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Éliminations dans les corps valués

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Résumé

Cette thèse est une contribution à la théorie des modèles des corps valués. Les principaux résultats de ce texte sont des résultats d'éliminations des quantificateurs et des imaginaires.
Le premier chapitre contient une étude des imaginaires dans les extensions finies de $\mathbb{Q}_p$. On y démontre que ces corps ainsi que leurs ultraproduits éliminent les imaginaires dans le langage géométrique. On en déduit un résultat de rationalité uniforme pour les fonctions zêta associées aux familles de relations d'équivalences définissables dans les extensions finies de $\mathbb{Q}_p$.

La motivation première du deuxième chapitre est l'étude de $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ en tant que corps valué analytique de différence. Plus généralement, on démontre un théorème d'élimination des quantificateurs de corps dans le langage **RV** pour les corps valués analytiques σ-Henséliens de caractéristique nulle. On donne aussi une axiomatisation de la théorie de $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ ainsi qu'une preuve qu'elle est NIP.
Dans le troisième chapitre, on prouve la densité des types définissables dans certains enrichissements d'ACVF. On en déduit un critère pour l'élimination des imaginaires et la propriété d'extension invariante. Ce chapitre contient aussi des résultats abstraits sur les ensembles extérieurement définissables dans les théories NIP.
Dans le dernier chapitre, les résultats du chapitre précédent sont appliqués à $\mathrm{VDF}_{\mathcal{EC}}$, la modèle complétion des corps valués munis d'une dérivation qui préserve la valuation, pour obtenir l'élimination des imaginaires dans le langage géométrique ainsi que la densité des types définissables et la propriété d'extension invariante. Ce chapitre contient aussi des considérations sur les fonctions définissables, les types et les groupes définissables dans $\mathrm{VDF}_{\mathcal{EC}}$.

Mots-clefs : Théorie des modèles, corps valués, élimination des quantificateurs, élimination des imaginaires, structure analytique, dérivation, différence, métastabilité.

Eliminations in valued fields

Abstract

This thesis is about the model theory of valued fields. The main results in this text are eliminations of quantifiers and imaginaries.
The first chapter is concerned with imaginaries in finite extensions of $\mathbb{Q}_p$. I show that these fields and their ultraproducts eliminate imaginaries in the geometric language. As a corollary, I obtain the uniform rationality of zeta functions associated to families of equivalence relations that are definable in finite extensions of $\mathbb{Q}_p$.

The motivation for the second chapter is to study $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ as an analytic difference valued field. More generally, I show a field quantifier elimination theorem in the **RV**-language for σ-Henselian characteristic zero valued fields with an analytic structure. I also axiomatise the theory of $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ and I show that this theory is NIP.
In the third chapter, I prove the density of definable types in certain enrichments of ACVF. From this result, I deduce a criterion for the elimination of imaginaries and the invariant property. This chapter also contains abstract results on externally definable sets in NIP theories.
In the last chapter, the previous chapter is applied to $\mathrm{VDF}_{\mathcal{EC}}$, the model completion of valued fields with a valuation preserving derivation, to obtain the elimination of imaginaries in the geometric language, as well as the density of definable types and the invariant extension property. This chapter also contains considerations about definable functions, types and definable groupes in $\mathrm{VDF}_{\mathcal{EC}}$.

Keywords : Model theory, valued fields, elimination of quantifiers, elimination of imaginaries, analytic structure, derivation, difference, metastability.

Exterminate! Exterminate!
Any random Dalek

Plus il y a d'emmental, plus il y a de trous ;
plus il y a de trous, moins il y a d'emmental ;
donc plus il y a d'emmental, moins il y a d'emmental.
Sagesse populaire

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[2]I would also like to thank Tom for giving me the idea of adding analytic functions everywhere.

