Imaginaries, invariant types and pseudo-p-adically closed fields

Samaria Montenegro* Silvain Rideau*

June 29, 2018

In this paper, we give a very general criterion for elimination of imaginaries using an abstract independent relation. We also study germs of definable functions at certain well-behaved invariant types. Finally we apply these results to prove the elimination of imaginaries in bounded pseudo-p-adically closed fields.

Introduction

Elimination of imaginaries is a positive answer to the question of finding definable moduli spaces for every definable family of definable sets; that is, given definable sets $X \subseteq Y \times Z$, finding a definable map $f\colon Y \to W$ such that for all $y_1, y_2 \in Y$, $f(y_1) = f(y_2)$ if and only if the fiber of X above y_1 is equal to the one above y_2. This is equivalent to the existence, for every set defined with parameters, of a smallest set of definition. In recent work of Hrushovksi [Hru14] on the elimination of imaginaries in algebraically closed valued fields, the focus is shifted to the local question of finding smallest sets of definition for definable types — and proving that there are enough definable types to deduce elimination of imaginaries.

But there are many structures of interest, among which the field of p-adic numbers, where there are too few definable types for this local approach to work. It is therefore tempting to work with invariant types instead. This presents a number of issues, the most basic being that, generally, invariant types cannot have smallest sets of definition due to their fundamentally infinitary nature. Some of them, for example, encode the cofinality of an infinite ordered set. In this paper, we choose focus on a class of tractable invariant types: those that are arbitrarily close to definable types; the set $\overline{\mathcal{D}}(M/A)$ of Definition **(1.1)**. The first part of this paper consists in providing the tools for an approach to elimination of imaginaries based on these definably approximable types.

The most important technical issue is the understanding of germs of definable functions at definably approximable types. At general invariant types, these germs are complicated hyperimaginaries. However, we show that, at a definably approximable type, germs of definable functions are encoded by the cofinality of a filtered ordered set of imaginary points. This

*Partially supported by ValCoMo (ANR-13-BS01-0006)

can then used to encode germs of functions in a theory using elimination of imaginaries in a reduct, cf. Proposition **(1.8)**. We also give necessary and sufficient conditions for every invariant type to be definably approximable — equivalently, in an NIP theory, for every non forking formula to contain a definable type. In dp-minimal theories, Simon and Starchenko [SS14] give a sufficient conditions for this density result to hold over models. We show that for the density result to hold over arbitrary algebraically closed basis, it suffices for the result to hold over models and for every definable set to contain a type definable almost over its code. In particular, the density result holds in algebraically closed valued fields (including imaginaries).
The other abstract contribution of this paper is to provide a very general criterion for weak elimination of imaginaries. Many proofs of elimination of imaginaries in tame unstable contexts follow the outline of Hrushovksi's approach in bounded pseudo-algebraically closed fields [Hru02, Proposition 3.2]. So it seemed interesting to isolate an abstract criterion pinpointing the exact ingredients of this proof. We show that if one can find an independence relation satisfying a strong form of extension and a certain amalgamation result, then weak elimination of imaginaries follows. For certain choices of independence relation (for example definable independence or invariant independence), this criterion reduces to previously known criteria of Hrushovski [Hru14, Lemma 1.17] or the second author [Rid, Proposition 9.1]. For other choices of independence relation, one recovers a large number of previously known elimination of imaginaries proofs, for examples the one in algebraically closed fields with a generic automorphism [CH99, §1.10].

The second part of this paper is devoted to applying these general methods to the class of bounded pseudo-p-adically closed fields. Note that the approach presented in this paper does not rely on the elimination of imaginaries in p-adic fields but rather reproves it, providing us with a new, slightly different, proof of [HMR, Theorem 2.6]. The new proof focuses, from the start, on constructing invariant types rather than reducing the problem to the elimination of unary imaginaries.
A field is said to be pseudo-algebraically closed if every absolutely irreducible variety over this field has a rational point. This class of fields first appeared in Ax's work on pseudo-finite fields [Ax68]. Their model theory has since been extensively studied, in parallel with the development of simplicity. Indeed bounded pseudo-algebraically closed fields — those that have only finitely many extensions of any given degree — provide the main examples of simple unstable fields. As interest in less restrictive tameness notions grew, for example notions like NTP_2 that do not preclude the existence of any definable order, it also became important to find algebraic examples, in particular enriched fields, that would provide us with study cases.
In [Mon17a] and [Mon17b], the first author thus started a neo-stability flavored study of two classes of large fields extending the class of pseudo-algebraically closed fields: pseudo-p-adically closed fields and pseudo-real closed fields. Those two classes consist of the fields over which any absolutely irreducible variety with a simple point over every p-adically closed (respectively real closed) extension has a rational point. These classes were defined by Basarav, Prestel, Grob, Jarden and Haran over 30 years ago in [Bas84; Pre82; Gro87; HJ88] and a number of their model theoretic properties, pertaining to model completeness and the description of types, had been worked out. In [Mon17b], the first author provided new tools to

study these fields, mostly in the bounded case, and proved a number of classification results. One particularity that stands out in the first author's work is that, although most results are proved for both classes, elimination of imaginaries was only proved for pseudo-real closed fields [Mon17a]. The initial motivation for this paper was to repair that asymmetry.

Since bounded pseudo-p-adically closed fields that are not pseudo-algebraically closed fields come with finitely many definable valuations, one cannot expect elimination of imaginaries in a language with just one sort for the field, contrary to what happens with pseudo-real closed fields. But since the work of [HHM06], we know how to circumvent that particular issue: we have to add, for each valuation, codes for certain definable modules over the valuation ring; that is, work in the so-called geometric language. The main question regarding the imaginaries in bounded pseudo-p-adically closed fields then becomes to prove that there are no imaginaries arising from the interaction between the various valuations and, therefore, that it suffices to add the geometric sorts for each of the valuations. The main result of this paper, Theorem **(2.25)**, is a positive answer to this question.

Our first step towards this result, and a core ingredient of the rest of the paper, is Proposition **(2.29)** which states that not only are the geometric sorts for each valuation orthogonal but also that the structure of any given geometric sort is the one induced by the relevant p-adic closure. We then proceed to deduce, from this strong statement on the independence of the valuations, a result on the structure of definable subsets of the valued field, where, at first sight, the valuations do interact. Since we are unable, in pseudo-p-adic fields, to give a canonical version of the first author's density result [Mon17b, Theorem 6.11], we choose to work at the level of types by proving, in Theorem **(2.44)**, that every type over algebraically closed sets of imaginary parameters is consistent with a global quantifier free type which is invariant over the geometric part of the parameters. The last ingredient needed to prove weak elimination of imaginaries is to extend the first author's amalgamation result [Mon17b, Theorem 3.21] to allow geometric parameters, cf. Theorem **(2.55)**. This in particular required proving, in Theorem **(2.16)**, existential closedness of pseudo-p-adicaly closed fields in a stronger language containing Macintyre's language for each p-adic valuation.

As is often the case, coding finite sets is mostly an independent issue. Here, we deduce it from the coding of finite sets in algebraically closed fields equipped with finitely many independent valuations. Our approached, inspired by Johnson's account [Joh, §6.2] of the coding of finite sets in algebraically closed valued fields, consists in first lifting tuples of geometric points by generically stable types of the valued field and then using the code of these types to encode finite sets.

The organization of the paper is as follows. Section 1 contains the more abstract results and Section 2 is devoted to pseudo-p-adically closed fields. We start, in Section 1.1, by recalling definitions related to imaginaries and their elimination. In Section 1.2, we consider invariant types that can be approximated by definable types and the germs of definable functions over such types. We conclude the abstract part of the paper, in Section 1.3, by proving a criterion for weak elimination of imaginaries and discussing various specific cases of that criterion.

Preliminaries on the model theory of valued fields and pseudo-p-adically closed fields can be found in Section 2.1. Section 2.2 contains the main theorem and various details regarding the language that we will be using. The orthogonality and purity of the geometric sorts is proved in Section 2.3. Section 2.4 is devoted to the description of the algebraic closure in bounded pseudo-p-adically closed fields, including geometric imaginaries, and in Section 2.5, we prove the existence of quantifier free invariant extensions of types over algebraically closed bases. Amalgamation over geometric points is proved in Section 2.6. Finally, in Section 2.7, we code finite sets.

The authors would like to thank Gabriel Conant for giving them the idea of working with an abstract independence relation in the criterion for elimination of imaginaries.

1 Imaginaries and germs of functions

1.1 Preliminaries

We will assume knowledge of standard model theoretic knowledge and notation. We refer the reader to [TZ12] for an introduction to model theory.

Let us start by recalling the basic definitions regarding elimination of imaginaries. Let X be a definable set in some $\mathcal{L}$-structure M. The tuple $b \in M$ is a canonical parameter for X (via the $\mathcal{L}$-formula $\phi(x, y)$) if for all tuple $b' \in M$, $X = \phi(M, b')$ if and only if $b' = b$. An $\mathcal{L}$-theory T is said to eliminate imaginaries if every definable set in any model of T has a canonical parameter (via some $\mathcal{L}$-formula). Equivalently, if T has a sufficiently many constants, T eliminates imaginaries if and only if, for every $\mathcal{L}$-formula $\phi(x, y)$, there exists an $\varnothing$-definable map f such that

$$T \vdash \forall y_1 \forall y_2 \, (f(y_1) = f(y_2) \leftrightarrow (\forall x \, (\phi(x, y_1) \leftrightarrow \phi(x, y_2)))).$$

Note that if ϕ defines an equivalence relation E in T, then we have that $T \vdash \forall x \forall y \, (xEy \leftrightarrow f(x) = f(y))$.

Given any language $\mathcal{L}$, we define the language $\mathcal{L}^{\mathrm{eq}}$ that contains, for each $\mathcal{L}$-formula $\phi(x, y)$, a new sort E_ϕ and a new function symbol f_ϕ from the product of sorts of y to E_ϕ. Given an $\mathcal{L}$-theory T, we define the $\mathcal{L}^{\mathrm{eq}}$-theory

$$T^{\mathrm{eq}} := T \cup \bigcup_\phi \{f_\phi \text{ is onto and } \forall y_1 \forall y_2 \, (f(y_1) = f(y_2) \leftrightarrow (\forall x \, (\phi(x, y_1) \leftrightarrow \phi(x, y_2))))\}.$$

The theory T^{eq} eliminates imaginaries and to any $M \models T$, we can associate a unique $M^{\mathrm{eq}} \models T^{\mathrm{eq}}$ whose reduct to $\mathcal{L}$ is M. We denote by $\mathrm{acl}^{\mathrm{eq}}$ (respectively $\mathrm{dcl}^{\mathrm{eq}}$) the algebraic (respectively definable) closure in M^{eq}. If X is an $\mathcal{L}(M)$-definable set, we define $\ulcorner X \urcorner = \mathrm{dcl}^{\mathrm{eq}}(b) \subseteq M^{\mathrm{eq}}$ for any choice of canonical parameter b of X.

1.2 Invariant types approximated by definable types

In this section, we want to show how being able to approximate invariant types by definable types can be helpful to compute "canonical bases" of invariant types. This will allow us to study the invariance of germs of functions over invariant types. We also give necessary and sufficient conditions, that hold in algebraically closed valued fields, for invariant types to be approximated by definable types.

We denote by $\mathcal{S}(M)$ the type space over M and by $\mathcal{S}_x(M)$ the type space over M in variable x.

Definition 1.1: *Let T be some $\mathcal{L}$-theory and $A \subseteq M \models T$ and $p(x) \in \mathcal{S}_x(M)$.*

1. *We say that p is $\mathrm{Aut}(M/A)$-invariant if for all $\sigma \in \mathrm{Aut}(M/A)$, any $\mathcal{L}$-formula $\phi(x, y)$ and tuple $a \in A$, $\phi(x, a) \in p$ if and only if $\phi(x, \sigma(a)) \in p$.*
2. *The type p is $\mathcal{L}(A)$-definable if for every $\mathcal{L}$-formula $\phi(x, y)$, there is some $\mathcal{L}(A)$-formula $\mathrm{d}_p x\, \phi(x, y) = \theta(y)$ such that for all $a \in A$, $p \vdash \phi(x, a)$ if and only if $M \models \mathrm{d}_p x\, \phi(x, a)$.*

Definition 1.2: *Let T be some $\mathcal{L}$-theory and let $A \subseteq M \models T$. We define:*

1. *$\mathcal{I}_x(M/A) \subseteq \mathcal{S}_x(M)$ the set of $\mathrm{Aut}(M/A)$-invariant types over M;*
2. *$\mathcal{D}_x(M/A) \subseteq \mathcal{S}_x(M)$ the set of $\mathcal{L}(A)$-definable types over M;*
3. *$\overline{\mathcal{D}}_x(M/A)$ the set of types $p \in \mathcal{S}_x(M)$ such that for all $\mathcal{L}(M)$-formula $\phi(x) \in p$, there exists $q \in \mathcal{D}_x(M/A)$ such that $\phi \in q$.*

Remark 1.3: We have $\mathcal{D}(M/A) \subseteq \overline{\mathcal{D}}(M/A) \subseteq \mathcal{I}(M/A)$ and the last two sets are closed in $\mathcal{S}(M)$. Moreover, as the notation indicates, $\overline{\mathcal{D}}(M/A)$ is the closure of $\mathcal{D}(M/A)$.

Fact 1.4: *Let T be some $\mathcal{L}$-theory, let $A \subseteq M \models T$ and let $p \in \overline{\mathcal{D}}_x(M/A)$. Then there exists a sequence $(q_i)_{i \in I} \in \mathcal{D}_x(M/A)$, indexed by a directed set of size at most $2^{|\mathcal{L}(A)|}$, such that for all $\mathcal{L}(M)$-formula $\phi(x)$, $\phi \in p$ if and only if $\phi \in q_i$ for almost all i.*

By almost all i, we mean that it holds for all i greater that some $i_0 \in I$. This is just the characterization of closure by nets.

