Prime and atomic models

Lemma 4.2: 10/02

Let M, N be structures, $f : M \to N$ a partial elementary embedding with finite domain A and $c \in M$. Assume M is atomic, then f can be extended to a partial elementary embedding defined at c.

Proof. Choose an enumeration $A = \{a_i : i \leqslant n\}$. Since M is atomic, $\mathrm{tp}(ac)$ is isolated by some formula $\varphi(x, y)$. We have $M \vDash \exists y\ \varphi(a, y)$ and thus $N \vDash \exists y\ \varphi(f(a), y)$. Let $d \in N^y$ be such that $N \vDash \varphi(f(a), d)$. Since φ isolates $\mathrm{tp}(ac)$, it follows that $\mathrm{tp}(f(a)d) = \mathrm{tp}(ac)$, so, extending f by $c \mapsto d$, we get the required partial elementary embedding. ■

Corollary 4.3:

Let M be an atomic structure, then $M$is $\aleph_0$-homogeneous.

Corollary 4.4:

Assume $|\mathcal{L}| = \aleph_0$. Let T be complete theory and $M \vDash T$. The following are equivalent:

(i) M is prime;

(ii) M is atomic and countable.

Corollary 4.5:

Assume $|\mathcal{L}| = \aleph_0$. Prime models, of some theory T, are unique up to isomorphism.

Proposition 4.6:

Assume $|\mathcal{L}| = \aleph_0$. Let T be a complete theory. The following are equivalent:

(i) T has a prime model;

(ii) T has an atomic model;

(iii) For all finite tuple of variable x, isolated types are dense in $\mathrm{S}_x(T)$.

Remark 4.7:
Condition (iii) is equivalent to: for all formula $\varphi(x)$ such that $T \vDash \exists x\ \varphi(x)$, there exists $p \in \langle\varphi\rangle \subseteq \mathrm{S}_x(T)$ which is isolated.

Proposition 4.8:
Assume $|\mathcal{L}| = \aleph_0$. Let T be a theory. Assume that for all finite x, $|\mathrm{S}_x(T)| < 2^{\aleph_0}$, then T has a prime model.

The converse is not true, cf. $(\mathbb{Q}, <)$.

Remark 4.9:
A similar proof shows that if $|\mathrm{S}_x(T)| > \aleph_0$, then $|\mathrm{S}_x(T)| = 2^{\aleph_0}$

4.3. ω-categorical theories

10/03

Recall that a theory is ω-categorical if it has a unique countable model up to isomorphism. The following theorem characterizes ω-categorical by properties of their space of types or their automorphism group. For all finite tuple of variables x and structure M, the action of $\mathrm{Aut}(M)$ on M^x is given by $\sigma \star a = \sigma(a)$. A subset $X \in M^x$ is said to be $\mathrm{Aut}(M)$-invariant if for all $\sigma \in \mathrm{Aut}(M)$, $\sigma(X) = X$.

Theorem 4.10 (Engeler,Ryll–Nardzewski,Svenonius)**:**

Assume $|\mathcal{L}| = \aleph_0$. Let T be a complete theory with infinite models. The following are equivalent:

- *(i) T is ω-categorical;*
- *(ii) for every finite x, any $p \in \mathrm{S}_x(T)$ is isolated;*
- *(ii') for every finite x, $|\mathrm{S}_x(T)| < \infty$;*
- *(ii") for every finite x, there are only finitely many $\equiv$-classes of formulas $\varphi(x)$, where $\varphi(x) \equiv \psi(x)$ if $T \models \forall x\ \varphi(x) \leftrightarrow \psi(x)$;*
- *(iii) Any countable $M \models T$ is saturated.*
- *(iv) for any countable $M \models T$ and any finite x, the action of $\mathrm{Aut}(M)$ on M^x has finitely many orbits.*
- *(iv') there exists a countable $M \models T$ such that, for all finite x, the action of $\mathrm{Aut}(M)$ on M^x has finitely many orbits.*
- *(v) for any countable $M \models T$ and $X \subseteq M^x$, if X is $\mathrm{Aut}(M)$-invariant, X is $\varnothing$-definable.*
- *(v') there exists a countable $M \models T$ such that any $\mathrm{Aut}(M)$-invariant $X \subseteq M^x$ is $\varnothing$-definable.*
- *(vi) there exists a countable $M \models T$ realizing only finitely many types in $\mathrm{S}_x(T)$, for any finite x.*

An action with finitely many orbits on any Cartesian power (as in (iv)), is called an oligomorphic action.