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Introduction

Le Logicien
Permettez moi de me présenter... Logicien professionnel : voici ma carte d'identité.
E. Ionesco, *Rhinocéros*, Acte I

Ce texte a pour sujet l'étude modèle théorique des corps valués. La théorie des modèles est l'étude des structures et des ensembles définissables au premier ordre dans ces structures. On distingue souvent la théorie des modèles pure, qui est l'étude abstraite de théories et de la combinatoire des ensembles définissables, et la théorie des modèles appliquée dont le but est d'étudier certaines structures concrètes provenant du reste des mathématiques ; ici, les corps valués. Mais la distinction est relativement arbitraire, les questions de théorie des modèles pure provenant souvent d'applications et les considérations plus « pures » permettant souvent de mieux comprendre les cas concrets qui nous intéressent.
De fait, ce texte est clairement un exemple de théorie des modèles appliquée mais certains des résultats présentés répondent à des préoccupations beaucoup plus « pures ».

Les corps valués et le principe d'Ax-Kochen-Eršov

Parmi les premiers résultats de théorie des modèles des corps valués, il y en a un en particulier qui préfigure le développement ultérieur du domaine : le théorème d'Ax-Kochen-Eršov [AK65 ; Erš65]. S'il serait déraisonnable de négliger l'aspect fondateur des travaux d'Abraham Robinson [Rob77] sur les corps valués algébriquement clos (ACVF), le théorème d'Ax-Kochen-Eršov, cependant, contient en germe un principe qui s'est révélée d'une extrême importance : un corps valué est une structure « contrôlée » d'une part par un groupe ordonné (son groupe de valeur) et d'autre part par un corps (son corps résiduel). Les corps de Hahn illustrent parfaitement cette idée : à partir de n'importe quel corps k et de n'importe quel groupe abélien ordonné Γ, on peut construire le corps valué $k((t^\Gamma))$ dont les éléments sont les séries formelles $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ à coefficient dans k telles que l'ensemble $\{\gamma \in \Gamma : a_\gamma \neq 0\}$ est bien ordonné. Le corps résiduel de $k((t^\Gamma))$ est exactement k et son groupe de valeur est exactement Γ.
Certes, tout corps valué K de groupe de valeur Γ et de corps résiduel k n'est pas isomorphe à $k((t^\Gamma))$. Mais, d'après le théorème d'Ax-Kochen-Eršov, s'il est Hensélien (c'est-à-dire que la valuation peut être étendue de façon unique à toute extension algébrique) et si k est de caractéristique nulle, la théorie au premier ordre de K est contrôlée par celle de k et Γ :

Théorème 0.1 (théorème d'Ax-Kochen-Eršov, 1965) **:**
Soient K et L des corps valués Henséliens d'équicaractéristique nulle. Les corps valués K et L sont élémentairement équivalents[3] si et seulement si leurs corps résiduels et leurs groupes de valeur le sont.