Definition 1.5: *Let M be an $\mathcal{L}$-structure $p \in \mathcal{S}(M)$ and f, g be $\mathcal{L}(M)$-definable functions, defined at p. We say that f and g have the same p-germ if $p(x) \vdash f(x) = g(x)$. We denote by $[f]_p$ the class of f for this equivalence relation.*

A priori, $[f]_p$ is an hyperimaginary: an equivalence class for an invariant equivalence relation. If p is definable, it can be identified with an imaginary point. The core of the following proposition is that, if $p \in \overline{\mathcal{D}}(M/A)$, then the hyperimaginary $[f]_p$ is coded by the "limit" of a sequence of imaginaries.
In what follows, let T_0 be some $\mathcal{L}_0$-theory eliminating quantifiers and imaginaries whose sorts we denote $\mathcal{R}_0$. Let T be a complete $\mathcal{L}$-theory containing the universal part of T_0. Let $M \models T$ be sufficiently saturated and homogeneous, $A \subseteq M^{\mathrm{eq}}$ be such that $\mathcal{R}_0(\mathrm{acl}_{\mathcal{L}}(A)) \subseteq A$ and $M_0 \models T_0$ containing $\mathcal{R}_0(M)$ be such that any automorphism of M extends to M_0. When we say that something is $\mathcal{L}_0$-definable, it will mean that it is definable in M_0. Assume that:

(†) for all $\epsilon \in \mathrm{dcl}^{M_0}_{\mathcal{L}_0}(\mathcal{R}_0(M))$, there exists a tuple $\eta \in \mathcal{R}_0(M)$ such that ϵ and η are interdefinable in the pair (M_0, M).

Proposition 1.6: *Let $p \in \overline{\mathcal{D}}^{\mathcal{L}_0}_x(M_0/\mathcal{R}_0(A))$ and f be an $\mathcal{L}_0(\mathcal{R}_0(M))$-definable function defined at p. If $[f]_p$ has a finite orbit under $\mathrm{Aut}_{\mathcal{L}}(M/A)$, then it is $\mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$-invariant.*

Proof. Let $(q_i)_{i \in I} \in \mathcal{D}^{\mathcal{L}_0}_x(M_0/\mathcal{R}_0(A))$ be as in Fact **(1.4)** with respect to p and let $\epsilon_i := [f]_{q_i} \in M_0$ and η_i be as in Hypothesis † with respect to ϵ_i. Let $F = (F_m)_m$ be an $\mathcal{L}_0$-definable family of functions such that $f = F_{m_0}$ for some m_0. Note that $p(x) \vdash F_{m_1}(x) = F_{m_2}(x)$ if and only if $q_i \vdash F_{m_1}(x) = F_{m_2}(x)$ for almost all i. In particular, for any $\sigma \in \mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$, $\sigma([f]_p) = [f]_p$ if and only if $\sigma(\epsilon_i) = \epsilon_i$ for almost all i.

Claim 1.7: *$\eta_i \in \mathcal{R}_0(A)$, for almost all $i \in I$.*

Proof. As $\eta_i \in \mathcal{R}_0(M)$, it suffices to prove that $\eta_i \in A$ for almost all i. If not, there is an unbounded subset $\mathcal{J} \subseteq I$ such that for all $j \in \mathcal{J}$, η_j has an infinite $\mathrm{Aut}_{\mathcal{L}}(M/A)$-orbit and hence $\mathrm{Aut}_{\mathcal{L}}(M/A\eta_j)$ has infinite index. By Neumann's lemma, for all choice of $j_1, \ldots, j_n \in \mathcal{J}$, there exists $\tau_1, \ldots \tau_n \in \mathrm{Aut}_{\mathcal{L}}(M/A)$ such that, for all k, the $\tau_l(\eta_{j_k})$ are all distinct. By saturation and homogeneity, there exists $(\tau_l)_{l\in\omega} \in \mathrm{Aut}_{\mathcal{L}}(M/A)$ such that, for all $j \in \mathcal{J}$, the $\tau_l(\eta_j)$ are all distinct and hence, extending τ_l to M_0, so are all the $\tau_l([f]_{q_j})$. Since the set $\mathcal{J}$ is unbounded, it follows that the $\tau_l([f]_p)$ are all distinct, a contradiction. ◊

We have proved that $\mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$ fixes almost all ϵ_i and hence it fixes $[f]_p$. □

Let $\mathcal{H}$ be a subset of $\mathcal{R}_0$. We now also assume that:

($\star$) For all tuple $a \in \mathcal{H}(N)$, where $N \succcurlyeq M$, $\mathcal{R}_0(\mathrm{acl}_{\mathcal{L}}(Ma)) \subseteq \mathrm{acl}_{\mathcal{L}_0}(\mathcal{R}_0(Ma))$.

Proposition 1.8: *Let $a \in \mathcal{H}(N)$, where $N \succcurlyeq M$, and $c \in \mathcal{R}_0(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa))$ be tuples. Assume that $p := \mathrm{tp}_{\mathcal{L}_0}(\mathcal{R}_0(a)/M_0) \in \overline{\mathcal{D}}(M_0/\mathcal{R}_0(A))$ Then, there exists an $\mathcal{L}_0(M)$-definable map F such that $[F]_p$ is $\mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$-invariant and $c \in \mathrm{acl}_{\mathcal{L}_0}(F(a))$.*

Proof. Since $c \in \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa)$, we can find an $\mathcal{L}^{\mathrm{eq}}(A)$-definable function f such that $f(a)$ encodes a finite set containing c. By ($\star$), we have that $f(a) \subseteq \mathcal{R}_0(\mathrm{acl}_{\mathcal{L}}(Ma)) \subseteq \mathrm{acl}_{\mathcal{L}_0}(Ma)$. It follows that we can find an $\mathcal{L}_0(M)$-definable function F such that $F(a)$ encodes (in M_0) a finite set containing $f(a)$. Note that we have $c \in f(a) \subseteq \mathrm{acl}_{\mathcal{L}_0}(F(a))$.
By compactness (and replacing F with a finite union), we may assume that $f(x) \subseteq F(x)$ always holds. The cardinality of F is constant on realizations of p. We may assume it is minimal among all possible F. Let $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M/A)$, then $f(a) \subseteq F(a) \cap F^{\sigma}(a)$. By minimality, we must have $|F(a) \cap F^{\sigma}(a)| = |F(a)| = |F^{\sigma}(a)|$ and hence $F(a) = F^{\sigma}(a)$. We have just proved that $[F]_p$ is $\mathrm{Aut}_{\mathcal{L}}(M/A)$-invariant. By Proposition **(1.6)**, it is, in fact, $\mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$-invariant. □

If T_0 is stable — more precisely if p is generically stable (cf. [ACP14, Definition 1.6]) — we can deduce a stronger consequence of this results that shows quite clearly that what we have been encoding (germs of) functions:

Corollary 1.9: *In the setting of Proposition **(1.8)**, assume p is generically stable. Then*

$$\mathcal{R}_0(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa)) \subseteq \mathrm{acl}_{\mathcal{L}_0}(\mathcal{R}_0(A)a).$$

Proof. Pick any $c \in \mathcal{R}_0(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa)$ and let F be as in Proposition **(1.8)**. Recall that p being generically stable is definable and therefore that $[F]_p \in \mathrm{dcl}_{\mathcal{L}_0}(\mathcal{R}_0(A)) \subseteq \mathcal{R}_0(A)$. Also, by [ACP14, Theorem 2, 2], there exists an $\mathcal{L}_0(\mathcal{R}_0(A))$-definable map G such that $[G]_p = [F]_p$. Thus, $c \in \mathrm{acl}_{\mathcal{L}_0}(F(a)) = \mathrm{acl}_{\mathcal{L}_0}(G(a)) \subseteq \mathrm{acl}_{\mathcal{L}_0}(\mathcal{R}_0(A)a)$. □

For technical reasons, we will need to involve a third intermediary language. Let $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}$, $T_{0,\forall} \subseteq T_{1,\forall} \subseteq T$ and $M \subseteq M_1 \subseteq M_0$ be such that $M_1 \models T_1$ and every automorphism of M_1 extends to an automorphism of M_0. Assume T_1 is NIP.

Corollary 1.10: *Let $N \succcurlyeq M$ and $a \in \mathcal{H}(N)$ a tuple. If $p := \mathrm{tp}_{\mathcal{L}_0}(\mathcal{R}_0(a)/M_0) \in \overline{\mathcal{D}}(M_0/\mathcal{R}_0(A))$ and $q := \mathrm{tp}_{\mathcal{L}_1}(\mathcal{R}_0(a)/M_1) \in \mathcal{I}(M_1/\mathcal{R}_0(A))$, then $\mathrm{tp}_{\mathcal{L}_1}(\mathcal{R}_0(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa))/M_1) \in \mathcal{I}(M_1/\mathcal{R}_0(A))$.*

Proof. By Proposition **(1.8)**, we can find F such that $[F]_p$ is $\mathrm{Aut}_{\mathcal{L}_0}(M_0/\mathcal{R}_0(A))$-invariant and $\mathcal{R}_0(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}}(Aa)) \subseteq \mathrm{acl}_{\mathcal{L}_0}(F(a)) \subseteq \mathrm{acl}_{\mathcal{L}_1}(F(a))$. Since automorphisms of M_1 extend to automorphisms of M_0, $[F]_q$ is $\mathrm{Aut}_{\mathcal{L}_1}(M_1/\mathcal{R}_0(A))$-invariant and hence $\mathrm{tp}_{\mathcal{L}_1}(F(a)/M_1) \in \mathcal{I}(M_1/\mathcal{R}_0(A))$. The corollary now follows from [HP11, Lemma 2.12]. □

Note that, in the above corollary, we only build an invariant type and not, *a priori*, a definably approximable one. Let us therefore conclude with examples of theories in which $\mathcal{I}(M/A) = \overline{\mathcal{D}}(M/A)$, in which case we will be able to continue with the induction and build invariant types in all arities, cf. Theorem **(2.44)**.

Proposition 1.11: *Let T be an $\mathcal{L}$-theory that weakly eliminates imaginaries and M be sufficiently saturated. Assume:*
(i) *for all $N \preccurlyeq M \models T$, $\mathcal{I}(M/N) = \overline{\mathcal{D}}(M/N)$;*
(ii) *for all $A = \mathrm{acl}(A) \subseteq M \models T$ any unary $\mathcal{L}(A)$-definable set is consistent with a type $p \in \mathcal{D}(M/A)$.*

Then, for all $A = \mathrm{acl}(A) \subseteq M \models T$, $\mathcal{I}(M/A) = \overline{\mathcal{D}}(M/A)$.

Note that the converse is obvious.

Proof. Let us start with some classic lemmas.

Claim 1.12: *Let $A = \mathrm{acl}(A) \subseteq M$. Assume $\mathrm{tp}(a/M) \in \mathcal{D}(M/A)$ and $c \subseteq \mathrm{acl}(Aa)$, then $\mathrm{tp}(c/M) \in \mathcal{D}(M/A)$.*

Proof. We may assume that c is a finite tuple. Let $\phi(y,x)$ be an $\mathcal{L}(A)$-formula such that $\phi(y,a)$ algebrizes c over Aa. Let $p = \mathrm{tp}(a/M)$ and pick $\psi(y,z)$ any $\mathcal{L}$-formula. We define $z_1 E z_2$ to hold if $M \models \mathrm{d}_p x\,(\forall y\,\phi(y,x) \to (\psi(y,z_1) \leftrightarrow \psi(y,z_2)))$. This is an $\mathcal{L}(A)$-definable finite equivalence and its classes are $\mathcal{L}(A)$-definable by elimination of imaginaries. Moreover, $\phi(c,M)$ is a finite union of E-classes. It is therefore $\mathcal{L}(A)$-definable. ◊

Claim 1.13: *Pick $A \subseteq N \preccurlyeq M$, $p \in \mathcal{D}(N/A)$, $a \models p$ in M, $q \in \mathcal{D}(M/Aa)$ and $c \models q$. Then $\mathrm{tp}(ac/N) \in \mathcal{D}(N/A)$.*

Proof. Pick $\psi(x,y,z)$ any $\mathcal{L}$-formula. Let $\theta(y,z,a)$ be an $\mathcal{L}(A)$-formula such that, for all b, $d \in M$, $\psi(b,y,d) \in q$ if and only if $M \models \theta(b,d,a)$. Then, for all $b \in N$, $M \models \psi(a,c,b)$ if an only if $M \models \mathrm{d}_p x\,\theta(x,b,x)$. ◊

Claim 1.14: *For all $N \preccurlyeq M$ containing A, there exists $N^\star \preccurlyeq M$ containing A such that $\mathrm{tp}(N^\star/N) \in \mathcal{D}(N/A)$.*

Proof. Let $\{\phi_i(x_i) : i \in \kappa\}$ be an enumeration of all consistent $\mathcal{L}(A)$-formulas in one variable x_i. By induction, for all i, we build $a_i \in M$ such that $M \models \phi_i(a_i)$, $c_i = \mathrm{acl}(Ac_{<i}a_i)$ and $\mathrm{tp}(c_i/N) \in \mathcal{D}(N/A)$. The element a_i is found by Hypothesis **1.11.(ii)** applied to $Ac_{<i} = \mathrm{acl}(Ac_{<i})$. Then by Claim **(1.13)**, $\mathrm{tp}(c_{<i}a_i/N) \in \mathcal{D}(N/A)$. Let $c_i = \mathrm{acl}(Ac_{<i}a_i)$. By Claim **(1.12)**, $\mathrm{tp}(c_i/N) \in \mathcal{D}(N/A)$. Let $d_0 = A$ and $d_1 = c_{<\kappa} = \mathrm{acl}(Ad_1)$. By repeating the above construction, we obtain $(d_i)_{i\in\omega}$ such that, for all $i \in \omega$, $\mathrm{tp}(d_i/N) \in \mathcal{D}(N/A)$ and any consistent $\mathcal{L}(d_{<i})$-formula in one variable is realized in d_i. By Tarski-Vaught, $d_{<\omega} \preccurlyeq M$. Moreover, $A \subseteq d_{<\omega}$ and $\mathrm{tp}(d_{<\omega}/N) \in \mathcal{D}(N/A)$. ◊

Pick $p \in \mathcal{I}(M/A)$. We have to show that for all $\mathcal{L}(M)$-formula $\phi(x)$, if $p \vdash \phi$, then there exists $q \in \mathcal{D}(M/A)$ such that $q \vdash \phi$. Actually, it suffices to find such a $q \in \mathcal{D}(N/A)$ for some $N \preccurlyeq M$ containing A and such that $\phi \in \mathcal{L}(N)$. By Claim **(1.14)**, we find $A \subseteq N^\star \subseteq M$ such that $\mathrm{tp}(N^\star/N) \in \mathcal{D}(N/A)$. By Hypothesis **1.11.(i)**, we find $q \in \mathcal{D}(M/N^\star)$ consistent with ϕ. Let $c \vDash q$, by Claim **(1.13)**, $\mathrm{tp}(N^\star c/N) \in \mathcal{D}(N/A)$. In particular, $\phi \in q|_N = \mathrm{tp}(c/N) \in \mathcal{D}(N/A)$. □

Remark 1.15: Let $\mathcal{H}$ be a set of dominant sorts in T, i.e. any other sort is the image of a $\varnothing$-definable function whose domain is a product of sorts in $\mathcal{H}$. Then, in Proposition **(1.11)**, it suffices to assume Hypothesis **1.11.(ii)** for definable subsets of a single sort in $\mathcal{H}$.

