Imaginaries and invariant types in existentially closed valued differential fields

Silvain Rideau*

August 31, 2015

We answer two open questions about the model theory of valued differential fields introduced by Scanlon [Sca00]. We show that they eliminate imaginaries in the geometric language introduced by Haskell, Hrushovski and Macpherson and that they have the invariant extension property. These two result follow from an abstract criterion for the density of definable types in enrichments of algebraically closed valued fields. Finally, we show that this theory is metastable.

In [Sca00], Scanlon showed that the model theory of equicharacteristic zero fields equipped with both a valuation and a contractive derivation (i.e. a derivation ∂ such that for all x, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$) is reasonably tractable. Indeed, Scanlon proved a quantifier elimination theorem for certain of these fields and showed the existence of an elementary class of existentially-closed such valued differential fields, which we will, from now on, denote $\mathrm{VDF}_{\mathcal{EC}}$. In this paper, we wish to investigate further their model theoretic properties.
One of the model theoretic questions we address is the elimination of imaginaries. A theory is said to eliminate imaginaries if for every $\varnothing$-definable set D and every $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f such that xEy if and only if $f(x) = f(y)$; in other words, a theory eliminates imaginaries if the category of definable sets is closed under quotients. In [HHM06], Haskell, Hrushovski and Macpherson proved that, although algebraically closed valued fields (ACVF) cannot eliminate imaginaries in any of the "usual" languages, it suffices to add certain quotients, the so-called "geometric sorts" to obtain elimination of imaginaries. By analogy with the fact that differentially closed fields of characteristic zero (DCF_0) have no more imaginaries than algebraically closed fields (ACF), it was conjectured that $\mathrm{VDF}_{\mathcal{EC}}$ also eliminates imaginaries in the geometric language with a symbol added for the derivation.
To prove their elimination results Haskell, Hrushovski and Macpherson developed the theory of "stable domination" and "metastability", an attempt at formalising the idea that, if we ignore the value group, algebraically closed valued fields behave in a very stable-like way — see Section 1.3 for precise definitions. Few examples of metastable theories are known, but

*Partially supported by ValCoMo (ANR-13-BS01-0006)

$\mathrm{VDF}_{\mathcal{EC}}$ seemed like a promising candidate. Once again, the analogy with differentially closed fields is tempting. Among stable fields, algebraically closed fields are extremely well understood but are too tame (they are strongly minimal) for any of the more subtle behaviour of stability to appear. The theory DCF_0 of differentially closed fields in characteristic zero, on the other hand, is still quite tame (it is ω-stable) but some pathologies begin to show and in studying DCF_0 one gets a better understanding of stability. The theory $\mathrm{VDF}_{\mathcal{EC}}$ could play a similar role with respect to ACVF: it is a more complicated but still very tractable theory in which to experiment with metastability, provided we prove it is metastable.

Nevertheless, it was quickly realised that the metastability of $\mathrm{VDF}_{\mathcal{EC}}$ was still an open question, one of the difficulties being to prove the invariant extension property. A theory has the invariant extension property if, as in stable theories, every type over an algebraically closed set A has a "nice" global extension: an extension which is preserved under all automorphisms that fix A (Definition **(1.14)**).

In Theorem **(1.18)**, we solve these three questions, by showing that $\mathrm{VDF}_{\mathcal{EC}}$ eliminates imaginaries in the geometric language, has the invariant extension property and is metastable over its value group.

Following the general idea of [Hru14; Joh], elimination of imaginaries relative to the geometric sorts is obtained as a consequence, on the one hand, of the density of definable types over algebraically closed parameters and, on the other hand, of being able to control the canonical basis of definable types in $\mathrm{VDF}_{\mathcal{EC}}$. This second part of the problem is tackled in [RS]. Moreover, the invariant extension property is also a consequence of the density of definable types. Hence, one of the goals of this paper (which will occupy us from Sections 6 to 9) is to prove density of definable types in $\mathrm{VDF}_{\mathcal{EC}}$: given any A-definable set X in a model of $\mathrm{VDF}_{\mathcal{EC}}$, we find a type in X which is definable over the algebraic closure of A.

To be precise, we will be working in the more general context of a theory $\widetilde{T}$ enriching a theory T which is itself a C-minimal enrichment of ACVF, see Section 6 for precise definitions. But for the sake of clarity, in this introduction, we will focus on the example where T is ACVF and $\widetilde{T}$ is $\mathrm{VDF}_{\mathcal{EC}}$.

Let us fix some notations. Let $\mathcal{L}_{\mathrm{div}}$ be the one sorted language for ACVF and $\mathcal{L}_{\partial,\mathrm{div}} := \mathcal{L}_{\mathrm{div}} \cup \{\partial\}$ be the one sorted language for $\mathrm{VDF}_{\mathcal{EC}}$, where ∂ is a symbol for the derivation. It follows from quantifier elimination in $\mathrm{VDF}_{\mathcal{EC}}$ that, to describe the $\mathcal{L}_{\partial,\mathrm{div}}$-type of x (denoted p), it suffices to give the $\mathcal{L}_{\mathrm{div}}$-type of $\partial_\omega(x) := (\partial^n(x))_{n<\omega}$ (denoted $\nabla_\omega(p)$) and that p is consistent with X if and only if $\nabla_\omega(p)$ is consistent with $\partial_\omega(X)$, the image of X under the map ∂_ω. Note that $\nabla_\omega(p)$ is the pushforward of p by ∂_ω restricted to $\mathcal{L}_{\mathrm{div}}$ and $\nabla_\omega(p)$ is definable if and only if p is. Therefore it will be enough to find a "generic" definable $\mathcal{L}_{\mathrm{div}}$-type q consistent with $\partial_\omega(X)$.

A definable $\mathcal{L}_{\mathrm{div}}$-type is a consistent collection of definable Δ-types where Δ is a finite set of $\mathcal{L}_{\mathrm{div}}$-formulas and so we can ultimately reduce to finding, for any such finite Δ a "generic" definable Δ-type consistent with some $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable set (see Proposition **(9.5)**). It follows that most of the preparatory work in the second part of this paper (Sections 6 to 8) will attempt to better understand Δ-types for finite Δ in ACVF.

An example of this somewhat convoluted back and forth between two languages $\mathcal{L}_{\mathrm{div}}$ and $\mathcal{L}_{\partial,\mathrm{div}}$ is essentially underlying the proof of elimination of imaginaries in DCF_0; in that case the back and forth is between the language of rings and the language of differential rings,

although, in the classical proof, it may not appear clearly. One might think that this example is too simple, but it is, in fact, quite revealing of what is going on in what follows. Take any set X definable in DCF_0, let $X_n := \partial_n(X)$ where $\partial_n(x) := (\partial^i(x))_{0\leqslant i\leqslant n}$ and let Y_n be the Zariski closure of X_n. Now, choose a consistent sequence $(p_n)_{n<\omega}$ of ACF-types such that p_n has maximal Morley rank in Y_n. Because ACF is stable all the p_n are definable and, by elimination of imaginaries in ACF, they already have canonical bases in the fields itself. Then the complete type of points x such that $\partial_n(x) \vDash p_n$ is also definable with a canonical basis of field points and it is obviously consistent with X.
In ACVF, we cannot use the Zariski closure because we also need to take into account valuative inequalities. But the balls in ACVF are combinatorially well-behaved, and we can approximate sets definable in $\mathrm{VDF}_{\mathcal{EC}}$ by finite fibrations of balls over lower dimensional sets, i.e. cells in the C-minimal setting (see Section 9). Because C-minimality is really the core property of ACVF which we are using, the results presented here generalise naturally to any C-minimal extension of ACVF. Although that might seem like unnecessary generalisation, we hope it might lead in the future to a proof that $\mathrm{VDF}_{\mathcal{EC}}$ with analytic structure has the invariant extension property and has no more imaginaries than ACVF with analytic structure (denoted $\mathrm{ACVF}_{\mathcal{A}}$), even though we know that $\mathrm{ACVF}_{\mathcal{A}}$ does not eliminate imaginaries in the geometric language and we have no concrete idea of what those analytic imaginaries might be (see [HHM13]).

The paper is organised as follows. The first part contains model theoretic considerations about $\mathrm{VDF}_{\mathcal{EC}}$. In Section 1, we give some background and state Theorem **(1.18)**, our main new theorem about $\mathrm{VDF}_{\mathcal{EC}}$ whose proof uses most of what appears later in the paper. Section 2 consists in an exploration of the properties of an analogue of the prolongations on the type space. In Section 3, we prove that the constant field is stably embedded in models of $\mathrm{VDF}_{\mathcal{EC}}$. In Section 4, we study the definable and algebraic closures and finally in Section 5, we prove that metastability bases exist in $\mathrm{VDF}_{\mathcal{EC}}$.
As announced earlier, Sections 6 to 9 contain the proof of Theorem **(9.8)**, an abstract criterion for the density of definable types. In Section 6 we study certain "generic" Δ-types, for Δ finite, in a C-minimal expansion T of ACVF (see Definition **(6.12)**). In Section 7, we introduce the notion of *quantifiable* types and we show that the previously defined "generic" types are quantifiable. In Section 8, we consider definable families of functions into the value group, in ACVF and $\mathrm{ACVF}_{\mathcal{A}}$, and show that their germs are internal to the value group and in Section 9, we put everything together to prove Theorem **(9.8)**. Finally, in Section 10, we show how this density result can be used to give a criterion for elimination of imaginaries and the invariant extension property.
This paper also contains, as an appendix, improvements of known results on stable embeddedness in pairs of valued fields which are used in Section 3 and in order to apply the results of [RS].

I would like to thank my PhD advisors, Tom Scanlon and Élisabeth Bouscaren, for our discussions, all their corrections, and, most importantly, their endless support. I would also like to thank Pierre Simon and Martin Hils for our many enlightening discussions.

Model theory of valued differential fields

1. Background and main results

Let us now fix some notation. Whenever X is a definable set (or a union of definable sets) and A is a set of parameters, $X(A)$ will denote $X \cap A$. Usually in this notation there is an implicit definable closure, but we want to avoid that here because more often than not there will be multiple languages around and hence multiple definable closures which could be implicit. Similarly, if $\mathcal{S}$ is a set of definable sorts, we will write $\mathcal{S}(A)$ for $\bigcup_{S \in \mathcal{S}} S(A)$.
Also, the symbol $\subset$ will denote strict inclusion.

For all the definitions concerning stability or the independence property, we refer the reader to [Sim].

1.1. Valued differential fields

We will mostly study *equicharacteristic zero* valued fields in the leading term language. It consists of three sorts $\mathbf{K}$, $\mathbf{RV}$ and $\mathbf{\Gamma}$ with the ordered group language on $\mathbf{\Gamma}$, the ring language on $\mathbf{RV}$ and $\mathbf{K}$ and maps $\mathrm{rv} : \mathbf{K} \to \mathbf{RV}$ and $\mathrm{val}_{\mathbf{RV}} : \mathbf{RV} \to \mathbf{\Gamma}$. Note that, although the group structure on $\mathbf{RV}$ will be denoted multiplicatively, on $\mathbf{\Gamma}$ it will be denoted additively.
A valued field (K, val) has a canonical $\mathcal{L}^{\mathbf{RV}}$-structure given by interpreting $\mathbf{\Gamma}$ as its value group and $\mathbf{RV}$ as $(K^\star/(1+\mathfrak{M})) \cup \{0\}$ where $\mathfrak{M}$ denotes the maximal ideal of the valuation ring $\mathcal{O} \subseteq K$. The map rv is interpreted as the canonical projection $K^\star \to \mathbf{RV}^\star := K^\star/(1+\mathfrak{M})$ extended to K by $\mathrm{rv}(0) = 0$. The function $\cdot$ on $\mathbf{RV}$ is interpreted as its group structure and we have a short exact sequence $1 \to \mathbf{k}^\star \to \mathbf{RV}^\star \to \mathbf{\Gamma} \to 0$ where $\mathbf{k} := \mathcal{O}/\mathfrak{M}$ is the residue field. The function $+$ is interpreted as the function induced by the addition on the fibres $\mathbf{RV}_\gamma := \mathrm{val}_{\mathbf{RV}}^{-1}(\gamma) \cup \{0\}$ (and for all x and $y \in \mathbf{RV}$ such that $\mathrm{val}_{\mathbf{RV}}(x) < \mathrm{val}_{\mathbf{RV}}(y)$, we define $x + y = y + x := x$). Note that $(\mathbf{RV}_\gamma, +, \cdot)$ is a one dimensional $\mathbf{k}$-vector space and that $\mathbf{RV}_0 = \mathbf{k}$. Although $\mathrm{val}_{\mathbf{RV}}(0)$ is usually denoted $+\infty \neq \gamma$, we consider here that 0 lies in each $\mathbf{RV}_\gamma$. It is in fact the unit of the group $(\mathbf{RV}_\gamma, +)$.
The valued fields we consider are also endowed with a derivation ∂ such that for all $x \in \mathbf{K}$, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$. They are usually called valued fields with a contractive derivation and were first studied by Scanlon in [Sca00]. We denote $\mathcal{L}_\partial^{\mathbf{RV}}$ the language $\mathcal{L}^{\mathbf{RV}}$ enriched with two new symbols $\partial : \mathbf{K} \to \mathbf{K}$ and $\partial_{\mathbf{RV}} : \mathbf{RV} \to \mathbf{RV}$. In a valued differential field with contractive derivation, we interpret ∂ as the derivation and $\partial_{\mathbf{RV}}$ as the function induced by ∂ on each $\mathbf{RV}_\gamma$. This function $\partial_{\mathbf{RV}}$ turns $\mathbf{RV}_\gamma$ into a differential $\mathbf{k}$-vector space and, moreover, for any x and $y \in \mathbf{RV}$, we have $\partial_{\mathbf{RV}}(x \cdot y) = \partial_{\mathbf{RV}}(x) \cdot y + x \cdot \partial_{\mathbf{RV}}(y)$. We will denote by $\mathcal{L}_{\partial,\mathbf{RV}}$ the restriction of $\mathcal{L}_\partial^{\mathbf{RV}}$ to the sorts $\mathbf{RV}$ and $\mathbf{\Gamma}$.
The valued fields we will consider are also "sufficiently closed" in the sense that they are ∂-Henselian. This notion can take many equivalent forms but we will give the one considered in [Sca00]. Let $\partial_n(x)$ denote $(x, \partial(x), \dots, \partial^n(x))$ and $\partial_\omega(x)$ denote $(\partial^i(x))_{i \in \mathbb{N}}$.

Definition 1.1 (∂-Henselian>)**:**
Let $(K, \mathrm{val}, \partial)$ *be a valued differential field.* K *is* ∂*-Henselian if for all* $P \in \mathcal{O}(K)[X_0, \ldots, X_n]$ *and* $a \in \mathcal{O}(K)$ *such that* $\mathrm{val}(P(\partial_n(a))) > 0$ *and* $\min_i\{\mathrm{val}(\frac{\partial}{\partial X_i}P(\partial_n(a)))\} = 0$*, there exists* $c \in \mathcal{O}$ *such that* $P(\partial_n(c)) = 0$ *and* $\mathrm{res}(c) = \mathrm{res}(a)$*.*

Definition 1.2 (Enough constants)**:**
Let $(K, \mathrm{val}, \partial)$ *be a valued differential field. We say that* K *has enough constants if* $\mathrm{val}(\mathrm{C}_K) = \mathrm{val}(K)$ *where* $\mathrm{C}_K := \{x \in K : \partial(x) = 0\}$ *denotes the field of constants.*

Let $\mathcal{L}_{\mathrm{div}}$ be the one sorted language for valued fields. It consists of the ring language enriched with a predicate $x|y$ interpreted as $\mathrm{val}(x) \leqslant \mathrm{val}(y)$. Let $\mathcal{L}_{\partial,\mathrm{div}} := \mathcal{L}_{\mathrm{div}} \cup \{\partial\}$ and $\mathrm{VDF}_{\mathcal{EC}}$ be the $\mathcal{L}_{\partial,\mathrm{div}}$-theory of valued fields with a contractive derivation which are ∂-Henselian with enough constants, such that the residue field is differentially closed of characteristic zero and the value group is divisible. Note that we call it $\mathrm{VDF}_{\mathcal{EC}}$ and not $\mathrm{VDF}_{\mathcal{EC},0}$ or $\mathrm{VDF}_{\mathcal{EC},0,0}$ because we mostly work in equicharacteristic zero in this text. Similarly we will not specify the characteristic when speaking of ACVF, but it is understood that we are speaking of $\mathrm{ACVF}_{0,0}$.

Example 1.3:
Let $(k, \partial) \vDash \mathrm{DCF}_0$ and $\mathbf{\Gamma}$ be a divisible ordered Abelian group. We endow the Hahn field $K = k[[t^{\Gamma}]]$ of power series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ with well ordered support and coefficients in k, with the derivation $\partial(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma) := \sum_{\gamma\in\Gamma} \partial(a_\gamma)t^\gamma$. Then $(K, \mathrm{val}, \partial) \vDash \mathrm{VDF}_{\mathcal{EC}}$.

As in the case of ACVF, $\mathrm{VDF}_{\mathcal{EC}}$ can also be considered in the one sorted, two sorted and the three sorted languages which are enrichments of the valued field versions with symbols for the derivation. For example, the three sorted language for valued differential fields consists of the sorts $\mathbf{K}$, $\mathbf{\Gamma}$ and $\mathbf{k}$ along with the differential ring language on $\mathbf{K}$ and $\mathbf{k}$, the ordered Abelian group language on $\mathbf{\Gamma}$ and maps $\mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}$ and $\mathrm{res} : \mathbf{K}^2 \to \mathbf{k}$ where $\mathrm{res}(x, y)$ is interpreted as $\mathrm{res}(xy^{-1})$.
Recall that an $\mathcal{L}$-definable set D in some $\mathcal{L}$-theory T is said to be stably embedded if, for all $M \vDash T$ and all $\mathcal{L}(M)$-definable set X, $X \cap D$ is $\mathcal{L}(D(M))$definable.

Theorem 1.4 ([Sca00; Sca03])**:**

(i) The theory $\mathrm{VDF}_{\mathcal{EC}}$ *eliminates quantifiers (and is complete) in the one sorted language, the two sorted language, the three sorted language and the leading term language;*

(ii) The value group $\mathbf{\Gamma}$ *is stably embedded and a pure divisible ordered Abelian group;*

(iii) The residue field $\mathbf{k}$ *is stably embedded and a pure model of* DCF_0*;*

(iv) The theory $\mathrm{VDF}_{\mathcal{EC}}$ *is* NIP *(does not have the independence property).*

Proof. By [Sca00, Theorem 7.1] we have field quantifier elimination in the three sorted language. The stable embeddedness and purity results for $\mathbf{k}$ and $\mathbf{\Gamma}$ follow (see, for example, [Rid, Remark A.10.2]). Now, the theory induced on $\mathbf{k}$ and $\mathbf{\Gamma}$ are, respectively, differentially

closed fields and divisible ordered Abelian groups. Both of these theories eliminate quantifiers. Quantifier elimination in the three sorted language follows and so does qualifier elimination in the one sorted and two sorted languages.
As for the leading term structure, by [Sca03, Corollary 5.8 and Theorem 6.3], $\mathrm{VDF}_{\mathcal{EC}}$ eliminates quantifiers relative to $\mathbf{RV}$ and hence one can easily check that $\mathbf{RV}$ is stably embedded and it is a pure $\mathcal{L}_{\partial,\mathbf{RV}}$-structure. Quantifier elimination for $\mathrm{VDF}_{\mathcal{EC}}$ in the leading term language now follows from quantifier elimination for the structure induced on $\mathbf{RV}$ which we prove in the following lemma:

Lemma 1.5:
Let $T_{\mathbf{RV}}$ be the $\mathcal{L}_{\partial,\mathbf{RV}}$-theory of short exact sequences of Abelian groups $1 \to \mathbf{k}^\star \to \mathbf{RV}^\star \to \mathbf{\Gamma} \to 0$ where $\mathbf{k} \vDash \mathrm{DCF}_0$ and $\mathbf{\Gamma}$ is a divisible ordered Abelian group, for all $\gamma \in \mathbf{\Gamma}$, $(\mathbf{RV}_\gamma, +, \cdot, \partial)$ is a differential $\mathbf{k}$-vector space and for all x and $y \in \mathbf{RV}$, $\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y)$.
Then T eliminates quantifiers.

Proof. If suffices to prove that for all M, $N \vDash T_{\mathbf{RV}}$ such that N is $|M|^+$-saturated and for all partial isomorphism $f : M \to N$, there exists an isomorphism $g : M \to N$ extending f defined on all of M.
Let A be the domain of f. We construct the extension step by step:

Claim 1.6: *We may assume that $\mathbf{\Gamma}(A) = \mathbf{\Gamma}(M)$*

Proof. By quantifier elimination in divisible ordered Abelian groups, there exists $g : \mathbf{\Gamma}(M) \to \mathbf{\Gamma}(N)$ defined on all of $\mathbf{\Gamma}(M)$ and extending $f|_{\mathbf{\Gamma}}$. It is easy to see that $f \cup g$ is a partial isomorphism. ▲

Claim 1.7: *We may assume that A is closed under inverses.*

Proof. For all a and $b \in \mathbf{RV}(A)$, we define $g(a^{-1} \cdot b) := f(a)^{-1} \cdot f(b)$. One can check that $g \cup f$ defines a partial isomorphism. ▲

Claim 1.8: *We may assume that $\mathbf{k}(M) \subseteq A$.*

Proof. By elimination of quantifiers in DCF_0, and saturation of N, $f|_{\mathbf{k}}$can be extended to $h : \mathbf{k}(M) \to \mathbf{k}(N)$. Now define $g(\lambda \cdot a) = h(\lambda) \cdot f(a)$ for all $\lambda \in \mathbf{k}$ and $a \in A$. As A is closed under inverse, this is well-defined and one can check that $f \cup g$ is indeed a partial isomorphism. ▲

Now, let $a \in M$ and $\gamma = \mathrm{val}_{\mathbf{RV}}(a)$. If $a \notin A$, then $\mathbf{RV}_\gamma(A) = \emptyset$.

Claim 1.9: *There exists $c \in \mathbf{RV}^\star_\gamma(M)$ such that $\partial(c) = 0$.*

This is an easy consequence of the fact that $\mathbf{k} \vDash \mathrm{DCF}_0$ and it is true of any finite dimensional differential $\mathbf{k}$-vector space. But let us give the proof in dimension one.

Proof. Pick any $c \in \mathbf{RV}^\star_\gamma(M)$. We have $\partial(c) = \lambda \cdot c$ for some $\lambda \in \mathbf{k}(M)$. We want to find $\mu \neq 0$ such that $\partial(\mu \cdot c) = 0$, i.e. $\partial(\mu) \cdot c + \mu\lambda \cdot c = 0$ and equivalently, $\partial(\mu) + \lambda\mu = 0$. But this equation has a solution in $\mathbf{k}(M)$ as it is differentially closed. ▲

Pick any such $c \in \mathbf{RV}^{\star}_{\gamma}(M)$. If there exist an $n \in \mathbb{N}_{>0}$ such that $c^n \in A$, let n_0 be the minimal such n, if such an n does not exist, let $n_0 = 0$. In both cases, let $b \in \mathbf{RV}_{f(\gamma)}(N)$ be such that $\partial(b) = 0$ and $b^{n_0} = f(c^{n_0})$. Note that if $n_0 = 0$, the last condition is empty and if $n_0 \neq 0$, the first condition follows from the second one.
Now, for all $a \in \mathbf{RV}(A)$ and $n \in \mathbb{N}$, define $g(a \cdot c^n) = f(a) \cdot b^n$. It is easy to check that $g \cup f$ is a partial isomorphism. Applying this last construction repetitively, we obtain a morphism $g : M \to N$. ♦

The fact that $\mathrm{VDF}_{\mathcal{EC}}$ is NIP is an easy consequence of elimination of quantifiers (in the one sorted language for example) and the fact that ACVF is NIP. ■

Remark 1.10:

1. If $M \models \mathrm{VDF}_{\mathcal{EC}}$, $\mathbf{K}(M)$ and $\mathrm{C}_{\mathbf{K}}(M)$ are algebraically closed, but $\mathbf{K}(M)$ is not differentially closed as the set $\{x \in \mathbf{K}(M) : \exists y\, \partial(y) = xy\}$ defines the valuation ring.
2. Scanlon also proved that $\mathrm{VDF}_{\mathcal{EC}}$ is the model completion of valued fields with a contractive derivation (see [Sca00, Theorem 7.1]). To be precise he proved that any structure A in the three sorted language for valued differential fields where $\mathbf{k}(A)$ is a differential field, $\mathbf{\Gamma}(A)$ is an ordered Abelian group and $(\mathbf{K}(A), \mathrm{val}, \mathrm{res})$ is a valued field with a contractive derivation (val and res are not assumed to be onto) can be embedded in a model of $\mathrm{VDF}_{\mathcal{EC}}$.

   A similar result in the leading term language holds: let A be an $\mathcal{L}^{\mathbf{RV}}_{\partial}$-structure such that $\mathbf{\Gamma}(A)$ is an Abelian ordered group, $\mathbf{k}(A)$ is a differential field, $\mathbf{RV}(A)$ is family of dimension one differential $\mathbf{k}(A)$-vector spaces indexed by a subgroup of $\mathbf{\Gamma}(A)$ with a symmetric bilinear group law $\cdot$ such that $\partial_{\mathbf{RV}}(x \cdot y) = \partial_{\mathbf{RV}}(x) \cdot y + x \cdot \partial_{\mathbf{RV}}(y)$ and $(\mathbf{K}(A), \mathrm{rv}, \partial)$ is a valued field with a contractive derivation (rv is not assumed to be onto). Then A can be embedded in a model of $\mathrm{VDF}_{\mathcal{EC}}$.

1.2. Elimination of imaginaries

Let us now recall some facts about elimination of imaginaries. A more thorough introduction can be found in [Poi85, Sections 16.d and 16.e]. An imaginary is a point in an interpretable set or equivalently a class of a definable equivalence relation. To every theory T we can associate a theory T^{eq} obtained by adding all the imaginaries. More precisely a new sort and a new function symbol are added for every $\varnothing$-interpretable set and they are interpreted respectively as the interpretable set itself and the canonical projection to the interpretable set. A model of T^{eq} is usually denoted M^{eq}. We write $\mathrm{dcl}^{\mathrm{eq}}$ and $\mathrm{acl}^{\mathrm{eq}}$ to denote the definable and algebraic closure in M^{eq}.
To every set X definable (with parameters in some model M of T), we can associate the set $\ulcorner X \urcorner \subseteq M^{\mathrm{eq}}$ which is the smallest definably closed set of parameters over which X is defined. We usually call $\ulcorner X \urcorner$ the code or canonical parameter of X.
Let T be a theory in a language $\mathcal{L}$ and $\mathcal{R}$ a set of $\mathcal{L}$-sorts. The theory T eliminates imaginaries up to $\mathcal{R}$ if every set X definable with parameters is in fact definable over $\mathcal{R}(\ulcorner X \urcorner)$; we say that X is coded in $\mathcal{R}$. If every X is only definable over $\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$, we say that T weakly elimi-

nates imaginaries. A theory eliminates imaginaries up to $\mathcal{R}$ if and only if it weakly eliminates imaginaries up to $\mathcal{R}$ and every finite set from the sorts $\mathcal{R}$ is coded in $\mathcal{R}$.
In [HHM06], Haskell, Hrushovski and Macpherson introduced the geometric language $\mathcal{L}^{\mathcal{G}}$ for valued fields. It consists of a sort $\mathbf{K}$ for the valued field itself, equipped with the ring language, for all $n \in \mathbb{N}$ the sorts $\mathbf{S}_n = \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ where $\mathcal{O}$ denotes the valuation ring of $\mathbf{K}$ and the sorts $\mathbf{T}_n = \mathrm{GL}_n(K)/\mathrm{GL}_{n,n}(\mathcal{O})$ where $\mathrm{GL}_{n,n}(\mathcal{O}) \leqslant \mathrm{GL}_n(\mathcal{O})$ consists of the matrices which are congruent modulo the maximal ideal $\mathfrak{M}$ to the matrix whose last columns contains only zeroes except for a one on the diagonal, and finally the canonical projections onto $\mathbf{S}_n$ and $\mathbf{T}_n$. We will denote by $\mathcal{G}$ the sorts of the geometric language. Note that $\mathbf{S}_1$ is exactly the value group and the canonical projection from $\mathbf{K}^{\star}$ onto $\mathbf{S}_1$ is the valuation.
The main "raison d'être" of this geometric language is the following theorem:

Theorem 1.11 ([HHM06, Theorem 1.0.1])**:**
The theory $\mathrm{ACVF}^{\mathcal{G}}$ of algebraically closed valued fields in the geometric language eliminates imaginaries.