Donc, si K est Hensélien d'équicaractéristique nulle il est élémentairement équivalent à $k((t^{\Gamma}))$, où k est son corps résiduel et Γ est son groupe de valeur.
Le théorème d'Ax-Kochen-Eršov s'étend tel quel à la caractéristique mixte non ramifiée, c'est-à-dire quand K est de caractéristique nulle, k est de caractéristique p positive et $\mathrm{val}(p)$ est le plus petit élément strictement positif. Avec quelques complications supplémentaires, il s'étend aussi à la caractéristique mixte finiment ramifiée (quand $\mathrm{val}(p)$ est un multiple fini du plus petit élément strictement positif du groupe de valeur). Dans la suite, on dira plus généralement qu'un corps valué de caractéristique nulle est finiment ramifié si sa caractéristique résiduelle est nulle ou s'il est de caractéristique mixte finiment ramifiée.
L'un des corollaires les plus marquants du théorème d'Ax-Kochen-Eršov est que, si $\mathfrak{U}$ est un ultrafiltre non principal sur l'ensemble des nombres premiers, alors les deux ultraproduits $\prod_p \mathbb{Q}_p/\mathfrak{U}$ et $\prod_p \mathbb{F}_p((t))/\mathfrak{U}$ sont élémentairement équivalents en tant que corps valués. Autrement dit, un énoncé (du premier ordre) est vérifié dans tout $\mathbb{Q}_p$ pour p assez grand, si et seulement s'il est vérifié dans tout $\mathbb{F}_p((t))$ pour p assez grand. La motivation première d'Ax et Kochen lorsqu'ils ont prouvé ce principe de transfert était de répondre à la question suivante d'Artin : est-il vrai que tout polynôme sur $\mathbb{Q}_p$ homogène de degré d en au moins $d^2 + 1$ variables a une racine non triviale. D'après un théorème de Lang, ce résultat est vrai dans $\mathbb{F}_p((t))$, pour tout p. On en déduit donc que pour tout d il existe un nombre premier p_0 tel que la réponse à la question d'Artin est positive dans tout $\mathbb{Q}_p$ tel que $p \geqslant p_0$. On sait depuis que ce résultat est « optimal » car on connait des contre-exemples pour p petit.
Récemment ce type de résultats de transfert a été étendu, au-delà des langages du premier ordre, aux égalités d'intégrales p-adiques contenant des caractères additifs [CL10], par le biais de l'intégration motivique. Cela a permis, par exemple, à Cluckers, Hales et Loeser [CHL11] de transférer le lemme fondamental du programme de Langlands de la caractéristique positive à la caractéristique nulle.
L'idée sous-jacente au théorème d'Ax-Kochen-Eršov de contrôle d'un corps valué Hensélien par son groupe de valeur et son corps résiduel a de nombreux avatars dans la théorie des modèles des corps valués telle qu'elle a été développée depuis. L'un de ceux-ci est le résultat de Delon [Del81] (en équicaractéristique nulle) et Bélair [Bél99] (en caractéristique mixte finiment ramifiée) sur la « modération » des corps valués finiment ramifiés.
Avant d'expliciter ce résultat, il est nécessaire d'expliquer ce qu'on entend ici par modération. Dans ses travaux sur la classification, Shelah a introduit, dès les années 1970, un certain nombre de classes de théories plus ou moins modérées, liées à la présence (ou non) de configurations combinatoires dans leurs modèles. La première qu'il ait définie, et de loin la plus étudiée de ces classes, est celle des théories *stables* :

Définition 0.2 (Théorie stable) **:**
Soient $\varphi(x,y)$ une $\mathcal{L}$-formule et M une $\mathcal{L}$-structure. On dit que φ a la propriété de l'ordre dans M s'il existe des suites de uples $(a_i)_{i\in\mathbb{N}}$ et $(b_i)_{i\in\mathbb{N}} \in M$ tels que $M \vDash \varphi(a_i, b_j)$ si et seulement si $i \leqslant j$.
Une $\mathcal{L}$-théorie T est dite stable si aucune $\mathcal{L}$-formule n'a la propriété de l'ordre dans aucun modèle de T.

[3] C'est-à-dire tout énoncé du premier ordre dans le langage des corps valués vrai dans l'un des corps valués est vrai dans l'autre.

Cette condition, qui semble particulièrement combinatoire, a, en fait, un grand nombre de conséquences, qui font de la stabilité un « paradis » de la théorie des modèles dans lequel de nombreux outils ont été développés (les types définissables, la déviation, l'indépendance, etc). L'exemple le plus classique de théorie stable est celui des corps algébriquement clos (ACF) mais il en existe d'autres. Certaines n'ont que peu de contenu algébrique comme la théorie des ensembles infinis dans le langage de l'égalité ou encore la théorie d'une relation d'équivalence avec un nombre infini de classes infinies. Mais d'autres, les modules sur un anneau donné, les corps séparablement clos (SCF) ou encore les corps différentiellement clos de caractéristique nulle (DCF_0), sont des structures algébriques. Cependant, de nombreuses structures qui apparaissent en mathématiques sont instables, limitant de fait les applications possibles des résultats obtenus grâce à la stabilité.
Par exemple, le corps $\mathbb{R}$ (plus généralement les corps réels clos) ou les corps valués ne peuvent pas être stables puisqu'on y trouve un ordre définissable. Mais depuis la définition de la stabilité, d'autres notions de modération ont été introduites. Nous ne parlerons, ici, que d'une seule d'entres elles, compatible avec la présence d'un ordre définissable, la notion de théorie NIP. Cette notion avait également été définie par Shelah, mais avait été peu étudiée jusqu'au début des années 2000. Elle a fait depuis l'objet d'une attention croissante.