Corollary 1.16: *Let T be any C-minimal theory which weakly eliminates imaginaries. Then, for all $A = \mathrm{acl}(A) \subseteq M \vDash T$ sufficiently saturated, $\mathcal{I}(M/A) = \overline{\mathcal{D}}(M/A)$.*

Proof. Hypothesis **1.11.(ii)** holds for definable subsets of $\mathbf{K}$ in C-minimal theories by taking the generic type of any outer ball of the swiss cheese decomposition. Hypothesis **1.11.(i)** then follows by [SS14, Theorem 5 and Lemma 6]. □

1.3 A criterion using amalgamation

The following criterion is an attempt at an abstract account of the proof given by Hrushovski in [Hru02, Proposition 3.2] and adapted in many various settings since then. Let T be an $\mathcal{L}$-theory with sorts $\mathcal{R}$ and $M \vDash T$ be sufficiently saturated and homogeneous.

Proposition 1.17: *Let $\downarrow$ be a ternary relation on small subsets of M. Assume:*
- *(i) for all $E = \mathrm{acl}^{\mathrm{eq}}(E) \subseteq M^{\mathrm{eq}}$, tuple $a \in M$ and $C \subseteq M$, there exists $a^\star$ such that $a^\star \equiv_{\mathcal{L}(E)} a$ and $a^\star \downarrow_{\mathcal{R}(E)} C$;*
- *(ii) for all $E = \mathrm{acl}(E) \subseteq M$ and tuples $a, b, c \in M$, if $b \downarrow_E a$, $c \downarrow_E ab$ and $a \equiv_{\mathcal{L}(E)} b$, then there exists $c^\star$ such that $ac \equiv_{\mathcal{L}(E)} ac^\star \equiv_{\mathcal{L}(E)} bc^\star$.*

Then T weakly eliminates imaginaries.

Proof. Pick any $e \in M^{\mathrm{eq}}$. There exists an $\varnothing$-definable map f and a tuple $a \in M$ such that $e = f(a)$. Let $\overline{E} = \mathrm{acl}^{\mathrm{eq}}(e)$ and $E = \mathcal{R}(\overline{E})$. We want $e \in \mathrm{dcl}^{\mathrm{eq}}(E)$. It suffices to prove that $\mathrm{tp}(a/E) \vdash f(a) = e$.
Pick any $b \equiv_{\mathcal{L}(E)} a$. By Hypothesis (i), there exists $b^\star \equiv_{\mathcal{L}(\overline{E})} b$ such that $b^\star \downarrow_E a$. Applying Hypothesis (i) again, we find $c \equiv_{\mathcal{L}(\overline{E})} a$ such that $c \downarrow_E ab$. Note that we have $f(c) = e = f(a)$. Now, applying Hypothesis (ii), we find $c^\star$ such that $ac \equiv_{\mathcal{L}(E)} ac^\star \equiv_{\mathcal{L}(E)} b^\star c^\star$. It follows that $e = f(a) = f(c) = f(c^\star) = f(b^\star)$. Since $b^\star \equiv_{\mathcal{L}(\overline{E})} b$ and $e \in \overline{E}$, we have that $f(b) = e$. □

The notion of independence that we will be using, in this paper, is quantifier free invariant independence.

Definition 1.18:
1. *Let $A \subseteq M$. A quantifier free type p over M is said to be $\mathrm{Aut}(M/A)$-invariant if for every quantifier free $\mathcal{L}$-formula $\phi(x, y)$, $a \in M$ and $\sigma \in \mathrm{Aut}(M/A)$, $\phi(x, a) \in p$ if and only if $\phi(x, \sigma(a)) \in p$.*

2. *Let $a \in M$ be a tuple and $C, B \subset M$. We define $a \downarrow^{\mathrm{i,qf}}_C B$ to hold if there exists an* $\mathrm{Aut}(M/A)$*-invariant quantifier free type p over M such that $a \models p|_{CB}$.*

We write $a \equiv^{\mathrm{qf}}_{\mathcal{L}(E)} b$ to say a and b have the same quantifier free type over E.

Lemma 1.19: *Hypothesis 1.17.(ii) for $\downarrow^{\mathrm{i,qf}}$ follows from:*

(ii') for all $E = \mathrm{acl}(E) \subseteq M$ and tuples $a_1, a_2, c_1, c_2, c \in M$, if $a_1 \downarrow^{\mathrm{i,qf}}_E a_2$, $c \downarrow^{\mathrm{i,qf}}_E a_1 a_2$ and $c_1 \equiv_{\mathcal{L}(E)} c_2$ and, for all i, $a_i c_i \equiv^{\mathrm{qf}}_{\mathcal{L}(E)} a_i c$, then there exists $c^\star$ such that $a_i c_i \equiv_{\mathcal{L}(E)} a_i c^\star$, for all i.

Proof. Let E, a, b and c be as in Hypothesis 1.17.(ii). Since $a \equiv_{\mathcal{L}(A)} b$, we can find d such that $ac \equiv_{\mathcal{L}(E)} bd$. Then we do have $a \downarrow^{\mathrm{i,qf}} b$, $c \downarrow^{\mathrm{i,qf}}_E ab$ and $c \equiv_{\mathcal{L}(E)} d$. Moreover, since a and b are $\mathrm{Aut}(M/E)$-conjugated and there exists an $\mathrm{Aut}(M/E)$-invariant quantifier free type p, it follows that $a \equiv^{\mathrm{qf}}_{\mathcal{L}(Ec)} b$ and hence $bd \equiv^{\mathrm{qf}}_{\mathcal{L}(E)} bc$. It follows that we can apply (ii') to find a $c^\star$ that $ac \equiv_{\mathcal{L}(E)} ac^\star$ and $bd \equiv_{\mathcal{L}(E)} bc^\star$. Since $bd \equiv_{\mathcal{L}(E)} ac$, we have the required conclusion. □

Remark 1.20: Note that if T eliminates quantifiers (or if we are working in the Morleyized language), (ii') and therefore (ii), holds trivially for $\downarrow^{\mathrm{i,qf}}$. In that case, Proposition **(1.17)** reduces to the statement that if every definable set X definable in models of T contains a type which is $\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-invariant, then T weakly eliminates imaginaries. A particular case of that statement is Hrushovski's criterion via definable types [Hru14, Lemma 1.17] where invariant is replaced by definable — and the coding of definable sets is built in.
One important difference of working with definable types, though, is that, as pointed out in Remark **(1.15)**, building types on some X which are definable almost over $\ulcorner X \urcorner$, without controlling in which sorts the canonical basis lies, can be done just for unary dominant definable sets. The general case follows by an easy induction. Also, contrary to definable types, invariant types do not have an imaginary canonical basis and thus it is much harder to compute their canonical basis, unless it is obvious from the construction of the invariant type, as in Proposition **(2.41)**.

2 Pseudo p-adically closed fields

2.1 Preliminaries

In this section we will give all the preliminaries on valued fields, p-adically closed fields and pseudo pseudo p-adically fields that are required throughout the paper.

Notation 2.1: Whenever F is a field, we denote by $\overline{F}^{\mathrm{a}}$ its algebraic closure.

2.1.1 Valued Fields

Let us start by fixing our notations regarding valued fields and introduce the geometric language.

Definition 2.2: *A valuation on a field F is a group morphism map $\mathrm{v} : F^\star \to \Gamma$, where $(\Gamma, +, 0, <)$ is a totally ordered abelian group, such that for all $a, b \in F$, $\mathrm{v}(a+b) \geq \min\{\mathrm{v}(a), \mathrm{v}(b)\}$. The set $\mathcal{O} := \{x \in F : \mathrm{v}(x) \geq 0\}$ is a subring of F that is called the valuation ring. The set $\mathfrak{M} := \{x \in F : \mathrm{v}(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}$. The field $\mathbf{k} := \mathcal{O}/\mathfrak{M}$ is called the residue field of F.*

Definition 2.3 (Geometric language)**:** *Let (F, v) be a valued field. For all $m \in \mathbb{Z}_{>0}$, We define*

$$\mathrm{S}_m(F) := \mathrm{GL}_m(F)/\mathrm{GL}_m(\mathcal{O}) \text{ and } \mathbf{T}_m(F) := \mathrm{GL}_m(F)/\mathrm{GL}_{m,m}(\mathcal{O}),$$

where $\mathrm{GL}_{m,m}(\mathcal{O})$ is the group of matrices $M \in \mathrm{GL}_m(\mathcal{O})$ whose last column reduces modulo $\mathfrak{M}$ to a columun of zero except for a 1 on the diagonal.
The geometric language *$\mathcal{L}^{\mathcal{G}}$ consists of the sort $\mathbf{K}$ equipped with the ring language, sorts S_m and $\mathbf{T}_m$, for all $m \in \mathbb{Z}_{>0}$, maps $s_m : \mathbf{K}^{m^2} \to \mathrm{S}_m$ and $t_m : \mathbf{K}^{m^2} \to \mathbf{T}_m$ interpreted as the canonical projections and the necessary predicates to have quantifier elimination in the $\mathcal{L}^{\mathcal{G}}$-theory $\mathrm{ACVF}^{\mathcal{G}}$ of algebraically closed valued fields.*

Remark 2.4:
1. The sort S_m is the moduli space of all rank m free $\mathcal{O}$-submodules of $\mathbf{K}^m$, i.e. $\mathcal{O}$-lattices. For all $s \in \mathrm{S}_m$, let $\Lambda(s)$ to be the lattice whose set of bases is s.
2. We can identify $\mathrm{S}_1 = \mathbf{K}/\mathcal{O}^\star$ with $\mathrm{v}(\mathbf{K}^\star)$ and s_1 with the valuation map.
3. For all $s \in \mathrm{S}_m$, the fiber of the natural map $\tau_m : \mathbf{T}_m \to \mathrm{S}_m$ above s can be identified (once we add a zero) with the dimension m $\mathbf{k}$-vector space $\Lambda(s)/\mathfrak{M}\Lambda(s)$.
4. When the valuation v is discrete, i.e. there are elements θ with minimal positive valuation, there is an $\varnothing$-definable map from $\mathbf{T}_m$ to S_{m+1}. It follows that, in that case, the sorts $\mathbf{T}_m$ are not necessary to obtain elimination of imaginaries.

The geometric sorts where introduced by Haskell, Hrushovski and Macpherson to prove:

Theorem 2.5 ([HHM06])**:** *The theory $\mathrm{ACVF}^{\mathcal{G}}$ of algebraically closed valued fields in the geometric language eliminates imaginaries.*

2.1.2 p-adically closed fields

Definition 2.6: *Let (F, v) be a valued field.*
1. *The valuation v is called p-adic if the residue field is $\mathbb{F}_p$ and $\mathrm{v}(p)$ is the smallest positive element of the value group $\mathrm{v}(F^\star)$.*
2. *We say that (F, v) is p-adically closed if (F, v) is p-adically valued and it has no proper p-adically valued algebraic extension.*
3. *A p-adic closure of (F, v) is an algebraic extension $(\overline{F}, \overline{\mathrm{v}})$, which is p-adically closed. It always exist when (F, v) is p-adically valued.*

Fact 2.7 (Properties of the theory of p-adically closed fields)**:**
- (i) *The class $p\mathrm{CF}$ of p-adically closed fields is elementary in the language $\mathcal{L}_{\mathrm{rg}} := \{+, -, \cdot, 0, 1\}$ of rings.*
- (ii) *Let $\mathcal{L}_{\mathrm{Mac}}$ be the language of rings to which we add a binary predicate $\mid$ and, for all $m \in \mathbb{Z}_{>0}$, unary predicates $\{\mathrm{P}_m : m > 1\}$. We interpret $x|y$ as $\mathrm{v}(x) \leq \mathrm{v}(y)$ and P_m as set of m-th powers. By [Mac76, Theorem 1], $p\mathrm{CF}$ eliminates quantifiers in $\mathcal{L}_{\mathrm{Mac}}$.*

2.1.3 Pseudo p-adically closed fields

We now define of pseudo p-adically closes fields and recall known results. We end this section by proving a new result regarding existential closedness of bounded p-adically closed fields. We refer the reader to [Jar91], [HJ88] and [Mon17b] for more details.

Definition 2.8: *Let $F \leq L$ be an extension of characteristic 0 fields.*

1. *$F \leq L$ is called totally p-adic if every p-adic valuation of F can be extended to a p-adic valuation of L.*
2. *$F \leq L$ is a regular extension if $L \cap \overline{F}^a = F$.*

Definition 2.9: *A field F of characteristic 0 is pseudo-p-adically closed (PpC) if it is is existentially closed, as an $\mathcal{L}_{\mathrm{rg}}$-structure, in every totally p-adic regular extension.*

Remark 2.10: By Lemma 13.9 of [HJ88] this is equivalent to every non-empty absolutely irreducible variety V defined over F having an F-rational point, provided that it has a simple rational point in each p-adic closure of F.

Fact 2.11 ([Jar91, Theorem 10.8])**:** *Let F be a PpC field and let v be a p-adic valuation on F. Then:*

(i) *The p-adic closure of F with respect to v is exactly its Henselization. In particular all p-adic closures of F with respect to v are F-isomorphic.*
(ii) *F is dense, for the v-topology, in its p-adic closure $\overline{F}^{\mathrm{p}}$.*
(iii) *$\mathrm{v}(F)$ is a $\mathbb{Z}$ group.*
(iv) *If v_1 and v_2 are distinct p-adic valuations on F, then v_1 and v_2 are independent, i.e. v_1 and v_2 generate different topologies.*

Remark 2.12: Let F be a PpC field and let $\mathrm{v}_1, \ldots, \mathrm{v}_n$ be different p-adic valuations on F. For all $i \leq n$, let U_i be a subset of F^r which is open in the topology associated to v_i. Then by [PZ78, Theorem 4.1] and Fact **(2.11)** we have that $\bigcap_i U_i \neq \varnothing$.