1.3. Metastability

Let T be a theory, $M \vDash T$ be sufficiently saturated and $A \subseteq M$. Recall that an $\mathcal{L}(A)$-definable set X is said to be stably embedded if for all $\mathcal{L}(M)$-definable set Y, $Y(M) \cap X(M)$ is of the form $Z(M) \cap X(M)$ where Z is $\mathcal{L}(A \cup X(M))$-definable. The set X is stable stably embedded if it is stably embedded and the $\mathcal{L}(A)$-induced structure on X is stable. We denote by St_A the structure whose sorts are the stable stably embedded sets which are $\mathcal{L}(A)$-definable, equipped with their $\mathcal{L}(A)$-induced structure. We will denote by $\mathop{\smile\!\!\!\!\!\mid}_C$ forking independence in St_C.

Definition 1.12 ($\star$-definable functions)**:**
Let $\mathcal{L}$ be a language, M an $\mathcal{L}$-structure and $x = (x_i)_{i\in I}$ and $y = (y_j)_{j\in J}$ be (potentially infinite) tuples of variables. Let S_i be the sort in which the variable x_i lives, and similarly for S_j. A partial function $f : \prod_{i\in I} S_i(M) \to \prod_{j\in J} S_j(M)$ is said to be $(\mathcal{L}, x, y)$-definable (or simply an $(\mathcal{L}, \star)$-definable function, if we do not want to specify x or y) if for all $j \in J$ there exists an $\mathcal{L}$-definable function $f_j : \prod_{i\in I} S_i \to S_j$ such that $f = \prod_{j\in J} f_j$.

Definition 1.13 (Stable domination)**:**
Let M be some $\mathcal{L}$-structure, $C \subseteq M$, f an $(\mathcal{L}(C), \star)$-definable map to St_C and $p \in \mathcal{S}(C)$. We say that p is stably dominated via f if for every $a \vDash p$ and $B \subseteq M$ such that $\mathrm{St}_C(\mathrm{dcl}(CB)) \mathop{\smile\!\!\!\!\!\mid}_C f(a)$,

$$\mathrm{tp}(B/f(a)) \vdash \mathrm{tp}(B/Ca).$$

We say that p is stably dominated if it is stably dominated via some map f. It is then stably dominated via any map enumerating $\mathrm{St}_C(\mathrm{dcl}(Ca))$ for any $a \vDash p$.

Definition 1.14 (Invariant extension property)**:**
Let T be an $\mathcal{L}$-theory that eliminates imaginaries, $A \subseteq M$ for some $M \vDash T$. We say that T has the invariant extension property over A if, for all $N \vDash T$, every type $p \in \mathcal{S}(A)$ can be extended to an $\mathrm{Aut}(N/A)$-invariant type.

We say that T has the invariant extension property if T has invariant extensions over any $A = \mathrm{acl}(A) \subseteq M \models T$.

Definition 1.15 (Metastability)**:**
Let T be a theory and $\mathbf{\Gamma}$ an $\emptyset$-definable stably embedded set. We say that T is metastable over $\mathbf{\Gamma}$ if:

(i) *The theory T has the invariant extension property.*

(ii) Existence of metastability bases: *For all $A \subseteq M$, there exists $C \subseteq M$ containing A such that for all tuples $a \in M$, $\mathrm{tp}(a/C\mathbf{\Gamma}(\mathrm{dcl}(Ca)))$ is stably dominated.*

In [HHM08], Haskell, Hrushovski and Macpherson showed that maximally complete fields are metastability bases in ACVF. Recall that a valued field (K, val) is maximally complete if every chain of ball contains a point or equivalently every pseudo-Cauchy sequence from K (a sequence $(x_\alpha)_{\alpha\in\kappa}$ such that for all $\alpha < \beta < \gamma$, $\mathrm{val}(x_\gamma - x_\beta) > \mathrm{val}(x_\beta - x_\alpha)$) have a limit in K, i.e.there exists $a \in K$ such that for all $\alpha < \beta$, $\mathrm{val}(a - x_\beta) > \mathrm{val}(a - x_\alpha)$.

To finish this section, let us introduce two other kinds of types which coincide with stably dominated types in NIP metastable theories.

Definition 1.16 (Stable genericity)**:**
Let M be some NIP $\mathcal{L}$-structure and $p \in \mathcal{S}(M)$. The type p is said to be generically stable if it is $\mathcal{L}(M)$-definable and finitely satisfiable in some (small) $N \preccurlyeq M$.

Definition 1.17 (Orthogonality to $\mathbf{\Gamma}$)**:**
Let $M \models T$ be sufficiently saturated, $C \subseteq M$, $\mathbf{\Gamma}$ be an $\mathcal{L}$-definable set and $p \in \mathcal{S}(M)$ be an $\mathrm{Aut}(M/C)$-invariant type. The type p is said to be orthogonal to $\mathbf{\Gamma}$ if for all $B \subseteq M$ containing A and $a \models p|_B$, $\mathbf{\Gamma}(\mathrm{dcl}(Ba)) = \mathbf{\Gamma}(\mathrm{dcl}(B))$.

1.4. New results about $\mathrm{VDF}_{\mathcal{EC}}$

Let $\mathcal{L}^{\mathcal{G}}_{\partial}$ be the language $\mathcal{L}^{\mathcal{G}}$ enriched with a symbol for the derivation $\partial : \mathbf{K} \to \mathbf{K}$ and let $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ be the $\mathcal{L}^{\mathcal{G}}_{\partial}$-theory of models of $\mathrm{VDF}_{\mathcal{EC}}$. One of the goals of this paper is to prove the following:

Theorem 1.18:
The theory $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ eliminates imaginaries, has the invariant extension property and is metastable. Moreover, over algebraically closed sets of parameters, definable types are dense.

By density of definable types, we mean that every definable set X is consistent with a global $\mathcal{L}^{\mathcal{G}}_{\partial}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-definable type p.

Proof. Density of definable types, elimination of imaginaries and the invariant extension property follow from Theorems **(9.8)** and **(10.7)** taking T to be $\mathrm{ACVF}^{\mathcal{G}}$ and $\widetilde{T}$ to be $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$. Hypothesis **9.8.(i)** follows from Theorem **1.4.(ii)** and **1.4.(iii)**. A theory eliminate $\exists^\infty$ if the cardinality of finite definable sets is uniformly bounded. In DCF_0, the algebraic closure is the field algebraic closure of the generated differential field. It follows that the cardinality of finite definable sets is (uniformly) bounded by the degree of the polynomials whose roots form the finite set. As for $\mathbf{\Gamma}$, it is a pure model of DOAG which is an o-minimal theory and o-minimal theories always eliminate $\exists^{\infty}$.

Hypothesis **9.8.(ii)** is an easy consequence of elimination of quantifiers: let $\varphi(x;s)$ be an $\mathcal{L}_{\partial}^{\mathcal{G}}$-formula such that x and s are tuples of field variables, then there exists an $\mathcal{L}_{\mathrm{div}}$-formula $\psi(u;t)$ and $n \in \mathbb{N}$ such that $\varphi(x;s)$ is equivalent modulo $\mathrm{VDF}_{\mathcal{EC}}$ to $\psi(\partial_n(x);\partial_n(s))$, i.e. for all $m \in \widetilde{N}$, ∂_n is an $\mathcal{L}_{\partial,\mathrm{div}}$-definable bijection between $\varphi(\widetilde{N};m)$ and $\psi(x,\partial_n(m)) \cap \partial_n(\mathbf{K}^{|x|})$.

Hypothesis **9.8.(iii)** follows from the fact that if $k \vDash \mathrm{DCF}_0$ then the Hahn field $k((t^{\mathbb{R}}))$ with the derivation $\partial(\sum_i a_i t^i) = \sum_i \partial(a_i)t^i$, i.e. $\partial(t) = 0$, is a model of $\mathrm{VDF}_{\mathcal{EC}}$. By Corollary **(A.8)** the underlying valued field is uniformly stably embedded in every elementary extension.

The fact that $\mathrm{ACVF}^{\mathcal{G}}$ admits $\mathbf{\Gamma}$-reparametrisations is proved in Proposition **(8.8)**.

Finally, as we have now proved the invariant extension property, to prove metastability, there only remains to prove the existence of metastability basis. However, in Proposition **(5.6)**, we show that any algebraically closed maximally complete differential field C is a metastability basis, hence there only remains to show that any $A \subseteq M \vDash \mathrm{VDF}_{\mathcal{EC}}$ is contained in such a (small) $C \subseteq \mathbf{K}(M)$.

This follows from a classical proof, but for the sake of completeness, let us sketch the argument. Taking any lifting in $\mathbf{K}$ of the points in A, we may assume that $A \subseteq \mathrm{dcl}_{\mathcal{L}_{\partial}^{\mathcal{G}}}(\mathbf{K}(A))$. We may also assume that $\mathbf{K}(A)$ is algebraically closed. If $\mathbf{K}(A)$ is not maximally complete, take (x_α) to be a maximal pseudo-convergent sequence with no pseudo-limit in $\mathbf{K}$ and such that the order-degree of the minimal differential polynomial P pseudo-solved by (x_α) is minimal among all such pseudo-convergent sequences. Then the extension by any root of P which is also a pseudo-limit a is immediate, see [Sca00, Proposition 7.32]. Iterating this last step as many times as necessary, we obtain an immediate extension C of A which is maximally complete. Because $\mathbf{k}(C) = \mathbf{k}(A)$ and $\mathbf{\Gamma}(C) = \mathbf{\Gamma}(A)$, $\mathbf{K}(C)$ is Henselian and $\mathbf{K}(A)$ is algebraically closed, $\mathbf{K}(C)$ is also algebraically closed. ∎

At the very end of [HHM08], an incorrect proof of the metastability of $\mathrm{VDF}_{\mathcal{EC}}$ (and in particular of the invariant extension property) is sketched. Because it overlooks major difficulties inherent to the proof of the invariant extension property, there can be no easy way to fix this proof and new techniques had to be developed. Furthermore, it seems that proving the existence of metastable bases in $\mathrm{VDF}_{\mathcal{EC}}$ is not as straightforward as one might hope either and the proof we give in Proposition **(5.6)** is surprisingly convoluted as it relies on a good understanding of the imaginaries of the structure induced on $\mathbf{RV}$.

2. Prolongation of the type space

The goal of this section is to study the relation between types in $\mathrm{VDF}_{\mathcal{EC}}$ and types in ACVF. This construction plays a fundamental role in the rest of this paper, be it to prove that there are metastability bases in $\mathrm{VDF}_{\mathcal{EC}}$ or that definable types are dense, although, in the proof of Theorem **(9.8)**, it appears in a more abstract setting.

For all $x \in \mathbf{K}$ or $x \in \mathbf{RV}$, let $\partial_\omega(x)$ denote, respectively, $(\partial^n(x))_{n\in\mathbb{N}}$ and $(\partial^n_{\mathbf{RV}}(x))_{n\in\mathbb{N}}$. If $x \in \mathbf{\Gamma}$, let $\partial_\omega(x)$ denote $(x)_{n\in\mathbb{N}}$. If x is a tuple of variables, we denote by x_∞ the tuple $(x^{(i)})_{i\in\mathbb{N}}$

where each $x^{(i)}$ is sorted like x.
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ be sufficiently saturated and $A \leqslant M$ be a substructure. We write $\mathcal{S}_x^{\mathcal{L}}(A)$ for the space of complete $\mathcal{L}$-types over A in the variable x.

Definition 2.1:
Let us assume that $A \subseteq \mathbf{K} \cup \mathbf{\Gamma} \cup \mathbf{RV}$. We define $\nabla_\omega : \mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A) \to \mathcal{S}_{x_\infty}^{\mathcal{L}^{\mathbf{RV}}}(A)$ to be the map which sends a complete type p to the complete type

$$\nabla_\omega(p) := \{\varphi(x_\infty, a) : \varphi \text{ is an } \mathcal{L}^{\mathbf{RV}}\text{-formula and } \varphi(\partial_\omega(x), a) \in p\}.$$

Proposition 2.2:
The function ∇_ω is a homeomorphism onto its image (which is closed).

Proof. As $\mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A)$ is compact and $\mathcal{S}_{x_\infty}^{\mathcal{L}^{\mathbf{RV}}}(A)$ is Hausdorff, it suffices to show that ∇_ω is continuous and injective. Let us first show continuity. Let $U = \langle \varphi(x_\infty, a) \rangle \subseteq \mathcal{S}_{x_\infty}^{\mathcal{L}^{\mathbf{RV}}}(A)$, then $\nabla_\omega^{-1}(U) = \langle \varphi(\partial_\omega(x), a) \rangle \subseteq \mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A)$. As for ∇_ω being injective, let p and $q \in \mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A)$ and let $\varphi(x, a)$ be an $\mathcal{L}_\partial^{\mathbf{RV}}$-formula in $p \smallsetminus q$. By quantifier elimination, we can assume that φ is of the form $\theta(\partial_\omega(x), a)$ for some $\mathcal{L}^{\mathbf{RV}}$-formula θ. Then $\theta(x_\infty, a) \in \nabla_\omega(p) \smallsetminus \nabla_\omega(q)$. ∎

In fact, we can describe exactly what the closed image is. For every differential ring R let $R\{X_1, \dots, X_m\}$ be the ring of differential polynomials in m variables (i.e. $R[X_i^{(j)} : j \in \mathbb{N} \text{ and } 1 \leqslant i \leqslant m]$). We also denote ∂ the derivation on this ring. When $P \in R\{X_1, \dots, X_m\}$, we will write $P^\star \in R[X_i^{(j)} : j \in \mathbb{N} \text{ and } 1 \leqslant i \leqslant m]$ for the underlying polynomial.

Proposition 2.3:
Let x be a tuple of $\mathbf{K}$-variables and $\mathcal{P}_x$ be the following set of $\mathcal{L}^{\mathbf{RV}}(A)$-formulas:

$$\begin{aligned} & \{\mathrm{val}(\partial(P)^\star(x_\infty)) \geqslant \mathrm{val}(P^\star(x_\infty)) : P \in \mathbf{K}(A)\{x\}\} \\ \cup\ & \{a = \mathrm{rv}((\tfrac{P}{Q})^\star(x_\infty)) \wedge \mathrm{val}(\partial(\tfrac{P}{Q})^\star(x_\infty)) = \mathrm{val}((\tfrac{P}{Q})^\star(x_\infty)) \\ & \quad \to \partial(a) = \mathrm{rv}(\partial(\tfrac{P}{Q})^\star(x_\infty)) : a \in \mathbf{RV}(A) \text{ and } P, Q \in \mathbf{K}(A)\{x\}\} \\ \cup\ & \{a = \mathrm{rv}((\tfrac{P}{Q})^\star(x_\infty)) \wedge \mathrm{val}(\partial(\tfrac{P}{Q})^\star(x_\infty)) > \mathrm{val}((\tfrac{P}{Q})^\star(x_\infty)) \\ & \quad \to \partial(a) = 0 : a \in \mathbf{RV}(A) \text{ and } P, Q \in \mathbf{K}(A)\{x\}\} \end{aligned}$$

Let $\mathcal{P}_x(A) \subseteq \mathcal{S}_{x_\infty}^{\mathcal{L}^{\mathbf{RV}}}(A)$ be the set of types over A containing $\mathcal{P}_x$. Then $\nabla_\omega(\mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathcal{P}_x(A)$.

There is a slight abuse of notation in the above formulas. To be precise, the expression $\mathrm{rv}(\partial(\frac{P}{Q})^\star(x_\infty))$ should read instead

$$\mathrm{rv}\left(\frac{Q^\star(x_\infty)\partial(P)^\star(x_\infty) - P^\star(x_\infty)\partial(Q)^\star(x_\infty)}{(Q^\star(x_\infty))^2}\right).$$

A similar description of $\nabla_\omega(\mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ can be given when x also contains $\mathbf{\Gamma}$ and $\mathbf{RV}$-variables but it is much more cumbersome.

Proof. For any $P \in \mathbf{K}(A)\{\overline{X}\}$ and tuple $c \in \mathbf{K}(M)$ of the right length, we have $\partial(P(c)) = \partial(P)(\partial_\omega(c))$. It follows that $\nabla_\omega(\mathcal{S}_x^{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) \subseteq \mathcal{P}_x(A)$. Let us now show that this inclusion is an equality. Let $p \in \mathcal{P}_x(A)$ and $\overline{c} \models p$. In particular $c = (c_{j,i})_{j\in\mathbb{N},0\leqslant i<|x|}$. Then, $\{P \in \mathbf{K}(A)\{X\} : P^\star(\overline{c}) = 0\}$ is a differential ideal as for all such P, $\mathrm{val}(\partial(P)^\star(\overline{c})) \geqslant \mathrm{val}(P^\star(\overline{c})) = \infty$. Hence $L = \mathbf{K}(A)(\overline{c})$ can be endowed with a (unique) differential ring structure such that $\partial(c_{j,i}) = c_{j+1,i}$.
For any $P, Q \in \mathbf{K}(A)\{X\}$, we have

$$\mathrm{val}(\partial(P)^\star(\overline{c})) \geqslant \mathrm{val}(P^\star(\overline{c})) \text{ and } \mathrm{val}(\partial(Q)^\star(\overline{c})) \geqslant \mathrm{val}(Q^\star(\overline{c})),$$

and hence

$$\mathrm{val}\left(\partial\left(\frac{P^\star(\overline{c})}{Q^\star(\overline{c})}\right)\right) - \mathrm{val}\left(\frac{P^\star(\overline{c})}{Q^\star(\overline{c})}\right) = \mathrm{val}\left(\frac{\partial(P)^\star(\overline{c})}{P^\star(\overline{c})} - \frac{\partial(Q)^\star(\overline{c})}{Q^\star(\overline{c})}\right) \geqslant 0.$$

Then $C = L \cup A$ is a valued field with a contractive derivation — in the broader sense where rv and val might not be onto. As $\mathrm{VDF}_{\mathcal{EC}}$ is the model completion of such structures (see Remark **1.10.2**), we can find an embedding $f : C \to M$ such that f fixes A. Let $q = \mathrm{tp}_{\mathcal{L}_\partial^{\mathbf{RV}}}((f(c_{0,i}))_{0\leqslant i<|x|}/A)$, then $\nabla_\omega(q) = p$. ∎

We will now look at how ∇_ω and its inverse behave with respect to various properties of types. One must beware though that however innocent these questions might seem, transferring some properties from p to $\nabla_\omega(p)$ actually presents real challenges. Proving Propositions **(2.4)** and **(2.5)** required the development of [RS]. Note that in [RS] the variables of the type p are in $\mathbf{K}$, the same proof applies if the variables are in $\mathbf{K}$, $\mathbf{RV}$ and $\mathbf{\Gamma}$ (we have to use elimination of quantifiers in $\mathcal{L}_\partial^{\mathbf{RV}}$ instead).

Proposition 2.4 ([RS, Corollary 3.3])**:**
Let $p \in \mathcal{S}^{\mathcal{L}_\partial^{\mathbf{RV}}}(M)$. Assume $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)$. The following are equivalent:

(i) p *is* $\mathcal{L}_\partial^{\mathbf{RV},\mathrm{eq}}(A)$*-definable;*

(ii) $\nabla_\omega(p)$ *is* $\mathcal{L}^{\mathcal{G}}(\mathcal{G}(A))$*-definable;*

(iii) p *is* $\mathcal{L}_\partial^{\mathcal{G}}(\mathcal{G}(A))$*-definable.*

Proposition 2.5 ([RS, Corollary 3.5])**:**
Let $p \in \mathcal{S}^{\mathcal{L}_\partial^{\mathbf{RV}}}(M)$. Assume $A = \mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)$. The following are equivalent:

(i) p *is* $\mathrm{Aut}_{\mathcal{L}_\partial^{\mathbf{RV},\mathrm{eq}}}(M/A)$*-invariant;*

(ii) $\nabla_\omega(p)$ *is* $\mathrm{Aut}_{\mathcal{L}^{\mathcal{G}}}(M/\mathcal{G}(A))$*-invariant;*

(iii) p *is* $\mathrm{Aut}_{\mathcal{L}_\partial^{\mathcal{G}}}(M/\mathcal{G}(A))$*-invariant.*

Proposition 2.6:
Let $p \in \mathcal{P}_x(A)$, f be an $(\mathcal{L}^{\mathcal{G}}(A),\star)$-definable map defined on p and let D be the image of f. Assume that $A \subseteq \mathbf{K} \cup \mathbf{\Gamma} \cup \mathbf{RV}$, p is stably dominated via f and that $D^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}^{\mathbf{RV}}_{\partial}}(A)) = D^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}^{\mathbf{RV}}}(A))$. Then $\nabla_\omega^{-1}(p)$ is also stably dominated (via $f \circ \partial_\omega$).

To be precise, it suffices to consider D^{eq} relative to the $\mathcal{L}^{\mathbf{RV}}$-induced structure (and not the $\mathcal{L}^{\mathbf{RV}}_{\partial}$-induced structure).

Proof. We will need the following result:

Claim 2.7: *Let D be $\mathcal{L}^{\mathcal{G}}(M)$-definable. If D is stable and stably embedded in* ACVF*, then it is also stable and stably embedded in* $\mathrm{VDF}_{\mathcal{EC}}$*.*

Proof. It follows from [HHM06, Lemma 2.6.2 and Remark 2.6.3] that $D \subseteq \mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(E \cup \mathbf{k})$ for some finite $E \subseteq D$. Because $\mathbf{k}$ also eliminates imaginaries, is stable and stably embedded in $\mathrm{VDF}_{\mathcal{EC}}$, it immediately follows that D is stably embedded and stable in $\mathrm{VDF}_{\mathcal{EC}}$ too. ♦

Now let $c \models \nabla_\omega^{-1}(p)$ and $B \subseteq \mathbf{K}$ be such that

$$\mathrm{St}_A^{\mathcal{L}^{\mathbf{RV}}_{\partial}}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}_{\partial}}(AB)) \downarrow_A^{\mathcal{L}^{\mathbf{RV}}_{\partial}} f(\partial_\omega(c))$$

where $\downarrow_A^{\mathcal{L}^{\mathbf{RV}}_{\partial}}$ denotes independence in $\mathrm{St}_A^{\mathcal{L}^{\mathbf{RV}}_{\partial}}$ over A. By hypothesis, $D^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}^{\mathbf{RV}}_{\partial}}(A)) = D^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}^{\mathbf{RV}}}(A))$, and hence

$$\mathrm{St}_A^{\mathcal{L}^{\mathbf{RV}}}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(A\partial_\omega(B))) \downarrow_A^{\mathcal{L}^{\mathbf{RV}}} f(\partial_\omega(c))$$

where $\downarrow_A^{\mathcal{L}^{\mathbf{RV}}}$ denotes independence in $\mathrm{St}_A^{\mathcal{L}^{\mathbf{RV}}}$ over A. Since $\partial_\omega(c) \models p$ and p is stably dominated via f,

$$\begin{aligned}
\mathrm{tp}_{\mathcal{L}^{\mathbf{RV}}_{\partial}}(B/f(\partial_\omega(c))) &\vdash \mathrm{tp}_{\mathcal{L}^{\mathbf{RV}}}(\partial_\omega(B)/f(\partial_\omega(c)))\\
&\vdash \mathrm{tp}_{\mathcal{L}^{\mathbf{RV}}}(\partial_\omega(B)/A\partial_\omega(c))\\
&\vdash \mathrm{tp}_{\mathcal{L}^{\mathbf{RV}}_{\partial}}(B/Ac).
\end{aligned}$$

Here the last implication comes from the fact that ∇_ω is one to one on the space of types. Hence, we have proved that $\nabla_\omega^{-1}(p)$ is stably dominated via $f \circ \partial_\omega$. ∎

Proposition 2.8:
Assume M sufficiently saturated and homogeneous and let $p \in \mathcal{S}_x^{\mathcal{L}^{\mathbf{RV}}_{\partial}}(M)$ be $\mathcal{L}^{\mathbf{RV}}_{\partial}(M)$-definable. The following are equivalent:

(i) *p is stably dominated;*

(ii) *p is generically stable;*

(iii) *p is orthogonal to $\mathbf{\Gamma}$;*

(iv) *$\nabla_\omega(p)$ is stably dominated;*

(v) *$\nabla_\omega(p)$ is generically stable;*

(vi) *$\nabla_\omega(p)$ is orthogonal to $\mathbf{\Gamma}$;*

Proof. The equivalence of (iv), (v) and (vi) is proved for ACVF in [HL, Proposition 2.8.1]. Moreover, the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold in any NIP theory where $\mathbf{\Gamma}$ is ordered.

We proved in Proposition **(2.6)** that (iv) implies (i). Let us now prove that (iii) implies (iv). By Proposition **(2.4)**, $\nabla_\omega(p)$ is $\mathcal{L}^{\mathbf{RV}}(A)$-definable for some $A \subseteq M$. Let $C \models \mathrm{VDF}_{\mathcal{EC}}$ be maximally complete, and contain A and let $c \models p|_C$. As p is orthogonal to $\mathbf{\Gamma}$, we have $\mathbf{\Gamma}(C) \subseteq \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(Cc)) \subseteq \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}_\partial}(Cc)) = \mathbf{\Gamma}(C)$. Because C is maximally complete and maximally complete fields are metastability bases in ACVF (see [HHM08, Theorem 12.18.(ii)]), $\mathrm{tp}(\partial_\omega(c)/C\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(Cc)))$ is stably dominated. But

$$\mathrm{tp}(\partial_\omega(c)/C\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(Cc))) = \mathrm{tp}(\partial_\omega(c)/C) = \nabla_\omega(p)|_C$$

and hence $\nabla_\omega(p)$ is also stably dominated. ■

3. Stable embeddedness of the field of constants

In this section we wish to study the stable embeddedness of the field of constants $\mathrm{C}_{\mathbf{K}}$ in $\mathrm{VDF}_{\mathcal{EC}}$. We will be using the results of Appendix A and we will be working in the one sorted language for valued differential fields $\mathcal{L}_{\partial,\mathrm{div}} = \mathcal{L}_{\mathrm{div}} \cup \{\partial\}$. Nevertheless, it is easy to see that the results we obtain here are valid in any choice of language.
Recall Definition **(A.1)**: an extension $K \subseteq L$ of valued field is said to be separated if any finite dimensional sub-K-vector space of L has a basis a such that for any tuple $\lambda \in K$, $\mathrm{val}(\sum_i \lambda_i a_i) = \min_i\{\mathrm{val}(\lambda_i a_i)\}$.

Proposition 3.1:
Any pair $K \subseteq L$ which is elementarily equivalent to a pair $K^\star \subseteq L^\star$, where $K^\star$ is maximally complete, is separated. In particular, in models of $\mathrm{VDF}_{\mathcal{EC}}$, the pair $\mathrm{C}_{\mathbf{K}} \subseteq \mathbf{K}$ is separated.

Proof. The first part of the corollary is an immediate consequence of Propositions **(A.2)** and the fact that separation is preserved by elementary equivalence of the pair. The rest of the corollary then follows because for any $k \models \mathrm{DCF}_0$ and $\Gamma \models \mathrm{DOAG}$, $k((t^\Gamma))$ equipped with the derivation $\partial(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma) = \sum_{\gamma\in\Gamma} \partial(a_\gamma) t^\gamma$ is a model of $\mathrm{VDF}_{\mathcal{EC}}$ whose constant field is $\mathrm{C}_k((t^\Gamma))$ which is maximally complete and, as $\mathrm{VDF}_{\mathcal{EC}}$ is complete, all the pairs $\mathrm{C}_{\mathbf{K}} \subseteq \mathbf{K}$ are elementary equivalent. ■

Proposition 3.2:
The field of constants $\mathrm{C}_{\mathbf{K}}$ is stably embedded in models of $\mathrm{VDF}_{\mathcal{EC}}$. It follows that it is a pure model of ACVF.