Proof.

(i)⇒(ii) If there exists $p \in \mathrm{S}_x(T)$ non-isolated, then, by compactness (and downwards Löwenheim-Skolem), we find $M \models T$ countable realizing p. By the omitting type theorem (cf. **??**), we also find $N \models T$ countable omitting p. The structures M and N cannot be isomorphic and hence T is not ω-categorical.

(ii)⇒(ii') By (ii) $\mathrm{S}_x(T) = \bigcup_{p \in \mathrm{S}_x(T)} \{p\}$ is an open cover. By compactness, $\mathrm{S}_x(T)$ is covered by finitely many of these open, i.e. it is finite.

(ii')⇒(ii) Finite Hausdorff topological spaces are discrete. Indeed, for every $p \neq q \in \mathrm{S}_x(T)$, there exits an open $U_{p,q}$ such that $p \in U_{p,q}$ but $q \notin U_{p,q}$. Then $\{p\} = \bigcup_{q \neq p} U_{p,q}$ is open.

(ii')⇒(ii") We have that $\varphi \equiv \psi$ if and only if $\langle\varphi\rangle = \langle\psi\rangle \subseteq \mathrm{S}_x(T)$. It follows that $2^{|\mathrm{S}_x(T)|}$ is a bound on the set of $\equiv$-classes.

(ii")⇒(ii') If $\varphi \equiv \psi$, then for any $p \in \mathrm{S}_x(T)$, $\varphi \in p$ if and only if $\psi \in p$. It follows that $2^{|\{\varphi/\equiv : \varphi(x) \text{ formula}\}|}$ is a bound on $|\mathrm{S}_x(T)|$.

(ii)⇒(iii) By (ii) every model of T is atomic. We saw (cf. **??**) that atomic models are $\aleph_0$-homogeneous and hence countable models of T are homogeneous. Let $M \vDash T$, $a \in M^y$, for some finite y, and $p \in \mathrm{S}_x(a)$. Let $q(x,y) = \{\varphi(x,y) : \varphi(x,a) \in p\} \in \mathrm{S}_{x,y}(T)$. By (ii), q is isolated by some formula $\varphi(x,y)$. Since p is finitely satisfiable and $\varphi(x,a) \in p$ by definition of q, we can find $c \in M^x$ such that $M \vDash \varphi(c,a)$. But then $\mathrm{tp}(ac) = q$ and, by definition of q, $\mathrm{tp}(c/a) = p$.

(iii)⇒(i) By **??**, countable saturated models of T are isomorphic.

(iii)∧(ii')⇒(iv) Let $M \vDash T$. Since M is saturated — and hence homogeneous —, any $a, c \in M^x$ are in the same orbit of $\mathrm{Aut}(M)$ if and only if they have the same type. So $|\mathrm{S}_x(T)|$ bounds the number of orbits.

(iv)⇒(iv') This is obvious.

(iii)∧(iv)⇒(v) As stated above, each orbit of $\mathrm{Aut}(M)$ is the set of realization of some $p \in \mathrm{S}_x(T)$. Since p is isolated, there exists $\varphi(x)$ such that $a \in M^x$ realizes p if and only if $M \vDash \varphi(a)$. Every orbit is, therefore, a definable set. Since an invariant $X \subseteq M^x$ is a union of orbits and there at most finitely many orbits in that union, X is $\varnothing$-definable.

(v)⇒(v') This is obvious.

(iv')⇒(vi) Let M be as in iv'. Since all the tuples in an orbit of $\mathrm{Aut}(M)$ on M^x have the same type, the number of orbits bounds the number of types realized in M.

(v')⇒(vi) Let $M \vDash T$ and let us assume that there exists distinct types $(p_i)_{i\in\omega} \in \mathrm{S}_x(T)$ all realized in M. For every $Y \subseteq \omega$, let $X = \{a \vDash p_i : i \in Y\}$. Since the sets of realizations of p_i are $\mathrm{Aut}(M)$-invariant, X is $\mathrm{Aut}(M)$-invariant. It follows that there are $2^{\aleph_0}$ distinct $\mathrm{Aut}(M)$-invariant subsets of M^x. But there are only $\aleph_0$ formulas $\varphi(x)$, so they cannot all be definable.