Definition 2.13: *Let F be a field, $n \geq 1$, and let $\mathrm{v}_1, \ldots, \mathrm{v}_n$ be n distinct p-adic valuations on F. The field $(F, \mathrm{v}_1, \ldots, \mathrm{v}_n)$ is n-pseudo p-adically closed (n–PpC) if F is a PpC field and $\mathrm{v}_1, \ldots, \mathrm{v}_n$ are the only p-adic valuations of F. If U_i is an open (respectively closed) set for the topology associated to the valuation v_i, we will say that U_i is i-open (respectively i-closed).*

Notation 2.14: Let F be a fields and for $0 < i \leq n$, let v_i be a p-adic valuation on F and let us fix a p-adic closure F_i of F with respect to v_i. Let also $\mathcal{L}_i = \mathcal{L}_{\mathrm{rg}} \cup \{\mathrm{P}^i_m : m \in \mathbb{Z}_{>0},\ 1 \leq i \leq n\}$. Let $\mathcal{L} := \bigcup_{i=1}^n \mathcal{L}_i$. We interpret P^i_m as :

$$F \vDash \mathrm{P}^i_m(a) \text{ if and only if } F_i \vDash \exists y\, y^m = a \wedge a \neq 0.$$

Recall that, by Fact **(2.11)**, if F is n–PpC, F_i is unique up to isomorphism, so the $\mathcal{L}$-structure of F does not depend on the choice of F_i. In particular, if L is a totally p-adic extension of F, the $\mathcal{L}$-structure induced by L on F is that unique $\mathcal{L}$-structure. Note also that $\mathrm{P}^i_m(F_i)$ is a i-clopen subset of $F_i^\star$.

Fact 2.15 ([EJ90, Lemma 3.6])**:** *Let $(F, \mathrm{v}_1, \dots, \mathrm{v}_n)$ be an n–PpC field and let V be an absolutely irreducible variety defined over F. For each $1 \leq i \leq n$, let $q_i \in V(F_i)$ be a simple point. Then V contains an F-rational point q, arbitrarily i-close to q_i, for all $i \in \{1, \dots, n\}$.*

Theorem 2.16: *Let $(F, \mathrm{v}_1, \dots \mathrm{v}_n)$ be n–PpC and let L be a regular totally p-adic extension of F. Then F is existentially closed in L as an $\mathcal{L}$-structure.*

Proof. Let $\varphi := \exists X \phi(X)$ be an existential $\mathcal{L}$-formula such that $L \vDash \varphi$. We need to show that $F \vDash \varphi$. Observe that φ is equivalent to a formula of the form:

$$\exists X \left(\bigvee_{l \leq k} (X \in V_l \wedge X \notin Z_l \wedge \bigwedge_{i \leq n} X \in U_{i,l}) \right),$$

where V_l and Z_l are Zariski closed over F and $U_{i,l}$ is an i-open quantifier free $\mathcal{L}_i(F)$-definable set. Since the disjunction symbol commutes with the existential quantifiers, we may assume that $k = 1$. Also, if $g(X)$ is a polynomial, then the formula $g(X) \neq 0$ is equivalent to $\exists Y\, (Yg(X) - 1 = 0)$. So we can assume that φ is of the form:

$$\exists X\, (X \in V \wedge \bigwedge_{i \leq n} X \in U_i),$$

where V is Zariski closed over F and U_i is an i-open quantifier free $\mathcal{L}_i(F)$-definable set. Let $a \in L$ be any tuple such that $L \vDash a \in V \wedge \bigwedge_{i \in I} a \in U_i$. Since $F \leq L$ is a regular extension, so is $F \leq F(a)$. By [FJ08, Corollary 10.2.2], there is an absolutely irreducible variety W, defined over F such that a is a generic point of W. Note that $W \subseteq V$.
For every i, let L_i be the p-adic closure used to define the $\mathcal{L}$-structure of L and let $F_i \preccurlyeq L_i$ be a p-adic closure of F. Since $L \subseteq L_i$,

$$L_i \vDash \psi_i(a) := a \text{ is a simple point of } W \wedge a \in U_i,$$

and hence we find $b_i \in F_i$ such that $F_i \vDash \psi_i(b_i)$. By Fact **(2.15)** there is $b \in W(F)$ such that b is arbitrarily i-close to b_i, for all i. In particular, we can choose $b \in \bigcap_i U_i$. Then we have $F \vDash \phi(b)$. □

Remark 2.17: For every $i \in \{1, \dots, n\}$ and $x, y \in F_i$, we have

$$\mathrm{v}_i(x) \leq \mathrm{v}_i(y) \text{ if and only if } x^m + py^m \in \mathrm{P}^i_m,$$

for any choice of m prime to p. So Theorem **(2.16)** also holds if we add the predicates $|_i$ to the language.

Remark 2.18: As a Corollary of Theorem **(2.16)**, we can generalize [Mon17b, Theorem 7.1] to obtain that if $n \geq 2$ and F is an n–PpC field, then $\mathrm{Th}_{\mathcal{L}_{\mathrm{rg}}}(F)$ is not NIP. The proof is exactly as in [Mon17b, Theoreme 7.1], the only difference is that we can work in $\mathcal{L}$, without using boundedness so that $\mathcal{L}$ is a definable expansion of $\mathcal{L}_{\mathrm{rg}}$.

2.2 Notations and new results

In this section, we describe the language in which we prove elimination of imaginaries for the bounded PpC fields. Theorem **(2.25)** is the main result of this paper.

Definition 2.19: *A field F is bounded if for any $n \in \mathbb{Z}_{>0}$, F has finitely many extensions of degree n.*

Let us fix some notation for the rest of the paper.

Notation 2.20: Fix a bounded PpC field F. By [Mon17b, Lemma 6.1] there is $n \in \mathbb{Z}_{>0}$ such that F has exactly n p-adic valuations $\{\mathrm{v}_1, \ldots, \mathrm{v}_n\}$. For each $i \in \{1, \ldots, n\}$ let $\mathcal{O}_i$ denote the valuation ring associated with v_i and let $\mathcal{O} := \bigcap_i \mathcal{O}_i$. Denote by $\mathfrak{M}_i$ the maximal ideal of $\mathcal{O}_i$. Let $F_0 \preccurlyeq F$ be a countable elementary substructure (in the language of rings). Let $\overline{\mathcal{L}}_i$ be a copie of $\mathcal{L}^{\mathcal{G}}(F_0)$ sharing the field sort $\mathbf{K}$ and constant symbols for the elements of F_0. Let $\mathcal{G}_i$ denote the sorts of $\overline{\mathcal{L}}_i$. For all $m \in \mathbb{Z}_{>0}$, we denote by $\mathbf{S}^i_m$ the sort interpreted as $\mathrm{GL}_m(\mathbf{K})/\mathrm{GL}_m(\mathcal{O}_i)$ and $\mathbf{T}^i_m$ the sort interpreted by $\mathrm{GL}_m(\mathbf{K})/\mathrm{GL}_{m,m}(\mathcal{O}_i)$. Let $\mathrm{ACVF}^{\mathcal{G}}_i$ denote the copie of ACVF in $\overline{\mathcal{L}}_i$. Let $\mathcal{L}_i$ denote a definable enrichment of $\overline{\mathcal{L}}_i$ in which $p\mathrm{CF}^{\mathcal{G}}_i$, the $\mathcal{L}_i$-theory of p-adically closed fields, eliminates quantifiers. Let $\mathcal{L} := \bigcup_{i \leq n} \mathcal{L}_i$ and $T := \mathrm{Th}_{\mathcal{L}}(F)$, where the $\mathcal{L}_i$-structure of F is induced by its p-adic closure for v_i.

Remark 2.21: Observe that $\overline{F}^{\mathrm{a}} = \overline{F_0}^{\mathrm{a}} F$ and hence, for all $M \models T$, $\overline{\mathbf{K}(M)}^{\mathrm{a}} = \overline{F_0}^{\mathrm{a}}\mathbf{K}(M)$. So $\mathbf{K}(M)$ is bounded. Moreover the extension $F_0 \leq \mathbf{K}(M)$ is regular.

Proposition 2.22: *In models of T, every $\mathcal{L}$-definable set is $\mathcal{L}_{\mathrm{rg}}(F_0)$-interpretable.*

Proof. Let us first prove that $\mathcal{L}_{\mathrm{Mac}}$ is definable in $\mathcal{L}_{\mathrm{rg}}(F_0)$:

Claim 2.23: *Let $M, N \models T$, $A = \overline{\mathbf{K}(A)}^{\mathrm{a}} \cap M \subseteq M$, $B = \overline{\mathbf{K}(B)}^{\mathrm{a}} \cap N \subseteq N$ and $f : A \to B$ be an $\mathcal{L}_{\mathrm{rg}}(F_0)$-isomorphism. Then f is an $\mathcal{L}$-isomorphism.*

Proof. By quantifier elimination in $\mathcal{L}_{\mathrm{Mac}}$ and the fact that $|_i$ can be defined wihout quantifiers using the predicates P^i_m of Notation **(2.14)**, it suffices to check that these predicates are preserved by f. Let $M_i \models p\mathrm{CF}^{\mathcal{G}}_i$ contain M. Then for any $a \in A$, $\mathrm{P}^i_m(a)$ holds if and only if there exists $y \in M_i$ such that $y^m = a$. Let $L \geq \mathbf{K}(M)$ be the compositum of all degree m extensions of $\mathbf{K}(M)$ inside M_i. Since $\mathbf{K}(M)$ is bounded, L is a finite extension and there exists $\alpha \in \overline{F_0}^{\mathrm{a}}$, such that $L = \mathbf{K}(M)[\alpha]$. Then, $\mathrm{P}^i_m(a)$ holds if and only if there exists $y \in \mathbf{K}(M)[\alpha]$ such that $y^m = x$. Identifying $\mathbf{K}(M)[\alpha]$ with $\mathbf{K}^l(M)$ for some l, we see that the coordinates of y are in $\overline{(F_0 a)}^{\mathrm{a}} \cap \mathbf{K}(M) \subseteq \mathbf{K}(A)$. It follows that $P_{m,i}(a)$ if and only if $P_{m,i}(f(a))$. ◊

It follows that every $\mathcal{O}_i$ is $\mathcal{L}_{\mathrm{rg}}(F_0)$-definable and hence that the geometric sorts for the valuation i are $\mathcal{L}_{\mathrm{rg}}(F_0)$-interpretable.
Now, let $\phi(x)$ be any $\overline{\mathcal{L}}_i$-formula. Let f be the canonical projection from some cartesian power of $\mathbf{K}$ to the sort of x. The formula $\psi(y) := \phi(f(y))$ is equivalent, in $\mathrm{ACVF}^{\mathcal{G}}_i$, to a formula in the language $\mathcal{L}_{\mathrm{rg}} \cup \{|_i\}$ and hence $\phi(x)$ is equivalent, in $\mathrm{ACVF}^{\mathcal{G}}_i$ to both $\forall y\, f(y) = x \to \psi(y)$ and $\exists y\, f(y) = x \wedge \psi(y)$. It follows that these two formulas also define the trace of ϕ in any model of $p\mathrm{CF}^{\mathcal{G}}_i$. So in models of $p\mathrm{CF}^{\mathcal{G}}_i$, every $\overline{\mathcal{L}}_i$-definable set is interpretable in $\mathcal{L}_{\mathrm{rg}}$, since $|_i$ is

definable in $\mathcal{L}_{\mathrm{rg}}$. The same argument, but starting with a $\mathcal{L}_i$-formula and looking at its trace on M allows us to conclude. □

Notation 2.24: For the rest of the paper, we fix $M \models T$ sufficiently saturated and homogeneous. Let $\overline{M_i} \models \mathrm{ACVF}_i^{\mathcal{G}}$ be the algebraic closure of M, along with an extension of v_i, and let M_i be the p-adic closure of M inside $\overline{M_i}$. We denote by acl, $\mathrm{acl}_{\overline{\mathcal{L}}_i}$, $\mathrm{acl}_{\mathcal{L}_i}$, dcl, $\mathrm{dcl}_{\overline{\mathcal{L}}_i}$ and $\mathrm{dcl}_{\mathcal{L}_i}$ the model theoretic algebraic and definable closures in M, $\overline{M_i}$ and M_i respectively.
Let $A \subseteq M$ and $a, b \in M$ be tuples. We denote by $a \equiv_{\mathcal{L}(A)} b$, $a \equiv_{\overline{\mathcal{L}}_i(A)} b$ and $a \equiv_{\mathcal{L}_i(A)} b$ the fact that a and b have the same type over A in M, $\overline{M_i}$ and M_i respectively. We also denote by tp, $\mathrm{tp}_{\overline{\mathcal{L}}_i}$ and $\mathrm{tp}_{\mathcal{L}_i}$ the type in M, $\overline{M_i}$ and M_i respectively.

Observe that $a \equiv_{\overline{\mathcal{L}}_i(A)} b$ (respectively $a \equiv_{\mathcal{L}_i(A)} b$) if and only if a and b have the same quantifier free $\overline{\mathcal{L}}_i$-type (respectively $\mathcal{L}_i$-type) in M over A. Note also that $a \equiv^{\mathrm{qf}}_{\mathcal{L}(A)} b$ if and only $a \equiv_{\mathcal{L}_i(A)} b$, for all i.
We can now state our main result:

Theorem 2.25: *The theory T eliminates imaginaries.*

Proof. We apply Proposition **(1.17)** and Lemma **(1.19)** to obtain weak elimination of imaginaries. Hypothesis (i) is proved in Theorem **(2.44)** and Hypothesis (ii') is proved in Theorem **(2.55)**. Since, by Corollary **(2.58)**, finite sets are coded, we obtain elimination of imaginaries. □

Remark 2.26: As noted above, for all m, $\mathbf{T}^i_m$ is $\mathcal{L}_i$-definably embedded in S^i_m, so the $\mathbf{T}^i_m$ are not necessary to eliminate imaginaries. Also the map sending a coset of $\mathrm{GL}_m(\mathcal{O})$ in $\mathrm{GL}_m(\mathbf{K})$ to the tuple of $\mathrm{GL}_m(\mathcal{O}_i)$ cosets it is contained in is an $\mathcal{L}$-definable bijection — the inverse is given by taking the intersection of the $\mathrm{GL}_m(\mathcal{O}_i)$ cosets. So T also eliminates imaginaries in the language with a sort for $\mathbf{K}$ and, for all $m \in \mathbb{Z}_{>0}$, a sort S_m for $\mathrm{GL}_m(\mathbf{K})/\mathrm{GL}_m(\mathcal{O})$.