Proof. Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ and $\varphi(x,m)$ be a formula with parameters in $\mathbf{K}(M)$. By quantifier elimination, we may assume that φ is of the form $\psi(\partial_n(x),\partial_n(m))$ where ψ is an $\mathcal{L}_{\mathrm{div}}$-formula and $n \in \mathbb{N}$. For all $x \in \mathrm{C}_K$, $\varphi(x,m)$ is equivalent to $\psi(x,0,\dots,0,\partial_n(m))$, i.e. an $\mathcal{L}_{\mathrm{div}}(\mathbf{K}(M))$-formula, hence it suffices to prove that $\mathrm{C}_{\mathbf{K}}(M)$ is stably embedded in $\mathbf{K}(M)$ as a valued field. But $\mathbf{K}(M)$ is algebraically closed, the pair $\mathrm{C}_{\mathbf{K}}(M) \subseteq \mathbf{K}(M)$is separated (Proposition **(3.1)**) and, because there are enough constants, $\mathrm{val}(\mathbf{K}(M)) = \mathrm{val}(\mathrm{C}_{\mathbf{K}}(M))$, hence we can apply Theorem **(A.6)**.
It now follows from quantifier elimination that any subset of $\mathrm{C}_{\mathbf{K}}$ definable in M is definable by a quantifier-free $\mathcal{L}_{\mathrm{div}}(\mathrm{C}_{\mathbf{K}}(M))$-formula and hence is defined in $\mathrm{C}_{\mathbf{K}}(M)$ by the same formula. ■

These results can be transposed easily to the Witt vectors over $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ with the lifting of the Frobenius $\mathrm{W}(\mathrm{Frob}_p)$. Recall that in this field there are definable angular component maps ac_n for all the residue rings $\mathbf{R}_n := \mathcal{O}/p^n\mathfrak{M}$ which are compatible with the automorphism. Indeed, $\mathrm{Fix}(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})) = \mathbb{Q}_p$ where angular component maps ac_n are definable and for all $x \in \mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ define $\mathrm{ac}_n(x) := \mathrm{res}_n(xy^{-1})\mathrm{ac}_n(y)$ for any $y \in \mathbb{Q}_p$ such that $\mathrm{val}(x) = \mathrm{val}(y)$. Thus we will consider the theory of this field in the three sorted language $\mathcal{L}^{\mathrm{ac}}_{\sigma,P}$ with the angular component maps described above and divisibility predicates P_n on the value group. Let WF_p denote $\mathrm{Th}_{\mathcal{L}^{\mathrm{ac}}_{\sigma,P}}(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}}), \mathrm{W}(\mathrm{Frob}_p), \mathrm{ac}_n)$.

Theorem 3.3 ([BMS07]):
The theory WF_p eliminates quantifiers (and is complete).

Proof. By [BMS07, Theorem 11.4], WF_p eliminates quantifiers in $\mathcal{L}^{\mathrm{ac}}_{\sigma,P}$ relative to $\mathbf{R} = \bigcup_n \mathbf{R}_n$ and $\mathbf{\Gamma}$. The theorem now follows from the fact that algebraically closed fields eliminate quantifiers and $\mathbb{Z}$-groups eliminate quantifiers once we add divisibility predicates. ∎

Proposition 3.4:
The fixed field $\mathrm{Fix}(\mathbf{K})$ is stably embedded in models of WF_p and is a pure valued field elementarily equivalent to $\mathbb{Q}_p$

Proof. It is essentially the same proof as in Proposition **(3.2)**. Let $M \models \mathrm{WF}_p$. By quantifier elimination, the intersection of any definable set in M with $\mathrm{Fix}(\mathbf{K})$ is the intersection with $\mathrm{Fix}(\mathbf{K})$ of a set definable in M as a valued field with angular components. Because $(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}}), \mathrm{W}(\mathrm{Frob}_p))$ is a model of WF_p whose fixed field is $\mathbb{Q}_p$ (which is also maximally complete), by Proposition **(3.1)**, in models of WF_p, the pair $\mathrm{Fix}(\mathbf{K}) \subseteq \mathbf{K}$ is separated. We can now apply Theorem **(A.7)** to conclude. The fact that it is a pure valued field now follows by elimination of quantifiers (and the fact that the angular component maps we chose are definable in $\mathrm{Th}(\mathbb{Q}_p)$). ∎

4. Definable and algebraic closure in $\mathrm{VDF}_{\mathcal{EC}}$

In this section, we investigate the definable and algebraic closures in $\mathrm{VDF}_{\mathcal{EC}}$. We show that they are not as simple as one might hope. In DCF_0, the definable closure of a is exactly the field generated by $\partial_\omega(a)$. In $\mathrm{VDF}_{\mathcal{EC}}$, we have, at least, to take in account the Henselianisation, but we show that the definable closure of a can be even larger than the Henselianisation of the field generated by $\partial_\omega(a)$. This fact was already known to Ehud Hrushovski and Thomas Scanlon but was never written down.
We will, again, be working in the leading term language and all the set of parameters that appear in this section will be living in the sorts $\mathbf{K} \cup \mathbf{\Gamma} \cup \mathbf{RV}$. We denote by $\langle A \rangle_\partial$ the $\mathcal{L}^{\mathbf{RV}}_\partial$-structure generated by A and $\langle A \rangle_{^{-1},\partial}$ the closure of A under both $\mathcal{L}^{\mathbf{RV}}_\partial$-terms and inverses on $\mathbf{K}$, $\mathbf{RV}$ and $\mathbf{\Gamma}$. Recall that $\subset$ denotes strict inclusion.

Fact 4.1:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *be sufficiently saturated. For all* $C \subseteq M$*, there exists* $A \subseteq M$*, such that* $C \subseteq A$ *and* $\mathbf{K}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(\langle A\rangle_\partial)) = \overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{h}} \subset \mathbf{K}(\mathrm{dcl}_{\mathcal{L}_{\partial,\mathrm{div}}}(A))$*. In fact, there exists* $a \in \mathbf{K}(\mathrm{dcl}_{\mathcal{L}_{\partial,\mathrm{div}}}(A))$ *which is transcendental over* $\langle A\rangle_\partial$*. In particular, we also have* $a \notin \mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\langle A\rangle_\partial)$*.*

We show that certain (linear) differential equations which have infinitely many solutions in differentially closed fields have only one solution in models of $\mathrm{VDF}_{\mathcal{EC}}$, due to the valuative restrictions imposed by having a contractive derivation. In other terms, we exhibit a two dimensional ACVF-definable set which contains a unique prolongation type that has only one realisation of the form $\partial_\omega(x)$.

Proof. Let $P(\overline{X}) \in \mathcal{O}(M)[\overline{X}]$, $a \in \mathcal{O}(M)$ and $\varepsilon \in \mathfrak{M}(M)$. Let $Q_a(\partial_\omega(x)) = x - a + \varepsilon P(\partial_\omega(x))$, then Q_a has a unique zero in M. Indeed $\mathrm{val}(Q_a(a)) > 0$, $\mathrm{val}(\frac{\partial Q_a}{\partial X_0}(a)) = \mathrm{val}(1) = 0$ and $\mathrm{val}(\frac{\partial Q_a}{\partial X_i}(a)) = \mathrm{val}(\varepsilon) + \mathrm{val}(\frac{\partial P}{\partial X_i}(a)) > 0$, hence σ-Henselianity applies. If $Q_a(x) = Q_a(y) = 0$, then $\mathrm{res}(x) = \mathrm{res}(a) = \mathrm{res}(y)$ and thus $\mathrm{val}(x-y) > 0$. Let $\eta := x - y$, we have

$$Q_a(y) = x + \eta - a + \varepsilon P(x+\eta) = x + \eta - \varepsilon(\sum_I P_I(a)\eta^I) = \eta + \varepsilon(\sum_{|I|>0} P_I(a)\eta^I).$$

But, if $\eta \neq 0$, $\mathrm{val}(\varepsilon P_I(a)\eta^I) > |I|\mathrm{val}(\eta) \geqslant \mathrm{val}(\eta)$ and hence $\mathrm{val}(Q_a(y)) = \mathrm{val}(\eta) \neq \infty$, a contradiction. Hence the equation $P(\partial_\omega(x)) = 0$ has a unique solution in M.
Let us now show that, in some cases, we do get new definable functions. We may assume that $C = \mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(\mathbf{K}(C))$. Let k be a differential field, $\widetilde{a} \in k$ be differentially transcendental and let us equip $k[[\varepsilon]]$ with the usual contractive derivation (i.e. the one such that $\mathrm{val}(\varepsilon) = 0$). We embed $k[[\varepsilon]]$ in M so that k and $\mathbf{k}(C)$ are independent and $\mathbf{K}(C)(\varepsilon)$ is a transcendental ramified extension of $\mathbf{K}(C)$. To avoid any confusion, let us denote by a the image of $\widetilde{a}$ by the embedding of k into $k[[\varepsilon]]$ and into M. One can check that for all $n \in \mathbb{N}$, $\mathrm{res}(\mathbf{K}(C)(\varepsilon, \partial_n(a))) = \mathbf{k}(C)(\partial_n(\widetilde{a}))$.
Let us now try to solve $x - a - \varepsilon\partial(x) = 0$ in $k[[\varepsilon]]$. Let $x = \sum x_i\varepsilon^i$ where $x_i \in k$, the equation can then be rewritten as:

$$\sum x_i\varepsilon^i = a\varepsilon^0 + \sum \partial(x_i)\varepsilon^{i+1},$$

and hence $x_0 = a$ and $x_{i+1} = \partial(x_i) = \partial^{i+1}(a)$. If $x \in \overline{\langle C, a, \varepsilon\rangle_{-1,\partial}}^{\mathrm{alg}}$ then for some $n \in \mathbb{N}$, we must have $x \in \overline{\mathbf{K}(C)(\partial_n(a), \varepsilon)}^{\mathrm{alg}}$. Any automorphism of $\sigma : k \cup \mathbf{k}(C)$ fixing $\mathbf{k}(C)$ can be lifted into an automorphism of $k[[\varepsilon]] \cup C$ fixing C and sending $\sum x_i\varepsilon^i \in k[[\varepsilon]]$ to $\sum \sigma(x_i)\varepsilon^i$. Because $\partial^{n+1}(\widetilde{a})$ is transcendental over $\mathbf{k}(C)(\partial_n(\widetilde{a}))$, it follows that x has an infinite orbit over $\mathbf{K}(C)(\partial_n(a), \varepsilon)$, a contradiction. ■

Nevertheless, by quantifier elimination, the definable closure in $\mathbf{\Gamma}$, $\mathbf{k}$ and $\mathbf{RV}$ is exactly what one could hope for. Let us begin with the easier cases of $\mathbf{\Gamma}$ and $\mathbf{k}$.

Proposition 4.2:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq M$*, then*

$$\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}_\partial}(A)) = \mathbf{\Gamma}(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}_\partial}(A)) = \mathbb{Q} \otimes \mathbf{\Gamma}(\langle A\rangle_{-1,\partial}),$$

$$\mathbf{k}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}_\partial}(A)) = \mathbf{k}(\langle A\rangle_{-1,\partial}) \text{ and } \mathbf{k}(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}_\partial}(A)) = \overline{\mathbf{k}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}}.$$

Proof. Let us first show that $\Gamma(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathbb{Q}\otimes\mathrm{val}(\langle A\rangle_{^{-1},\partial})$. By quantifier elimination in the leading term language, any formula with variables in $\mathbf{\Gamma}$ and parameters in A is of the form $\varphi(x,a)$ where $a\in\mathbf{\Gamma}(\langle A\rangle_\partial)$. In particular, any $\gamma\in\mathbf{\Gamma}(M)$ algebraic over A is algebraic, in $\mathbf{\Gamma}$, over $\mathbf{\Gamma}(\langle A\rangle_\partial)$ which is a pure divisible ordered Abelian group. It follows immediately that $\gamma\in\mathbb{Q}\otimes\mathbf{\Gamma}(\langle A\rangle_{^{-1},\partial})$. Finally, as $\mathbb{Q}\otimes\mathrm{val}(\langle A\rangle_{^{-1},\partial})$ is rigid over $\mathrm{val}(\langle A\rangle_{^{-1},\partial})\subseteq\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ the equality $\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathbf{\Gamma}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ also holds.
As for the results concerning $\mathbf{k}$, they are proved similarly. Indeed, any formula with variables in $\mathbf{k}$ and parameters in A is of the form $\varphi(x,a)$ where $a\in\mathbf{k}(\langle A\rangle_{^{-1},\partial})$ is a tuple. The proof of this fact requires a little more work than for $\mathbf{\Gamma}$ because formulas of the form $\sum_{i\in I}a_ix^i = 0$ where $a_i\in\mathbf{RV}(\langle A\rangle_{^{-1},\partial})$ are not immediately seen to be of the right form. But we may assume that all a_i have the same valuation (as only the monomials with minimal valuation are relevant to this equation). Hence, it is equivalent to $\sum_{i\in I}a_ia_{i_0}^{-1}x^i = 0$ which is of the right form.
The results now follow from the fact that in DCF_0 the definable closure is just the differential field generated by the parameters and the algebraic closure is its field theoretic algebraic closure. ∎

Let us now tackle the case of $\mathbf{RV}$.

Proposition 4.3:
Let $M\vDash\mathrm{VDF}_{\mathcal{EC}}$ *and* $A\subseteq M$. *Then*

$$\mathbf{RV}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathbf{RV}(\langle A\rangle_{^{-1},\partial}) \text{ and } \mathbf{RV}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathbf{RV}(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\langle A\rangle_\partial)).$$

Proof. Let $\widetilde{A} := (\mathbf{RV}\cup\mathbf{\Gamma})(\langle A\rangle_\partial)$. By quantifier elimination for $\mathrm{VDF}_{\mathcal{EC}}$ in the leading term language, any formula with variables in $\mathbf{RV}$ and parameters in A is of the form $\varphi(x,a)$ where $a\in\widetilde{A}$ is a tuple. In particular, $\mathbf{RV}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))\subseteq\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})$.

Claim 4.4: *Let* $\gamma\in\mathbf{\Gamma}\smallsetminus\mathbb{Q}\otimes\mathrm{val}_{\mathbf{RV}}(\mathbf{RV}(\widetilde{A}))$. *We have* $\mathbf{RV}_\gamma(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \mathbf{RV}_\gamma(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \varnothing$.

Proof. Pick any $(d_i)_{i\in\mathbb{N}}\in\mathbf{k}$ such that $d_{mn}^n = d_m$ and $\partial(d_m) = 0$ for all m and $n\in\mathbb{N}$. Let us write $\mathbf{\Gamma} = (\mathbb{Q}\otimes\mathbf{\Gamma}(\widetilde{A}))\bigoplus_{i\in I}\mathbb{Q}\gamma_i$ where one of the γ_i is γ. Define a group morphism $\sigma:\mathbf{\Gamma}\to\mathbf{k}(M)$ sending all of $\mathbb{Q}\otimes\mathbf{\Gamma}(\widetilde{A})$ and all $\gamma_i\neq\gamma$ to 0 and $p/q\cdot\gamma$ to d_q^p. For all $x\in\mathbf{RV}$, we now define $\tau(x) = \sigma(\mathrm{val}_{\mathbf{RV}}(x))\cdot x$. It is easy to check that τ is an $\mathcal{L}_{\partial,\mathbf{RV}}$-automorphism of $\mathbf{RV}$ and that τ fixes $\widetilde{A}$.
On the fibre $\mathbf{RV}_\gamma$, τ sends x to $d_1\cdot x$. It immediately follows that, because we have infinitely many choices for d_1 (as $\mathrm{C}_{\mathbf{k}}$ is algebraically closed), the orbit of x under $\mathrm{Aut}_{\mathcal{L}_{\partial,\mathbf{RV}}}(\mathbf{RV}/\widetilde{A})$ is infinite. Thus $\mathbf{RV}_\gamma(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \mathbf{RV}_\gamma(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \varnothing$. ♦

Claim 4.5: *Let* $\gamma\in\mathbb{Q}\otimes\mathrm{val}_{\mathbf{RV}}(\mathbf{RV}(\widetilde{A}))\smallsetminus\mathrm{val}_{\mathbf{RV}}(\mathbf{RV}(\widetilde{A}))$. *Then* $\mathbf{RV}_\gamma(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \varnothing$ *and* $\mathbf{RV}_\gamma(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\widetilde{A}))\neq\varnothing$.

Proof. Let n be minimal such that $\delta = \gamma^n\in\mathrm{val}_{\mathbf{RV}}(\mathbf{RV}(\widetilde{A}))$. Taking d_i as above, such that $d_1 = 1$, and defining σ such that $\sigma(p/q\delta) = d_q^p$, we obtain an $\mathcal{L}_\partial^{\mathbf{RV}}$-automorphism τ which

fixes $\widetilde{A}$ and acts on $\mathbf{RV}_\gamma$ by multiplying by d_n. As there are n choices for d_n, we obtain that $\mathbf{RV}_\gamma(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \varnothing$.
Now, let us show that $\mathbf{RV}_\gamma(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\widetilde{A})) \neq \varnothing$. Let $c \in \mathbf{RV}_\gamma$. We have $\mathrm{val}_{\mathbf{RV}}(c^n) = \gamma^n$ and there exists $\lambda \in \mathbf{k}$ such that $\lambda \cdot c^n \in \widetilde{A}$. Let $\mu \in \mathbf{k}$ be such that $\mu^n = \lambda$ and let $a = \mu \cdot c$. Then $a^n = \lambda \cdot c^n \in \widetilde{A}$. As, the kernel of $x \mapsto x^n$ is finite, we have $a \in \mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\widetilde{A})$. ♦

Now, let $c \in \mathbf{RV}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A}))$. By Claims **(4.5)** and **(4.4)**, $\mathrm{val}_{\mathbf{RV}}(c) = \mathrm{val}_{\mathbf{RV}}(a)$ for some $a \in \widetilde{A}$. It follows that $c \cdot a^{-1} \in \mathbf{k}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A}))$, which, by Proposition **(4.2)** is equal to $\mathbf{k}(\widetilde{A})$, and hence $c = (c \cdot a^{-1}) \cdot a \in \mathbf{RV}(\widetilde{A})$.
If $c \in \mathbf{RV}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathcal{G}}}(\widetilde{A}))$, by Claim **(4.4)** and **(4.5)**, $\mathrm{val}_{\mathbf{RV}}(c) = \mathrm{val}_{\mathbf{RV}}(a)$ for some $a \in \mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\widetilde{A})$, and hence that $c \cdot a^{-1} \in \mathbf{k}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(\widetilde{A})) = \mathbf{k}(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(\widetilde{A}))$. ■

Concerning the definable closure and algebraic closure in the sort $\mathbf{K}$, although the situation is not ideal, we nevertheless have some control over it:

Corollary 4.6:
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ and $A \subseteq \mathbf{K}(M)$, then $\overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}} \subseteq \mathbf{K}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ is an immediate extension.

Proof. We have $\mathrm{val}(\mathbf{K}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))) \subseteq \mathbf{\Gamma}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathrm{val}(\overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}})$ where the second equality comes from Proposition **(4.2)**. Similarly $\mathrm{res}(\mathbf{K}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))) \subseteq \mathbf{k}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathrm{res}(\overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}})$ and hence $\mathbf{K}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ is an immediate extension of $\overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}}$. ■

Corollary 4.7:
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ and $A \subseteq \mathbf{K}(M)$ then $\mathbf{K}(\langle A\rangle_{-1,\partial}) \subseteq \mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ is an immediate extension.

Proof. By Proposition **(4.2)**, $\mathrm{res}(\mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))) \subseteq \mathbf{k}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathrm{res}(\mathbf{K}(\langle A\rangle_{-1,\partial}))$ and $\mathrm{val}(\mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))) \subseteq \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \mathbb{Q} \otimes \mathrm{val}(\mathbf{K}(\langle A\rangle_{-1,\partial}))$. Let $L := \mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ and $F := \overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{h}}$. Let $c \in L$. We already know that $\mathrm{val}(c) \in \mathbb{Q} \otimes \mathrm{val}(F)$. Let n be minimal such that $n \cdot \mathrm{val}(c) = \mathrm{val}(a)$ for some $a \in F$. Let us show that $n = 1$. We have $\mathrm{res}(ac^{-n}) \in \mathrm{res}(L) = \mathrm{res}(F)$ and we can find $u \in F$ such that $\mathrm{res}(ac^{-n}) = \mathrm{res}(u)$. As L must be Henselian (indeed $\overline{L}^{\mathrm{h}} = \mathrm{dcl}_{\mathcal{L}_{\mathrm{div}}}(L) = L$), we can find $v \in L$ such that $v^n = ac^{-n}u^{-1}$, i.e. $(cv)^n = au^{-1} \in F$. Hence we may assume that c^n itself is in F.
But derivations have a unique extension to algebraic extensions and, as F is Henselian, the valuation also has a unique extension to the algebraic closure. It follows that any algebraic conjugate of c is also an $\mathcal{L}_{\partial,\mathrm{div}}$-conjugate of c. As $\mathbf{K}(M)$ is algebraically closed, it contains non trivial n-th roots of the unit and it follows that we must have $n = 1$.
We have just proved that $\mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A))$ is an immediate extension of $\overline{\mathbf{K}(\langle A\rangle_{-1,\partial})}^{\mathrm{h}}$ and hence of $\mathbf{K}(\langle A\rangle_{-1,\partial})$. ■

In the field of constants, though, we can describe both the definable closure and the algebraic closure.

Proposition 4.8:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq \mathrm{C}_{\mathbf{K}}(M)$, *then* $\mathbf{K}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \overline{\mathrm{Frac}(A)}^{\mathrm{h}}$ *and* $\mathbf{K}(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A)) = \overline{\mathrm{Frac}(A)}^{\mathrm{alg}}$.

Proof. It follows from the fact that the pair $\mathrm{C}_{\mathbf{K}} \subseteq \mathbf{K}$ is separated (see Proposition **(3.1)**) that $\mathrm{C}_{\mathbf{K}}$ does not have any immediate extension in $\mathbf{K}$. Hence $\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A) \subseteq \mathrm{C}_{\mathbf{K}}(M)$ and $\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(A) \subseteq \mathrm{C}_{\mathbf{K}}(M)$. The proposition now follows from the fact that, by Proposition **(3.2)** $\mathrm{C}_{\mathbf{K}}$ is a pure model of ACVF. ∎

5. Metastability in $\mathrm{VDF}_{\mathcal{EC}}$

In this section we prove that maximally complete models of $\mathrm{VDF}_{\mathcal{EC}}$ are metastability bases. Together with our later work on the existence of invariant extensions in $\mathrm{VDF}_{\mathcal{EC}}$, this will allow us to conclude that this theory is metastable.
As mentioned earlier, the existence of metastability bases in $\mathrm{VDF}_{\mathcal{EC}}$ is not as straightforward as one might hope. The main issue is that we can only prove Proposition **(2.6)** when we control the $\mathcal{L}^{\mathbf{RV}}$-algebraic closure of the parameters inside the stable part. Thus we cannot apply it blindly to sets of the form $C\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(Cc))$.
However, in ACVF, we have a more precise description of types over maximally complete fields:

Proposition 5.1 ([HHM08, Remark 12.19])**:**
Let $M \models \mathrm{ACVF}$, $C \subseteq M$ *be maximally complete and algebraically closed,* $a \in \mathbf{K}(M)$ *be a tuple and* $H := \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(Ca))$. *Then* $\mathrm{tp}(a/CH)$ *is stably dominated via* $\mathrm{rv}(C(a))$, *where* $\mathrm{rv}(x)$ *is seen as an element of* $\mathbf{RV}_{\mathrm{val}(x)} \subset \mathrm{St}_{CH}$.

It follows that to prove the existence of metastability basis, we only have to study the $\mathcal{L}_\partial^{\mathbf{RV}}$-algebraic closure in $\mathbf{RV}^{\mathrm{eq}}$. In Proposition **(4.3)**, we showed that we have control over the $\mathcal{L}_\partial^{\mathbf{RV}}$-algebraic closure hence it suffices to prove that $\mathbf{RV}$ with its $\mathcal{L}_\partial^{\mathbf{RV}}$-induced structure eliminates imaginaries. As a matter of fact, we only need to prove elimination for the $\mathcal{L}_\partial^{\mathbf{RV}}$-structure induced on $\mathbf{RV}_H = \bigcup_{\gamma\in H} \mathbf{RV}_\gamma$ where each fibre is a distinct sort.
In [Hru12], Hrushovski studies such structures. The main difference with [Hru12] is that, here, each fibre will also be endowed with a derivation. Let us now give a precise definition of these objects.

Definition 5.2 (Differential linear structure)**:**
Let H *be a group. We define* $\mathcal{L}_{H,\partial}$ *to be the language with a sort* $\mathbf{RV}_\gamma$ *for each* $\gamma \in H$. *The sort* $\mathbf{RV}_0 := \mathbf{k}$ *is equipped with the differential ring language and, for all other* γ, *the sort* $\mathbf{RV}_\gamma$ *is equipped with the language of* $\mathbf{k}$*-vector spaces (i.e. a map* $+_\gamma : \mathbf{RV}_\gamma^2 \to \mathbf{RV}_\gamma$, *a constant* 0_γ *and maps* $-_\gamma : \mathbf{RV}_\gamma \to \mathbf{RV}_\gamma$, $\cdot_\gamma : \mathbf{k} \times \mathbf{RV}_\gamma \to \mathbf{RV}_\gamma$ *and* $\partial_\gamma : \mathbf{RV}_\gamma \to \mathbf{RV}_\gamma$*). Moreover, we ask that for all* γ *and* $\delta \in H$ *there is a map* $\cdot_{\gamma,\delta} : \mathbf{RV}_\gamma \times \mathbf{RV}_\delta \to \mathbf{RV}_{\gamma+\delta}$.
To avoid cluttering the notations, we usually will not write the indexes of $+$, 0, $-$, ∂ *and* $\cdot$ *as they should be clear from the context.*
A differential linear structure is an $\mathcal{L}_{H,\partial}$*-structure such that* $\mathbf{RV}_0 = k$ *is a differential field, each*

$\mathbf{RV}_\gamma$ is a one dimensional differential $\mathbf{k}$-vector space and $\cdot_{\gamma,\varepsilon}$ is a bilinear map. Moreover, we ask that:

(i) For all $\gamma \in H$, $\cdot_{0,\gamma}$ coincides with the scalar multiplication (equivalently, for all $x \in \mathbf{RV}_\gamma$, $1 \cdot x = x$ where 1 denotes the unit in $\mathbf{k}$);

(ii) For all $\gamma, \varepsilon \in H$, all $x \in \mathbf{RV}_\gamma$ and $y \in \mathbf{RV}_\varepsilon$, $x \cdot y = y \cdot x$;

(iii) For all $\gamma, \varepsilon, \eta \in H$, all $x \in \mathbf{RV}_\gamma$, $y \in \mathbf{RV}_\varepsilon$ and $z \in \mathbf{RV}_\eta$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(iv) For all $\gamma, \varepsilon \in H$, all $x \in \mathbf{RV}_\gamma$ and $y \in \mathbf{RV}_\varepsilon$, $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$.

(v) For all $\gamma, \varepsilon \in H$, all $x \in \mathbf{RV}_\gamma$ and $y \in \mathbf{RV}_\varepsilon$, $\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y)$.

Let $T_{H,\partial}$ denote the $\mathcal{L}_{H,\partial}$-theory of differential linear structures. We denote by T_{H,DCF_0} the theory of models of $T_{H,\partial}$ where $\mathbf{k} \vDash \mathrm{DCF}_0$

Remark 5.3:

1. Because all the $\mathbf{RV}_\gamma$ have dimension 1, $\cdot_{\gamma,\varepsilon}$ induces a bijection between $\mathbf{RV}_\gamma \otimes \mathbf{RV}_\varepsilon$ and $\mathbf{RV}_{\gamma+\varepsilon}$ and $\cdot_{-\gamma,\gamma}$ induces a bijection between $\mathbf{RV}_{-\gamma}$ and the dual of $\mathbf{RV}_\gamma$.
2. It also follows that for all $x \in \mathbf{RV}_\gamma \smallsetminus \{0\}$ there exists (a necessarily unique) $y \in \mathbf{RV}_{-\gamma}$ such that $y \cdot x = 1$. We will denote it x^{-1}.
3. Let M be a model of $\mathrm{VDF}_{\mathcal{EC}}$, and $H \leqslant \mathbf{\Gamma}(M)$, then $\mathbf{RV}_H(M) \vDash T_{H,\mathrm{DCF}_0}$. Moreover, by quantifier elimination for $\mathrm{VDF}_{\mathcal{EC}}$ in the leading term language [Sca03, Corollary 5.8 and Theorem 6.3], $\mathbf{RV}_H(M)$ is stably embedded in M and a pure $\mathcal{L}_{H,\partial}$-structure.