Définition 0.3 (Théorie NIP) **:**
Soient $\varphi(x, y)$ une $\mathcal{L}$-formule et M une $\mathcal{L}$-structure. On dit que φ a la propriété de l'indépendance dans M s'il existe des suites de uples $(a_i)_{i\in\mathbb{N}}$ et $(b_J)_{J\subseteq\mathbb{N}} \in M$ tels que $M \vDash \varphi(a_i, b_J)$ si et seulement si $i \in J$.
Une $\mathcal{L}$-théorie T est dite NIP (*not the independence property), ou dépendante, si aucune $\mathcal{L}$-formule n'a la propriété de l'indépendance dans aucun modèle de T.*

Parmi les théories NIP, on trouve un grand nombre d'exemples naturels. Il y a d'abord toutes les théories stables, ainsi que la théorie des corps réels clos (RCF) ou plus généralement toute théorie o-minimale, comme par exemple la théorie de $\mathbb{R}$ muni de l'exponentielle et des fonctions analytiques à support compact. On y trouve aussi de nombreuses théories de corps valués : la théorie des corps valués algébriquement clos, la théorie de $\mathbb{Q}_p$ et plus généralement la théorie de toute extension finie de $\mathbb{Q}_p$.
Ceci nous ramène au « principe d'Ax-Kochen-Eršov » relatif à la modération des corps valués que l'on a déjà mentionné :

Théorème 0.4 ([Del81 ; Bél99]) **:**
Soit T une théorie de corps valués Henséliens de caractéristique nulle finiment ramifiés. Alors T est NIP *si et seulement si la théorie induite sur le corps résiduel l'est.*

Ce théorème est une instance d'un principe plus général qui peut s'énoncer ainsi : un corps valué Hensélien de caractéristique nulle finiment ramifié n'est pas plus compliqué que ne le sont son groupe de valeur et son corps résiduel. Toute mention du groupe de valeur a disparu du théorème ci-dessus parce que, d'après un théorème de [GS84], tous les groupes abéliens ordonnés sont NIP.
Comme on l'a déjà mentionné, les théories stables ont de très bonnes propriétés. L'une d'entre elles est que tous les types sont définissables. Rappelons qu'un type sur M est un

ultrafiltre sur l'algèbre de Boole des ensembles définissables à paramètres dans M. Il est dit *définissable* si pour toute formule $\varphi(x;s)$ il existe une formule $\theta(s)$ telle que, pour tout m, $\theta(m)$ est vrai si et seulement si $\varphi(x;m) \in p$. De manière équivalente, dans tout modèle M d'une théorie stable, tout ensemble extérieurement définissable[4], est en fait définissable dans M (avec paramètres). Cette propriété caractérise la stabilité et ne peut donc pas être vraie dans toutes les théories NIP. Mais les ensembles extérieurement définissables ont toujours de très bonnes propriétés dans ces dernières. La principale d'entre elles est l'existence des définitions honnêtes (uniformes), introduites par Chernikov et Simon [CS13 ; CS] et qui feront leur apparition dans la Section III.1 de ce texte.

On considère souvent qu'une théorie NIP est contrôlée par une partie stable et une partie ordonnée. Dans le cas des corps valués, ce principe qui n'est pas sans rappeler celui d'Ax-Kochen-Eršov, peut être illustré précisément. Dans les corps locaux de caractéristique nulle[5], le corps résiduel étant fini, la structure est essentiellement contrôlée par le groupe de valeur. On aimerait alors dire que ces théories sont « purement instables », ce qui est formalisé par Simon [Sim13] sous le nom de distalité.