2.2.1 A pseudo-real digression

Recall that a pseudo-real closed field is a field which is existentially closed in any regular extension to which every order extends. Elimination of imaginaries for bounded pseudo-really closed fields, in the language of rings with constants for an elementary subfield, was proved in [Mon17a]. However, there seem to be a error in the final arguments. As the proof of Lemma 4.5 is written, it implicitly uses elimination of imaginaries in bounded pseudo-real closed fields. Indeed, to be able to find a_1 and a_2 as stated, it is necessary to assume that $\mathrm{acl}^{\mathrm{eq}}(Ee) \cap M \subseteq E$. However, unless we already know elimination of imaginaries, this is not implied by $E = \mathrm{acl}(E) \subseteq \mathrm{acl}^{\mathrm{eq}} e \cap M$. But that weaker hypothesis is, *a priori*, the only one that remains true when, in Claim 1 of Theorem 4.8, E is replaced by $E(a')$.
However, this is easy to fix:

Lemma 2.27: *Let M be a bounded pseudo-real closed field and let X be M-definable. There exists $p \in \mathcal{D}(\overline{M}^{\mathrm{a}}/\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner) \cap M)$ consistent with M.*

Proof. Take p to be the generic type of any irreducible component of the Zariski closure of $X(M)$. By elimination of imaginaries in algebraically closed fields (and Galois theory), p is $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner) \cap M$-definable. □

In Claim 1 of the proof of [Mon17a, Theorem 4.8], one can, using Lemma **(2.27)**, first find $p \in \mathcal{D}(\overline{M}^{\mathrm{a}}/E)$ consistent with $f^{-1}(e)$ and then choose $a \models p$. It then follows from Corollary **(1.9)** that for any $a' \subseteq a$, $E' := \mathrm{acl}^{\mathrm{eq}}(ea') \cap M = \mathrm{acl}(Ea')$ and hence that $\mathrm{acl}^{\mathrm{eq}}(E'e) \cap M \subseteq E'$. It follows that we can safely apply Lemma 4.5 to E' without knowing elimination of imaginaries in M.
Another, somewhat overkill, approach would be to adapt the general outline of the proof presented in this paper. The only result on bounded pseudo-p-adically closed fields which is proved in this paper and whose pseudo-real closed equivalent is not already proved in [Mon17b] is Proposition **(2.41)**. But the bounded pseudo-p-adically closed equivalent is an easy consequence of [Mon17b, Lemma 4.4] and Neumann's Lemma. The rest of the arguments can be copied *mutatis mutandis* replacing $\mathrm{ACVF}^{\mathcal{G}}$ and $p\mathrm{CF}^{\mathcal{G}}$ by the theory of real closed fields.
The proof presented in this paper could also be adapted to a number of other case: among other examples, bounded pseudo-algebraically closed valued fields with finitely many independent valuations.

2.3 Orthogonality of the geometric sorts

In this section we will prove that the S^i_m sorts are orthogonal. We also show that their structure is the pure structure induced by M_i.

Lemma 2.28: *Let $m \in \mathbb{Z}_{>0}$, $A \subseteq \mathbf{K}(M)$ and, for all $i \in \{1, \dots, n\}$ let $s_i \in \mathrm{S}^i_m(M)$. Assume M is $|A|^+$-saturated, then there exists $c \in \bigcap_i s_i(M)$ such that* $\mathrm{trdeg}(c/A) = m^2$.

Proof. For all i, $\mathcal{O}_i^\star$ is i-open in $\mathbf{K}(M)$ and hence $\mathrm{GL}_m(\mathcal{O}_i) \subseteq \mathbf{K}^{m^2}$ is i-open and so is s_i. Therefore, there exists a product of i-closed balls U_i included in S_i. So it suffices to find $c \in \bigcap_i U_i \subseteq \mathbf{K}^{m^2}(M)$ such that $\mathrm{trdeg}(c/A) = m^2$. This is easily seen to reduce (by induction on the dimension and translation) to showing that $\bigcap_i \mathcal{O}_i = \mathcal{O}$ contains transcendental elements over any small set of parameters and by compactness to showing that $\mathcal{O}$ is infinite. But this is an immediate consequence of Remark **(2.12)**. □

Proposition 2.29: *Let $A \subseteq \mathbf{K}(M)$ and, for all $i \leq n$, let s_i and $s'_i \in \mathrm{S}^i_m(M)$. If $s_i \equiv_{\mathcal{L}_i(A)} s'_i$, for all i, then $(s_1, \dots, s_n) \equiv_{\mathcal{L}(A)} (s'_1, \dots, s'_n)$.*

Proof. By Lemma **(2.28)**, there exists $c \in \bigcap_i s_i(M)$ such that $\mathrm{trdeg}(c/A) = m^2 = |c|$.

Claim 2.30: *There exists $c_i \in s'_i(M)$ such that $c_i \equiv_{\mathcal{L}_i(A)} c$.*

Proof. By compactness, we have to show that for all (quantifier free) $\mathcal{L}_i(A)$-formula $\phi(x)$ such that $M_i \models \phi(c)$, $s'_i \cap \phi(M) \neq \emptyset$. Note that $s_i \cap \phi(M_i)$ is an $\mathcal{L}_i(A)$-definable subset of $(\mathbf{K}(M_i))^{m^2}$ that contains an element of transcendence degree m^2 over A, so it is i-open. As $s_i \equiv_{\mathcal{L}_i(A)} s'_i$, $s'_i \cap \phi(M_i)$ is also i-open and non empty. By the density of M in M_i, this set has has a point in M. ◊

By [Mon17b, Lemma 6.12], $\mathrm{tp}(c/A) \cup \{x \in \bigcap_i s_i'\}$ is consistent in M. Let c' realise this type. Then there exists an $\mathcal{L}(A)$-automorphism of M sending c to c' and hence $s_i = c\mathrm{GL}_m(\mathcal{O}_i)$ to $s_i' = c'\mathrm{GL}_m(\mathcal{O}_i)$. □

Corollary 2.31: *Let $A \subseteq \mathbf{K}(M)$ and $X \subseteq \prod_i \mathbf{S}^i_{m_i}$ be $\mathcal{L}(A)$-definable. Then there exists finitely many quantifier free $\mathcal{L}_i(A)$-formulas $\phi_{i,j}(x_i)$ such that $X = \bigcup_j \prod_i \phi_{i,j}(M)$.*

Proof. By Proposition **(2.29)**, the following set is inconsistent:

$$\{\psi(x_i) \leftrightarrow \psi(x_i') : 0 < i \leq n \text{ and } \psi \text{ is an } \mathcal{L}_i(A)\text{-formula}\} \cup \{(x_i)_i \in X \wedge (x_i')_i \notin X\}.$$

By compactness, it follows that there are k_i formulas $\psi_{i,j}(x_i)$ such that

$$M \vDash \forall x_1 \dots x_n \, (\bigwedge_{0 \leq j < k_i} \psi_{i,j}(x_i) \leftrightarrow \psi_{i,j}(x_i')) \to ((x_i)_{i \leq n} \in X \leftrightarrow (x_i')_{i \leq n} \in X).$$

For all $\epsilon_i : k_i \to 2$, let $\theta_{i,\epsilon_i}(x_i) = \bigwedge_{0 \leq j < k_i} \psi(x_i)^{\epsilon_i(j)}$ and for all tuple $\epsilon = (\epsilon_i)_i$, $\theta_\epsilon(x) = \theta_{i,\epsilon_i}(x_i)$. Then for all ϵ, if $\theta_\epsilon(M) \cap X \neq \varnothing$, then $\theta_\epsilon(M) \subseteq X$. Let $E = \{\epsilon : \theta_\epsilon(M) \cap X \neq \varnothing\}$. Then

$$X = \bigcup_{\epsilon \in E} \theta_\epsilon(M) = \bigcup_{\epsilon \in E} \prod_i \theta_{i,\epsilon_i}(M).$$

This concludes the proof. □

Define $\mathcal{G}_i^{\mathrm{im}}$ to be the set of all $\mathcal{L}_i$-sorts but $\mathbf{K}$.

Corollary 2.32: *Let $A \subseteq M$, S_i be a product of sorts in $\mathcal{G}_i^{\mathrm{im}}$ and $X \subseteq \prod_i S_i$ be an $\mathcal{L}(A)$-definable subset. Then, there exists quantifier free $\mathcal{L}_i(\mathcal{G}_i(A))$-definable sets $X_{i,j} \subseteq S_i$ such that $X = \bigcup_j \prod_i X_{i,j}(M)$.*

Proof. It suffices to show the corollary for $A \subseteq \mathbf{K}(M)$, all the other parameters can be replaced by free variables and put back in afterwards. Because any finite product of sorts from $\mathcal{G}_i^{\mathrm{im}}$ can be encoded, in $p\mathrm{CF}_i^{\mathcal{G}}$, in any $\mathbf{S}^i_m$ for large enough m, we may also assume that $S_i = \mathbf{S}^i_m$ for some fixed m. We can now apply Corollary **(2.31)**. □

2.4 The algebraic closure

Let us now describe the algebraic closure in T.

Proposition 2.33: *Let $A \subseteq M$. Then, for all i, $\mathbf{S}^i_m(\mathrm{acl}(A)) \subseteq \mathrm{acl}_{\mathcal{L}_i}(\mathcal{G}_i(A))$.*

Proof. By Corollary **(2.32)**, any (finite) $\mathcal{L}(A)$-definable subset of $\mathbf{S}^i_m$ is quantifier free $\mathcal{L}_i(\mathcal{G}_i(A))$-definable. The proposition follows. □

The sorts $\mathbf{K}$ and $\mathbf{S}^i_m$ are obviously not orthogonal as there are functions with infinite image from $\mathbf{K}$ to $\mathbf{S}^i_m$, but the converse is not true.

Proposition 2.34: *Let $A \subseteq M$. Then* $\mathbf{K}(\mathrm{acl}(A)) \subseteq \overline{\mathbf{K}(A)}^{\mathrm{a}}$.

Proof. Let $c \in \mathbf{K}(\mathrm{acl}(A))$. As in the proof of Corollary **(2.32)**, we may assume that there exists $s_i \in \mathbf{S}^i_m$, for some fixed m, such that $c \in \mathrm{acl}(\mathbf{K}(A)(s_i)_{i \leq n})$. By Lemma **(2.28)**, there exists

$e \in \bigcap_i s_i(M)$ such that $\mathrm{trdeg}(e/\mathbf{K}(A)c) = m^2 = |e|$. Then by Lemma 2.6 of [Mon17b] $c \in \mathrm{acl}(\mathbf{K}(A)e) \subseteq \overline{\mathbf{K}(A)e}^{\mathrm{a}}$. But e is algebraically independent from c over A. It follows that $c \in \overline{\mathbf{K}(A)}^{\mathrm{a}}$. □

Corollary 2.35: *Let $A \subseteq M$. Then,*

$$\mathrm{acl}(A) \subseteq \bigcup_i \mathrm{acl}_{\mathcal{L}_i}(\mathcal{G}_i(A)).$$

Proof. This is an easy consequence of Propositions **(2.33)** and **(2.34)**. □

Corollary 2.36: *Let S be one of the sorts in $\mathcal{G}_i^{\mathrm{im}}$ and S' be a sort in $\mathcal{G}_j$ for some $j \neq i$. Then any $\mathcal{L}(M)$-definable function $S \to S'$ has finite image.*

Proof. If $S' = \mathbf{S}^j_m$, then this follows immediately from Corollary **(2.32)**. If $S' = \mathbf{K}$, it follows from Proposition **(2.34)** and compactness. □

Corollary 2.37: *Let $A \subseteq \mathbf{K}(M)$. Then $\mathrm{acl}(A) \subseteq \bigcup_i \mathrm{acl}_{\overline{\mathcal{L}}_i}(A)$.*

Proof. By Corollary **(2.35)**, $\mathrm{acl}(A) \subseteq \bigcup_i \mathrm{acl}_{\mathcal{L}_i}(A)$. Since the relative algebraic closure of a field inside a p-adic field is its p-adic closure (cf. [HMR, Section 4.(i)]), we have $\mathrm{acl}_{\mathcal{L}_i}(C) \subseteq \mathrm{acl}_{\overline{\mathcal{L}}_i}(C)$. □

We have proved that hypothesis $(\star)$ of Section 1.2 holds of $\mathrm{ACVF}_i^{\mathcal{G}}$ and T. Let us now prove that (†) also holds:

Lemma 2.38: *Let $\epsilon \in \mathrm{dcl}_{\overline{\mathcal{L}}_i}(M)$, for some $i \leq n$. Then there exists $\eta \in M$ such that ϵ and η are interdefinable in the pair $(\overline{M_i}, M)$.*

Proof. We know by Fact **(2.11)** that the Henselization of $\mathbf{K}(M)$ with respect to v_i is the p-adic closure $\mathbf{K}(M_i)$ of $\mathbf{K}(M)$. It follows that if $\epsilon \in \mathbf{K}(\mathrm{dcl}_{\overline{\mathcal{L}}_i}(M))$, then $\epsilon \in \mathbf{K}(M_i)$. As $\overline{\mathbf{K}(M)}^{\mathrm{a}} = \overline{F_0}^{\mathrm{a}}\mathbf{K}(M)$, and $\mathbf{K}(M_i) \subseteq \overline{\mathbf{K}(M)}^{\mathrm{a}}$ there is $b \in \overline{F_0}^{\mathrm{a}}$ such that $\mathbf{K}(M)[\epsilon] = \mathbf{K}(M)[b]$. Since $\epsilon \in M_i$, it follows that $b \in \overline{F_0}^{\mathrm{a}} \cap \mathbf{K}(M_i) := F_1 \subseteq \mathrm{dcl}_{\overline{\mathcal{L}}_i}(\varnothing)$. Let η be the unique tuple in M such that $\epsilon = \sum_i \eta_i b^i$. Then η and ϵ are interdefinable in the pair $(\overline{M_i}, M)$.
The case $\epsilon \in \mathbf{S}^i_m \cup \mathbf{T}^i_m$ is tackled as in [HMR, (ii) p. 30]. Let us first assume that $\epsilon \in \mathbf{T}^i_m$. Find a finite extension L of $\mathbf{K}(M_i)$, of degree r, in which the lattice coded by $\tau_m(\epsilon)$, denoted $\Lambda(\epsilon)$, has a basis and the coset coded by ϵ has a point c. Since $\overline{\mathbf{K}(M_i)}^{\mathrm{a}} = \overline{F_1}^{\mathrm{a}}\mathbf{K}(M_i)$, we can find $a \in \overline{F_0}^{\mathrm{a}}$ such that $\mathcal{O}(L) = \mathcal{O}(\mathbf{K}(M_i))[a]$. Let $f_a : (\mathbf{K}(M_i))^r \to L$ be the map $r \mapsto \sum_i r_i a^i$. Then $f_a^{-1}(c + \mathfrak{M}_i\Lambda(\epsilon)) \subseteq (\mathbf{K}(M_i))^{mr}$ is coded by some $\eta \in \mathbf{T}^i_{mr}(M_i) = \mathbf{T}^i_{mr}(M)$. If a and a' have the same $\overline{\mathcal{L}}_i(F_0)$-orbit, then they have the same $\overline{\mathcal{L}}_i(F_1)$-orbit and hence the same $\overline{\mathcal{L}}_i(M_i)$-orbit. In particular, there exists $\tau \in \mathrm{Aut}_{\overline{\mathcal{L}}_i}(\overline{M_i})$ fixing M_i pointwise such that $\tau(a) = a'$. Then $f_a^{-1}(c + \mathfrak{M}_i\Lambda(\epsilon)) = f_{\tau(a)}^{-1}(c + \mathfrak{M}_i\Lambda(\epsilon)) = f_{a'}^{-1}(c + \mathfrak{M}_i(\Lambda(\epsilon)))$. It follows that any automorphism of $\overline{M_i}$ that stabilizes M globally fixes ϵ if and only if it fixes η. Note that we did not use in the proof that $\overline{M_i}$ was the algebraic closure of M, so the same argument works in a sufficiently saturated and homogeneous model of the pair $(\overline{M_i}, M)$. So η and ϵ are interdefinable. If $\epsilon \in \mathbf{S}^i_m$, since ϵ and $\mathfrak{M}_i\Lambda(\epsilon)$ are interdefinable in $\overline{M_i}$, we can apply the previous case to $\mathfrak{M}_i\Lambda(\epsilon)$. □

We can now apply Corollary **(1.10)** in our setting. If we already knew elimination of imaginaries in T, then the following result would be a rather immediate corollary of the description of the algebraic closure and the fact that pCF and ACVF are NIP. But since we do not know elimination of imaginaries yet, this is where the encoding of (germs of) functions happens.