Proposition 5.4:
The theory T_{H,DCF_0} eliminates quantifiers and $\mathbf{k}$ is stably embedded and a pure model of DCF_0.

Proof. The proof is the same as for Lemma **(1.5)**, except that Claim **(1.6)** is not needed here. As for the stable embeddedness and purity claim they follow from quantifier elimination. ∎

Proposition 5.5:
The theory T_{H,DCF_0} eliminates imaginaries.

Proof. Let us first prove that T_{H,DCF_0} is stable. Let $M \vDash T_{H,\mathrm{DCF}_0}$, $A \subseteq M$. At the cost of enlarging A, we may assume that for all $\gamma \in H$, $\mathbf{RV}_\gamma(A) \neq \{0\}$. Then, for all $\gamma \in H$, $\mathbf{RV}_\gamma$ is in $\mathcal{L}_H(A)$-definable bijection with $\mathbf{k} \vDash \mathrm{DCF}_0$. It follows that T_{H,DCF_0} is indeed stable. In particular, (global) definable types are dense over any set of algebraically closed parameters. Now, to prove the elimination of imaginaries in T_{H,DCF_0}, by Proposition **(10.2)**, it suffices to show that (definable) types have real canonical bases. But, by quantifier elimination, the canonical basis of $\mathrm{tp}_{\mathcal{L}_H}(c/M)$ is equal to the canonical basis of $\mathrm{tp}_{\mathcal{L}_H^\star}(\partial_\omega(c)/M)$ where $\mathcal{L}_H^\star := \mathcal{L}_H \smallsetminus \{\partial_\gamma : \gamma \in H\}$; and this type has a real canonical basis by elimination of imaginaries in ACF-linear structures with flags [Hru12, Lemma 5.6]. Note that, as every $\mathbf{RV}_\gamma$ is one dimensional, models of T_{H,DCF_0} trivially have flags. ∎

We can now show the existence of metastable bases in $\mathrm{VDF}_{\mathcal{EC}}$.

Proposition 5.6:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$, $C \subseteq \mathbf{K}(M)$ be a maximally complete algebraically closed differential subfield and $a \in \mathbf{K}(M)$. Then, the type $\mathrm{tp}_{\mathcal{L}_\partial^{\mathbf{RV}}}(a/C\boldsymbol{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(Ca)))$ is stably dominated.

Proof. Let $H := \boldsymbol{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(Ca))$. By Proposition **(5.1)**,

$$H = \mathbb{Q} \otimes \boldsymbol{\Gamma}(\langle Ca\rangle_{-1,\partial}) = \boldsymbol{\Gamma}(\mathrm{dcl}_{\mathcal{L}^{\mathbf{RV}}}(C\partial_\omega(a))).$$

By Proposition **(5.1)**, $\mathrm{tp}_{\mathcal{L}^{\mathbf{RV}}}(\partial_\omega(a)/CH)$ is stably dominated via $\mathrm{rv}(C(a)) \subseteq \mathbf{RV}_H$. By Propositions **(4.3)** and **(5.5)**,

$$\mathbf{RV}_H^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}_{\mathcal{L}_\partial^{\mathbf{RV}}}(CH)) = \mathbf{RV}_H(\mathrm{acl}_{\mathcal{L}_\partial^{\mathbf{RV}}}(CH)) = \mathbf{RV}_H(\mathrm{acl}_{\mathcal{L}^{\mathbf{RV}}}(CH)).$$

By Proposition **(2.6)**, we can now conclude that $\mathrm{tp}_{\mathcal{L}_\partial^{\mathbf{RV}}}(a/CH)$ is stably dominated. ∎

Definable types in enrichments of ACVF

6. Types and uniform families of balls

Let $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ and $T \supseteq \mathrm{ACVF}$ be an $\mathcal{L}$-theory that eliminates imaginaries. We assume that T is C-minimal, i.e. every $\mathcal{L}$-sort is the image of an $\mathcal{L}$-definable map with domain some $\mathbf{K}^n$ (we say that $\mathbf{K}$ is dominant) and for all $M \models T$, every $\mathcal{L}(M)$-definable unary set $X \subseteq \mathbf{K}$ is a Boolean combination of balls. For a more extensive introduction to C-minimal theories, one can refer to [Cub14].
In this section, we wish to make precise the idea that in C-minimal theories, $n+1$-types can be viewed as generic types of balls parametrised by realisations of an n-type. This is an obvious higher dimensional generalisation of the unary notion of genericity in a ball (see [HHM06, Definition 2.3.4] or [HMR, Section 3]). To do so, we introduce a class of Δ-types (see Definition **(6.12)**) for Δ a finite set of $\mathcal{L}$-formulas that will play a central role in the rest of this text. We also show that at the cost of enlarging Δ, we may assume that all types are of this specific form.
We take the convention that points in $\mathbf{K}$ are closed balls of radius $+\infty$ and $\mathbf{K}$ itself is an open ball of radius $-\infty$.

Definition 6.1 ($\mathbf{B}^{[l]}$ and $\mathbf{B}^{[l]}_{\mathrm{st}}$)**:**
Let $\overline{\mathbf{B}}$ be the set of all closed balls (potentially with radius $+\infty$), $\dot{\mathbf{B}}$ be the set of all open balls (potentially with radius $-\infty$), $\mathbf{B} := \overline{\mathbf{B}} \cup \dot{\mathbf{B}}$ and $l \in \mathbb{N}_{>0}$. We define $\mathbf{B}^{[l]} := \{B \subseteq \mathbf{B} : |B| \leqslant l\}$. We also define $\mathbf{B}^{[l]}_{\mathrm{st}} := \{B \in \mathbf{B}^{[l]}$: all the balls in B have the same radius and they are either all open or all closed$\}$.

Notation 6.2:
For all $B \in \mathbf{B}^{[l]}$, we will be denoting by $\mathbb{S}(B)$ the set $\bigcup_{b \in B} b$, i.e. the set of valued field points in the balls of B. Because the balls can be nested, $\mathbb{S}$ is not an injective function. But in each fibre of $\mathbb{S}$ there is a unique element with minimal cardinality — the one where there is no

intersection between the balls. We will denote by $\mathbb{B}$ this canonical section of $\mathbb{S}$.

Remark 6.3:

1. As $\mathbf{B}$ is the disjoint union of the sets of codes for open balls and the set of codes for closed ones, one can decide whether a given code is the code of an open or a closed ball and hence $\mathbf{B}_{\mathrm{st}}^{[l]}$ is indeed an interpretable set. In fact, one can also recognise if a ball b is open or closed by looking if the set $\{\mathrm{val}(x-y) : x, y \in b\}$ has a smallest element or not.
2. Note that $\varnothing \in \mathbf{B}_{\mathrm{st}}^{[l]}$
3. Points in $\mathbf{B}_{\mathrm{st}}^{[l]}$ behave more or less like balls. For example if B_1 and $B_2 \in \mathbf{B}_{\mathrm{st}}^{[l]}$ are such that $\mathbb{S}(B_1) \subset \mathbb{S}(B_2)$, where $\subset$ denotes the strict inclusion, then either all the balls in B_1 have smaller radius than the balls in B_2 or if they have equal radiuses, then the balls in B_1 must be open and those in B_2 must be closed, or else the inclusion would not be strict.

Definition 6.4 (Generalised radius)**:**
Let $B \in \mathbf{B}_{\mathrm{st}}^{[l]} \smallsetminus \{\varnothing\}$. We define the generalised radius of B (denoted $\mathrm{grad}(B)$) to be the pair $(\gamma, 0)$ when the balls in B are closed of radius γ and $(\gamma, 1)$ when they are open of radius γ. The set of generalised radiuses, a subset of $(\mathbf{\Gamma} \cup \{-\infty, +\infty\}) \times \{0,1\}$, is ordered lexicographically.
We also define the generalised radius of $\varnothing$ to be $(+\infty, 1)$, i.e. greater than any generalised radius of non empty $B \in \mathbf{B}_{\mathrm{st}}^{[l]}$.

Proposition 6.5:
Let $(B_i)_{i\in I} \subseteq \mathbf{B}_{\mathrm{st}}^{[l]}$. Assume that there exists i_0 such that the balls in B_{i_0} have generalised radius greater or equal than all the other B_i. In particular this holds if I is finite. Then $\mathbb{B}(\bigcap_i \mathbb{S}(B_i)) \subseteq B_{i_0}$. Moreover, there exists $(i_j)_{0<j\leqslant l} \in I$ such that $\bigcap_i \mathbb{S}(B_i) = \bigcap_{j=0}^{l} \mathbb{S}(B_{i_j})$.

Proof. For any $b \in B_{i_0}$, if $\bigcap_i \mathbb{S}(B_i) \cap b \neq \varnothing$ then $b \subseteq \bigcap_i \mathbb{S}(B_i)$. Hence $\bigcap_i \mathbb{S}(B_i) = \mathbb{S}(\{b \in B_{i_0} : b \cap \bigcap_i B_i \neq \varnothing\})$ and $\mathbb{B}(\bigcap_i \mathbb{S}(B_i)) \subseteq B_{i_0}$. Moreover, if $\bigcap_i \mathbb{S}(B_i) \cap b = \varnothing$, then there exists i_b such that $b \cap \mathbb{S}(B_{i_b}) = \varnothing$ and $\bigcap_i \mathbb{S}(B_i)$ can be obtained by intersecting B_{i_0} with the B_{i_b} of which there are at most l. ∎

Definition 6.6 ($d_i(B_1, B_2)$)**:**
Let b_1 and $b_2 \in \mathbf{B}$. When $b_1 \cap b_2 = \varnothing$, we define $d(b_1, b_2)$ to be $\mathrm{val}(x_1 - x_2)$, where $x_i \in b_i$, which does not depend on the choice of the x_i. When $b_1 \cap b_2 \neq \varnothing$, we define $d(b_1, b_2) = \min\{\mathrm{rad}(b_1), \mathrm{rad}(b_2)\}$, where $\mathrm{rad}(b)$ is the radius of the ball b.
For all B_1 and $B_2 \in \mathbf{B}^{[l]}$, let us define $D(B_1, B_2) := \{d(b_1, b_2) : b_1 \in B_1$ and $b_2 \in B_2\}$ and let us list the elements in $D(B_1, B_2)$ as $d_1 > d_2 > \cdots > d_k$. For all $i \leqslant k$, we define $d_i(B_1, B_2) := d_i$.

This definition coincides with the definition in Section 8 for finite sets of points. When B_1 and $B_2 \in \mathbf{B}_{\mathrm{st}}^{[l]}$, we also define $d_0(B_1, B_2) := \min\{\mathrm{rad}(B_1), \mathrm{rad}(B_2)\}$; it is equal to $d_1(B_1, B_2)$ when $\mathbb{S}(B_1) \cap \mathbb{S}(B_2) \neq \varnothing$. Later, for coding purposes we might want $d_i(B_1, B_2)$ to be defined for all $i \leqslant l^2$ in which case, for $i > k$, we set $d_i(B_1, B_2) = d_k$.

Let $M \vDash T$, $F = (F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ (in particular Λ is an $\mathcal{L}(M)$-definable set) and $\Delta(x,y;t)$ be a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$, $y \in \mathbf{K}$ and t is a tuple of variables. To simplify notations, we will be denoting $\mathbb{S}(F_\lambda(x))$ by $F_\lambda^{\mathbb{S}}(x)$. We define $\Psi_{\Delta,F}(x,y;t,\lambda)$ to be the set $\Delta(x,y;t) \cup \{y \in F_\lambda(x) \wedge \lambda \in \Lambda\}$.
Note that if $n = 0$ all of what we prove in this section and in Section 7 still hold (and is in fact much more straightforward because we are considering fixed balls instead of parametrised balls).

Definition 6.7 (Δ adapted to F)**:**
We say that Δ is adapted to F if there are $\lambda_\varnothing$ and $\lambda_{\mathbf{K}} \in \Lambda$ such that for all $x \in \mathbf{K}^n$, $F_{\lambda_\varnothing}(x) = \varnothing$ and $F_{\lambda_{\mathbf{K}}}(x) = \{\mathbf{K}\}$ and for all $p \in \mathcal{S}_{x,y}^{\Delta}(M)$, $p(x,y)$ decides:

(i) For all $\square \in \{=,\subseteq\}$, all λ and $(\mu_i)_{0\leqslant i<l} \in \Lambda(M)$, whether $F_\lambda^{\mathbb{S}}(x) \square \bigcup_{0\leqslant i<l} F_{\mu_i}^{\mathbb{S}}(x)$ (respectively $F_\lambda(x) \square \bigcup_{i<l} F_{\mu_i}(x)$);

(ii) For all λ_1, all λ_2 and $\mu \in \Lambda(M)$, whether $F_\mu^{\mathbb{S}}(x) = F_{\lambda_1}^{\mathbb{S}}(x) \cap F_{\lambda_2}^{\mathbb{S}}(x)$;

(iii) For all $\lambda \in \Lambda(M)$, whether the balls in $F_\lambda(x)$ are closed;

(iv) For all $\square \in \{=,\leqslant\}$, all λ, μ_1 and $\mu_2 \in \Lambda(M)$ and all $i \leqslant l^2$, whether $\mathrm{rad}(F_{\lambda_1}(x)) \square d_i(F_{\mu_1}(x), F_{\mu_2}(x))$.

Note that none of the above formulas actually depend on y so what is really relevant is not p but the closed set induced by p in $\mathcal{S}_x^{\mathcal{L}}(M)$.

Until Proposition **(6.15)**, let us assume that Δ is adapted to F and let $p \in \mathcal{S}_{x,y}^{\Delta}(M)$.

Definition 6.8 (Generic intersection)**:**
We say that F is closed under generic intersection over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$, there exists $\mu \in \Lambda(M)$ such that

$$p(x,y) \vdash F_\mu^{\mathbb{S}}(x) = F_{\lambda_1}^{\mathbb{S}}(x) \cap F_{\lambda_2}^{\mathbb{S}}(x).$$

Let us assume, until Proposition **(6.15)**, that F is closed under generic intersection over p.

Definition 6.9 (Generic irreducibility)**:**
For all $\lambda \in \Lambda(M)$, we say that F_λ is generically irreducible over p if for all $\mu \in \Lambda(M)$, if $p(x,y) \vdash F_\mu(x) \subseteq F_\lambda(x)$ and $p(x,y) \vdash F_\mu(x) \neq \varnothing$ then $p(x,y) \vdash F_\mu(x) = F_\lambda(x)$.
We say that F is generically irreducible over p if for every $\lambda \in \Lambda(M)$, F_λ is generically irreducible over p.

Let us now show that generically irreducible families of balls behave nicely under generic intersection.

Proposition 6.10:
Let λ_1 and $\lambda_2 \in \Lambda(M)$ be such that F_{λ_1} and F_{λ_2} are generically irreducible over p and $p(x,y)$ implies that the balls in $F_{\lambda_1}(x)$ have smaller or equal generalised radius than the balls in $F_{\lambda_2}(x)$. Then either $p(x,y) \vdash F_{\lambda_1}^{\mathbb{S}}(x) \cap F_{\lambda_2}^{\mathbb{S}}(x) = \varnothing$ or $p(x,y) \vdash F_{\lambda_1}^{\mathbb{S}}(x) \cap F_{\lambda_2}^{\mathbb{S}}(x) = F_{\lambda_1}^{\mathbb{S}}(x)$.

Proof. Let $(a,c) \vDash p$. By Proposition **(6.5)**, we have that $\mathbb{B}(F^{\$}_{\lambda_1}(a) \cap F^{\$}_{\lambda_2}(a)) \subseteq F^{\$}_{\lambda_1}(a)$. By generic intersection, there exists μ such that $p(x,y) \vdash F^{\$}_{\mu}(x) = F^{\$}_{\lambda_1}(x) \cap F^{\$}_{\lambda_2}(x)$. Then $F_{\mu}(a) \subseteq F_{\lambda_1}(a)$ and hence, if $F_{\mu}(a) \neq \emptyset$, $F_{\mu}(a) = F_{\lambda_1}(a)$. ∎

Corollary 6.11:
Assume p is $\mathcal{L}(M)$-definable. Then $\Lambda_p := \{\lambda \in \Lambda : F_\lambda$ is generically irreducible over $p\}$ is $\mathcal{L}(M)$-definable and the $\mathcal{L}(M)$-definable family $(F_\lambda)_{\lambda\in\Lambda_p}$ is closed under generic intersection over p.

Proof. The definability of Λ_p is a consequence of the definability of p and the closure of $(F_\lambda)_{\lambda\in\Lambda_p}$ under generic intersection follows from Proposition **(6.10)**. ∎

Until Proposition **(6.15)**, let us also assume that F is generically irreducible over p.

Definition 6.12 ((Δ, F)-generic type of E over p)**:**
Let $E \subset \Lambda(M)$. We define $\alpha_{E/p}(x,y)$, the (Δ, F)-generic type of E over p, to be the following $\Psi_{\Delta,F}$-type over M:

$$\begin{aligned} p(x,y) \quad &\cup \quad \{y \in F^{\$}_{\lambda}(x) : \lambda \in E\} \\ &\cup \quad \{y \notin F^{\$}_{\mu}(x) : \mu \in \Lambda(M) \text{ and for all } \lambda \in E,\ p(x,y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)\}. \end{aligned}$$

Note that most of the time, Δ and F will be obvious from the context and it will not be an issue that the notation $\alpha_{E/p}$ mentions neither Δ nor F.

Proposition 6.13:
Let $E \subset \Lambda(M)$ be such that $\alpha_{E/p}$ is consistent, then $\alpha_{E/p}$ generates a complete $\Psi_{\Delta,F}$-type over M.

Therefore, when it is consistent, we will identify $\alpha_{E/p}$ with the type it generates.

Proof. Pick any $\mu \in \Lambda(M)$. Either there exists $\lambda \in E$ such that $p(x,y) \vdash F^{\$}_{\mu}(x) \cap F^{\$}_{\lambda}(x) = \emptyset$, in which case $\alpha_{E/p}(x,y) \vdash y \notin F^{\$}_{\mu}(x)$, or there exists $\lambda \in E$ such that $p(x,y) \vdash F^{\$}_{\lambda}(x) \subseteq F^{\$}_{\mu}(x)$, and then $\alpha_{E/p}(x,y) \vdash y \in F^{\$}_{\mu}(x)$, or for all $\lambda \in E$, $p(x,y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)$ and hence $\alpha_{E/p}(x,y) \vdash y \notin F^{\$}_{\mu}(x)$. ∎

Remark 6.14:
Any $q \in \mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(M)$ is of the form $\alpha_{E/p}$. Indeed let $p := q|_{\Delta}$ and $E = \{\lambda \in \Lambda(M) : q(x,y) \vdash y \in F_\lambda(x)\}$, then, quite clearly, $q = \alpha_{E/p}$.

Although this will not be used afterwards, when y does not appear in Δ, we can prove consistency under some obvious hypothesis:

Proposition 6.15:
Assume that y does not appear in any of the formulas in Δ and let $E \subseteq \Lambda(M)$ be such that for all λ_1 and $\lambda_2 \in E$, $p(x) \vdash F_{\lambda_1}(x) \cap F_{\lambda_2}(x) \neq \emptyset$. Then $\alpha_{E/p}$ is consistent and generates a complete $\Psi_{\Delta,F}$-type over M.

Proof. Let us show consistency. Completeness then follows from Proposition **(6.13)**. If this type is not consistent, there exists finitely many $\lambda_i \in E$ and finitely many $\mu_j \in \Lambda(M)$ such that for all $\lambda \in E$, $p(x) \vdash F^{\$}_{\mu_j}(x) \subset F^{\$}_{\lambda}(x)$ and $p(x) \vdash \bigcap_i F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$. Replacing $\bigcap_i F^{\$}_{\lambda_i}(x)$ by their generic intersection, we find $\lambda \in E$ such that $p(x) \vdash F^{\$}_{\lambda}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$. Let $a \vDash p$ and b be one of the balls in $F_\lambda(a)$. This ball is covered by finitely many subballs from $\bigcup_j F_{\mu_j}(a)$ and, as the residue field is infinite, it must be included in one of those balls. Let us assume that $b \subseteq F^{\$}_{\mu_1}(a)$. Then the balls of $F_\lambda(a)$ must have smaller or equal generalised radius than those of $F_{\mu_1}(a)$. Hence by Proposition **(6.10)**, $F^{\$}_{\mu_1}(a) \cap F^{\$}_{\lambda}(a) = F^{\$}_{\lambda}(a)$, i.e. $F^{\$}_{\lambda}(a) \subseteq F^{\$}_{\mu_1}(a)$, a contradiction. ∎

Now that we have found finite sets Θ of $\mathcal{L}$-formulas, namely those of the form $\Psi_{\Delta,F}$, for which we understand the Θ-types, let us show that any finite set of formulas with variables in $\mathbf{K}^{n+1}$ can be decided by some $\Psi_{\Delta,F}$ for well chosen Δ and F.

Proposition 6.16 (Reduction to $\Psi_{\Delta,F}$-types)**:**
Let $\Theta(x, y; t)$ be a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $y \in \mathbf{K}$. Then there exists an $\mathcal{L}$-definable family $(F_\lambda)_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$ and a finite set of $\mathcal{L}$-formulas $\Delta(x; s)$ such that any $\Psi_{\Delta,F}$-type decides all the formulas in Θ.

Proof. Let $\varphi(x, y; t)$ be a formula in Θ. As T is C-minimal, for all tuples $a \in \mathbf{K}$ and $c \in M$, the set $\varphi(a, M; c)$ has a canonical representation as Swiss cheeses, i.e. it is of the form $\bigcup_i(b_i \smallsetminus b_{i,j})$ where the b_i and $b_{i,j}$ are algebraic over ac. In particular, there exists $l \in \mathbb{N}_{>0}$ and $\mathcal{L}(c)$-definable functions $H_{\varphi,c} : \mathbf{K}^n \to \mathbf{B}^{[l]}$ and $G_{\varphi,c} : \mathbf{K}^n \to \mathbf{B}^{[l]}$ such that $M \vDash \forall y\,(y \in H^{\$}_{\varphi,c}(a) \smallsetminus G^{\$}_{\varphi,c}(a) \iff \varphi(a, y; c))$. By compactness, we can find finitely many $\mathcal{L}$-definable families $(H_{i,\varphi,c})_{c\in M}$ and $(G_{i,\varphi,c})_{c\in M}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l_{i,\varphi}]}$ such that for any choice of c and a there is an i such that $\varphi(a, y; c) \iff y \in H^{\$}_{i,\varphi,c}(a) \smallsetminus G^{\$}_{i,\varphi,c}(a)$. Choosing l to be the maximum of the $l_{i,\varphi}$ and using any coding trick, one can find an $\mathcal{L}$-definable family $(F_\lambda)_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$ such that for any $\varphi \in \Theta$, i and c we find μ and $\nu \in \Lambda$ such that $H_{i,\varphi,c} = F_\mu$ and $G_{i,\varphi,c} = F_\nu$.
Now let $\Delta(x; t, \mu, \nu) = \{\forall y\,(\varphi(x, y; t) \iff y \in F^{\$}_{\mu}(x) \smallsetminus F^{\$}_{\nu}(x)) : \varphi \in \Theta\}$. Then for any $p \in \mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(M)$, $\varphi \in \Theta$ and tuple $c \in M$, there exists μ and $\nu \in \Lambda(M)$ such that $p(x, y) \vdash \varphi(x, y; c) \iff y \in F^{\$}_{\mu}(x) \smallsetminus F^{\$}_{\nu}(x)$ and either $p(x, y) \vdash y \in F^{\$}_{\mu}(x) \wedge y \notin F^{\$}_{\nu}(x)$ in which case $p(x, y) \vdash \varphi(x, y; c)$ or not, in which case $p(x, y) \vdash \neg\varphi(x, y; c)$. ∎

And now let us show that we can refine any Δ and F into a family verifying all previous hypotheses.

Proposition 6.17 (Reduction to $\mathbf{B}^{[l]}_{\mathrm{st}}$)**:**
Let $A \subseteq M$ and $(F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$. Then there exists an $\mathcal{L}(A)$-definable family $(G_\omega)_{\omega\in\Omega}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ such that for all λ there exist $(\omega_i)_{0\leqslant i<l}$ such that $F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$ and for all ω there exists λ such that $G_\omega(x) \subseteq F_\lambda(x)$.

Proof. For all $\lambda \in \Lambda$, $0 < i \leqslant l$ and $j = 0, 1$, we define $G_{\lambda,i,j}(x) := \{b \in F_\lambda(x) : b$ is open if $j = 0$, closed otherwise and b has the i-th smallest radius among the balls in $F_\lambda(x)\}$. As i and j only take finitely many values, $G = (G_\omega)_{\omega\in\Omega}$ can indeed be viewed as an $\mathcal{L}(A)$-definable family. Then for all x, $G_\omega(x) \in \mathbf{B}^{[l]}_{\mathrm{st}}$ and for all x and λ, $G_{\lambda,i,j}(x) \subseteq F_\lambda(x)$ and $F_\lambda(x) = \bigcup_{i,j} G_{\lambda,i,j}(x)$ and at most l of them are non empty. ∎

Definition 6.18 (Generic complement)**:**
We say that F is closed under generic complement over p if for all λ and $\mu \in \Lambda(M)$ such that $p(x) \vdash F_\mu(x) \subseteq F_\lambda(x)$, there exists $\kappa \in \Lambda(M)$ such that

$$p(x) \vdash F_\lambda(x) = F_\mu(x) \uplus F_\kappa(x).$$

Note that p can indeed decide any such statement because it is equivalent to $F_\lambda(x) = F_\mu(x) \cup F_\kappa(x)$ and $F^{\$}_\mu(x) \cap F^{\$}_\kappa(x) = \varnothing$.

Lemma 6.19:
Let $F = (F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$, $\Delta(x;t)$ a finite set of $\mathcal{L}$-formulas adapted to F and $p \in \mathcal{S}^{\Delta}_x(M)$. Assume that F is closed under generic complement over p. Let $\Lambda_p := \{\lambda \in \Lambda : F_\lambda$ is generically irreducible over $p\}$, then for all $\lambda \in \Lambda(M)$ there exists $(\lambda_i)_{0\leqslant i<l} \in \Lambda_p(M)$ such that $p(x) \vdash F_\lambda(x) = \bigcup_i F_{\lambda_i}(x)$.

Proof. Let $x \vDash p$. We work by induction on $|F_\lambda(x)|$. If there exists $\mu \in \Lambda(M)$ such that $F_\mu(x) \subset F_\lambda(x)$ and $F_\mu(x) \neq \varnothing$, then there exists $\kappa \in \Lambda(M)$ such that $F_\lambda(x) = F_\mu(x) \uplus F_\kappa(x)$. We now apply the induction hypothesis to $F_\mu(x)$ and $F_\kappa(x)$. Finally, because $|F_\lambda(x)| \leqslant l$, we cannot cut it in more than l distinct pieces. ∎

Proposition 6.20 (Reduction to irreducible families)**:**
Let $A \subseteq M$, $(F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and $\Delta(x;t)$ a finite set of $\mathcal{L}$-formulas. Then, there exists an $\mathcal{L}(A)$-definable family $(G_\omega)_{\omega\in\Omega}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and a finite set of $\mathcal{L}$-formulas $\Theta(x;t,s) \supseteq \Delta(x;t)$ such that Θ is adapted to G and for any $p \in \mathcal{S}^{\Theta}_x(M)$:

(i) G is closed under generic intersection and complement over p;

(ii) For all $\omega \in \Omega(M)$ there exists $\lambda \in \Lambda(M)$ such that $p(x) \vdash G_\omega(x) \subseteq F_\lambda(x)$;

(iii) For all $\lambda \in \Lambda(M)$, there exists $\omega \in \Omega(M)$ such that $p(x) \vdash F_\lambda(x) = G_\omega(x)$;

(iv) For all $\omega \in \Omega(M)$, there exists $(\omega_i)_{0\leqslant i<l} \in \Omega_p(M)$ such that $p(x) \vdash G_\omega(x) = \bigcup_i G_{\omega_i}(x)$;

where $\Omega_p := \{\omega \in \Omega : G_\omega$ is generically irreducible over $p\}$.