(vi)⇒(ii') Let M be as in (vi). The set $R = \{p : \exists a \in M^x \ M \vDash p(a)\} \subseteq \mathrm{S}_x(M)$ is dense in $\mathrm{S}_x(T)$. Indeed any formula φ such that $T \vDash \exists x\ \varphi(x)$ has a realization in M. Since $\mathrm{S}_X(T)$ is Hausdorff, singletons are closed and hence R is closed. So $\mathrm{S}_X(T) = R$ is finite. ■

Example 4.11:

1. The theory DLO is ω-categorical. There is one 1-type and three 2-types.
2. Infinite sets without structure are ω-categorical.
3. If k is finite, $k-\mathrm{VS}$ are ω-categorical. If k is finite, they are not. There are ω non isomorphic countable models.

Definition 4.12:
Let M be a structure, $A \subseteq M$ and $c \in M$. We say that c is algebraic over A if there exists a formula $\varphi(x,y)$ and $a \in A^y$ such that $|\varphi(M,a)| < \infty$ and $M \vDash \varphi(c,a)$. We define the algebraic closure of A to be $\operatorname{acl}(A) := \{c \in M : c$ *algebraic over* $A\}$.

Proposition 4.13:
Let T be ω-categorical. Then, for all finite x and variable y, there exist $m \in \mathbb{Z}_{\geqslant 0}$ such that for all $M \vDash T$ and $a \in M^x$, $|\operatorname{acl}(a)^y| \leqslant m$.

Proof. Fix a tuple x, a variable y. By **??**, $r = |\mathrm{S}_{yx}(T)| < \infty$ and every type in $\mathrm{S}_{yx}(T)$. For every $p \in \mathrm{S}_y(T)$ let m_p be the minimal, if it exists — otherwise let $m_p = 0$—, such that there exists $\varphi(y,x)$ with $\forall y_0 \dots y_{m_p} \bigwedge_i \varphi(y_i,x) \to \bigvee_{i \neq j} y_i = y_j$ in p. Let $m := \sum_p m_p$. Pick any $a \in M^x$ and $b \in \operatorname{acl}(a)^y$ and let $p = \operatorname{tp}(ba)$. Then $m_p > 0$ and there are at most m_p b' such that $\operatorname{tp}(b'a) = p$. It follows that $|\operatorname{acl}(a)^y| \leqslant m$. ■

Corollary 4.14:
Let K be an infinite field. Then, in any language containing the ring language, $\operatorname{Th}(K)$ *is not ω-categorical.*

Proof. By compactness, there exists $t \in N \succcurlyeq K$ transcendent over the prime field. Then $\langle t \rangle$ contains (an isomorphic copy of) the ring of polynomials over the prime field, which is not finite. ■

There is a very weak converse to **??**:

Proposition 4.15:
Assume $\mathcal{L}$ is a finite language . Let T be a complete theory that eliminates quantifiers. Assume that for all finite tuple $n \in \mathbb{Z}_{\geqslant 0}$, there exist $m \in \mathbb{Z}_{\geqslant 0}$ such that for all $M \vDash T$ and $A \subseteq M$ with $|A| \leqslant n$, $|\langle A \rangle| \leqslant m$. Then, T is ω-categorical.

Proof. Since T elimiates quantifiers, the type of a tuple a is entirely determined by the $\mathcal{L}(a)$-isomorphism type of $\langle a \rangle$. Since the language is finite there are is finite number $k_{m,x}$ of $\mathcal{L}(c)$-structures of size m where c is a new tuple of constants sorted as x. Fix a finite tuple of variables x, then, by hypothesis, the structure generated by any tuple sorted as x has size bounded by some m. There are, therefore, at most $\sum_{r \leqslant m} k_{r,x}$ elements in $\mathrm{S}_x(T)$. ■

5. Elimination of imaginaries

10/04

6. Morley rank

11/14

Proposition 6.1:
Let $\varphi(x,y)$ be a formula and $a \in M^y$, $b \in M^y$ be such that $\operatorname{tp}(a) = \operatorname{tp}(b)$. *Then* $\operatorname{MR}(\varphi(x,a)) = \operatorname{MR}(\varphi(x,b))$.