Corollary 2.39: *Let $i \in \{1, \ldots, n\}$. Let $A \subseteq M^{\mathrm{eq}}$ containing $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(A))$ and $a \in \mathbf{K}(N)$, where $N \succcurlyeq M$, be a tuple such that $\mathrm{tp}_{\overline{\mathcal{L}}_i}(a/\overline{M}_i) \in \mathcal{I}(\overline{M}_i/\mathcal{G}_i(A))$ and $\mathrm{tp}_{\mathcal{L}_i}(\mathcal{G}_i(a)/M_i) \in \mathcal{I}(M_i/\mathcal{G}_i(A))$. Then $\mathrm{tp}_{\overline{\mathcal{L}}_i}(\mathcal{G}_i(\mathrm{acl}^{\mathrm{eq}}(Aa))/\overline{M}_i) \in \mathcal{I}(\overline{M}_i/\mathcal{G}_i(A))$ and $\mathrm{tp}_{\mathcal{L}_i}(\mathcal{G}_i(\mathrm{acl}^{\mathrm{eq}}(Aa))/M_i) \in \mathcal{I}(M_i/\mathcal{G}_i(A))$.*

Proof. We apply Corollary **(1.10)** twice — once with $T_1 = \mathrm{ACVF}_i^{\mathcal{G}}$ and a second time with $T_1 = p\mathrm{CF}_i^{\mathcal{G}}$. □

2.5 A local density result

Definition 2.40: *Let $\overline{M}_i \models \mathrm{ACVF}_i^{\mathcal{G}}$. We denote by $\mathbf{B}_i(\overline{M}_i)$ the set of balls in $\overline{M}_i$. We consider points to be closed balls of infinite radius and the whole field to be an open ball of radius $-\infty$.*
Let $P \subseteq \mathbf{B}(\overline{M}_i)$. We define α_P, the generic type of $\bigcap_{b \in P} b$ as:

$$\alpha_P(x) := \{x \in b : b \in P\} \cup \{x \notin b' : \forall b \in P, b' \subset b\}.$$

By C-minimality of ACVF, when it is consistent, α_P generates a complete type. We will not distinguish α_P from the complete type it generates. Note that, if $P \subseteq \mathbf{B}_i(A)$, then $\alpha_P \in \mathcal{I}(\overline{M}_i/A)$.

Proposition 2.41: *Let $A \subseteq M^{\mathrm{eq}}$ containing $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(A))$ and $c \in \mathbf{K}(M)$. For each $i \leq n$, let $P_i = \{b \in \mathbf{B}_i(A) : c \in b\}$. Then the partial type:*

$$\mathrm{tp}(c/A) \cup \bigcup_i \alpha_{P_i}$$

is consistent.

Proof. Assume that this type is not consistent. By compactness, there exists a $\mathcal{L}(A)$-formula $\phi(x)$, balls $b_i \in P_i$ and $b_{i,j} \in \mathbf{B}_i(\overline{M}_i)$ with $b_{i,j} \subset b_i$ for all $b_i \in P_i$, such that $M \models \forall x\, \phi(x) \wedge \bigwedge_i x \in b_i \to \bigvee_{i,j} x \in b_{i,j}$. Because the valuation v_i is p-adic, any ball is covered by finitely many subballs and, because A contains $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(A))$, P_i cannot have a minimal element. It follows that replacing the $b_{i,j}$ by the smallest ball covering them, we may assume that there is only one $b_{i,j}$ denoted b'_i. For all tuple of balls $b'_2, \ldots, b'_n$ (with finite radius) where $b'_i \in \mathbf{B}_i$, let $f_1(b'_2, \ldots, b'_n)$ be the minimal ball covering $(\phi(M) \cap \bigcap b_i(M)) \smallsetminus \bigcup_{i \geq 2} b'_i(M)$. This ball exists because, by Fact **(2.11)**, $\mathrm{v}_i(M)$ is a pure $\mathbb{Z}$-group.
By Corollary **(2.36)**, $f_1(b'_2, \ldots, b'_n) \in A$. Removing (the set encoded by) $f_1(b'_2, \ldots, b'_n)$ from $\phi(x)$, we still have $M \models \phi(c)$, but now $M \models \forall x\, \phi(x) \wedge \bigwedge_i x \in b_i \to \bigvee_{i \geq 2} b'_i$. By induction, removing $\mathcal{L}^{\mathrm{eq}}(A)$-definable sets from $\phi(x)$, we can get to a situation where $M \models \phi(c)$ and $\phi(M) \cap \bigcap_i b_i(M) = \varnothing$, a contradiction. □

This result is local version, for n–PpC, of [Mon17a, Theorem 4.4]. In fact, this theorem would follows from the n–PpC version of [Mon17a, Theorem 4.4] (which is not known to hold) that must take in account the geometric sorts. The converse is far from clear.

Question 2.42: Does the n–PpC version of [Mon17a, Theorem 4.4] hold?

Proposition 2.43: *Let $A \subseteq M^{\mathrm{eq}}$ containing $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(A))$ and $c \in \mathbf{K}(M)$. Then we can find types $p_i \in \mathcal{I}(\overline{M_i}/\mathcal{G}_i(A))$ and $q_i \in \mathcal{I}(M_i/\mathcal{G}_i(A))$ such that the partial type:*

$$\mathrm{tp}(c/A) \cup \bigcup_i p_i \cup \bigcup_i q_i$$

is consistent.

Proof. By Proposition **(2.41)**, we can find $P_i \subseteq \mathbf{B}_i(A)$ such that $\mathrm{tp}(c/A) \cup \bigcup_i \alpha_{P_i}$ is consistent. If any of the α_{P_i} is a realized type, then $c \in \mathbf{K}(A)$ and we are done.
We may assume that $c \vDash \alpha_{P_i}$. Pick any $d \in \bigcap_{b \in P_i} b(M_i)$ and let $f_m \in \overline{F_0}^{\mathrm{a}} \cap M$ be such that $f_m(c-d) \in \mathbf{K}(M_i)^m$. Then for any $d' \in \bigcap_{b \in P_i} b(M_i)$, $f_m(c-d') \in \mathbf{K}(M_i)^m$ since d and d' are much closer to each other than to c. Note also that for any $e \notin \bigcap_{b \in P_i} b(M_i)$, $f_m(c-e) \in \mathbf{K}(M_i)^m$ if and only if $f_m(d-e) \in \mathbf{K}(M_i)^m$, a value that does not depend on the choice of c.
So $\alpha_{P_i}(x)|_{M_i} \cup \{f_m(x-d) \in \mathbf{K}(M_i)^m : m \in \mathbb{Z}_{>0}\}$ generates $q_i(x) := \mathrm{tp}_{\mathcal{L}_i}(c/M_i)$, which is therefore in $\mathcal{I}(M_i/\mathcal{G}_i(A))$. □

The higher dimension version of Proposition **(2.43)** follows formally by induction from Corollary **(2.39)**.

Theorem 2.44: *Let $A \subseteq M^{\mathrm{eq}}$ containing $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(A))$ and $c \in \mathbf{K}^m(M)$. Then we can find types $p_i \in \mathcal{I}(\overline{M_i}/\mathcal{G}_i(A))$ and $q_i \in \mathcal{I}(M_i/\mathcal{G}_i(A))$ such that the partial type:*

$$\mathrm{tp}(c/A) \cup \bigcup_i p_i \cup \bigcup_i q_i$$

is consistent.

Proof. We proceed by induction on m. Assume that we have $p_i(x) \in \mathcal{I}(\overline{M_i}/\mathcal{G}_i(A))$, $q_i \in \mathcal{I}(M_i/\mathcal{G}_i(A))$ and $a \vDash \bigcup_i p_i \cup \bigcup_i q_i$. Pick some $c \in \mathbf{K}(M)$. We want to find types $r_i(x,y) \in \mathcal{I}(\overline{M_i}/\mathcal{G}_i(A))$ and $s_i(x,y) \in \mathcal{I}(M_i/\mathcal{G}_i(A))$ such that $\mathrm{tp}(ac/A) \cup \bigcup_i r_i \cup \bigcup_i s_i$ are consistant.
Let $E_i = \mathcal{G}_i(\mathrm{acl}^{\mathrm{eq}}(Aa))$. By Corollary **(2.39)**, we find $p'_i \in \mathcal{I}(\overline{M_i}/\mathcal{G}_i(A))$ and $q'_i \in \mathcal{I}(M_i/\mathcal{G}_i(A))$ such that $E_i \vDash p'_i \cup q'_i$. Let $E := A \cup \bigcup_i E_i$, then $\mathcal{G}(\mathrm{acl}^{\mathrm{eq}}(E)) \subseteq E$. Let $N \succcurlyeq M$ containing $M \cup E$, N_i and $\overline{N_i}$ as in Notation **(2.24)**. Applying Proposition **(2.41)**, we can find $p''_i \in \mathcal{I}(\overline{N_i}/E_i)$, $q''_i \in \mathcal{I}(M_i/E_i)$ and $c^* \vDash \mathrm{tp}(c/E) \cup \bigcup_i p''_i \cup q''_i$. Let $r_i(x,y) := \{\phi(x,y) : \phi(x,a) \in p''_i\}$ and $s_i(x,y) := \{\phi(x,y) : \phi(x,a) \in q''_i\}$. It is easy to check that these two types have the required properties. □

Remark 2.45: Note that, so far, we have not used [HMR, Theorem 2.6] — elimination of imaginaries in p-adically closed fields. Moreover, if M is p-adically closed, then there is just one p-adic valuation on M and the type q_1 that we constructed is a complete type. We have just reproved a strong version of the invariant extension property (cf. [HMR, Corollary 4.7]), of which, by Remark **(1.20)**, weak elimination of imaginaries follows. The main difference between the two proofs is that the proof presented here focuses from the start on finding invariant extensions. The arguments in this paper could also easily be carried out in finite extensions of p-adically closed fields.

2.6 Amalgamation over geometric points

The goal of this section is to improve Montenegro's amalgamation result [Mon17b, Theorem 3.21] to allow amalgamation over bases of geometric points.

Fact 2.46: *Let $E \leq A \leq C$ be field extensions.*
(i) *If $E \leq C$ is regular, then $\overline{E}^{a} A \cap C = EA$.*
(ii) *If $A \leq C$ is regular, then $\overline{E}^{a} C \cap \overline{A}^{a} = \overline{E}^{a} A$*

Proof.
(i) Since C is linearly disjoint from $\overline{E}^{a}$ over E, then C is linearly disjoint from $\overline{E}^{a} A$ over A.
(ii) Since $\overline{A}^{a}$ is linearly disjoint from C over A, and $A \leq \overline{E}^{a} A \leq \overline{A}^{a}$, we also have that $\overline{A}^{a}$ is linearly disjoint from $\overline{E}^{a} AC = \overline{E}^{a} C$ over $\overline{E}^{a} A$. □

Lemma 2.47: *Let $A \subseteq \mathbf{K}(M)$. If $\overline{A}^{a} \cap \mathbf{K}(M) \subseteq A$, then $\overline{A}^{a} = \overline{F_0}^{a} A$.*

Proof. We have $\overline{A}^{a} \subseteq \overline{\mathbf{K}(M)}^{a} = \overline{F_0}^{a}\mathbf{K}(M)$. So $\overline{A}^{a} \subseteq \overline{F_0}^{a}\mathbf{K}(M) \cap \overline{A}^{a} = \overline{F_0}^{a} A$. The equality follows from Fact 2.46.(ii) and the fact that $A \leq \mathbf{K}(M)$ is regular. □

Lemma 2.48: *Let $A \subseteq M \preccurlyeq N \models T$ and $\mathbf{K}(A) \subseteq C \subseteq \mathbf{K}(N)$. Assume that $\mathbf{K}(\mathrm{dcl}(A)) \cap M \subseteq A$ and that, for some $i \leq n$, $\mathrm{qftp}_{\mathcal{L}_i}(C/M)$ is $\mathrm{Aut}_{\mathcal{L}}(M/A)$-invariant. Then C and and $\mathbf{K}(M)$ are algebraically disjoint over $\mathbf{K}(A)$.*

Proof. Let m be any finite tuple in C and V be its algebraic locus over $\mathbf{K}(M)$. We have to show that V is defined over $\mathbf{K}(B)$. Pick any $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M/A)$. Since $\mathrm{qftp}_{\mathcal{L}_i}(C/M)$ is $\mathrm{Aut}_{\mathcal{L}}(M/A)$-invariant, we also have $m \in \sigma_i(V) = \sigma(V)$ and hence $V \subseteq \sigma(V)$. It follows, using σ^{-1}, that $V = \sigma(V)$ and, by elimination of imaginaries in ACF, V is defined over $\mathbf{K}(\mathrm{dcl}(B)) = \overline{\mathbf{K}(B)}^{a} \cap M = \mathbf{K}(B)$. □