Proof. Adding them if necessary, we may assume that F contains the constant functions equal to $\varnothing$ and $\{\mathbf{K}\}$ respectively. For all $\overline{\lambda} \in \Lambda^{l+1}$, let $H_{\overline{\lambda}}(x) := \mathbb{B}(\bigcap_{0\leqslant i\leqslant l} F^{\$}_{\lambda_i}(x))$. It follows from Proposition **(6.5)**, that $H = (H_{\overline{\lambda}})_{\overline{\lambda}\in\Lambda^{l+1}}$ is well-defined and that **6.20.(ii)** holds for H. Adding finitely many formulas to $\Delta(x;t)$, we obtain $\Xi(x;s)$ which is adapted to H. Let $p \in \mathcal{S}^{\Xi}_x(M)$. Proposition **(6.5)** also implies that for a given x, the intersection of any number of $F^{\$}_\lambda(x)$ is given by the intersection of $l+1$ of them and hence is an instance of H. As Ξ is adapted to H, we have proved that H is closed under generic intersection over any Ξ-type p. Condition **6.20.(iii)** also clearly holds for H.
Let $B \in \mathbf{B}^{[l]}_{\mathrm{st}}$, we define B^1 to be B and B^0 to be its complement (in $\mathbf{B}$). As previously, to simplify notations, for $\varepsilon \in \{0,1\}$, we will write $H^{\varepsilon}_\mu(x)$ for $(H_\mu(x))^{\varepsilon}$.

Claim 6.21: *Let $B \in \mathbf{B}^{[l]}_{\mathrm{st}}$. Any Boolean combination of sets $(C_i)_{i\leqslant r} \subseteq B$ (where we take the* *complement in B, i.e. $C^0 \cap B$) lives in $\mathbf{B}_{\mathrm{st}}^{[l]}$ and can be written as $\bigcap_{j<l} \bigcup_{k<l} (C_{j,k}^{\varepsilon_{j,k}} \cap B)$ where the $C_{j,k}$ are taken among the C_i and $\varepsilon_{j,k} \in \{0,1\}$.*

Proof. Such a Boolean combination lives in $\mathbf{B}_{\mathrm{st}}^{[l]}$ because it is a subset of B. The fact that it can be written as $\bigcap_j \bigcup_k (C_{j,k}^{\varepsilon_{j,k}} \cap B)$ is just the existence of the conjunctive normal form. Moreover, as in Proposition **(6.5)**, any intersection $\bigcap_k C_{j,k}^{\varepsilon_{j,k}} \cap B$ for fixed j can be rewritten as the intersection of at most l of then (for each ball from B missing from the intersection, choose a k such that this ball is not in $C_{j,k}^{\varepsilon_{j,k}} \cap B$). Similarly, the union can be rewritten as the union of at most l of them by choosing, for every $b \in B$ which appears in the union a j such that b appears in $\bigcup_k (C_{j,k}^{\varepsilon_{j,k}} \cap B)$. ♦

For all $\nu \in \Lambda^{l+1}$, $\overline{\mu} \in (\Lambda^{l=1})^{l^2}$ and $\overline{\varepsilon} \in 2^{l^2}$, we define $G_{\nu,\overline{\mu},\overline{\varepsilon}}(x) = \bigcap_{i<l} \bigcup_{j<l} ((H_{\mu_{i,j}}^{\varepsilon_{i,j}}(x)) \cap H_\nu(x))$ whenever all the $H_{\mu_{i,j}} \subseteq H_\nu(x)$ and $G_{\nu,\overline{\mu},\overline{\varepsilon}}(x) = H_\nu(x)$ otherwise. Adding some more formulas to Ξ, we obtain a finite set of formulas $\Theta(x;t,s,u)$ which is adapted to G. It is clear that **6.20.(ii)** and **6.20.(iii)** still hold. Furthermore,

$$G_{\nu,\overline{\mu},\overline{\varepsilon}}^{\mathbb{S}}(x) \cap G_{\sigma,\overline{\tau},\overline{\eta}}^{\mathbb{S}}(x) = \bigcap_{i,k} \bigcup_{j,r} (\mathbb{S}(H_{\mu_{i,j}}^{\varepsilon_{i,j}}(x)) \cap \mathbb{S}(H_{\tau_{k,r}}^{\eta_{k,r}}(x)) \cap H_\nu^{\mathbb{S}}(x) \cap H_\sigma^{\mathbb{S}}(x)).$$

As H is closed under generic intersection there exists ρ such that $H_\rho^{\mathbb{S}}(x) = H_\nu^{\mathbb{S}}(x) \cap H_\sigma^{\mathbb{S}}(x)$. By Proposition **(6.5)**, we have both $\mathbb{B}(\mathbb{S}(H_{\mu_{i,j}}^{\varepsilon_{i,j}}(x)) \cap H_\rho^{\mathbb{S}}(x)) \subseteq H_\rho(x)$ and $\mathbb{B}(\mathbb{S}(H_{\tau_{k,r}}^{\eta_{k,r}}(x)) \cap H_\rho^{\mathbb{S}}(x)) \subseteq H_\rho(x)$ and we can conclude by Claim **(6.21)** that G is also closed under generic intersection over p. Similarly we show that whenever $G_{\nu,\overline{\mu},\overline{\varepsilon}}(x) \subseteq G_{\sigma,\overline{\tau},\overline{\eta}}(x)$ then $G_{\nu,\overline{\mu},\overline{\varepsilon}}^0(x) \cap G_{\sigma,\overline{\tau},\overline{\eta}}(x)$ is also an instance of G, i.e. G is closed under generic complement over p and hence **6.20.(iv)** is proved in Lemma **(6.19)**. ■

7. Quantifiable types

Let us begin with the example that motivates the definition of quantifiable types. Let b be an open ball in some model of ACVF and α_b be its generic type — the type of points which are in b but avoid all its strict subballs. Let X be any set definable in an enrichment of ACVF. Then all realisations of α_b are in X, i.e. $\alpha_b \vdash x \in X$, if and only if there exists $b' \in \mathbf{B}$ such that $b' \subset b$ and $b \setminus b' \subseteq X$. Thus, although for most definable sets X, both X and its complement are consistent with α_b, if it happens that all realisations of α_b are in X, then there is a formula which says so. We have just shown that α_b is quantifiable as a partial $\widetilde{\mathcal{L}}$-type (see Definition **(7.1)**) for any enrichment $\widetilde{\mathcal{L}}$ of ACVF. If $(b_i)_{i \in I}$ is a strict chain of balls, i.e. $P := \bigcap_i b_i$ is not a ball, the exact same proof shows that the generic type of P is also quantifiable as a partial $\widetilde{\mathcal{L}}$-type.

If b is a closed ball, the situation is somewhat more complicated because $\alpha_b(x) \vdash x \in X$ if and only if there exists finitely many maximal open subballs $(b_i)_{0 \leqslant i < k}$ of b such that for all $x \in \mathbf{K}$, $x \in b \setminus \bigcup_i b_i$ implies $x \in X$. Because the set of maximal open subballs of a given ball is internal to the residue field, to obtain that α_b is quantifiable (as a partial $\widetilde{\mathcal{L}}$-type), we need to know that the $\widetilde{\mathcal{L}}$-induced structure on $\mathbf{k}$ eliminates $\exists^\infty$ to bound the number of maximal open subballs we have to remove. Recall that an $\mathcal{L}$-theory T eliminates $\exists^\infty$ if for every $\mathcal{L}$-formula $\varphi(x;s)$ there is an $n \in \mathbb{N}$ such that for all $M \models T$ and $m \in M$, if $|\varphi(M;m)| < \infty$ then $|\varphi(M;m)| \leqslant n$.

The notion of quantifiable type will play a fundamental role in Section 9. The main result of this section is Corollary **(7.12)** which says that, under some more hypothesis on the families of parametrised balls we consider, the types of the form $\alpha_{E/p}$ (see Definition **(6.12)**) are quantifiable if E is definable and p is quantifiable. The proof is essentially a parametrised version of the argument above. We then prove that we can refine families of parametrised balls so that they have the necessary properties.

Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure.

Definition 7.1 (Quantifiable partial $\mathcal{L}$-types)**:**
Let p be a partial $\mathcal{L}(M)$-type. We say that p is quantifiable if for all $\mathcal{L}$-formulas $\varphi(x;s)$ there exists an $\mathcal{L}(M)$-formula $\theta(s)$ such that for all tuples $m \in M$,

$$M \models \theta(m) \text{ if and only if } p(x) \vdash \varphi(x;m).$$

Let $A \subseteq M$. If we want to specify that θ is an $\mathcal{L}(A)$-formula, we will say that p is $\mathcal{L}(A)$-quantifiable.

Remark 7.2:

1. A type p(x) is quantifiable if we can quantifiy universally and existentially over realisations of p, that is for every formula $\varphi(x)$, "for all $x \models p$, $\varphi(x)$ holds" and "there exists an $x \models p$ such that $\varphi(x)$ holds" are both first order formulas. Hence the name.

2. The notion of quantifiability of a type generalises definability of types to partial types. If p is a complete $\mathcal{L}$-type then it is definable if and only if it is quantifiable.

3. In fact, there are various possible ways in which to extend definability to partial types depending on two things: do we want the defining scheme to be open, closed or clopen (i.e. ind-definable, pro-definable or definable) and do we want the closure under implication of the partial type also to be definable? Quantifiable partial types correspond to the case where the closure under implication of the type has a definable defining scheme. Although these different notions have often been indistinctively called definability, we feel that it is better to try and distinguish them and quantifiability seems to be a notion that naturally concerns the closure under implication of a partial type.

4. The partial types we will consider here are (complete) Δ-types for some set $\Delta(x;t)$ of $\mathcal{L}$-formulas. Note that if $p \in \mathcal{S}^{\Delta}_x(M)$ is $\mathcal{L}(A)$-quantifiable, it is in particular $\mathcal{L}(A)$-definable as a Δ-type, i.e. for any formula $\varphi(x;t) \in \Delta$, there is an $\mathcal{L}(A)$-formula $d_p x\varphi(x;t) = \theta(t)$ such that for all tuples $m \in M$, $\varphi(x;m) \in p$ if and only if $M \models d_p x\varphi(x;m)$. In particular, p has a canonical extension $p|_N$ to any $N \geqslant M$ defined using the same defining scheme, i.e. for all $\varphi(x;t) \in \Delta$ and tuple $m \in N$, $\varphi(x;m) \in p|_N$ if and only if $N \models d_p x\varphi(x;m)$

Let us now prove some results on quantifiable types which will not be needed afterwards but which shed some light on this notion.

Proposition 7.3:
Let $\Delta(x;t)$ be a set of $\mathcal{L}$-formulas and $p \in \mathcal{S}^{\Delta}_{x}(M)$ be quantifiable. Assume that M is $(\aleph_0+|\Delta|)^+$-saturated, then for all $N \succcurlyeq M$, $p|_N$ is quantifiable, using the same formulas.

Proof. Let $\varphi(x;s)$ be any $\mathcal{L}$-formula. By quantifiability of p, there exists $\theta(s)$ such that for all tuples $m \in M$, $M \vDash \theta(m)$ if and only if $p(x) \vdash \varphi(x;m)$, which in turn is equivalent to the existence of a finite number of $\psi_i(x;m_i) \in p$ such that $M \vDash \forall x \bigwedge_i \psi_i(x;m_i) \to \varphi(x;m)$. Let $d_p x\, \psi_i(x;t_i)$ be the $\mathcal{L}(M)$-formula in the defining scheme of p relative to ψ_i, we have:

$$\theta(s) \to \bigvee_{\overline{\psi}\in\Delta} \exists \bar{t}\,(\bigwedge_{i<|\overline{\psi}|} d_p x\, \psi_i(x;t_i) \wedge (\forall x \bigwedge_{i<|\overline{\psi}|} \psi_i(x;t_i) \to \varphi(x;s))).$$

Because there are at most $\aleph_0+|\Delta|$ parameters involved in the formulas above and M is $(\aleph_0+|\Delta|)^+$-saturated, there exists finitely many tuples $(\overline{\psi}_j)_{0\leqslant j<k}$ such that

$$\theta(s) \to \bigvee_{0\leqslant j<k} \exists \bar{t}\,(\bigwedge_{i<|\overline{\psi}_j|} d_p x\, \psi_{j,i}(x;t_{j,i}) \wedge (\forall x \bigwedge_{i<|\overline{\psi}_j|} \psi_{j,i}(x;t_{j,i}) \to \varphi(x;s))).$$

It follows that in any $N \succcurlyeq M$, the same implication holds and hence for all $m \in N$, $N \vDash \theta(m)$ implies that $p|_N \vdash \varphi(x;m)$.
Now assume that there exists $m \in N$ such that $N \vDash \neg\theta(m)$ but $p|_N \vdash \varphi(x;m)$. Then there exists $(\psi_i(s;m_i))_{0\leqslant i<k} \in p|_N$ such that $N \vDash \forall x \bigwedge \psi_i(x;m_i) \to \varphi(x;m)$. Therefore

$$N \vDash \exists s\, \neg\theta(s) \wedge \exists \bar{t}\,(\bigwedge_i d_p x\, \varphi_i(x;t_i) \wedge (\forall x \bigwedge \psi_i(x;m_i) \to \varphi(x;m))).$$

Because $N \succcurlyeq M$, this also holds in M, contradicting the quantifiability of p. ■

Remark 7.4:
The saturation hypothesis is not superfluous. Indeed, let $\mathcal{L} := E$, M be the $\mathcal{L}$-structure where E is an equivalence relation with exactly one class of every finite cardinality. Let $\Delta(x;t) := \{x = t\}$ and $p := \{x \neq m : m \in M\}$. Then by quantifier elimination and the fact that $\{m \in M : p(x) \vdash xEm\} = \varnothing$ is definable, p is quantifiable. But for all $N \succcurlyeq M$, $\{n \in N : p|_N(x) \vdash \neg xEn\} = M$ is not definable if $N \neq M$.

Proposition 7.5:
Let $A \subseteq M$. Assume M is $|A|^+$-saturated and strongly $|A|^+$-homogeneous. If p is quantifiable and $\mathrm{Aut}(M/A)$-invariant, then it is $\mathcal{L}(A)$-quantifiable.

Recall that a structure M is strongly κ-homogeneous if every partial elementary isomorphism whose domain has cardinality $< \kappa$ can be extended to an automorphism of M.

Proof. Let $\varphi(x;s)$ be any $\mathcal{L}$-formula and $\theta(s)$ be the $\mathcal{L}$-formula such that for all tuples $m \in M$, $M \vDash \theta(m)$ if and only if $p(x) \vdash \varphi(x;m)$. Let $\sigma \in \mathrm{Aut}(M/A)$ and $m \in M$ be such that $M \vDash \theta(m)$. Then $p = \sigma(p) \vdash \theta(x;\sigma(m))$ and hence $M \vDash \theta(\sigma(m))$, i.e. $\theta(M)$ is stabilised globally by $\mathrm{Aut}(M/A)$. ■

As previously, let now $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$, $T \supseteq$ ACVF be a C-minimal $\mathcal{L}$-theory which eliminates imaginaries, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $\widetilde{\mathcal{L}}$ be an enrichment of $\mathcal{L}$, $\widetilde{T}$ an $\widetilde{\mathcal{L}}$-theory containing T, $\widetilde{M} \models \widetilde{T}$ and $M := \widetilde{M}|_{\mathcal{L}}$. We will also be assuming that $\mathbf{k}$ is stably embedded in $\widetilde{T}$ and that the induced theory on $\mathbf{k}$ eliminates $\exists^\infty$. Until the end of the section, quantifiability of types will refer to quantifiability as partial $\widetilde{\mathcal{L}}$-types.
Let $\widetilde{A} \subseteq \widetilde{M}$, $A := \mathcal{R}(\widetilde{A})$, $F = (F_\lambda)_{\lambda \in \Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ and $\Delta(x, y; t)$ a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $y \in \mathbf{K}$, $p \in \mathcal{S}^{\Delta}_{x,y}(M)$ be definable. Assume that Δ is adapted to F and that F is closed under generic intersection over p and is generically irreducible over p.

Definition 7.6 (Generic covering property)**:**
We say that F has the generic covering property over p if for any $E \subseteq \Lambda(M)$ and any finite set $(\lambda_i)_{0 \leqslant i < k} \in \Lambda(M)$ such that for all $\mu \in E$, $p(x, y) \vdash F^{\$}_{\lambda_i}(x) \subset F^{\$}_{\mu}(x)$, there exists $(\kappa_j)_{0 \leqslant j < l} \in \Lambda(M)$ such that:

(i) For all j, $p(x, y) \vdash$ "the balls in $F_{\kappa_j}(x)$ are closed";

(ii) For all $\mu \in E$ and j, $p(x, y) \vdash F^{\$}_{\kappa_j}(x) \subseteq F^{\$}_{\mu}(x)$;

(iii) For all i, $p(x, y) \vdash F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\kappa_j}(x)$;

Note that if $E = \{\lambda_0\}$ and $p(x, y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are closed", then the generic covering property holds trivially as it suffices to take all $\kappa_j = \lambda_0$. It will only be interesting if $p(x, y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are open" or E does not have a smallest element over p, i.e. for all $\lambda \in E$ there exists $\mu \in E$ such that $p(x, y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)$.
Let $\mathcal{E} \subseteq \Lambda$ be $\widetilde{\mathcal{L}}(\widetilde{A})$-definable.

Proposition 7.7:
Assume that one of the following holds:

(i) $\mathcal{E}(\widetilde{M})$ does not have a smallest element over p;

(ii) there is a $\lambda_0 \in \mathcal{E}(\widetilde{M})$ such that for all $\lambda \in \mathcal{E}(\widetilde{M})$, $p(x, y) \vdash F^{\$}_{\lambda_0}(x) \subseteq F^{\$}_{\lambda}(x)$ and $p(x, y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are open".

Assume also that p is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable and F has the generic covering property over p, then $\alpha_{\mathcal{E}(\widetilde{M})/p}$ is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable.

Proof. Let $\varphi(x, y; t)$ be an $\widetilde{\mathcal{L}}$-formula. Then, for all tuples $m \in \widetilde{M}$ such that $\alpha_{\mathcal{E}(\widetilde{M})/p}(x, y) \vdash \varphi(x, y; m)$, there exists $\lambda_0 \in \mathcal{E}(\widetilde{M})$ and a finite number of $(\lambda_i)_{0<i<k} \in \Lambda(M)$ such that for all $\mu \in \mathcal{E}(\widetilde{M})$ and $i > 0$, $p(x, y) \vdash F^{\$}_{\lambda_i}(x) \subset F^{\$}_{\mu}(x)$ and $p(x, y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{i>0} F^{\$}_{\lambda_i}(x) \to \varphi(x, y; m)$. By the generic covering property, we can find $(\kappa_j)_{0 \leqslant j < l} \in \Lambda(M)$ such that, for all j, $p(x, y) \vdash$ "the balls in $F_{\kappa_j}(x)$ are closed", for all $\mu \in \mathcal{E}(\widetilde{M})$ and j, $p(x, y) \vdash F^{\$}_{\kappa_j}(x) \subseteq F^{\$}_{\mu}(x)$ and for all $i > 0$, $p(x, y) \vdash F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\kappa_j}(x)$.
If $\mathcal{E}(\widetilde{M})$ does not have a smallest element over p, for all $\mu \in \mathcal{E}(\widetilde{M})$ and j, we have that $p(x, y) \vdash F^{\$}_{\kappa_j}(x) \subset F^{\$}_{\mu}(x)$. If $\mathcal{E}(\widetilde{M})$ has a smallest element, because the balls in $F_{\lambda_0}(x)$

are open and those in $F_{\kappa_i}(x)$ are closed, we also have $p(x,y) \vdash F^{\$}_{\kappa_j}(x) \subset F^{\$}_{\lambda_0}(x)$. As the $\bigcup_j F^{\$}_{\kappa_j}(x)$ covers $\bigcup_i F^{\$}_{\lambda_i}(x)$, it follows that:

$$p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{0 \leqslant j < l} F^{\$}_{\kappa_j}(x) \to \varphi(x,y;m).$$

By $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiability of p there exists an $\widetilde{\mathcal{L}}(\widetilde{A})$-formula $\delta_1(\kappa,\mu)$ equivalent to $p(x,y) \vdash F^{\$}_{\kappa}(x) \subset F^{\$}_{\mu}(x)$ and an $\widetilde{\mathcal{L}}(\widetilde{A})$-formula $\delta_2(\lambda_0,\overline{\kappa},m)$ equivalent to $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{j<l} F^{\$}_{\kappa_j}(x) \to \varphi(x,y;m)$. We have just shown that, for all tuples $m \in \widetilde{M}$, $\alpha_{\mathcal{E}(\widetilde{M})/p}(x,y) \vdash \varphi(x,y;m)$ implies that:

$$\widetilde{M} \vDash \exists \lambda_0 \in \mathcal{E}\, \exists \overline{\kappa} \in \Lambda \bigwedge_{j<l} \forall \mu \in \mathcal{E}\, \delta_1(\kappa_j,\mu) \wedge \delta_2(\lambda_0,\overline{\kappa},m).$$

The converse is trivial. ∎

Definition 7.8 (Maximal open subball property)**:**
We say that F has the maximal open subball property over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$ such that $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subset F^{\$}_{\lambda_2}(x)$, there exists $(\mu_i)_{0 \leqslant i < l} \in \Lambda(M)$ such that:

(i) *For all i, $p(x,y) \vdash$ "the balls in $F_{\mu_i}(x)$ are open";*

(ii) *For all i, $p(x,y) \vdash \mathrm{rad}(F_{\lambda_2}(x)) = \mathrm{rad}(F_{\mu_i}(x))$.*

(iii) *$p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_i F^{\$}_{\mu_i}(x)$;*

Note that when the balls in $F_{\lambda_2}(x)$ are open, it suffices to take all $\mu_i = \lambda_2$. Hence this property is only useful when the balls in $F_{\lambda_2}(x)$ are closed.

Proposition 7.9:
Assume that there is a $\lambda_0 \in \mathcal{E}(\widetilde{M})$ such that for all $\lambda \in \mathcal{E}(\widetilde{M})$, $p(x,y) \vdash F^{\$}_{\lambda_0}(x) \subseteq F^{\$}_{\lambda}(x)$ and that $p(x,y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are closed". Assume also that p is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable and that F has the maximal open subball property over p, then the type $\alpha_{\mathcal{E}(\widetilde{M})/p}$ is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable.

Proof. If the balls in $F_{\lambda_0}(x)$ have radius $+\infty$, they are singletons. By irreducibility, $F_{\lambda_0}(x)$ does not have any strict subset of the form $F_{\lambda}(x)$ and $\alpha_{\mathcal{E}(\widetilde{M})/p} \vdash \varphi(x,y;m)$ if and only if $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \to \varphi(x,y;m)$. We can conclude immediately by $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable of p. We may now assume that the balls in $F_{\lambda_0}(x)$ have a radius different from $+\infty$. Let us begin with some preliminary results.

Claim 7.10: *Let $(Y_{\omega,x})_{\omega\in\Omega,x\in\mathbf{K}^n}$ be a definable family of sets such that for all ω and x, $Y_{\omega,x} \subseteq \{b : b$ is a maximal open subball of some $b' \in F_{\lambda_0}(x)\}$. Then there exists $k \in \mathbb{N}$ such that for all $\omega \in \Omega$ and $x \in \mathbf{K}^n$, either $|Y_{\omega,x}| \geqslant \infty$ or $|Y_{\omega,x}| \leqslant k$.*

Let $\overline{\mathcal{B}}_{\gamma}(a)$ denote the closed ball of radius γ around a.

Proof. Let $Y_{1,\omega,x,a,c} := \{b \in \mathbf{B} : b \in Y_{\omega,x}, b$ is a maximal open subball of $\overline{\mathcal{B}}_{\mathrm{val}(c)}(a)\}$. Note that for any maximal open subball b of $\overline{\mathcal{B}}_{\mathrm{val}(c)}(a)$, the set $\{(x-a)/c : x \in b\}$ is a coset of $\mathfrak{M}$

in $\mathcal{O}$, i.e. an element of $\mathbf{k}$ which we denote $\mathrm{res}_{a,c}(b)$. The function $\mathrm{res}_{a,c}$ is one to one. Let $Y_{2,\omega,x,a,c} := \mathrm{res}_{a,c}(Y_{1,\omega,x,a,c})$.
Then $Y_2 = (Y_{2,\omega,x,a,c})_{\omega,x,a,c}$ is an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable family of subsets of $\mathbf{k}$ and hence by stable embeddedness of $\mathbf{k}$ in T (as well as compactness and some coding) there exists an $\widetilde{\mathcal{L}}(\mathbf{k}(\widetilde{M}))$-definable family $(X_d)_{d\in D}$ where $D \subseteq \mathbf{k}^r$ for some r such that for all (ω, x, a, c), there exists $d \in D$ such that $Y_{2,\omega,x,a,c} = X_d$. Moreover as the theory induced on $\mathbf{k}$ eliminates $\exists^\infty$, there exists $s \in \mathbb{N}$ such that for all $d \in D$, either $|X_d| \geqslant \infty$ or $|X_d| \leqslant s$. It follows that for all (ω, x, a, c), either $|Y_{1,\omega,x,a,c}| \geqslant \infty$ or $|Y_{1,\omega,x,a,c}| \leqslant s$. But, as there are at most l balls in $F_{\lambda_0}(x)$ and that each of these balls contains infinitely or at most s maximal open subballs from $Y_{\omega,x}$, we have that for all x and ω, $|Y_{\omega,x}| \geqslant \infty$ or $|Y_{\omega,x}| \leqslant ls$. ♦

Let $X_m := \{\lambda \in \Lambda : p(x,y) \nvdash y \in F^{\$}_\lambda(x) \to \varphi(x,y;m)$ and $p(x,y) \vdash$ "the balls in $F_\lambda(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$" $\}$. By quantifiability of p, X_m is an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable family. Let $Y_{m,x} := \{b : \exists\lambda \in X_m,\ b \in F_\lambda(x)\}$. Then by Claim **(7.10)**, there exists k such that for all m and x, $|Y_{m,x}| < \infty$ implies $|Y_{m,x}| \leqslant k$.
Let us now assume that $\alpha_{\mathcal{E}(\widetilde{M})/p}(x,y) \vdash \varphi(x,y;m)$. Then there exists a finite number of $(\mu_i)_{0\leqslant i<r} \in \Lambda(M)$ such that $p(x,y) \vdash F^{\$}_{\mu_i}(x) \subset F^{\$}_{\lambda_0}(x)$ and $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_i F^{\$}_{\mu_i}(x) \to \varphi(x,y;m)$. As F has the maximal open subball property over p and is closed under generic intersection, we may assume that $p(x,y) \vdash$ "the balls in the $F_{\mu_i}(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$".

Claim 7.11: $X_m(M) \subseteq \{\lambda \in \Lambda(M) :$ *for some* i, $p(x,y) \vdash F_\lambda(x) = F_{\mu_i}(x)\}$. *In particular* $|Y_{m,x}| < \infty$ *and hence* $|Y_{m,x}| \leqslant k$.

Proof. Let $\lambda \in X_m$. There exists $x, y \vDash p$ such that $y \in F^{\$}_\lambda(x)$, the balls in $F_\lambda(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$ and $\vDash \neg\varphi(x,y;m)$. Hence $y \in \bigcup_i F^{\$}_{\mu_i}(x)$. We may assume that $y \in F^{\$}_{\mu_0}(x)$ and hence that $F^{\$}_{\mu_0}(x) \cap F^{\$}_\lambda(x) \neq \varnothing$. By Proposition **(6.10)**, we must have $F^{\$}_{\mu_0}(x) \cap F^{\$}_\lambda(x) = F^{\$}_\kappa(x)$ for both $\kappa = \lambda$ and $\kappa = \mu_0$, i.e. $F_\lambda(x) = F_{\mu_0}(x)$ and because such an equality is decided by p this holds for all realisations of p.
It follows that $Y_{m,x} \subseteq \bigcup_i F_{\mu_i}(x)$ and $|Y_{m,x}| \leqslant rl < \infty$. ♦

Thus for all $(x,y) \vDash p$, only k balls among the ones in $\bigcup_i F_{\mu_i}(x)$ cover $\varphi(x, F^{\$}_{\lambda_0}(x); m)$. By similar arguments as in Proposition **(6.5)**, we may assume that for all i, $F_{\mu_i}(x) \subseteq \bigcup_{j=1}^k F_{\mu_j}(x)$. It follows that:

$$p(x,y) \vdash \bigwedge_{j=1}^{k} F^{\$}_{\mu_j}(x) \subset F^{\$}_{\lambda_0}(x) \wedge (y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{i=1}^{k} F^{\$}_{\mu_i}(x) \to \varphi(x,y;m))$$

where k does not depend on m. We can now conclude as in Proposition **(7.7)**. ■

Corollary 7.12:
If p is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable and F has the generic covering property and the maximal open subball property over p, then $\alpha_{\mathcal{E}(\widetilde{M})/p}$ is $\widetilde{\mathcal{L}}(\widetilde{A})$-quantifiable.