Proof. ■

Remark 6.2:
If M is $\aleph_0$-saturated, then we can compute the Morley rank of any $\mathcal{L}(M)$-formula using onky parameters from M.

Definition 6.3:
Let α be an ordinal. We say that an $\mathcal{L}(M)$-formula φ is α-strongly minimal if for all $N \succcurlyeq M$ and $\mathcal{L}(N)$-formula $\psi(x)$, $\mathrm{MR}(\varphi \wedge \psi) < \alpha$ *or* $\mathrm{MR}(\varphi \wedge \neg\psi) < \alpha$.

Definition 6.4:
Let α be an ordinal and let $\varphi(x)$ and $\psi(x)$ be $\mathcal{L}(M)$-formulas. We write $\varphi(x) \sim_\alpha \psi(x)$ (and we say φ and ψ are α-equivalent) if $\mathrm{MR}(\varphi\Delta\psi) < \alpha$, *where* $\varphi\Delta\psi = \varphi\vee\psi\wedge(\neg(\varphi\wedge\psi))$.

Proposition 6.5:
Let M be $\aleph_0$-saturated and φ be an $\mathcal{L}(M)$-formula. Assume that $\mathrm{MR}(\varphi(x)) = \alpha < \infty$. *Then*

(i) *there exists pairwise inconsistent $\mathcal{L}(M)$-formulas $\varphi_i(x)$ for $0 \leqslant i < n$ such that φ_i is α-strongly minimal and $M \vDash \forall x\ (\varphi \leftrightarrow \bigvee_i \varphi_i)$;*

(ii) *Moreover, if there are pairewise inconsistent $\mathcal{L}(M)$-formulas $\psi_j(x)$ for $0 \leqslant j < m$ such that ψ_j is α-strongly minimal and $M \vDash \forall x\ (\varphi \leftrightarrow \bigvee_j \varphi_j)$, then $m = n$ and up to permutation of the ψ_i, we have that $\psi_i \sim_\alpha \varphi_i$.*

The integer n is called the Morley degree of φ. We write $\mathrm{MD}(\varphi) = n$ and $\mathrm{MRD}(\varphi) = (\alpha, n)$

Proof. ∎

Lemma 6.6:
Let φ and ψ be $\mathcal{L}(M)$-formulas.

(i) *If* $\mathrm{MR}(\varphi) < \mathrm{MR}(\psi) < \infty$, *then* $\mathrm{MD}(\varphi \vee \psi) = \mathrm{MD}(\psi)$;

(ii) *If* $\mathrm{MR}(\varphi) = \mathrm{MR}(\psi) < \infty$ *and they are inconsistent, then* $\mathrm{MD}(\varphi \vee \psi) = \mathrm{MD}(\varphi) + \mathrm{MD}(\psi)$

Proof. ∎

6.1. Morley rank is strongly minimal theories

11/17

Proposition 6.7:
Let $A \subseteq M$, and $a, b \in M$ be tuples.

1. *If* $b \in \mathrm{acl}(Aa)$, *then* $\mathrm{MR}(a/A) \geqslant \mathrm{MR}(b/A)$;
2. $\mathrm{MR}(ab/A) \geqslant \mathrm{MR}(a)$.

Proof. We prove ∎

A. Set theoretic preliminaries

We will need a few set theoretical notions in the beginning. Here is what you should know.