Let us first generalize the characterisation of types in bounded pseudo-p-adically closed fields (cf. [Jar91, Proposition 10.4] and [Mon17b, §6.4]) to allow geometric parameters:

Lemma 2.49: *Let $M, N \models T$, $A \subseteq M$, $B \subseteq N$ and $f : A \to B$ be an $\mathcal{L}$-isomorphism. Assume that $\overline{\mathbf{K}(A)}^{a} \cap M \subseteq A$ and $\overline{\mathbf{K}(B)}^{a} \cap N \subseteq B$ and N is $|M|^{+}$-saturated. Then there exists an $\mathcal{L}$-embedding $g : M \to N$ such that $\overline{\mathbf{K}(g(M))}^{a} \cap N \subseteq g(M)$.*

Proof. Let $\mathfrak{U}_i \models p\mathrm{CF}_i^{\mathcal{G}}$ be a sufficiently saturated and homogeneous and contain $\mathcal{G}_i(M) \cup \mathcal{G}_i(N)$. Let $A_i := \mathrm{acl}_{\mathcal{L}_i}(A)$, $B_i := \mathrm{acl}_{\mathcal{L}_i}(B)$ and $f_i : A_i \to B_i$ be an $\mathcal{L}_i$-isomorphism extending f. Also, for each i, let $p_i \in \mathcal{I}(\mathfrak{U}_i/B_i)$ extend $(f_i)_{\star}\mathrm{tp}_{\mathcal{L}_i}(\mathbf{K}(M)/A_i) = \{\phi(x, f_i(a)) : \phi(x, a) \in \mathrm{tp}_{\mathcal{L}_i}(\mathbf{K}(M)/A_i)\}$. Let $F_i \models p_i|_{NA_i}$. Note that the fields F_i are all isomorphic as fields (over $\mathbf{K}(N)\mathbf{K}(B)$) so we may identify them. Let $M^{\star}$ be the $\mathcal{L}$-structure whose underlying field is F_i and whose $\mathcal{L}_i$-structure is induced by $\mathfrak{U}_i$. Note that there is an $\mathcal{L}$-isomorphism between M and $M^{\star}$ extending f.
Note that since any automorphism of M extends to $\mathfrak{U}_i$, $\mathrm{qftp}_{\mathcal{L}_i}(M^{\star}/N)$ is $\mathrm{Aut}_{\mathcal{L}}(N/B)$-invariant. It follows, by Lemma **(2.48)**, $\mathbf{K}(M^{\star})$ is algebraically independent from $\mathbf{K}(N)$ over $\mathbf{K}(B)$. Also, the extension $\mathbf{K}(A) \leq \mathbf{K}(M)$ is regular, hence so are the extensions $\mathbf{K}(B) \leq \mathbf{K}(M^{\star})$ and $\mathbf{K}(N) \leq \mathbf{K}(M^{\star})\mathbf{K}(N)$. Since, $\mathbf{K}(M^{\star})\mathbf{K}(N) \leq \mathfrak{U}_i$, for all $i \leq n$, this extension is totally p-adic. Let $N_0 \preccurlyeq N$ be a small model containing B. As N is pseudo-p-adically closed, there exists

$M^\diamond \leq N$ which is $\mathcal{L}_{\mathrm{rg}}(N_0)$-isomorphic to $M^\star$. Since $\mathcal{L}$ is a definable enrichment of $\mathcal{L}_{\mathrm{rg}}(F_0)$, it follows that $M^\diamond$ is $\mathcal{L}(B)$-isomorphic to $M^\star$. Composing with the $\mathcal{L}$-isomorphism between M and $M^\star$, we get an $\mathcal{L}$-isomorphism $g : M \to M^\diamond$ extending f.

Claim 2.50: $\overline{\mathbf{K}(M^\diamond)}^{\mathrm{a}} \cap N = \mathbf{K}(M^\diamond)$

Proof. Note that $M^\diamond$ is $\mathcal{L}$-isomorphic to M and hence $M^\diamond \models T$. It follows that $\overline{\mathbf{K}(M^\diamond)}^{\mathrm{a}} = \overline{F_0}^{\mathrm{a}}\mathbf{K}(M^\diamond)$. Since $F_0 \leq N$ is regular, by Fact 2.46.(i), $\overline{\mathbf{K}(M^\diamond)}^{\mathrm{a}} \cap N = \overline{F_0}^{\mathrm{a}}\mathbf{K}(M^\diamond) \cap N = \mathbf{K}(M^\diamond)$. ◊

This concludes the proof. □

Proposition 2.51: *Let $M, N \models T$, $A \subseteq M$, $B \subseteq N$ and $f : A \to B$ be an $\mathcal{L}$-isomorphism. Assume that $\overline{\mathbf{K}(A)}^{\mathrm{a}} \cap M \subseteq A$ and $\overline{\mathbf{K}(B)}^{\mathrm{a}} \cap N \subseteq B$. Then f is elementary.*

Proof. Assume M and N are sufficiently saturated. We proved in Lemma **(2.49)** that the set of $\mathcal{L}$-isomorphisms between (small) $A \subseteq M$ and $B \subseteq N$ such that $\overline{\mathbf{K}(A)}^{\mathrm{a}} \cap M \subseteq A$ and $\overline{\mathbf{K}(B)}^{\mathrm{a}} \cap N \subseteq B$ has the back-and-forth (if it is not empty). It follows that any such isomorphism is elementary. □

Using the results of Section 2.3 and and the fact that the $\mathcal{L}$-structure on the sort $\mathbf{K}$ in T is a definable expansion of the ring language, we can improve this last result:

Corollary 2.52: *Let $F_0 \subseteq E \subseteq M$ and $a \in M$ be a tuple, then*

$$\mathrm{qftp}_{\mathcal{L}}(a/E) \cup \mathrm{qftp}_{\mathcal{L}_{\mathrm{rg}}(F_0)}(\overline{\mathbf{K}(E)\mathbf{K}(a)}^{\mathrm{a}} \cap M/\mathbf{K}(E)) \vdash \mathrm{tp}(a/E).$$

Proof. It suffices to prove that if we have $A, B \subseteq M$ containing F_0,$f : A \to B$ an $\mathcal{L}$-isomorphism and $g : \overline{\mathbf{K}(A)}^{\mathrm{a}} \cap M \to \overline{\mathbf{K}(B)}^{\mathrm{a}} \cap M$ an $\mathcal{L}_{\mathrm{rg}}(F_0)$-isomorphism such that $f|_{\mathbf{K}(A)} = g|_{\mathbf{K}(A)}$, then f is an elementary $\mathcal{L}$-isomorphism. By Claim **(2.23)**, g is an $\mathcal{L}$-isomorphism. By Proposition **(2.51)**, g is an elementary $\mathcal{L}$-isomorphism which extends to $\sigma \in \mathrm{Aut}_{\mathcal{L}}(M)$. Let $C := \sigma(A)$. It suffices to prove that $C \equiv_{\mathcal{L}} B$. We know that $\mathbf{K}(C) = \mathbf{K}(B)$ so it suffices to prove that $\mathcal{G}^{\mathrm{im}}(C) \equiv_{\mathcal{L}(\mathbf{K}(B))} \mathcal{G}^{\mathrm{im}}(B)$, where $\mathcal{G}^{\mathrm{im}}$ is the set of all $\mathcal{L}$-sorts except for $\mathbf{K}$. By hypothesis, $C \equiv^{\mathrm{qf}}_{\mathcal{L}} A \equiv^{\mathrm{qf}}_{\mathcal{L}} B$ so $\mathcal{G}^{\mathrm{im}}(C) \equiv^{\mathrm{qf}}_{\mathcal{L}(\mathbf{K}(B))} \mathcal{G}^{\mathrm{im}}(B)$. But, by Proposition **(2.29)**, this is equivalent to $\mathcal{G}^{\mathrm{im}}(C) \equiv_{\mathcal{L}(\mathbf{K}(B))} \mathcal{G}^{\mathrm{im}}(B)$. □

Lemma 2.53: *Let L_1, $L_2 \models p\mathrm{CF}_\forall$ and k_0 be a common $\mathcal{L}_{\mathrm{Mac}}$-substructure of L_1 and L_2 such that L_1 and L_2 are linearly disjoint over k_0. Then L_1L_2 can be made into an $\mathcal{L}_{\mathrm{Mac}}$-structure extending that L_1 and L_2 such that $L_1L_2 \models p\mathrm{CF}_\forall$.*

Proof. Since pCF eliminates quantifiers in $\mathcal{L}_{\mathrm{Mac}}$, there exists $\mathfrak{U} \models pCF$ containing both L_1 and L_2 as $\mathcal{L}_{\mathrm{Mac}}$-substructures. By induction, it suffices to consider the case where $L_1 = k_0(a)$. If a is algebraic over k_0, then the minimal polynomial of a over k_0 and L_2 coincide. So $k_0(a)$ and L_2 are linearly independent over k_0 (as subfields of $\mathfrak{U}$) and we are done. If a is transcendental over k_0, let $c \in \mathfrak{U} \setminus \overline{L_2}^{\mathrm{a}}$ realize $\mathrm{tp}_{\mathcal{L}_{\mathrm{Mac}}}(a/k_0)$. Then $k_0(c)$ is linearly independent from L_2 over k_0 and we are also done. □

Remark 2.54: The above proof is not really about p-adically closed fields. It holds of any theory of (enriched) fields that eliminates quantifiers and is algebraically bounded.

Theorem 2.55: *Let $M \models T$, $E \subseteq M$ and a_1, a_2, c_1, c_2, $c \in \mathbf{K}(M)$ be such that $\overline{\mathbf{K}(E)}^{\mathrm{a}} \cap M \subseteq E$, $\overline{\mathbf{K}(E)(a_1)}^{\mathrm{a}} \cap \overline{\mathbf{K}(E)(a_2)}^{\mathrm{a}} = \overline{\mathbf{K}(E)}^{\mathrm{a}}$, $c \mathop{\downarrow}^{\mathrm{i,qf}}_{E} a_1 a_2$, $c_1 \equiv_{\mathcal{L}(E)} c_2$, $c \equiv^{\mathrm{qf}}_{\mathcal{L}(Ea_1)} c_1$ and $c \equiv^{\mathrm{qf}}_{\mathcal{L}(Ea_2)} c_2$. Then $\mathrm{tp}(c_1/Ea_1) \cup \mathrm{tp}(c_2/Ea_2) \cup \mathrm{qftp}_{\mathcal{L}}(c/Ea_1a_2)$ is satisfiable.*

Proof. Let p be a quantifier free $\mathrm{Aut}_{\mathcal{L}}(M/E)$-invariant type extending the quantifier free type of c over Ea_1a_2. Choosing $c \models p$ in some $N \succcurlyeq M$, we may assume that $c \mathop{\downarrow}_E M$. By Lemma **(2.48)**, $\mathbf{K}(E)(c)$ is algebraically disjoint from $\mathbf{K}(M)$ over $\mathbf{K}(E)$. Let $A_j := \overline{\mathbf{K}(E)a_j}^{\mathrm{a}} \cap M$, $C_j := \overline{\mathbf{K}(E)c_j}^{\mathrm{a}} \cap M$, $C := \overline{\mathbf{K}(E)c}^{\mathrm{a}} \cap M$, $F = \overline{A_1A_2}^{\mathrm{a}} \cap M$ and $B_j := \overline{A_jC_j}^{\mathrm{a}} \cap M$. Since $c \equiv^{\mathrm{qf}}_{\mathcal{L}(Ea_j)} c_j$ and $\mathrm{acl}_{\mathcal{L}_i} = \mathrm{dcl}_{\mathcal{L}_i}$, we have that $C \equiv^{\mathrm{qf}}_{\mathcal{L}(A_j)} C_j$. So there exists an $\mathcal{L}(A_j)$-isomorphism $\phi_j : A_jC_j \to A_jC$ sending C_j to C. We can extend this isomorphism to an $\mathcal{L}_{\mathrm{rg}}(F_0)$-isomorphism $\overline{\phi}_j : \overline{A_jC_j}^{\mathrm{a}} \to \overline{A_jC}^{\mathrm{a}}$. Let $D_j := \overline{\phi}_j(B_j)$. A picture of the involved fields might help:

$$
\begin{tikzcd}
B_1 \arrow[r, "\overline{\phi}_1"] \arrow[d, dash] & D_1 \arrow[d, dash] & F \arrow[d, dash] & D_2 \arrow[d, dash] & B_2 \arrow[l, "\overline{\phi}_2"'] \arrow[d, dash] \\
A_1C_1 \arrow[r, "\phi_1"] \arrow[d, dash] \arrow[dr, dash] & A_1C \arrow[d, dash] \arrow[dr, dash] & A_1A_2 \arrow[dl, dash] \arrow[dr, dash] & A_2C \arrow[d, dash] \arrow[dl, dash] & A_2C_2 \arrow[l, "\phi_2"'] \arrow[d, dash] \arrow[dl, dash] \\
C_1 \arrow[d, dash] & A_1 \arrow[d, dash] & C \arrow[d, dash] & A_2 \arrow[d, dash] & C_2 \arrow[d, dash] \\
\mathbf{K}(E)(c_1) \arrow[drr, dash] & \mathbf{K}(E)(a_1) \arrow[dr, dash] & \mathbf{K}(E)(c) \arrow[d, dash] & \mathbf{K}(E)(a_2) \arrow[dl, dash] & \mathbf{K}(E)(c_2) \arrow[dll, dash] \\
 & & \mathbf{K}(E) & &
\end{tikzcd}
$$

By [Cha02, Lemma 2.5.(2)],

$$\overline{CA_1}^{\mathrm{a}} \cap \overline{CA_2}^{\mathrm{a}}\overline{A_1A_2}^{\mathrm{a}} = \overline{\overline{C}^{\mathrm{a}}(\overline{A_1}^{\mathrm{a}} \cap \overline{A_2}^{\mathrm{a}})}^{\mathrm{a}}\overline{A_1}^{\mathrm{a}} = \overline{C}^{\mathrm{a}}\overline{A_1}^{\mathrm{a}} = \overline{\mathbf{K}(E)}^{\mathrm{a}}CA_1.$$