Proof. This follows immediately from Propositions **(7.7)** and **(7.9)** and the fact that either $\mathcal{E}(\widetilde{M})$ is non empty and has no smallest element or it has a smallest element which consists of open balls or it has a smallest element which consists of closed balls or it is empty in which case we could also take $\mathcal{E}$ to consist of all the $\lambda \in \Lambda$ such that F_λ is constant equal to $\mathbf{K}$. ■

Let us conclude this section by showing that, as previously, we can find families of balls verifying all the necessary hypotheses. But because both the generic covering property and the maximal open subball property are instances of, more generally, being able to find large balls in the family, let us first consider the following definition. Recall that $d_i(B_1, B_2)$ is the i-th distance between balls of B_1 and balls of B_2 (see Definition **(6.6)**)

Definition 7.13 (Generic large ball property)**:**
We say that F has the generic large ball property over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$ and $i \in \mathbb{N}$, there exists $(\mu_j)_{0 \leqslant j < l} \in \Lambda(M)$ such that:

(i) For all j, $p(x,y) \vdash$ "the balls in $F_{\mu_j}(x)$ are closed";

(ii) For all j, $p(x,y) \vdash \mathrm{rad}(F_{\mu_j}(x)) = d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$.

(iii) $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$;

and, if $p(x,y) \vdash$ "the balls in $F_{\lambda_1}(x)$ are open" or $p(x,y) \vdash \mathrm{rad}(F_{\lambda_1}(x)) < d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$, there exists $(\rho_j)_{j<l} \in \Lambda(M)$ such that:

(i) For all j, $p(x,y) \vdash$ "the balls in $F_{\rho_j}(x)$ are open";

(ii) For all j, $p(x,y) \vdash \mathrm{rad}(F_{\rho_j}(x)) = d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$.

(iii) $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\rho_j}(x)$;

Definition 7.14 (Good representation)**:**
Let $\Delta(x,y;t)$ and $\Theta(x,y;s)$ be two finite sets of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $(F_\lambda)_{\lambda\in\Lambda}$ and $(G_\omega)_{\omega\in\Omega}$ be two $\mathcal{L}$-definable families of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$. We say that (Θ, G, x) is a good representation of (Δ, F, x) if for all $\mathcal{L}(M)$-definable $p \in \mathcal{S}^{\Theta}_x(M)$:

(i) Θ is adapted to G;

(ii) $(G_\omega)_{\omega\in\Omega_p}$ is closed under generic intersection over p;

(iii) $(G_\omega)_{\omega\in\Omega_p}$ has the generic large ball property over p;

(iv) p decides all formulas in Δ;

(v) For all $\lambda \in \Lambda(M)$, there exists a finite number of $(\omega_i)_{0\leqslant i<l} \in \Omega_p(M)$ such that $p(x,y) \vdash F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$.

where $\Omega_p := \{\omega \in \Omega : G_\omega$ is generically irreducible over $p\}$.

If we only want to say that **7.14.(i)** to **7.14.(iii)** hold we will say that (Θ, G, x) is a good representation.

Proposition 7.15 (Existence of good representations)**:**
Let $(F_\lambda)_{\lambda\in\Lambda}$ be any $\mathcal{L}$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ and $\Delta(x;t)$ any finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$. Then, there exists a good representation (Ψ, G, x) of (Δ, F, x).

Proof. Let us begin with some lemmas.

Lemma 7.16:
There exists $(H_\rho)_{\rho\in\mathrm{P}}$ an $\mathcal{L}$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ and $\Xi(x;t,s) \supseteq \Delta(x;t)$ a finite set of $\mathcal{L}$-formulas adapted to H such that H has the generic large ball property over any Ξ-type and for all $\lambda \in \Lambda$, there exists $\rho \in \mathrm{P}$ such that $H_\rho = F_\lambda$.

Proof. For all λ, μ and $\eta \in \Lambda$ and $i \leqslant l^2$, define $H_{\lambda,\mu,\eta,i,1}(x)$ to be the closed balls with radius $\min\{d_i(F_\mu(x), F_\eta(x)), \mathrm{rad}(F_\lambda(x))\}$ around the balls in $F_\lambda(x)$. If the balls in $F_\lambda(x)$ are open or if they are closed of radius strictly smaller than $d_i(F_\mu(x), F_\eta(x))$, define $H_{\lambda,\mu,\eta,i,0}(x)$ to be the set open balls with radius $d_i(F_\mu(x), F_\eta(x))$ around the balls in $F_\lambda(x)$. Otherwise, define $H_{\lambda,\mu,\eta,i,0}(x)$ to be the closed balls with radius $\min\{d_i(F_\mu(x), F_\eta(x)), \mathrm{rad}(F_\lambda(x))\}$ around the balls in $F_\lambda(x)$. By usual coding tricks, we may assume that H is an $\mathcal{L}$-definable family of functions. Adding finitely many formulas to Δ we obtain $\Xi(x;t,s)$ which is adapted to H. Let $p \in \mathcal{S}_x^{\Xi}(M)$ and $x \vDash p$.
Let us first show the closed ball case of the generic large ball property. For all λ_k, μ_k and $\eta_k \in \Lambda(M)$ and i_k and $j_k \in \mathbb{N}$ for $k \in \{1,2\}$ and $r \in \mathbb{N}$, $d := d_r(H_{\lambda_1,\mu_1,\eta_1,i_1,j_1}(x), H_{\lambda_2,\mu_2,\eta_2,i_2,j_2}(x))$ is either the radius of the balls in $H_{\lambda_k,\mu_k,\eta_k,i_k,j_k}(x)$, i.e. $d_{i_k}(F_{\mu_k}(x), F_{\eta_k}(x))$ or $\mathrm{rad}(F_{\lambda_k}(x))$, or the distance between two disjoint balls from the $H_{\lambda_k,\mu_k,\eta_k,i_k,j_k}(x)$ in which case it is also the distance between some disjoint balls in the $F_{\lambda_k}(x)$. If $d = d_{i_k}(F_{\mu_k}(x), F_{\eta_k}(x))$, it is easy to check that $H_{\lambda_1,\eta_k,\mu_k,i_k,1}$ has all the suitable properties; and that this one instance suffices. Otherwise there exists some m such that $H_{\lambda_1,\lambda_1,\lambda_2,m,1}(x)$ is suitable.
The same reasoning applies to the open ball case (the extra conditions under which we have to work are just here to ensure that the balls in $F_{\lambda_1}(x)$ are indeed smaller than those we are trying to build around them). ♦

Lemma 7.17:
Assume that F has the generic large ball property over any Δ-type. Let $(G_\omega)_{\omega\in\Omega}$ be any $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ and $\Theta(x;s)$ be any finite set of $\mathcal{L}$-formulas adapted to G such that for all $p \in \mathcal{S}_x^{\Theta}(M)$, we have:

(i) For all $\omega \in \Omega(M)$, there exists $\lambda \in \Lambda(M)$ such that $p(x) \vdash G_\omega(x) \subseteq F_\lambda(x)$;

(ii) For all $\lambda \in \Lambda(M)$, there exists $(\omega_i)_{0\leqslant i<l} \in \Omega(M)$ such that $p(x) \vdash F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$.

Then G also has the generic large ball property over any Θ-type.

Proof. Let ω_1 and $\omega_2 \in \Omega(M)$, $i \in \mathbb{N}_{>0}$ and $x \vDash p$. Then there exists λ_1 and $\lambda_2 \in \Lambda(M)$ such that $G_{\omega_k}(x) \subseteq F_{\lambda_k}(x)$. Then $d_i(G_{\omega_1}(x), G_{\omega_2}(x))$ is either the radius of one of the balls involved and hence is the radius of one of $F_{\lambda_k}(x)$ or the distance between a ball in $G_{\omega_1}(x)$ and a ball in $G_{\omega_2}(x)$, i.e. the distance between a ball in $F_{\lambda_1}(x)$ and one in $F_{\lambda_2}(x)$. In both cases, the large closed ball property in F allows us to find $(\mu_j)_{0\leqslant j<l} \in \Lambda(M)$ such

that $G^{\$}_{\omega_1}(x) \subseteq F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$, for all j, the balls in $F_{\mu_j}(x)$ are closed and their radius is $d_i(G_{\omega_1}(x), G_{\omega_2}(x))$. But, by hypothesis there are $(\rho_{j,k})_{0\leqslant k<l} \in \Omega(M)$ such that $F_{\mu_j}(x) = \bigcup_k G_{\rho_{j,k}}(x)$. By picking one $\rho_{j,k}$ per ball in $G_{\omega_1}(x)$, we see that l of them are enough to cover $G_{\omega_1}(x)$ and we are done. The open ball case is proved similarly as the extra conditions hold for G_{ω_1} and G_{ω_2} if and only if they hold for F_{λ_1} and F_{λ_2}. ♦

Adding them if we have to, we may assume that there is an instance of F constant equal to $\varnothing$ and another constant one equal to $\{\mathbf{K}\}$. Let $(H_\rho)_{\rho\in\mathrm{P}}$ and Ξ be as in Lemma **(7.16)**, $(G_\omega)_{\omega\in\Omega}$ and $\Theta(x;u)$ be as given by Proposition **(6.20)** applied to H. Let $p \in \mathcal{S}^{\Theta}_x(M)$. Then Conditions **7.14.(i)**, **7.14.(iv)** and **7.14.(v)** hold. Condition **7.14.(ii)** also holds, by Corollary **(6.11)**, and by Lemma **(7.17)** applied to $(G_\omega)_{\omega\in\Omega_p}$, **7.14.(iii)** also holds. ■

Proposition 7.18:
Let $(\Delta(x;t), (F_\lambda)_{\lambda\in\Lambda}, x)$ be a good representation and $p \in \mathcal{S}^{\Delta}_x(M)$ be $\mathcal{L}(M)$-definable. Then $F_p := (F_\lambda)_{\lambda\in\Lambda_p}$ has the generic covering property and the maximal open subball property over p, where $\Lambda_p := \{\lambda \in \Lambda : F_\lambda$ is generically irreducible over $p\}$

Proof. Let $x \vDash p$, λ_1 and $\lambda_2 \in \Lambda_p(M)$ be such that $F^{\$}_{\lambda_1}(x) \subset F^{\$}_{\lambda_2}(x)$. Then by the generic large ball property (because $F^{\$}_{\lambda_1}(x) \subset F^{\$}_{\lambda_2}(x)$, the necessary conditions hold), there exists $\mu_j \in \Lambda_p(M)$ such that the balls in $F_{\mu_j}(x)$ are open of radius $\mathrm{rad}(F_{\lambda_2}(x))$ and $F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$. We have proved the maximal open subball property.
Let now $E \subseteq \Lambda_p(M)$ and $(\lambda_i)_{0\leqslant i<k} \in \Lambda_p(M)$ be such that for all $\mu \in E$, $F^{\$}_{\lambda_i}(x) \subset F^{\$}_{\mu}(x)$. For any two μ_1 and $\mu_2 \in E$, if the balls in $F_{\mu_1}(x)$ are smaller than the balls in $F_{\mu_2}(x)$, by irreducibility, as $F^{\$}_{\mu_1}(x) \cap F^{\$}_{\mu_2}(x) \supseteq F_{\lambda_0}(x) \neq \varnothing$, we must have $F^{\$}_{\mu_1}(x) \subseteq F^{\$}_{\mu_2}(x)$. Let us define the following equivalence relation on $\bigcap_{\lambda\in E} F^{\$}_{\lambda}(x)$: $y_1 \equiv y_2$ if for all $\mu \in E$, y_1 and y_2 are in the same ball from $F_\mu(x)$. If we take two non equivalent points y_1 and y_2, there exists $\mu \in E$ such that y_1 and y_2 are not in the same ball from $F_\mu(x)$ and in fact this also holds for any η such that $F^{\$}_{\eta}(x) \subseteq F^{\$}_{\mu}(x)$. In particular it follows that there are at most l equivalence classes and that there exists μ_0 such that each equivalence class is contained in a different ball from $F_{\mu_0}(x)$. Moreover each of these equivalence classes is in fact the intersection of balls from the $F_\mu(x)$ for $\mu \in E$. We will denote these equivalence classes by $(P_j)_{j\in J}$.
For any j, let $B_j = \{b \in \bigcup_i F_{\lambda_i}(x) : b \subseteq P_j\}$. Then the set $R_j := \{d(b_1, b_2) : b_1, b_2 \in B_j\} \cup \{\mathrm{rad}(b) : b \in B_j\}$ is finite and hence has a minimum γ. By the generic large ball property, there exists $\mu_j \in \Lambda_p(M)$ such that the balls in $F_{\mu_j}(x)$ are closed of radius γ and one of its balls (call it b_0) contains one of the balls in B_j. In fact b_0 contains all of them as γ is the minimum of R_j. For all $\kappa \in E$, all $b \in B_j$ are such that $b \subset F^{\$}_{\kappa}(x)$. If $\mathrm{rad}(b_0) = d(b_1, b_2)$ for some some b_1 and $b_2 \in B_j$ then , because b_1 and b_2 are in the same ball from $F_\kappa(x)$, $\mathrm{rad}(b_0) = d(b_1, b_2) \leqslant \mathrm{rad}(F_\kappa(x))$. If $\mathrm{rad}(b_0) = \mathrm{rad}(b)$ for some $b \in B_j$, then because b is inside one of the balls from $F_\kappa(x)$, $\mathrm{rad}(b_0) = \mathrm{rad}(b) \leqslant \mathrm{rad}(F_\kappa(x))$. In both cases, $b_0 \subseteq F^{\$}_{\mu}(x)$. Let η_j be such that $F^{\$}_{\eta_j}(x) = F^{\$}_{\mu_j}(x) \cap \bigcap_{\kappa\in E} F^{\$}_{\kappa}(x)$. Such an η_j exists by generic intersection and because, by Proposition **(6.5)**, this intersection is given by the intersection of a finite numbers of its elements.
Then, as $F_{\eta_j}(x) \subseteq F_{\mu_j}(x)$, the balls in $F_{\eta_j}(x)$ are closed. Obviously, for all $\kappa \in E$, $F^{\$}_{\eta_j}(x) \subseteq F^{\$}_{\kappa}(x)$. Moreover, for all i, $F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$ and for all $\kappa \in E$, $F^{\$}_{\lambda_i}(x) \subseteq F^{\$}_{\kappa}(x)$, hence we

also have $F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\eta_j}(x)$. As there are at most l of the η_j, we are done. ■

8. Γ-reparametrisations

Let $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$, $T \supseteq \mathrm{ACVF}$ be an $\mathcal{L}$-theory which eliminates imaginaries. Assume that T is C-minimal. The two main examples of such theories are $\mathrm{ACVF}^{\mathcal{G}}$ and $\mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$ where $\mathcal{A}$ is some separated Weierstrass system (for example $\bigcup_{m,n} \mathbb{Z}[[X_0, \ldots, X_n]][Y_0, \ldots, Y_m]$) and $\mathrm{ACVF}_{\mathcal{A}}$ denotes the theory of algebraically closed valued fields with $\mathcal{A}$-analytic structure (see [CL11] or [Rid, Section 3]). This structure is considered in the language $\mathcal{L}_{\mathcal{A},\mathcal{Q}} \coloneqq \mathcal{L}_{\mathrm{div}} \cup \mathcal{A} \cup \{^{-1}\}$.

Remark 8.1:
The value group $\mathbf{\Gamma}$ is stably embedded and o-minimal in T. As $\mathbf{\Gamma}$ is an o-minimal group, the induced structure on $\mathbf{\Gamma}$ eliminates imaginaries.

Proof. Let $M \vDash T$ and $X \subseteq \mathbf{\Gamma}$ be a unary $\mathcal{L}(M)$-definable set. The set $\mathrm{val}^{-1}(X)$ is both a (potentially infinite) union of annuli around 0 and a finite union of Swiss cheeses, hence it is a finite union of annuli around 0 and X must be a finite union of intervals. Therefore, $\mathbf{\Gamma}$ is o-minimal in T and by [HO10], $\mathbf{\Gamma}$ is stably embedded in models of T. ■

Let $M \vDash T$, $f = (f_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$, $\Delta(x;t)$ be a finite set of $\mathcal{L}$-formulas and $p \in \mathcal{S}^{\Delta}_x(M)$. We wish to study the family f and in particular its germs over p (see Definition **(8.4)**), to show that they are internal to $\mathbf{\Gamma}$. This is later used as a partial elimination of imaginaries result in enrichments $\widetilde{T}$ of T where $\mathbf{\Gamma}$ is stably embedded: any subset of these germs definable in $\widetilde{T}$ is coded in $\mathbf{\Gamma}^{\mathrm{eq}}$ (where eq is taken relative to the theory induced by $\widetilde{T}$). We only achieve this goal in $\mathrm{ACVF}^{\mathcal{G}}$ and $\mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$. The idea of the proof is to reparametrise the family of functions (see Definition **(8.2)**).
Let $g = (g_\gamma)_{\gamma\in G}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$, where $G \subset \mathbf{\Gamma}^k$ for some k.

Definition 8.2 ($\mathbf{\Gamma}$-reparametrisation)**:**
We say that g $\mathbf{\Gamma}$-reparametrises f over p if for all $\lambda \in \Lambda(M)$, there is $\gamma \in G(M)$ such that

$$p(x) \vdash f_\lambda(x) = g_\gamma(x).$$

We say that T admits $\mathbf{\Gamma}$-reparametrisations if for every $\mathcal{L}(M)$-definable family $f = (f_\lambda)_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ there exists a finite set of $\mathcal{L}$-formulas $\Delta(x;s)$ such that for all $M \vDash T$ and $p \in \mathcal{S}^{\Delta}_x(M)$, there exists an $\mathcal{L}(M)$-definable family $g = (g_\gamma)_{\gamma\in G}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ which $\mathbf{\Gamma}$-reparametrises f over p.

We will also be needing a stronger form of reparametrisation. Let $g = (g_{\omega,\gamma})_{\omega\in\Omega,\gamma\in G}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ where $G \subseteq \mathbf{\Gamma}^k$ for some k.

Definition 8.3 (Uniform $\mathbf{\Gamma}$-reparametrisation)**:**
We say that g uniformly $\mathbf{\Gamma}$-reparametrises f over Δ-types if for every $p \in \mathcal{S}^{\Delta}_x(M)$ there exists $\omega_0 \in \Omega(M)$ such that for all $\lambda \in \Lambda(M)$, there is $\gamma \in G(M)$ such that

$$p(x) \vdash f_\lambda(x) = g_{\omega_0,\gamma}(x).$$

We say that T admits uniform $\mathbf{\Gamma}$-reparametrisations if for every $\mathcal{L}(M)$-definable family $f = (f_\lambda)_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ there exists a finite set of $\mathcal{L}$-formulas $\Delta(x;s)$ and an $\mathcal{L}(M)$-definable family $g = (g_{\omega,\gamma})_{\omega\in\Omega,\gamma\in G}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ which uniformly $\mathbf{\Gamma}$-reparametrises f over Δ-types.

We will say that Δ is adapted to f (respectively to g) when any Δ-type decides when $f_{\lambda_1}(x) = f_{\lambda_2}(x)$ (respectively $g_{\gamma_1}(x) = g_{\gamma_2}(x)$).

Definition 8.4 (p-germ)**:**
Assume that Δ is adapted to f and that p is $\mathcal{L}(M)$-definable. We say that f_{λ_1} and f_{λ_2} have the same p-germ if $p(x) \vdash f_{\lambda_1}(x) = f_{\lambda_2}(x)$. Let us denote $\partial_p f_\lambda \in M$ the code of the equivalence class of λ under the equivalence relation "having the same p-germ".

Proposition 8.5:
Let us assume that g is a $\mathbf{\Gamma}$-reparametrisation of f over p, that Δ is adapted to both f and g and that p is $\mathcal{L}(M)$-definable. The set $\{\partial_p f_\lambda : \lambda \in \Lambda\}$ is internal to $\mathbf{\Gamma}$, i.e. there is an $\mathcal{L}(M)$-definable one to one map from this set into some Cartesian power of $\mathbf{\Gamma}$.

Proof. As γ is a tuple from $\mathbf{\Gamma}$ and $\mathbf{\Gamma}$ is stably embedded in T and eliminates imaginaries (see Remark **(8.1)**), we may assume that $\partial_p g_\gamma \in \mathbf{\Gamma}$. Now pick any λ. Let γ be such that $p(x) \vdash f_\lambda(x) = g_\gamma(x)$. Then $\partial_p g_\gamma$ only depends on $\partial_p f_\lambda$ and not on λ or γ. It follows that the set $\{\partial_p f_\lambda : \lambda \in \Lambda\}$ is in $\mathcal{L}(M)$-definable one to one correspondence with a subset of the set $\{\partial_p g_\gamma : \gamma \in G\}$ which is itself a subset of some Cartesian power of $\mathbf{\Gamma}$. ∎

If Z_1 and $Z_2 \subseteq \mathbf{K}$ are finite sets, we will denote $D(Z_1, Z_2) := \{\mathrm{val}(z_1 - z_2) : z_1 \in Z_1$ and $z_2 \in Z_2\}$. Let us order the elements in $D(Z_1, Z_2)$ as $d_1 > d_2 > \cdots > d_k$ and let $d_i(Z_1, Z_2) := d_i$. If $Z_1 = \{z\}$ is singleton we will write $d_i(z, Z_2)$.

Proposition 8.6:
Let $t(x,y,\lambda) : \mathbf{K}^{n+1+l} \to \mathbf{K}$ be an $\mathcal{L}|_{\mathbf{K}}(M)$-term polynomial in y, i.e. $t = \sum_{i=0}^{d} t_i(x,\lambda)y^i$, where $|x| = n$, $|y| = 1$ and $|\lambda| = l$. Let $Z_\lambda(x) := \{y : t(x,y,\lambda) = 0\}$. Then there exists an $\mathcal{L}(M)$-definable family $q = (q_\eta)_{\eta\in\mathrm{H}}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ such that for all $N \succcurlyeq M$, $x \in \mathbf{K}^n(N)$ and $y \in \mathbf{K}(N)$, there exists $\mu_0 \in \Lambda(M)$ such that for all $\lambda \in \Lambda(M)$ there exists $\eta \in \mathrm{H}(M)$ and n smaller than the degree of t in y such that:

$$\mathrm{val}(t(x,y,\lambda)) = q_\eta(x) + n \cdot d_1(y, Z_{\mu_0}(x)).$$

Proof. For all $\alpha \in Z_\lambda(x)$, let m_α be its multiplicity and let us define:

$$u(x,\lambda) := \frac{t(x,y,\lambda)}{\prod\limits_{\alpha\in Z_\lambda(x)} (y-\alpha)^{m_\alpha}}$$

which is $\mathcal{L}$-definable and does not depend on y, and

$$q_{\lambda,k,\bar{j},\eta}(x) := \mathrm{val}(u(x,\lambda)) + \sum_{i=0}^{k} d_{j_i}(Z_\lambda(x), Z_\eta(x)).$$

where k is at most the degree of t in y and $j_i \leqslant l^2$. Note that because we can code disjunctions on a finite number of integers, q can be considered as an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$.
Let $N \geqslant M$, $x \in \mathbf{K}^n(N)$ and $y \in \mathbf{K}(N)$ and let us first assume that there exists $\mu_0 \in \Lambda(M)$ such that $d_1(y, Z_{\mu_0}(x)) = \max_\mu\{d_1(y, Z_\mu(x))\}$ and let $\alpha_0 \in Z_{\mu_0}(x)$ be such that $\mathrm{val}(y-\alpha_0) = d_1(y, Z_{\mu_0}(x))$. Now pick any $\lambda \in \Lambda(M)$ and $\alpha \in Z_\lambda(x)$.

Claim 8.7: *Either* $\mathrm{val}(y-\alpha) = d_1(y, Z_{\mu_0}(x))$ *or* $\mathrm{val}(y-\alpha) = d_{j_\alpha}(Z_\lambda(x), Z_{\mu_0}(x))$ *for some* j_α.

Proof. If $\mathrm{val}(y-\alpha) \neq d_1(y, Z_{\mu_0}(x))$, then $\mathrm{val}(y-\alpha) < d_1(y, Z_{\mu_0}(x))$ and $\mathrm{val}(y-\alpha) = \mathrm{val}(\alpha-\alpha_0) = d_j(Z_\lambda(x), Z_{\mu_0}(x))$ for some j. ♦

Let $Z_1 := \{\alpha \in Z_\lambda(x) : \mathrm{val}(y-\alpha) = d_1(y, Z_{\mu_0}(x))\}$ and $n := \sum_{\alpha \in Z_1} m_\alpha$. We have:

$$\begin{aligned} \mathrm{val}(t(x,y,\lambda)) &= \mathrm{val}(u(x,\lambda)) + \sum_{\alpha \in Z_\lambda(x)} m_\alpha \mathrm{val}(y-\alpha) \\ & \quad \mathrm{val}(u(x,\lambda)) + \sum_{\alpha \notin Z_1} m_\alpha d_{j_\alpha}(Z_\lambda(x), Z_{\mu_0}(x)) + n \cdot d_1(y, Z_{\mu_0}(x)) \\ & \quad q_{\lambda,k,\overline{j},\eta}(x) + n \cdot d_1(y, Z_{\mu_0}(x)) \end{aligned}$$

for some k and $\overline{j}$.
In the other case, if there does not exist a maximum in $\{d_1(y, Z_\mu(x))\}$, for any $\lambda \in \Lambda(M)$, there exists $\eta \in \Lambda(M)$ such that $d_1(y, Z_\eta(x)) > d_1(y, Z_\lambda(x))$. Let α_0 be such that $\mathrm{val}(y-\alpha_0) = d_1(y, Z_\eta(x))$, then for all $\alpha \in Z_\lambda(x)$, $\mathrm{val}(y-\alpha) = \mathrm{val}(\alpha-\alpha_0) = d_{j_\alpha}(Z_\lambda(x), Z_\eta(x))$ for some j_α. It follows that:

$$\begin{aligned} \mathrm{val}(t(x,y,\lambda)) &= \mathrm{val}(u(x,\lambda)) + \sum_{\alpha \in Z_\lambda(x)} m_\alpha d_{j_\alpha}(Z_\lambda(x), Z_\eta(x)) \\ &= q_{\lambda,k,\overline{j},\mu_0}(x) \end{aligned}$$

for some k and $\overline{j}$. ■

Proposition 8.8:
The theories $\mathrm{ACVF}^{\mathcal{G}}$ *and* $\mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$ *admit uniform* $\mathbf{\Gamma}$*-reparametrisations.*

Proof. Let $f = (f_\lambda)_{\lambda \in \Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$. We work by induction on n. The case $n = 0$ is trivial as f is nothing more than a family of points in $\mathbf{\Gamma}$ that can be reparametrised by themselves. Let us now assume that $n = m+1$ and $x = (y, z)$ where $|z| = 1$. Because $\mathbf{K}$ is dominant, we may assume up to reparametrisation that λ is a tuple from $\mathbf{K}$. If $T = \mathrm{ACVF}^{\mathcal{G}}$, the graph of f_λ is given by an $\mathcal{L}^{\mathcal{G}}(M)$-formula. If $T = \mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$, by [Rid, Corollary 5.5] there exists an $\mathcal{L}^{\mathcal{G}}(M)$-formula $\psi(z, \overline{w}, \gamma)$ and $\mathcal{L}|_{\mathbf{K}}$-terms $\overline{r}(x, \lambda)$ such that $M \vDash f_\lambda(y,z) = \gamma$ if and only if $M \vDash \psi(z, \overline{r}(y,\lambda), \gamma)$. Taking $\overline{r}$ to be the identity, the graph of f_λ also has this form when $T = \mathrm{ACVF}^{\mathcal{G}}$. By elimination of quantifiers in $\mathrm{ACVF}^{\mathcal{G}}$ (or in the two sorted language), we know that $\psi(z, \overline{w}, \gamma)$ is of the form $\chi((\mathrm{val}(P_i(z, \overline{w})))_{0 \leqslant i < k}, \gamma)$ where χ is an $\mathcal{L}^{\mathcal{G}}|_{\mathbf{\Gamma}}$-formula and $P_i \in \mathbf{K}(M)[Y, \overline{W}]$. We may also assume that χ defines a function $h : \mathbf{\Gamma}^k \to \mathbf{\Gamma}$.