Pour ce qui est des corps valués algébriquement clos, il y a une partie ordonnée (le groupe de valeur Γ) et une partie extrêmement stable, le corps résiduel qui est algébriquement clos. Haskell, Hrushovski et Macpherson [HHM06 ; HHM08] ont formalisé cette intuition en introduisant la notion de *métastabilité* : pour tout uple a le type de a sur Γ est « dominé » par la partie stable. Dans [Hrub], Hrushovski utilise la métastabilité pour dévisser les groupes définissables dans ACVF à partir de groupes internes au groupe de valeur, de groupes internes au corps résiduel et de schémas en groupe sur l'anneau de valuation.

Remarquons enfin qu'une forme du principe d'Ax-Kochen-Eršov apparaît dans le cadre de l'intégration motivique, dont l'un des buts est de trouver des invariants additifs des variétés algébriques définies sur des corps valués. En particulier, en suivant Hrushovski et Kazhdan [HK06], on peut trouver un tel invariant en associant à une variété sa classe dans le semi-anneau de Grothendieck des définissables dans ACVF[6]. Ils démontrent alors que ce semi-anneau est essentiellement isomorphe au produit tensoriel du semi-anneau de Grothendieck du corps résiduel et de celui du groupe de valeur.

Élimination des quantificateurs

Le théorème d'Ax-Kochen-Eršov **(0.1)** est, en fait, une conséquence naturelle de résultats d'élimination des quantificateurs ; bien qu'Ax, Kochen et Eršov ne donnent jamais explicitement de tels résultats.

Plus généralement, la question de l'élimination des quantificateurs est une question fondamentale lorsqu'on étudie une théorie spécifique. En effet, la classe des ensembles définissables dans une structure donnée est obtenue à partir d'ensembles de base, déterminés

[4]C'est-à-dire les M-points d'un ensemble définissable avec paramètres dans une extension élémentaire de M.

[5]Ici, par corps local de caractéristique nulle, on veut dire corps local non-archimédien de caractéristique nulle, autrement dit une extension finie de $\mathbb{Q}_p$ pour un certain premier p.

[6]C'est-à-dire le semi-anneau des classes d'isomorphismes définissables d'ensembles définissables dans ACVF muni de la somme disjointe comme addition et du produit comme multiplication.

par le langage choisi, par des combinaisons booléennes finies et des projections. Mais les projections sont des opérations beaucoup plus compliquées que les opérations booléennes et il est idéal, pour avoir une bonne compréhension des ensembles définissables, de savoir que les opérations booléennes suffisent ; c'est exactement le contenu de l'élimination des quantificateurs. Par exemple dans le cas d'ACF, les ensembles des bases sont les variétés algébriques affines et les combinaisons booléennes d'ensembles de base sont les ensembles constructibles dans la topologie de Zariski. L'élimination des quantificateurs pour ACF est donc équivalente à un théorème de Chevalley selon lequel l'image par une fonction régulière d'un ensemble constructible est encore constructible.

Dans le cas des corps valués, il existe une grande diversité de résultats d'élimination des quantificateurs. Le premier d'entre eux est celui qui est essentiellement prouvé dans les travaux d'Abraham Robinson [Rob77] sur les corps valués algébriquement clos dans le langage le plus simple possible pour les corps valués[7]. Dans ce cas, comme dans beaucoup d'autres, ce résultat d'élimination des quantificateurs permet de démontrer des résultats de modération. En l'occurrence, il permet de démontrer que la théorie ACVF est C-minimale. La *C-minimalité* est une notion qui généralise à la fois la notion de forte minimalité et celle d'o-minimalité et qui comme toute notion de minimalité concerne les ensembles définissables en une variable. Dans les théories fortement minimales on demande que les ensembles définissables unaires soient les mêmes que dans l'ensemble infini sans structure (autrement dit ils sont soit finis soit cofinis). Dans les théories o-minimales, on demande que ces ensembles soient les mêmes que ceux définis sans quantificateurs dans un ordre (c'est-à-dire des unions finies d'intervalles). Enfin dans les théories C-minimales, on demande que ces ensembles soient les mêmes que ceux définis sans quantificateurs dans les feuilles d'un arbre. Dans le cas d'un corps valué, cet arbre est l'arbre des boules fermées et la C-minimalité requiert que tout ensemble définissable unaire soit une combinaison booléenne de boules. Une notion de modération n'est intéressante que parce qu'elle donne des outils avec lesquels travailler. Dans le cas de la C-minimalité, on obtient, par exemple, une notion de décomposition cellulaire ainsi qu'une notion, associée, de dimension.