The last equality follows from Lemma **(2.47)**. Since $\mathbf{K}(E) \leq C_2F \leq \mathbf{K}(M)$ is regular, it follows $\mathbf{K}(E) \leq D_2F$ is regular and hence that D_2F is linearly disjoint from $\overline{\mathbf{K}(E)}^{\mathrm{a}}CA_1$ over CA_1. In conclusion, we have that $\overline{CA_1}^{\mathrm{a}} \cap D_2F = \overline{\mathbf{K}(E)}^{\mathrm{a}}CA_1 \cap D_2F = CA_1$, i.e. the extension $CA_1 \leq D_2F$ is regular. By a symmetric argument, the extension $CA_2 \leq D_1F$ is also regular. Since D_2F is linearly disjoint from $D_1 \leq \overline{CA_1}^{\mathrm{a}}$ over CA_1 and the extension $CA_1 \leq D_2F$ is regular, then so is the extension $D_2F \leq D_1D_2F$. Since $\mathbf{K}(E) \leq F$ is regular, it follows that D_1D_2F is linearly disjoint from $\overline{F}^{\mathrm{a}} = \overline{\mathbf{K}(E)}^{\mathrm{a}}F$ over F, i.e. the extension $F \leq D_1D_2F$ is regular.
Note that $CF \subseteq N$ can be made into a model of $\bigcup_i p\mathrm{CF}_{i,\forall}$. Also, B_j can be made into a model of $\bigcup_i p\mathrm{CF}_{i,\forall}$ and hence so does D_j by transfert. Note that since ϕ_j is an $\mathcal{L}$-isomorphism, the $\mathcal{L}_i$-structures induced by D_j and CF on CA_j coincide. Note also, that since $CA_j \leq D_lF$, where

$l \neq j$ is regular, D_j and CF are linearly disjoint over CA_j. By Lemma **(2.53)**, it follows that D_jF can be made into a model of $p\mathrm{CF}_{i,\forall}$ whose $\mathcal{L}_i$-structure extends that of CF. Since $CA_1 \leq D_2F$ is regular, we also have that D_1F and D_2F are linearly disjoint over CF. By Lemma **(2.53)** again, D_1D_2F can be made into a model of $p\mathrm{CF}_{i,\forall}$ whose $\mathcal{L}_i$-structure extends that of F.
Recall that $\mathbf{K}(E)(c)$ is algebraically disjoint from $\mathbf{K}(M)$ over $\mathbf{K}(E)$. It follows that CF is algebraically disjoint from $\mathbf{K}(M)$ over F and thus that $D_1D_2F \subseteq \overline{CF}^{\mathrm{a}}$ is algebraically disjoint from $\mathbf{K}(M)$ over F. Since $F \leq \mathbf{K}(M)$ is regular, $D_1D_2F \subseteq \overline{CF}^{\mathrm{a}}$ and $\mathbf{K}(M)$ are in fact linearly disjoint over F and, since $F \leq D_1D_2F$ is also regular, $M \leq D_1D_2\mathbf{K}(M)$ is regular. Moreover, by Lemma **(2.53)**, D_1D_2M can be made into a model of $p\mathrm{CF}_{i,\forall}$ for all i, i.e. the extension if totally p-adic. Since $M \models n\text{–P}p\mathrm{C}$, by Theorem **(2.16)**, there exists $M^\star \succcurlyeq M$ containing D_1D_2F as an $\mathcal{L}$-subtrucrture.
Since $\overline{B_j}^{\mathrm{a}} = \overline{\mathbf{K}(E)}^{\mathrm{a}}B_j$, we also have $\overline{D_j}^{\mathrm{a}} = \overline{\mathbf{K}(E)}^{\mathrm{a}}B_j$. As $\mathbf{K}(E)$ is regular in $\mathbf{K}(M^\star)$, it then follows that $\overline{D_j}^{\mathrm{a}} \cap M^\star = D_j$. Note that, since the $\mathcal{L}$-structure on $M(c)$ coincides with the one we built on D_1D_2F and hence with the one in $N(c)$, $\mathrm{qftp}_{\mathcal{L}}^{M^\star}(c/Ea_1a_2) = \mathrm{qftp}_{\mathcal{L}}^{N}(c/Ea_1a_2)$ has not changed and we still have that $c \equiv^{\mathrm{qf}}_{\mathcal{L}(Ea_j)} c_j$. By construction, we also have $D_j \equiv^{\mathrm{qf}}_{\mathcal{L}(A_j)} B_j$. It follows, from Corollary **(2.52)**, that $c \equiv_{\mathcal{L}(Ea_j)} c_j$. □

2.7 Finite sets

Our goal, in this section, is to prove that finite sets are coded. We first prove that finite sets are coded in the algebraic closure of a n–PpC field equipped with extensions of the geometric language for each valuation and then we conclude with Lemma **(2.38)**. The proof technique is inspired by Johnson's account of elimination of imaginaries in algebraically closed valued fields [Joh, §6.2].
In what follows, let F be an algebraically closed field and v_i be n independent valuations on F. Recall Notation **(2.20)**. Let $\mathcal{O}_i$ also denote the valuation ring for v_i and $\mathcal{O} = \bigcap_i \mathcal{O}_i$. Let $\overline{\mathcal{L}} := \bigcup_i \overline{\mathcal{L}}_i$. The field F can naturally be made into an $\overline{\mathcal{L}}$-structure.
A quantifier free $\overline{\mathcal{L}}$-type $p(x)$ over F is said to be *definable* if for each quantifier free $\overline{\mathcal{L}}_i$-formula $\phi(x,y)$, there exists a quantifier free $\overline{\mathcal{L}}_i$-formula $\theta(y)$ such that $\phi(x,a) \in p$ if and only if $\models \theta(a)$. If $p(x)$ and q are quantifier free $\overline{\mathcal{L}}$-types over F, then we define $p \otimes q$ to be the quantifier free definable $\overline{\mathcal{L}}$-type of tuples ab where $a \models p$ and $b \models q|_{Fa}$. It is also a quantifier free definable $\overline{\mathcal{L}}$-type. The type p is said to be *symmetric* if for any quantifier free definable $\overline{\mathcal{L}}$-type q, $p \otimes q = q \otimes p$, i.e. if $a \models p$ and $b \models q|_{Fb}$, then $a \models p|_{Fb}$. This happens if and only if, for all i, $p|_{\overline{\mathcal{L}}_i}$ is a generically stable type in $\mathrm{ACVF}_i^{\mathcal{G}}$.

Lemma 2.56: *Pick any $s \in \mathrm{S}_m(F) := \mathrm{GL}_m(F)/\mathrm{GL}_m(\mathcal{O})$. There exists a quantifier free symmetric s-definable $\overline{\mathcal{L}}$-type q_s such that $q_s(x) \vdash x \in s$.*

Proof. Let k_i be the residue field for the valuation v_i and p_i be the generic type of $\mathrm{GL}_m(k_i)$. Let q_i be the generically stable type of matrices in $\mathrm{GL}_m(\mathcal{O})$ whose reduct modulo $\mathfrak{M}$ realizes p_i. Note that, by independence of the valuations, $q := \bigcup_i q_i$ is consistent and also that, since q_i is invariant by multiplication in $\mathrm{GL}_m(\mathcal{O}_i)$, q is invariant by multiplication in $\mathrm{GL}_m(\mathcal{O})$. Pick any $M \in s(F)$. The quantifier free definable $\overline{\mathcal{L}}$-type q_s of elements MN for some $N \models q$ does not depend on the choice of M and it is quantifier free symmetric s-definable. □

Lemma 2.57: *Let C be a finite subset of $\mathrm{S}_m(F) \times F^l$ for some m and l. Then C is coded in $\mathcal{G} := \bigcup_i \mathcal{G}_i$.*

Proof. By Lemma **(2.56)**, for all $c \in C$, there exists a quantifier free symmetric c-definable type p_c concentrating on c, i.e. if $c = c_1 c_2$ with $c_2 \in \mathbf{K}^l$, then $p_c(x_1, x_2) \vDash x_1 \in c_1 \wedge x_2 = c_2$. Let s be the definable map sending a finite subset of $\mathbf{K}^{|x_1|+|x_2|}$ of size $|C|$ to its code. Note that we can choose the image of s to be a subset of some $\mathbf{K}^m$. Let $p_C = s_\star(\bigotimes_{c \in C} p_c)$. Note that since each p_c is symmetric, the type p_C does not depend on a choice of enumeration for C. So p_C is a $\ulcorner C \urcorner$-definable quantifier free type and C is $\overline{\mathcal{L}}(\ulcorner p_C \urcorner)$-definable. By elimination of imaginaries in ACVF_i, each $p_C|_{\overline{\mathcal{L}}_i}$ has a canonical basis in $\mathcal{G}_i$ and since $p_C = \bigcup_i p_C|_{\overline{\mathcal{L}}_i}$, p_C, and hence C, is $\mathcal{G}(\ulcorner C \urcorner)$-definable. □

Corollary 2.58: *Let $M \vDash T$, and C be a finite subset of $\mathcal{G}^k(M)$. Then C is coded in $\mathcal{G}$.*

Proof. Any product of sorts from $\mathcal{G}_i^{\mathrm{im}}$ can be $\mathcal{L}_i$-definably embedded in S_m^i for large enough m. Also, there is an $\mathcal{L}$-definable injection $\mathrm{S}_m^i(\mathbf{K}) \to \mathrm{S}_m(\mathbf{K})$ sending a coset of $\mathrm{GL}_m(\mathcal{O}_i)$ to its intersection with $\bigcap_{j \neq i} \mathrm{GL}_m(\mathcal{O}_j)$. So we may assume that C is a subset of $\mathrm{S}_m \times \mathbf{K}^l$ for some m and l. By Lemma **(2.57)**, C is coded by some $\epsilon \in \overline{M}^{\mathrm{a}}$. Note that since C is $\overline{\mathcal{L}}(M)$-definable, $\mathcal{G}_i(\epsilon) \subseteq \mathrm{dcl}_{\overline{\mathcal{L}}_i}(M)$. Thus, by Lemma **(2.38)**, we can find $\eta_i \in M$ such that $\mathcal{G}_i(\epsilon)$ and η_i are interdefinable in the pair $(\overline{M_i}, M)$. It follows that $\bigcup_i \eta_i$ is a code for C in M. □

References

[ACP14] H. Adler, E. Casanovas, and A. Pilay. Generic stability and stability. *J. Symb. Log.* 79 (2014), pp. 179–185.

[Ax68] J. Ax. The elementary theory of finite fields. *Ann. of Math. (2)* 88 (1968), pp. 239–271.

[Bas84] Ş. A. Basarab. Definite functions on algebraic varieties over ordered fields. *Rev. Roumaine Math. Pures Appl.* 29 (1984), pp. 527–535.

[Cha02] Z. Chatzidakis. Properties of forking in ω-free pseudo-algebraically closed fields. *J. Symb. Log.* 67.3 (2002), pp. 957–996.

[CH99] Z. Chatzidakis and E. Hrushovski. Model theory of difference fields. *Trans. Amer. Math. Soc.* 351.8 (1999), pp. 2997–3071.

[EJ90] I. Efrat and M. Jarden. Free pseudo p-adically closed fields of finite corank. *J. Algebra* 133.1 (1990), pp. 132–150.

[FJ08] M. D. Fried and M. Jarden. *Field arithmetic*. Third. Vol. 11. Ergeb. Math. Grenzgeb. (3). Revised by Jarden. Springer-Verlag, Berlin, 2008, pp. xxiv+792.

[Gro87] C. Grob. Die Entscheidbarkeit der Theorie der maximalen pseudo p-adisch abgeschlossenen Körper. *Konstanzer Dissertationen* 202 (1987).

[HJ88] D. Haran and M. Jarden. The absolute Galois group of a pseudo p-adically closed field. *J. Reine Angew. Math.* 383 (1988), pp. 147–206.

[HHM06] D. Haskell, E. Hrushovski, and D. Macpherson. Definable sets in algebraically closed valued fields: elimination of imaginaries. *J. Reine Angew. Math.* 597 (2006), pp. 175–236.

[Hru02] E. Hrushovski. Pseudo-finite fields and related structures. In: *Model theory and applications*. Vol. 11. Quad. Mat. Aracne, Rome, 2002, pp. 151–212.

[Hru14] E. Hrushovski. Imaginaries and definable types in algebraically closed valued fields. In: *Valuation Theory in Interaction*. Ed. by A. Campillo Lopez, F.-V. Kuhlmann, and B. Teissier. EMS Ser. Congr. Rep. 10. Eur. Math. Soc., Zürich, 2014, pp. 297–319.

[HMR] E. Hrushovski, B. Martin, and S. Rideau. Definable equivalence relations and zeta functions of groups. With an appendix by R. Cluckers. To appear in J. Eur. Math. Soc. (JEMS).

[HP11] E. Hrushovski and A. Pillay. On NIP and invariant measures. *J. Eur. Math. Soc. (JEMS)* 13.4 (2011), pp. 1005–1061.

[Jar91] M. Jarden. Algebraic realization of p-adically projective groups. *Contemp. Math.* 79.1 (1991), pp. 21–62.

[Joh] W. Johnson. On the proof of elimination of imaginaries in algebraically closed valued fields. arXiv: 1406.3654.

[Mac76] A. Macintyre. On definable subsets of p-adic fields. *J. Symb. Log.* 41.3 (1976), pp. 605–610.

[Mon17a] S. Montenegro. Imaginaries in bounded pseudo real closed fields. *Ann. Pure Appl. Logic* 168.10 (2017), pp. 1866–1877.

[Mon17b] S. Montenegro. Pseudo real closed fields, pseudo p-adically closed fields and NTP_2. *Ann. Pure Appl. Logic* 168.1 (2017), pp. 191–232.

[Pre82] A. Prestel. *Pseudo real closed fields, Set Theory and Model Theory*. Lecture Notes in Math. 872. Springer, 1982.

[PZ78] A. Prestel and M. Ziegler. Model-theoretic methods in the theory of topological fields. *J. Reine Angew. Math.* 299-300 (1978), pp. 318–341.

[Rid] S. Rideau. Imaginaries in valued differential fields. To appear in J. Reine Angew. Math.

[SS14] P. Simon and S. Starchenko. On forking and definability of types in some dp-minimal theories. *J. Symb. Log.* 79.4 (2014), pp. 1020–1024.

[TZ12] K. Tent and M. Ziegler. *A Course in Model Theory*. Assoc. Symbol. Logic, Chicago, IL. Lect. Notes Log. 40. Cambridge Univ. Press, Cambridge, 2012.