Let $t_i(y,z,\lambda) = P_i(z,\overline{r}(y,\lambda))$ and $q_i = (q_{i,\eta})_{\eta\in \mathrm{H}_i}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^m \to \mathbf{\Gamma}$ as in Proposition **(8.6)** with respect to t_i. By the usual coding tricks we may assume that there is only one family $q = (q_\eta)_{\eta\in\mathrm{H}}$ such that for all i and $\eta \in \mathrm{H}_i$ there exists $\varepsilon \in \mathrm{H}$ such that $q_{i,\eta} = q_\varepsilon$. By induction, there exists a uniform $\mathbf{\Gamma}$-reparametrisation for q, i.e. there exist a finite set of $\mathcal{L}$-formulas $\Xi(y;s)$ and an $\mathcal{L}(M)$-definable family $(u_{\varepsilon,\delta})_{\varepsilon\in\mathrm{E},\delta\in D}$ of functions $\mathbf{K}^m \to \mathbf{\Gamma}$, where $D \subseteq \mathbf{\Gamma}^l$ for some l, such that for any $p \in \mathcal{S}^{\Xi}_y(M)$, for some $\varepsilon_0 \in \mathrm{E}(M)$, $(u_{\varepsilon_0,\delta})_{\delta\in D}$ is a $\mathbf{\Gamma}$-reparametrisation of q. Let $Z_{i,\lambda}(y) := \{z : P_i(y,z,\lambda) = 0\}$ and

$$g_{\varepsilon,\overline{\mu},\overline{\delta},\overline{n}}(y,z) := h((u_{\varepsilon,\delta_i}(x) + n_i \cdot d_1(z, Z_{i,\mu_i}(y)))_{0\leqslant i<k}).$$

Let also $\varphi_{\overline{n}}(y,z;\lambda,\varepsilon,\mu,\overline{\delta})$:="$f_\lambda(y,z) = g_{\varepsilon,\mu,\overline{\delta},\overline{n}}(y,z)$" and $\Delta(y,z;s,\lambda,\varepsilon,\mu,\overline{\delta},\overline{n}) := \Xi(y;s) \cup \{\varphi_{\overline{n}}(y,z;\lambda,\varepsilon,\mu,\overline{\delta}) : \overline{n} \in \mathbb{N}\}$. For all $p \in \mathcal{S}^{\Delta}_{y,z}(M)$, there exists $\varepsilon_0 \in \mathrm{E}(M)$ such that $(u_{\varepsilon_0,\delta})_{\delta\in D}$ $\mathbf{\Gamma}$-reparametrises q over $p|_\Xi$. Let $(y,z) \vDash p$. By Proposition **(8.6)** there exists a tuple $\overline{\mu}_0 \in \Lambda(M)$ such that for all $\lambda \in \Lambda(M)$, there exists tuples $\overline{\eta} \in \mathrm{H}(M)$ and $\overline{n}$ such that

$$\mathrm{val}(t_i(y,z,\lambda)) = q_{\eta_i}(y) + n_i \cdot d_1(y, Z_{i,\mu_{0,i}}(x)).$$

As $y \vDash p|_\Xi$, there exists $\delta_i \in D(M)$ such that $q_{\eta_i}(y) = u_{\varepsilon_0,\delta_i}(y)$ and hence

$$\begin{aligned} f_\lambda(y,z) &= h((\mathrm{val}(t_i(y,z,\lambda)))_{0\leqslant i<k}) \\ &= h((u_{\varepsilon_0,\delta_i}(y) + n_i \cdot d_1(y, Z_{i,\mu_{0,i}}(x)))_{0\leqslant i<k}) \\ &= g_{\varepsilon_0,\overline{\mu_0},\overline{\delta},\overline{n}}(y,z). \end{aligned}$$

Because p decides such equalities, this holds in fact for all realisations of p. We have just shown that $(g_{\varepsilon_0,\overline{\mu_0},\overline{\delta},\overline{n},})_{\overline{\delta}\in D,\overline{n}\in\mathbb{N}}$ reparametrises f over p. But because $\overline{\delta}$ is a tuple from $\mathbf{\Gamma}$ and disjunctions on a finite number of bounded integers can be coded in $\mathbf{\Gamma}$, it is in fact a $\mathbf{\Gamma}$-reparametrisation. ∎

Question 8.9: Do all C-minimal extensions of ACVF admit uniform $\mathbf{\Gamma}$-reparametrisations?

9. Approximating sets with balls

As before, let $\widetilde{\mathcal{L}} \supseteq \mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ be languages, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory which eliminates imaginaries and admits $\mathbf{\Gamma}$-reparametrisations, $\widetilde{T}$ a complete $\widetilde{\mathcal{L}}$-theory containing T, $\widetilde{N} \vDash \widetilde{T}$, $N := \widetilde{N}|_{\mathcal{L}}$ and $\widetilde{A} = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}) \subseteq \widetilde{N}^{\mathrm{eq}}$. Let us assume that $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded in $\widetilde{T}$ and that the induced theories on $\mathbf{k}$ and $\mathbf{\Gamma}^{\mathrm{eq}}$ eliminate $\exists^\infty$ (where "eq" is taken relative to the $\widetilde{\mathcal{L}}$-induced structure on $\mathbf{\Gamma}$).

In this section we bring together all the work we have done in Sections 6, 7 and 8 to actually construct definable types, in order to prove Theorem **(9.8)**. The core of the work is done in Lemma **(9.1)** where we show that we can enrich a quantifiable partial $\widetilde{\mathcal{L}}$-type with formulas of the form $y \in F_\lambda(x)$, where F_λ is an $\mathcal{L}$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$, while maintaining consistency with a given $\widetilde{\mathcal{L}}$-definable set. Once this is done, it is only a question of proving the various reductions sketched in the introduction. In Proposition **(9.5)**, we show that we can enrich a quantifiable partial $\widetilde{\mathcal{L}}$-type with arbitrary formulas while maintaining

consistency with a given $\widetilde{\mathcal{L}}$-definable set. Finally, in Proposition (9.7), we show that every strict $(\widetilde{\mathcal{L}},\star)$-definable set X (see Definition (9.6)) is consistent with a definable $\mathcal{L}$-type.
Note that, even though all the type which are constructed in this section are $\mathcal{L}$-types (or Δ-types for some set Δ of $\mathcal{L}$-formulas), they are definable using $\widetilde{\mathcal{L}}(\widetilde{N})$-formulas: for every $\varphi(x;t)\in\Delta$, there exists an $\widetilde{\mathcal{L}}(\widetilde{N})$-formula $d_p x\varphi(x;t)$ such $\varphi(x;m)\in p$ if and only if $\widetilde{N}\vDash d_p x\varphi(x;m)$. One of the goals of [RS] is to show that, under some more hypotheses, such types are indeed $\mathcal{L}(N)$-definable.

Lemma 9.1:
*Let $Y\subset \mathbf{K}^{n+1}$ be a non empty $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable set. Let $(\Delta(x,y;t),(F_\lambda)_{\lambda\in\Lambda},x)$ be a good representation where $x\in\mathbf{K}^n$. Let $p(x,y)\in\mathcal{S}^{\Delta}_{x,y}(N)$ be $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable (as a partial $\widetilde{\mathcal{L}}^{eq}$-type) and consistent with Y. Assume that there exists an $\mathcal{L}(N)$-definable family $g=(g_\gamma)_{\gamma\in G}$ of functions $\mathbf{K}^n\to\mathbf{\Gamma}$ which $\mathbf{\Gamma}$-reparametrises the family $(\mathrm{rad}\circ F_\lambda)_{\lambda\in\Lambda}$ over p.
Then there exists a type $q(x,y)\in\mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(N)$ which is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with p and Y.*

We are looking for a type $q=\alpha_{E/p}$ (see Definition (6.12)) so most of the work consists in finding the right E.

Proof. We define the preorder $\trianglelefteq$ on $\Lambda_p:=\{\lambda\in\Lambda : F_\lambda$ is generically irreducible over $p\}$ by $\lambda\trianglelefteq\mu$ if and only if $p(x,y)\vdash(y\in F^{\$}_\lambda(x)\wedge(x,y)\in Y)\to y\in F^{\$}_\mu(x)$. Note that, by $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiability of p, $\trianglelefteq$ is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. Let $\sim$ be the associated equivalence relation, i.e $\lambda\sim\mu$ if and only if $p(x,y)\vdash(y\in F^{\$}_\lambda(x)\wedge(x,y)\in Y)\iff(y\in F^{\$}_\mu(x)\wedge(x,y)\in Y)$. Then $\trianglelefteq$ induces a (partial) order on $\Lambda_p/{\sim}$ that we will also denote $\trianglelefteq$. For any λ, let us denote by $\widehat{\lambda}\subseteq\Lambda_p$ the $\sim$-class of λ. The set $\Lambda_p/{\sim}$ has a greatest element given by the class of any $\lambda\in\Lambda_p(N)$ such that $F_\lambda(x)=\{\mathbf{K}\}$ for all x and a smallest element given by the class of any $\lambda\in\Lambda_p(N)$ such that $F_\lambda(x)=\varnothing$ for all x. Let $\widehat{\mathbf{K}}$ be the greatest element of $\Lambda_p/{\sim}$ and $\widehat{\varnothing}$ be its smallest element. Because p is consistent with Y, $\widehat{\mathbf{K}}\neq\widehat{\varnothing}$.

Claim 9.2: *Let $\lambda\in\Lambda_p\smallsetminus\widehat{\varnothing}$, then $\trianglelefteq$ totally orders $\{\widehat{\mu}:\mu\in\Lambda_p\wedge\lambda\trianglelefteq\mu\}$.*

Proof. Let μ_1 and $\mu_2\in\Lambda_p(N)$ such that $\lambda\trianglelefteq\mu_i$. Because $\lambda\notin\widehat{\varnothing}$ there exists $(x,y)\vDash p$ such that $y\in F^{\$}_\lambda(x)$ and $(x,y)\in Y$. As $\lambda\trianglelefteq\mu_i$, we also have $y\in F^{\$}_{\mu_i}(x)$ and hence $F^{\$}_{\mu_1}(x)\cap F^{\$}_{\mu_2}(x)\neq\varnothing$. By Proposition (6.10), we have $F^{\$}_{\mu_1}(x)\subseteq F^{\$}_{\mu_2}(x)$ or $F^{\$}_{\mu_2}(x)\subseteq F^{\$}_{\mu_1}(x)$ and we may assume that the first one holds. Then $\mu_1\trianglelefteq\mu_2$. ♦

Hence $((\Lambda_p/{\sim})\smallsetminus\{\widehat{\varnothing}\},\trianglelefteq)$ is a tree with the root on the top. Let us now show that the branches of this tree are internal to $\mathbf{\Gamma}$. Let $h(\lambda):=(\partial_p\mathrm{rad}(F_\lambda),0)$ if $p(x,y)$ implies that the balls in $F_\lambda(x)$ are closed and $h(\lambda):=(\partial_p\mathrm{rad}(F_\lambda),1)$ otherwise (the p-germ ∂_p of a function is defined in Definition (8.4)). By Proposition (8.5), we may assume (after adding some parameters) that the image of h is in some Cartesian power of $\mathbf{\Gamma}$. Let us also define $h_\star:\widehat{\lambda}\mapsto\ulcorner h(\widehat{\lambda})\urcorner$. By stable embeddedness of $\mathbf{\Gamma}$, $h_\star$ takes its values in $\mathbf{\Gamma}^{\mathrm{eq}}$.

Claim 9.3: *Pick any $\lambda\in\Lambda_p\smallsetminus\widehat{\varnothing}$, then the function $h_\star$ is injective on $\{\widehat{\mu}:\lambda\trianglelefteq\mu\}$.*

Proof. Let μ_1 and μ_2 be such that $\lambda\trianglelefteq\mu_i$. We have seen in Claim (9.2), that we may assume for all $(x,y)\vDash p$, $F^{\$}_{\mu_1}(x)\subseteq F^{\$}_{\mu_2}(x)$. Let $(x,y)\vDash p$. If $\widehat{\mu_1}\neq\widehat{\mu_2}$ then we must have $F^{\$}_{\mu_1}(x)\subset$

$F^{\mathbb{S}}_{\mu_2}(x)$. Hence either $\mathrm{rad}(F_{\mu_1}(x)) < \mathrm{rad}(F_{\mu_2}(x))$ or the balls in $F_{\mu_1}(x)$ are open and those in $F_{\mu_2}(x)$ are closed. In any case, $h(\mu_1) \neq h(\mu_2)$.
In fact for all $\omega_i \in \widehat{\mu_i}$ we obtain by the same argument that $h(\omega_1) \neq h(\omega_2)$ and hence $h_\star(\widehat{\mu_1}) \neq h_\star(\widehat{\mu_2})$. ♦

Let $\lambda \in \Lambda_p(N)$ be such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$. If $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$, the generic type of $\widehat{\lambda}(\widetilde{N})$ over p, is consistent with Y, it is, in particular, consistent and consistent with p. By Proposition **(6.13)**, it is a complete $\Psi_{\Delta,F}$-type. By Corollary **(7.12)**, $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable. It follows that taking $q = \alpha_{\widehat{\lambda}(\widetilde{N})/p}$ works. Therefore, it suffices to find a $\lambda \in \Lambda_p(N)$ such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is consistent with Y.

Claim 9.4: *Let $\lambda \in \Lambda_p(N)$. If $\widehat{\lambda} \neq \widehat{\varnothing}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is not consistent with Y, there exists μ such that $\widehat{\mu}$ is an immediate $\trianglelefteq$-predecessor of $\widehat{\lambda}$ and $\ulcorner\widehat{\mu}\urcorner \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}\ulcorner\widehat{\lambda}\urcorner)$.*

Proof. If $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is not consistent with Y, there exists $\mu_0 \in \widehat{\lambda}(\widetilde{N})$ and $(\mu_i)_{0<i<k} \in \Lambda_p(N)$ such that for all $\mu \in \widehat{\lambda}(\widetilde{N})$, $p(x,y) \vdash F^{\mathbb{S}}_{\mu_i}(x) \subset F^{\mathbb{S}}_{\mu}(x)$ and

$$p(x,y) \vdash y \in F^{\mathbb{S}}_{\mu_0}(x) \wedge (x,y) \in Y \to y \in \bigcup_{i=1}^{k} F^{\mathbb{S}}_{\mu_i}(x).$$

In particular $\mu_i \triangleleft \lambda$. Removing some μ_i, we may assume that for all i, $\mu_i \notin \widehat{\varnothing}$ and that $p(x,y) \vdash F^{\mathbb{S}}_{\mu_i}(x) \cap F^{\mathbb{S}}_{\mu_j}(x) = \varnothing$ for all $i \neq j$.
Let $\kappa \in \Lambda_p(N)$ be such that $\mu_{i_0} \trianglelefteq \kappa \trianglelefteq \lambda$ for some i_0. As $\kappa \trianglelefteq \lambda$, we have $p(x,y) \vdash (y \in F^{\mathbb{S}}_{\kappa}(x) \wedge (x,y) \in Y) \to (y \in F^{\mathbb{S}}_{\lambda}(x) \wedge (x,y) \in Y) \to \bigvee_{i=1}^{k}(y \in F^{\mathbb{S}}_{\mu_i}(x) \wedge (x,y) \in Y)$. Because $\mu_{i_0} \trianglelefteq \kappa$, we have $p(x,y) \vdash F^{\mathbb{S}}_{\kappa}(x) \cap F^{\mathbb{S}}_{\mu_{i_0}}(x) \neq \varnothing$. If $p(x,y) \vdash F^{\mathbb{S}}_{\kappa}(x) \subseteq F^{\mathbb{S}}_{\mu_{i_0}}(x)$ then $\kappa \trianglelefteq \mu_{i_0}$ and hence $\kappa \sim \mu_i$.
Otherwise, for any $i \neq i_0$, if $p(x,y) \vdash F^{\mathbb{S}}_{\kappa}(x) \cap F^{\mathbb{S}}_{\mu_i}(x) \neq \varnothing$ then we must have $p(x,y) \vdash F^{\mathbb{S}}_{\mu_i}(x) \subseteq F^{\mathbb{S}}_{\kappa}(x)$ and hence for $I = \{i : F^{\mathbb{S}}_{\mu_i}(x) \cap F_{\kappa}(x)^{\mathbb{S}} \neq \varnothing\}$ we have $p(x,y) \vdash (y \in F^{\mathbb{S}}_{\kappa}(x) \wedge (x,y) \in Y) \iff \bigvee_{i\in I}(y \in F^{\mathbb{S}}_{\mu_i}(x) \wedge (x,y) \in Y)$.
It follows that the set $\{\widehat{\kappa} : \mu_i \trianglelefteq \kappa \trianglelefteq \lambda$ for some $i\}$ is finite. In particular we could choose μ_i such that there is no κ such that $\widehat{\mu_i} \triangleleft \widehat{\kappa} \triangleleft \widehat{\lambda}$. The $\widehat{\mu_i}$ are then the (finitely many) direct $\trianglelefteq$-predecessors of $\widehat{\lambda}$ and for all i, $\widehat{\mu_i} \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}\ulcorner\widehat{\lambda}\urcorner)$. ♦

Let us assume that there does not exist λ such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is consistent with Y. Pick any $\lambda_0 \in \widehat{\mathbf{K}}(\widetilde{N})$, then $\widehat{\lambda_0} = \widehat{\mathbf{K}} \in \widetilde{A}$ and we can construct by induction, using Claim **(9.4)**, a sequence $(\lambda_i)_{i\in\omega}$ such that $\widehat{\lambda}_{i+1}$ is a direct $\trianglelefteq$-predecessor of $\widehat{\lambda}_i$. For all i, we have $|\{\widehat{\mu} : \widehat{\lambda_i} \trianglelefteq \widehat{\mu}\}| = i + 1 = |h_\star(\{\widehat{\mu} : \widehat{\lambda_i} \trianglelefteq \widehat{\mu}\})|$ and that contradicts the elimination of $\exists^\infty$ in $\mathbf{\Gamma}^{\mathrm{eq}}$. This concludes the proof. ∎

In the following proposition, the most general case is when $|x| = 0$, but we need this more complicated statement to carry out the induction.

Proposition 9.5:
Let $Y \subseteq \mathbf{K}^{n+m}$ be an $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable set, $\Delta(x,y;t)$ and $\Theta(y;s)$ be finite sets of $\mathcal{L}$-formulas where $|x| = n$ and $|y| = m$. Let $p \in \mathcal{S}^{\Delta}_{x,y}(N)$ be $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with Y. Then

there exists a finite set of $\mathcal{L}$-formulas $\Xi(x,y;s,t,r) \supseteq \Delta \cup \Theta$ and a type $q \in \mathcal{S}^{\Xi}_{x,y}(N)$ which is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with p and Y.

As noted in Remark **7.2.4**, q is in particular $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable as a Ξ-type.

Proof. We proceed by induction on $|y|$. The case $|y| = 0$ is trivial. Let us now assume that $y = (z,w)$ where $|w| = 1$. By Proposition **(6.16)** there exists $\Phi(z;u)$ a finite set of $\mathcal{L}$-formulas and $F = (F_\lambda)_{\lambda\in\Lambda}$ an $\mathcal{L}$-definable family of functions $\mathbf{K}^{m-1} \to \mathbf{B}^{[l]}$ such that $\Psi_{\Phi,F}$ decides any formula in Θ. By Propositions **(6.17)** and **(7.15)** we can assume that $F_\lambda : \mathbf{K}^{m-1} \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and (Φ,F,z) is a good representation. We can easily make F into an $\mathcal{L}$-definable family of functions $\mathbf{K}^{n+m-1} \to \mathbf{B}^{[l]}_{\mathrm{st}}$ by setting $G_\lambda(x,z) = F_\lambda(z)$. As T admits $\mathbf{\Gamma}$-reparametrisations, there exists $\Upsilon(x,z;v)$ such that for any $p \in \mathcal{S}^{\Upsilon}_y(N)$, there exists a $\mathbf{\Gamma}$-reparametrisation $(g_\gamma)_\gamma$ of $(\mathrm{rad}\circ G_\lambda)_{\lambda\in\Lambda}$ over p.
By induction applied to $\Delta_0((x,w),z;t) := \Delta(x,z,w;t)$, $\Theta_0(z;u,v) := \Phi(z;u) \cup \Upsilon(z;v)$ and p, we obtain a finite set of $\mathcal{L}$-formulas $\Omega(x,w,z;r) \supseteq \Delta \cup \Phi \cup \Upsilon$ and a type $q_1 \in \mathcal{S}^{\Omega}_{x,z,w}(N)$ which is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with p and Y. Let $g = (g_\gamma)_\gamma$ $\mathbf{\Gamma}$-reparametrise $(\mathrm{rad}\circ G_\lambda)_{\lambda\in\Lambda}$ over $q_1|_{\Upsilon}$. We can now apply Lemma **(9.1)** to Y, $(\Omega, G, (x,z))$, q_1 and g to find a type $q_2 \in \mathcal{S}^{\Psi_{\Omega,G}}_{x,w,z}(N)$ which is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with q_1 and Y. As all the formulas in Θ are decided by $\Psi_{\Phi,F}$ and hence by $\Psi_{\Omega,G}$, we may assume that q_2 is in fact a $(\Psi_{\Omega,G}\cup\Theta)$-type and we can take $\Xi = \Psi_{\Omega,G}\cup\Theta$ and $q = q_2$. ■

Definition 9.6 (Strict $\star$-definable sets)**:**
Let $\mathcal{L}$ be a language, N an $\mathcal{L}$-structure and $x = (x_i)_{i\in I}$ a (potentially infinite) tuple of variables. Let P be a set of $\mathcal{L}$-formulas with variables x. The set $P(N) := \{m \in N^x : \forall\varphi \in P,\ N \models \varphi(m)\}$ is said to be $(\mathcal{L},x)$-definable, or simply $(\mathcal{L},\star)$-definable if we do not want to specify x. We say that an $(\mathcal{L},\star)$-definable set is strict $(\mathcal{L},\star)$-definable if the projection on any finite subset of x is $\mathcal{L}$-definable.

When x is finite and P is infinite, $P(N)$ is usually called an ∞-definable set.

Proposition 9.7:
Let X be non empty strict $(\widetilde{\mathcal{L}}^{eq}(\widetilde{A}),x)$-definable where all the variables in x are $\mathbf{K}$-variables and $|x| \leqslant \aleph_0$. Assume also that $|\mathcal{L}| \leqslant \aleph_0$. Then there exists an $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable type $p \in \mathcal{S}^{\mathcal{L}}_x(N)$ consistent with X.

Note that this is the only proof where we need a cardinality hypothesis on $\mathcal{L}$.

Proof. Let $\{\varphi_j(x_j;t_j) : j < \omega\}$ be an enumeration of all $\mathcal{L}$-formulas such that x_j is a tuple of variables from x. Let $\Delta_{-1} := \varnothing$ and $p_{-1} := \varnothing$ and we construct by induction on j a finite set $\Delta_j(x_{\leqslant j};s_j)$ of $\mathcal{L}$-formulas and a type $p_j \in \mathcal{S}^{\Delta_j}_{\leqslant x_j}(N)$ such that for all $j < \omega$, $\Delta_j \cup \{\varphi_j\} \subseteq \Delta_{j+1}$, p_{j+1} is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable and consistent with p_j and X. Let us assume that p_j and Δ_j have been constructed. Let Y_{j+1} be the projection of X on the variables $x_{\leqslant j+1}$. Then Y_{j+1} is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. We can then apply Proposition **(9.5)** to $\Delta_j(x_{\leqslant j};s_j)$, $\{\varphi(x_{j+1};t_{j+1})\}$, p_j and Y_{j+1} in order to obtain p_{j+1}. As Y_{j+1} is the projection of X on the variables which appear in p_j and p_{j+1}, and that p_1, p_{j+1} and Y are consistent, it follows that p_j, p_{j+1} and X are also consistent.

We can now take $p := \bigcup_{j<\omega} p_j$. As each p_j is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable (as Δ_j-type), so is p (as a complete $\mathcal{L}$-type). ■

Note that p might not be quantifiable anymore (as a $\widetilde{\mathcal{L}}^{eq}$-type). Although the fact that p_j is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiable is necessary to carry out the induction, we will not need $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-quantifiability afterwards, except if we were to continue the induction. This is exactly why we cannot prove Proposition **(9.7)**, and hence Theorem **(9.8)**, if $\mathcal{L}$ is not countable. Nevertheless, we will see later that Theorem **(9.8)** is stronger than what is needed to prove elimination of imaginaries which we will be able to show even when $\mathcal{L}$ is not countable.
We now prove the main result we have been aiming for.

Theorem 9.8:

Let $\widetilde{\mathcal{L}} \supseteq \mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ be languages such that $|\widetilde{\mathcal{L}}| \leqslant \aleph_0$, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory which eliminates imaginaries and admits $\mathbf{\Gamma}$-reparametrisations and $\widetilde{T}$ a complete $\widetilde{\mathcal{L}}$-theory containing T such that $\mathbf{K}$ is dominant in $\widetilde{T}$ and:

(i) The sets $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded in $\widetilde{T}$ and the induced theories on $\mathbf{k}$ and $\mathbf{\Gamma}^{\mathrm{eq}}$ eliminate $\exists^\infty$;

(ii) For any $\widetilde{N} \vDash \widetilde{T}$, $A = \mathbf{K}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \widetilde{N}$ and any $\widetilde{\mathcal{L}}(A)$-definable set $X \subseteq \mathbf{K}^n$, there exists an $\widetilde{\mathcal{L}}$-definable bijection $f : \mathbf{K}^n \to Y$ such that $f(X) = Y \cap Z$ where Z is $\mathcal{L}(A)$-definable; note that f has to be defined without parameters.

Then for all $\widetilde{N} \vDash \widetilde{T}$ and every non empty $\widetilde{\mathcal{L}}(\widetilde{N})$-definable set X, there exists $p \in \mathcal{S}^{\widetilde{\mathcal{L}}}(\widetilde{N})$ which is consistent with X and $\widetilde{\mathcal{L}}^{eq}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner))$-definable.
If, moreover, the following also holds:

(iii) There exists $\widetilde{M} \vDash \widetilde{T}$ such that $\widetilde{M}|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension;

then, the type p can be assumed to be $\widetilde{\mathcal{L}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)))$-definable

This really is a result on the density of definable types with canonical basis in $\mathcal{R}$. Indeed let $\widetilde{A} = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})$, the conclusion of Theorem **(9.8)** states that the set $\{p \in \mathcal{S}^{\widetilde{\mathcal{L}}}(\widetilde{A}) : p \text{ is } \widetilde{\mathcal{L}}(\mathcal{R}(\widetilde{A}))\text{-definable}\}$ is dense in $\mathcal{S}^{\widetilde{\mathcal{L}}}(\widetilde{A})$.

Proof. Let $\widetilde{A} := \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)$. We may assume that $X \subset \mathbf{K}^n$ for some n. Indeed, let S_i be the sorts such that $X \subseteq \prod S_i$. Since $\mathbf{K}$ is dominant, there is an $\widetilde{\mathcal{L}}$-definable surjection $\pi : \mathbf{K}^n \to \prod S_i$. If we find p consistent with $Y := \pi^{-1}(X)$ and $\widetilde{\mathcal{L}}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner Y \urcorner))$-definable, then $\pi_\star p$ is consistent with X and $\widetilde{\mathcal{L}}(\widetilde{A})$-definable.
Let $F := \{f \text{ is an } \widetilde{\mathcal{L}}\text{-definable bijection whose domain is } \mathbf{K}^n\}$ and $\partial_\omega(x) := (f(x))_{f \in F}$. Then $\partial_\omega(X)$ is strict $(\widetilde{\mathcal{L}}^{eq}(\widetilde{A}), I)$-definable for some I with $|I| \leqslant \aleph_0$. By Proposition **(9.7)**, there exists an $\mathcal{L}(\widetilde{A})$-definable type $p \in \mathcal{S}^{\mathcal{L}}_I(N)$ consistent with $\partial_\omega(X)$.
Let $q = \{x : \partial_\omega(x) \vDash p\}$. Then q is consistent with X. There only remains to show that it is a complete type and that it is $\widetilde{\mathcal{L}}(\widetilde{A})$-definable. Let $\varphi(x; s)$ be an $\widetilde{\mathcal{L}}$-formula where $x \in \mathbf{K}^n$.