A.1. Ordinals

A well-order is a total order in which every subset has a minimal element. An initial segment of an ordered set $(X,<)$ is a set $Y \subseteq X$ such that for all $y \in Y$ and $x \in X$, if $x \leqslant y$, then $x \in Y$.
The class of ordinals is a class of representative for the isomorphism classes of well-ordered set. It does not really matter how we build them. We say that an ordinal α is smaller than an ordinal β if α is isomorphic to an initial segment of β. This is a well-order[1]. If α is an ordinal $\alpha+1$ is the ordinal isomorphic to $\alpha \cup \{\star\}$ where $\star$ is a new element larger than all the elements in α. It is smallest ordinal strictly above α. If $(\alpha_i)_{i\in I}$ is a chain of ordinals — i.e. I is totally ordered and, for all $i \leqslant j \in I$, $\alpha_i \leqslant \alpha_j$, then $\bigcup_{i\in I} \alpha_i$ denotes the smallest upper bound of the set $\{\alpha_i : i \in I\}$.
There is an induction principle on ordinals. If for all ordinal α some property (or construction) holding for all $\beta < \alpha$ implies that it holds for α, then that property (or construction) holds for all ordinal. In particular, since any ordinal is either successor — of the form $\alpha+1$ — or limit — the smallest upper bound of all strictly smaller ordinals —, to prove that a property holds of all ordinals, it suffices to show that if it holds at α then it holds at $\alpha+1$ and if it holds on all ordinals strictly smaller than some limit α, then it holds at α.
The smallest infinite ordinal is denoted ω, it is isomorphic $\mathbb{Z}_{\geqslant 0}$.

A.2. Cardinals

The cardinals are all the ordinals that are not in bijection with any of their initial segments. By the axiom of choice, any set X can be well-ordered and hence is in bijection with a unique cardinal $\kappa =: |X|$. Any cardinal κ has a cardinal successor — the smallest cardinal λ such that $\lambda > \kappa$ — denoted κ^+. Note that unless κ is finite, $\kappa+1$, the ordinal successor of κ is not a cardinal and hence κ^+ is not $\kappa+1$. Similarly given a chain of cardinals $(\kappa_i)_{i\in I}$, their upper bound $\bigcup_i \kappa_i$ is a cardinal.
Given cardinals κ and λ, $\kappa+\lambda$ denotes the cardinal of their disjoint union, with is also the cardinal of their product, which is equal to their maximum. Also, 2^κ denotes the cardinal of the power set of κ. We know that $\kappa < 2^\kappa$ and the generalized continuum hypothesis states that $2^\kappa = \kappa^+$ for all cardinal κ.
We define, by induction, the following function from ordinals to infinite cardinals which is strictly increasing and onto: $\aleph_0 = \omega$, for all α, $\aleph_{\alpha+1} = \aleph_\alpha^+$ and for all limit β, $\aleph_\beta = \bigcup_{\alpha<\beta} \aleph_\alpha$.

[1] To be precise, this is not a set, it is only a class, but we can safely ignore that distinction here.

A.3. Cofinality

Let $f : X \to Y$ be a function into some ordered set Y. We say that f is cofinal in Y if for all $y \in Y$, there exists $x \in X$ such that $f(x) \geqslant y$. The cofinality of an ordered set X is the largest ordinal α such that there is a cofinal map from α into X. The cofinality of X is a cardinal denoted $\mathrm{cof}(X)$.
A cardinal κ is said to be regular if $\kappa = \mathrm{cof}(\kappa)$. Every successor cardinal is regular.

B. General topology

Definition B.1:
A topological space is a set X with a collection of $\mathcal{O}$ of subsets of X such that $X, \varnothing \in \mathcal{O}$ and $\mathcal{O}$ is closed finite intersections and arbitrary unions.

Subsets of $\mathcal{O}$ are called open sets and complement of subsets of $\mathcal{O}$ are called closed sets.

Definition B.2:
A basis of open sets is a collection of $\mathcal{B}$ of subsets of X which is closed under finite union and finite intersection.
Given a basis of open sets $\mathcal{B}$ we can consider the topology on X whose open sets are arbitrary unions of elements of $\mathcal{B}$ — it is called the topology generated by $\mathcal{B}$.

For example the usual topology on $\mathbb{R}$ is the topology whose basis of open sets is the open intervals.

Definition B.3:
Let X be a topological space.

(i) X is Hausdorff if for all $x, y \in X$, there are open sets U and V such that $U \cap V = \varnothing$, $x \in U$ and $y \in V$.

(ii) X is compact if whenever $X = \bigcup_{i \in I} U_i$ where the U_i are open, there exists a finite $I_0 \subseteq I$ such that $X = \bigcup_{i \in I_0} U_i$.

(iii) A Hausdorff compact topological space X is totally disconnected if for all x, $y \in X$, there exists a clopen set U such that $x \notin U$ and $y \notin U$.

Definition B.4:
Let X and Y be two topological spaces, a map $f : X \to Y$ is continuous if for any open $U \subseteq Y$, $f^{-1}(U) \subseteq X$ is open.