Pour ce qui est de la théorie de $\mathbb{Q}_p$, on peut montrer que le langage de Robinson ne suffit pas pour éliminer les quantificateurs, autrement dit, il faut rajouter de nouveaux ensembles de base pour espérer éliminer les quantificateurs. Macintyre [Mac76] montre qu'il suffit de rajouter les ensembles de la forme $P_n(K) := \{x \in K : \exists y,\, x = y^n\}$ pour tout n.

Plus récemment, des résultats d'élimination des quantificateurs ont été démontré pour une classe de corps valués beaucoup plus large : les corps valués Henséliens de caractéristique nulle (non ramifiés). Ces résultats viennent essentiellement en deux grandes familles suivant les ensembles de base que l'on choisit : les résultats avec *composantes angulaires* et les résultats avec *termes dominants*. Pour suivre le développement historique du domaine, commençons par les composantes angulaires. Une *composante angulaire* d'un corps valué K est un morphisme de groupe $\mathrm{ac} : K^\star \to k^\star$ tel que la restriction de ac à $\mathcal{O}^\star$ coïncide avec le résidu $\mathrm{res} : \mathcal{O} \to k$, où $\mathcal{O} := \{x \in K : \mathrm{val}(x) \geqslant 0\}$ est l'anneau de valuation de K. On peut munir les séries de Laurent sur k (ou plus généralement sur tout corps de Hahn), les corps

[7]Les ensembles de base dans ce langage sont de la forme $\{x : \mathrm{val}(P(x)) \geqslant \mathrm{val}(Q(x))\}$, où P et Q sont des polynômes en plusieurs variables.

locaux et les vecteurs de Witt sur un corps parfait, d'une composante angulaire naturelle, mais tout corps valué ne peut pas être muni d'une composante angulaire. Toutefois, il suffit de supposer que le corps valué est $\aleph_1$-saturé pour en construire une.
D'après un théorème de Pas [Pas89], les corps valués Henséliens d'équicaractéristique nulle éliminent les quantificateurs de corps dans le langage (à trois sortes $\mathbf{K}$, $\mathbf{k}$ et $\mathbf{\Gamma}$) des corps valués muni d'une composante angulaire, aussi connu sous le nom de langage de Denef-Pas. Si l'on rajoute des « composantes angulaires d'ordre supérieur », ce théorème s'étend aux corps valués Henséliens de caractéristique mixte finiment ramifiés.
Le principal défaut des composantes angulaires est, qu'en général, même quand des composantes angulaires naturelles existent, elles ne sont pas définissables dans le langage des corps valués ; il existe quelques exceptions notables comme les corps locaux de caractéristique nulle où le système classique de « composantes angulaires d'ordre supérieur » est définissable. Les langages avec *termes dominants*, aussi connus sous le nom de congruences amc ou encore fonctions $\mathbf{RV}$, ont pour but de pallier ce manquement des composantes angulaires en considérant un langage qui ne rajoute aucun ensemble définissable sur le corps valué lui-même. On définit le groupe $\mathbf{RV} := \mathbf{K}^\star/(1+\mathfrak{M})$. Il existe une suite exacte courte $1 \to \mathbf{k}^\star \to \mathbf{RV} \to \mathbf{\Gamma} \to 0$ définissable dans tout corps valué. Il s'avère alors, par des travaux de Basarab et Kuhlmann [Bas91 ; BK92], qu'avec ces termes dominants, on peut éliminer les quantificateurs de corps dans les corps valués Henséliens d'équicaractéristique nulle. Comme précédemment, on obtient qu'en caractéristique mixte (possiblement infiniment ramifié), on peut éliminer les quantificateurs de corps, quitte à rajouter des « termes dominants d'ordre supérieur ».
Au cours des vingt-cinq dernières années, ces théorèmes d'élimination des quantificateurs ont été étendus à divers enrichissements des corps valués apparaissant naturellement en mathématiques. Tout d'abord, la plupart des corps valués que l'on considère, que ce soient des corps locaux ou des corps de séries formelles, sont complets[8]. Ils sont donc naturellement munis d'une structure analytique donnée par la spécialisation, sur leur domaine de convergence, de séries formelles. Les premiers résultats d'élimination des quantificateurs pour les corps valués munis d'une telle structure analytique remontent aux travaux de Denef et van den Dries [DD88] sur le corps $\mathbb{Q}_p$. Ils ont été suivis d'un grand nombre de travaux dans un cadre de plus en plus général, parmi lesquels on peut citer ceux de van den Dries, Haskell, Macpherson, Lipshitz, Robinson et Cluckers [Dri92 ; DHM99 ; LR00 ; LR05 ; CLR06 ; CL11]. Par exemple, dans [Dri92], van den Dries démontre l'élimination des quantificateurs de corps pour tout corps Hensélien d'équicaractéristique nulle avec structure analytique dans un langage avec composantes angulaires et en déduit un théorème d'Ax-Kochen-Eršov analytique.
L'autre enrichissement de corps valué qu'il est naturel de considérer consiste à rajouter un opérateur, que ce soit une dérivation ou un automorphisme de corps valué[9]. Les premiers résultats (construction de modèle-complétion et élimination des quantificateurs de corps) remontent à Scanlon [Sca00] où il étudie, sous certaines hypothèses techniques supplémentaires, les corps valués munis d'un opérateur D (notion qui généralise à la fois le cas