As $\mathbf{K}$ is dominant we may assume s is a tuple of variables from $\mathbf{K}$ too. By (iii), for all tuples $m \in \mathbf{K}(\widetilde{N})$, there exists $(f : \mathbf{K}^n \to Y) \in F$ and an $\widetilde{\mathcal{L}}$-definable map g (into $\mathbf{K}^l$ for some l) such that $f(\varphi(\widetilde{N};m)) = Y(\widetilde{N}) \cap Z(\widetilde{N})$ where Z is $\mathcal{L}(g(m))$-definable. As $\widetilde{N}$ is arbitrary, we may assume that it is sufficiently saturated and by compactness there exists a finite number of $(f_i : \mathbf{K}^n \to Y_i) \in F$, $\widetilde{\mathcal{L}}$-definable maps g_i and $\mathcal{L}$-formulas $\psi_i(y_i;t)$ such that for any tuple $m \in \mathbf{K}(\widetilde{N})$ there exists i_0 such that $f_{i_0}(\varphi(\widetilde{N};m)) = \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$.
Let c_1 and $c_2 \models q$ (i.e. $\partial_\omega(c_j) \models p$, for $j \in \{1,2\}$) and assume that $\models \varphi(c_1;m)$. Then $f_{i_0}(c_1) \in \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$. As $f_{i_0}(c_1) \equiv_{\mathcal{L}(N)} f_{i_0}(c_2)$ and $f_{i_0}(c_2) \in Y_{i_0}(\widetilde{N})$ we also have $f_{i_0}(c_2) \in \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N}) = f_{i_0}(\varphi(\widetilde{N};m))$ and, because f_{i_0} is a bijection, $\models \varphi(c_2;m)$. As for definability, we have just shown that $\varphi(x;m) \in q$ if and only if $\psi_{i_0}(y_i;g_{i_0}(m)) \in p$ for some i_0 such that $f_{i_0}(\varphi(\widetilde{N};m)) = \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$ but that can be stated with an $\widetilde{\mathcal{L}}(\widetilde{A})$-formula.
Moreover, if Hypothesis **9.8.(iii)** holds, by [RS, Corollary 1.6], the type q is in fact $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable and hence, p is $\widetilde{\mathcal{L}}(\mathcal{R}(\widetilde{A}))$-definable ∎

10. Imaginaries and invariant extensions

In this section, we investigate the link between the density of definable types, elimination of imaginaries and the invariant extension property (see Definition **(1.14)**). I am very much indebted to [Hru14; Joh] for making me realise that the density of definable types could play an important role in proving elimination of imaginaries. To be precise, we will show that both the elimination of imaginaries and the invariant extension property follow from the density of types invariant over real parameters.

Remark 10.1:
Because types definable over some parameters A are also A-invariant, the density of definable types over some parameters implies the density of invariant types. But the converse is false even in NIP theories. Consider $M \equiv \mathbb{Q}_p$ in the three sorted language with angular components. Assume M is $\aleph_0$-saturated. Let $\gamma \in \mathbf{\Gamma}(M)$ be such that $\gamma > n \cdot \mathrm{val}(p)$ for all $n \in \mathbb{N}$ and $b := \{x \in \mathbf{K}(M) : \mathrm{val}(x) = \gamma \wedge \mathrm{ac}_1(x) = 1\}$. Note that b is a ball. Because the residue field is finite, in $\mathrm{acl}(\ulcorner b \urcorner)$ there are all the balls $b' \subseteq b$ such that $\mathrm{rad}(b') - \mathrm{rad}(b) \in \mathbb{Z} \cdot \mathrm{val}(p)$. Let $(b_i)_{i \in \mathbb{N}}$ be a chain of balls such that $b_i \subseteq b$ and $\mathrm{rad}(b_i) = \mathrm{rad}(b) + i\mathrm{val}(p)$. Then for all x, $y \in \bigcap_i b_i$, $\mathrm{val}(x) = \mathrm{val}(y) = \gamma$ and $\mathrm{ac}_n(x) = \mathrm{ac}_n(y)$.
Let $P = \sum_i a_i X^i \in \mathbb{Q}[X]$, let i_0 be minimal such that $a_i \neq 0$, then for all $i \neq i_0$, $\mathrm{val}(a_{i_0}x^{i_0}) = \mathrm{val}(a_{i_0}y^{i_0}) < \mathrm{val}(a_i x^i) = \mathrm{val}(a_i y^i)$. In fact, $\mathrm{val}(a_i x^i) - \mathrm{val}(a_{i_0} x_{i_0}) \geqslant n \cdot \mathrm{val}(p)$ for all $n \in \mathbb{N}$. Thus $\mathrm{val}(P(x)) = \mathrm{val}(P(y))$ and $\mathrm{ac}_n(P(x)) = \mathrm{ac}_n(P(y))$. It now follows from field quantifier elimination that $x \equiv_\varnothing y$ and because any automorphism sending x to y must fix b they have the same type over b. Thus, there cannot be any ball in $\bigcap_i b_i$ algebraic over b and hence, by [HMR, Proposition 3.9], x and y have the same type over $\mathrm{acl}(b)$. Every type in b is of this form and none of them can be definable because $\bigcap_i b_i$ is a strict intersection.
Nevertheless, by [HMR, Remark 4.7], $\mathrm{Th}(\mathbb{Q}_p)$ has the invariant extension property and hence by Proposition **(10.5)**, invariant types are dense over algebraically closed sets.

In the following proposition, we show that the density of Δ-types invariant over real parameters for finite Δ suffices to prove weak elimination of imaginaries.

Proposition 10.2:
Let T be an $\mathcal{L}$-theory and $\mathcal{R}$ a set of its sorts such that for all $N \models T$, all non empty $\mathcal{L}(N)$-definable sets X and all $\mathcal{L}$-formulas $\varphi(x;s)$ (where x is sorted as X), there exists $p \in \mathcal{S}_x^{\varphi}(N)$ which is consistent with X and $\mathrm{Aut}(N/\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)))$-invariant. Then T weakly eliminates imaginaries up to $\mathcal{R}$.

Proof. Let M be a sufficiently saturated and homogeneous model of T, E be any $\mathcal{L}$-definable equivalence relation, X be one of its classes in M, $\varphi(x,y)$ be an $\mathcal{L}$-formula defining E and $A = \mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner))$. By hypothesis, there exists an $\mathrm{Aut}(N/A)$-invariant type $p \in \mathcal{S}_x^{\varphi}(M)$ consistent with X. Because X is defined by an instance of φ, we have in fact $p(x) \vdash x \in X$. For all $\sigma \in \mathrm{Aut}(N/A)$, $\sigma(X)$ is another E-class and $\sigma(p) = p \vdash x \in X$. It follows that $X \cap \sigma(X) \neq \varnothing$ and hence $X = \sigma(X)$. We have just proved that $\ulcorner X \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)))$, i.e. X is weakly coded in $\mathcal{R}$. ■

Corollary 10.3:
In the setting of Theorem (9.8), $\widetilde{T}$ eliminates imaginaries.

Proof. It follows from Theorem **(9.8)**, that the hypothesis of Proposition **(10.2)** holds in $\widetilde{T}$ and hence that $\widetilde{T}$ weakly eliminates imaginaries up to the sorts $\mathcal{R}$. But because any finite sets in $\mathcal{R}$ are also definable in T and hence are coded in T, $\widetilde{T}$ eliminates imaginaries up to the sorts $\mathcal{R}$. ■

In view of further applications (to valued fields with analytic structure for example), on may note that the cardinality assumption of Theorem **(9.8)** is not needed to obtain elimination of imaginaries (it is only needed to construct complete types and hence to obtain the invariant extension property).

Proposition 10.4:
In the setting of Theorem (9.8), without any cardinality assumption on $\widetilde{\mathcal{L}}$, $\widetilde{T}$ eliminates imaginaries.

Proof. As in Corollary **(10.3)**, it suffices to prove weak elimination of imaginaries. Let $\widetilde{N}$ be a sufficiently saturated and homogeneous model of $\widetilde{T}$, $N := \widetilde{N}|_{\mathcal{L}}$, E be any $\widetilde{\mathcal{L}}$-definable equivalence relation, X be one of its classes in $\widetilde{N}$, $\varphi(x,y)$ be an $\widetilde{\mathcal{L}}$-formula defining E, $\widetilde{A} = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)$ and $A = \mathcal{R}(\widetilde{A})$. By domination of $\mathbf{K}$, we may assume that $X \subseteq \mathbf{K}^n$ for some n and by Hypothesis **9.8.(ii)**, there exists an $\widetilde{\mathcal{L}}$-definable bijection $f : \mathbf{K}^n \to Y$ and and $\mathcal{L}(N)$-definable set $Z = \psi(M;m)$ such that $f(X) = Y \cap Z$. Note that $\ulcorner Y \cap Z \urcorner \in \widetilde{A}$.
By Proposition **(9.5)**, there exists an $\widetilde{\mathcal{L}}^{\mathrm{eq}}(\widetilde{A})$-definable type $p \in \mathcal{S}_y^{\psi}(N)$ which is consistent with $Y \cap Z$. By Hypothesis **9.8.(iii)** and by [RS, Corollary 1.6], the type p is in fact $\mathcal{L}(A)$-definable. Let c realise $p \cap Y \cap Z$ and $a \in X$ be such that $f(a) = c$. For all $\sigma \in \mathrm{Aut}_{\widetilde{\mathcal{L}}}(\widetilde{N}/A)$, $\sigma(c) \models \sigma(p) = p \vdash y \in Z$. Because Y is $\mathcal{L}$-definable, we also have $\sigma(c) \in Y$. It immediately follows that $\sigma(a) = f^{-1}(\sigma(c)) \in X$ and thus that $X \cap \sigma(X) \neq \varnothing$. Because both are E-equivalence classes, we must have $X = \sigma(X)$ and hence X is $\widetilde{\mathcal{L}}(A)$-definable, i.e. it is weakly coded. ■

Let us now consider the invariant extension property.

Proposition 10.5:
Let T be an $\mathcal{L}$-theory, $A \subseteq M$ for some $M \models T$. The following are equivalent:

(i) *For all $\mathcal{L}(A)$-definable non empty sets X and $N \models T$, $N \subseteq A$, there exists $p \in \mathcal{S}(N)$ such that p is $\mathrm{Aut}(N/A)$-invariant and is consistent with X;*

(ii) *T has the invariant extension property over A.*

Proof. Let us first show that (ii) implies (i). Let $N \models T$, X be $\mathcal{L}(A)$-definable and $p \in \mathcal{S}(A)$ be any type containing X. Let $q \in \mathcal{S}(N)$ be an $\mathrm{Aut}(N/A)$-invariant extension of p. Then q is consistent with X.
Let us now assume (i) and let $\mathrm{Inv}(N/A) = \{p \in \mathcal{S}(N) : p \text{ is } \mathrm{Aut}(N/A)\text{-invariant }\}$.

Claim 10.6: *The set $\mathrm{Inv}(N/A) \subseteq \mathcal{S}(N)$ is closed and hence compact.*

Proof. Let $p \in \mathcal{S}(N) \setminus \mathrm{Inv}(N/A)$. There exists $\varphi(x;s)$, a tuple $m \in N$ and $\sigma \in \mathrm{Aut}(N/A)$ such that $\varphi(x;m) \in p$ and $\varphi(x;\sigma(m)) \notin p$. Then, the set $\{q \in \mathcal{S}(N) : \varphi(x;m) \in q$ and $\varphi(x;\sigma(m)) \notin q\} = \{q \in \mathcal{S}(N) : \varphi(x;m) \wedge \neg\varphi(x;\sigma(m)) \in q\}$ is open and has empty intersection with $\mathrm{Inv}(N/A)$. ♦

Let $p \in \mathcal{S}(A)$. By hypothesis, for all $\mathcal{L}(A)$-definable sets $X \neq \varnothing$, there exists $q_X \in \mathrm{Inv}(N/A)$ which is consistent with X. It follows that for all $\mathcal{L}(A)$-definable sets X, the closed set $F_X := \langle X \rangle \cap \mathrm{Inv}(N/A) \neq \varnothing$. Moreover, for any finite number of $X_i \in p$, $\cap_i F_{X_i} = F_{\bigwedge_i X_i}$ is non empty. As $\mathrm{Inv}(N/A)$ is compact, there exists $q \in \bigcap_{X \in p} F_X$. Then $q \in \mathrm{Inv}(N/A)$ and for all $X \in p$, $q \in F_X \subseteq \langle X \rangle$ so q does extend p. ■

To conclude:

Theorem 10.7:

*In the setting of Theorem **(9.8)**, $\widetilde{T}$ eliminates imaginaries and has the invariant extension property.*

Proof. Elimination of imaginaries is proved in Corollary **(10.3)** and the invariant extension property then follows from Theorem **(9.8)** and Proposition **(10.5)**. ■

Appendix

A. Uniform stable embeddedness of Henselian valued fields

The goal of this section is to study stable embeddedness in pairs of valued fields and, in particular, to show that there exist models of ACVF uniformly stably embedded in every elementary extension. These models are used to prove that there are models of $\mathrm{VDF}_{\mathcal{EC}}$ whose underlying valued field is stably embedded in every elementary extension in the proof of Theorem **(1.18)**. Not all the results proved in this section are necessary to attain this goal though, but they are of a similar nature and are used in Section 3. Some of these results are valid in all characteristic, so unless explicitly stated, we do not assume that we are working in characteristic zero.

Following Baur, let us first introduce the notion of a separated pair of valued fields.

Definition A.1 (Separated pair)**:**
Let $K \subseteq L$ be an extension of valued fields. Call a tuple $a \in L$ K-separated if for any tuple $\lambda \in K$, $\mathrm{val}(\sum_i \lambda_i a_i) = \min_i\{\mathrm{val}(\lambda_i a_i)\}$. The pair $K \subseteq L$ is said to be separated if any finite dimensional sub-K-vector space of L has a K-separated basis.

Recall that a maximally complete field is a field where every chain of balls has a point. Let us now recall a well known result of [Bau82].

Proposition A.2:
Let K be a maximally complete field. Then any extension $K \subseteq L$ is separated.

Following [CD; Del89], let us give the links between separation of the pair $K \subseteq L$ and uniform stable embeddedness of K in L. But first let us define this last notion.

Definition A.3 (Uniform stable embeddedness)**:**
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that A is uniformly stably embedded if for all formulas $\varphi(x;t)$ there exists a formula $\chi(x;s)$ such that for all tuples $b \in M$ there exists a tuple $a \in A$ such that $\varphi(A,b) = \chi(A,a)$.

The proof of Proposition **(A.4)** is taken almost word for word from the one in [CD], although we have slightly different assumptions and we put more emphasis on uniformity here. Let $\mathcal{L}$ be an $\mathbf{RV}$-extension of $\mathcal{L}^{\mathbf{RV}}$ and let $\mathrm{T}_{\mathrm{Hen}}$ be the $\mathcal{L}^{\mathbf{RV}}$-theory of Henselian valued fields of *characteristic zero*. Recall that $\mathbf{RV}_n^\star = \mathbf{K}^\star/(1+n\mathfrak{M})$ and that $\mathrm{T}_{\mathrm{Hen}}$ resplendently eliminates field quantifiers (see [Bas91; BK92]).

Proposition A.4:
Let $M \vDash \mathrm{T}_{\mathrm{Hen}}$ be an $\mathcal{L}$-structure and $\varphi(x;s)$ an $\mathcal{L}$-formula. There exists an $\mathcal{L}|_{\mathbf{RV}}$-formula $\psi(y;u)$ and polynomials $Q_i \in \mathbb{Z}[\overline{X},\overline{T}]$ such that for any $N \preccurlyeq M$, where the pair $\mathbf{K}(N) \subseteq \mathbf{K}(M)$ is separated, and any $a \in M$ there exists $b \in \mathbf{K}(N)$ and $c \in \mathbf{RV}(M)$ such that $\varphi(N;a) = \psi(\mathrm{rv}_n(\overline{Q}(N,b));c)$.

Proof. By (resplendent) elimination of field quantifiers (and the fact that $\mathbf{K}$ is dominant), we may assume that $\varphi(x;a)$ is of the form $\theta(\mathbf{RV}_n(\overline{P}(x)))$ where $\overline{P} \in \mathbf{K}(M)[\overline{X}]$, $n \in \mathbb{N}$ and ψ is an $\mathcal{L}|_{\mathbf{RV}}$-formula. Let us write each P_i as $\sum_\mu a_{i,\mu}\overline{X}^\mu$. As the pair $\mathbf{K}(N) \subseteq \mathbf{K}(M)$ is separated, the $\mathbf{K}(N)$-vector space generated by the $a_{i,\mu}$ is generated by a $\mathbf{K}(N)$-separated tuple $\overline{d} \in \mathbf{K}(M)$. Note that $|\overline{d}| \leqslant |\overline{a}|$ and adding zeros to $\overline{d}$ we may assume $|\overline{d}| = |\overline{a}|$. For each i and μ, find $\lambda_{i,\mu,j} \in \mathbf{K}(N)$ such that $a_{i,\mu} = \sum_j \lambda_{i,\mu,j}d_j$. We can rewrite each P_i as $\sum_j d_jQ_{i,j}(\overline{X},\overline{\lambda})$, where $Q_{i,j} \in \mathbb{Z}[\overline{X},\overline{T}]$ does not depend on $\overline{a}$, and for all $x \in K(N)$ we have $\mathrm{val}(P_i(x)) = \min_j\{\mathrm{val}(d_jQ_{i,j}(x,\overline{\lambda}))\}$. As $\mathrm{rv}_n(x+y) = \mathrm{rv}_n(x) +_{n,n} \mathrm{rv}_n(y)$ whenever $\mathrm{val}(x+y) = \min\{\mathrm{val}(x),\mathrm{val}(y)\}$, it follows immediately that

$$\mathrm{rv}_n(P_i(x)) = \sum_{j \in J_i(x)} \mathrm{rv}_n(d_j)\mathrm{rv}_n(Q_{i,j}(x,\overline{\lambda}))$$

where $J_i(x) = \{j : \mathrm{val}(d_j)\mathrm{val}(Q_{i,j}(x,\overline{\lambda})) \text{ is minimal}\}$.
The proposition now follow easily with $b = \overline{\lambda}$ and $c = \mathrm{rv}_n(\overline{d})$. ■

Let $\mathcal{L}_{og}$ be the language of ordered groups. If M is algebraically closed, we can obtain a stronger statement.

Proposition A.5:
Let $M \models \mathrm{ACVF}$ and $\varphi(x,y)$ an $\mathcal{L}_{\mathrm{div}}$-formula. There is an $\mathcal{L}_{og}$-formula $\psi(y;u)$ and polynomials $Q_i \in \mathbb{Z}[\overline{X},\overline{T}]$ such that for any $N \leqslant M$, where the pair $\mathbf{K}(N) \subseteq \mathbf{K}(M)$ is separated, and any $a \in M$ there exists $b \in \mathbf{K}(N)$ and $c \in \mathbf{\Gamma}(M)$ such that $\varphi(N;a) = \psi(\mathrm{val}(\overline{Q}(N,b));c)$.

Proof. The proof is essentially the same as for Proposition **(A.5)** except that we use the quantifier elimination for algebraically closed valued fields in the two sorted language. ■

Let us now give the two consequences of these computations which we use in the paper.

Theorem A.6 (Algebraically closed case in any characteristic)**:**
Let $K \subseteq L$ be a separated pair of valued fields such that L is algebraically closed. Then K is stably embedded in L if and only if $\mathbf{\Gamma}(K)$ is stably embedded in $\mathbf{\Gamma}(L)$, as an ordered Abelian group.
Moreover, if $\mathbf{\Gamma}(K)$ is uniformly stably embedded in $\mathbf{\Gamma}(L)$, then K is uniformly stably embedded in L.

Proof. This follows immediately from Proposition **(A.5)**. ■

We will now be considering angular components in mixed characteristic $(0,p)$. Recall that $\mathbf{R}_n = \mathcal{O}/(p^n\mathfrak{M})$, $\mathbf{R} = \bigcup_n \mathbf{R}_n$ and that angular component maps are compatible systems of group morphisms $\mathrm{ac}_n : \mathbf{K}^\star \to \mathbf{R}_n^\star$ such that $\mathrm{ac}_n|_{\mathcal{O}^\star} = \mathrm{res}_n|_{\mathcal{O}^\star}$.

Theorem A.7 (Unramified mixed characteristic with ac)**:**
Let $K \subseteq L$ be a separated pair of unramified mixed characteristic valued fields with angular component maps such that L is Henselian and $\mathbf{R}_0(L)$ is perfect. Then K is stably embedded in L if and only if $\mathbf{\Gamma}(K)$ is stably embedded in $\mathbf{\Gamma}(L)$ (as an ordered Abelian group) and $\mathbf{R}_0(K)$ is stably embedded in $\mathbf{R}_0(L)$ (as a ring).
Moreover if $\mathbf{\Gamma}(K)$ is uniformly stably embedded in $\mathbf{\Gamma}(L)$ and $\mathbf{R}_0(K)$ is uniformly stably embedded in $\mathbf{R}_0(L)$, then K is uniformly stably embedded in L.

Proof. An angular component is nothing more than a section of the short exact sequence $\mathbf{R}_n^\star \to \mathbf{RV}_n^\star \to \mathbf{\Gamma}$. Therefore, it follows from Proposition **(A.4)** that we only need to prove that $\mathbf{R} \cup \mathbf{\Gamma}(K)$ is (uniformly) stably embedded in $\mathbf{R} \cup \mathbf{\Gamma}(L)$. Because L is unramified, it follows from the quantifier elimination in the language with angular components that $\mathbf{\Gamma}$ and $\mathbf{R}$ are orthogonal, i.e. any definable subset of $\mathbf{R}^n \times \mathbf{\Gamma}^m$ is a union of products $X \times Y$ where $X \subseteq \mathbf{R}^n$ is definable in $\mathbf{R}$ and $Y \subseteq \mathbf{\Gamma}^m$ is definable in $\mathbf{\Gamma}$. Hence it suffices to prove that $\mathbf{\Gamma}(K)$ is (uniformly) stably embedded in $\mathbf{\Gamma}(L)$ and that $\mathbf{R}(K)$ is (uniformly) stably embedded in $\mathbf{R}(L)$.
For all n, the canonical projection $\mathrm{res}_{0,n} : \mathbf{R}_n \to \mathbf{R}_0$ has a definable section defined by $\tau_n(x)$ is the only y such that $\mathbf{R}_{0,n}(y) = x$ and y is a p^n-th power. Using this residual version of the

Teichmüller liftings, one can show that $\mathbf{R}_n(L)$ is $\varnothing$-definably isomorphic to $\mathrm{W}_n(\mathbf{R}_0(L))$ and hence that $\mathbf{R}(K)$ is (uniformly) stably embedded in $\mathbf{R}(L)$ if and only if $\mathbf{R}_0(K)$ is (uniformly) stably embedded in $\mathbf{R}_0(L)$. ■

Corollary A.8:
Let k be any algebraically closed field. The Hahn field $K := k((t^{\mathbb{R}}))$ is uniformly stably embedded (as a valued field) in any elementary extension.

Proof. The field K is maximally complete, as are all Hahn fields, and so it is Henselian. Moreover its residue field k is algebraically closed and its value group $\mathbb{R}$ is divisible. It follows that K is algebraically closed. By Proposition **(A.2)**, any extension $K \subseteq L$ is separated. By Theorem **(A.6)**, it suffices to show that $\mathbb{R}$ is uniformly stably embedded (as an ordered group) in any elementary extension. But that follows from the fact that $(\mathbb{R}, <)$ is complete and $(\mathbb{R}, +, <)$ is o-minimal, see [CS15, Corollary 64]. ■

List of Notations

References

[Bas91] S. A. Basarab. Relative elimination of quantifiers for Henselian valued fields. *Ann. Pure Appl. Logic* 53.1 (1991), pp. 51–74.

[BK92] S. A. Basarab and F.-V. Kuhlmann. An isomorphism theorem for Henselian algebraic extensions of valued fields. *Manuscripta Math.* 77 (1992), pp. 113–126.

[Bau82] W. Baur. "Die Theorie der Paare reell abgeschlossner Körper". In: *Logic and Algorithmic: An International Symposium Held in Honour of Ernst Specker*. Monogr. Enseign. Math. 30. Kundig, 1982, pp. 25–34.

[BMS07] L. Bélair, A. Macintyre, and T. Scanlon. Model theory of the Frobenius on the Witt vectors. *Amer. J. Math.* 129.3 (2007), pp. 665–721.

[CS15] A. Chernikov and P. Simon. Externally definable sets and dependent pairs II. *Trans. Amer. Math. Soc.* 367.7 (2015), pp. 5217–5235.

[CL11] R. Cluckers and L. Lipshitz. Fields with analytic structure. *J. Eur. Math. Soc. (JEMS)* 13.4 (2011), pp. 1147–1223.

[Cub14] P. Cubides Kovacsics. "An introduction to C-minimal structures and their cell decomposition theorem". In: *Valuation theory and interaction*. Ed. by A. Campillo, F.-V. Kuhlmann, and B. Teissier. EMS Ser. Congr. Rep. 10. Eur. Math. Soc., Zürich, 2014, pp. 167–207.

[CD] P. Cubides Kovacsics and F. Delon. Definable types in algebraically closed valued fields. arXiv:1410.3589. To appear in MLQ Math. Log. Q.

[Del89] F. Delon. Définissabilité avec paramètres extérieurs dans $\mathbb{Q}_p$ et $\mathbb{R}$. *Proc. Amer. Math. Soc.* 106.1 (1989), pp. 193–198.

[HHM06] D. Haskell, E. Hrushovski, and D. Macpherson. Definable sets in algebraically closed valued fields: elimination of imaginaries. *J. Reine Angew. Math.* 597 (2006), pp. 175–236.

[HHM08] D. Haskell, E. Hrushovski, and D. Macpherson. *Stable Domination and Independance in Algebraically Closed Valued Fields*. Assoc. Symbol. Logic, Chicago, IL. Lect. Notes Log. 30. Cambridge Univ. Press, Cambridge, 2008.

[HHM13] D. Haskell, E. Hrushovski, and D. Macpherson. Unexpected imaginaries in valued fields with analytic structure. *J. Symbolic Logic* 78.2 (2013), pp. 523–542.

[HO10] A. Hasson and A. Onshuus. Embedded o-minimal structures. *Bull. Lond. Math. Soc.* 42.1 (2010), pp. 64–74.

[Hru12] E. Hrushovski. Groupoids, imaginaries and internal covers. *Turkish J. Math.* 36.2 (2012), pp. 173–198.

[Hru14] E. Hrushovski. Imaginaries and definable types in algebraically closed valued fields. In: *Valuation Theory in Interaction*. Ed. by A. Campillo Lopez, F.-V. Kuhlmann, and B. Teissier. EMS Ser. Congr. Rep. 10. Eur. Math. Soc., Zürich, 2014, pp. 297–319.

[HL] E. Hrushovski and F. Loeser. Non-archimedean tame topology and stably dominated types. To appear in Ann. of Math. Stud.

[HMR] E. Hrushovski, B. Martin, and S. Rideau. Definable equivalence relations and zeta functions of groups. arXiv:math/0701011.

[Joh] W. Johnson. On the proof of elimination of imaginaries in algebraically closed valued fields. arXiv:1406.3654.

[Poi85] B. Poizat. *Cours de théorie des modèles*. Une introduction à la logique mathématique contemporaine. Nur al-Mantiq wal-Ma'rifah, 1985, pp. vi+584.

[Poi00] B. Poizat. *A Course in Model Theory*. Universitext. An introduction to contemporary mathematical logic, Translated from the French by Moses Klein and revised by the author. Springer-Verlag, New York, 2000, pp. xxxii+443.

[Rid] S. Rideau. Some properties of analytic difference fields. arXiv:1401.1765. To appear in J. Inst. Math. Jussieu.

[RS] S. Rideau and P. Simon. Definable and invariant types in enrichments of NIP theories. arXiv:1507.06863.

[Sca00] T. Scanlon. A model complete theory of valued D-fields. *J. Symbolic Logic* 65.4 (2000), pp. 1758–1784.

[Sca03] T. Scanlon. "Quantifier elimination for the relative Frobenius". In: *Valuation Theory and Its Applications, Conference proceedings of the International Conference on Valuation Theory (Saskatoon, 1999)*. Vol. II. Fields Inst. Commun. 33. Amer. Math. Soc., Providence, RI, 2003, pp. 323–352.

[Sim] P. Simon. A Guide to NIP Theories. To appear in Lect. Notes Log.