[8]Ceci étant, la complétude n'est pas une propriété exprimable au premier ordre.
[9]Un automorphisme de corps tel que $\sigma(\mathcal{O}) = \mathcal{O}$.

d'une dérivation et celui d'un automorphisme) qui préserve la valuation, c'est-à-dire tel que pour tout x, $\mathrm{val}(D(x)) \geqslant \mathrm{val}(x)$. Il montre en particulier l'existence de la modèle-complétion des corps valués différentiels dont la dérivation préserve la valuation ($\mathrm{VDF}_{\mathcal{EC}}$). L'étude de cette théorie a motivé les Chapitres III et IV de ce texte.

Dans des travaux ultérieurs, Bélair, Macintyre, Scanlon, Durhan et van den Dries [Sca03 ; BMS07 ; AD10] étendent les résultats d'élimination des quantificateurs de corps aux isométries, c'est-à-dire aux automorphismes de corps tels que pour tout x, $\mathrm{val}(\sigma(x)) = \mathrm{val}(x)$. Par ailleurs, des automorphismes qui ne sont pas des isométries, mais tels que pour tout x tel que $\mathrm{val}(x) > 0$ et tout $n \in \mathbb{N}$, $\mathrm{val}(\sigma(x)) > n\mathrm{val}(x)$, qu'on appelle des automorphismes ω-croissants, apparaissent dans les travaux de Hrushovski [Hrua] sur les approximations tordues de Lang-Weil. L'élimination des quantificateurs de corps pour les corps valués munis d'un tel automorphisme est démontrée par Durhan dans [Azg10]. Pal a ensuite étendu dans [Pal12] ces résultats à des automorphismes plus généraux et enfin Durhan et Onay [DO] ont récemment montré que ces résultats sont vérifiés pour un automorphisme quelconque de corps valué.

Le but du Chapitre II de ce texte est de réconcilier ces deux grandes tendances en étudiant les corps valués munis à la fois d'une structure analytique et d'un automorphisme de corps valué. On y démontre, entre autres choses, un résultat d'élimination des quantificateurs de corps pour les corps σ-Henséliens analytiques. La motivation première de ces résultats est l'étude modèle théorique du corps des vecteurs de Witt sur $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ muni à la fois de sa structure analytique et du relèvement du Frobenius. Une bonne connaissance de cette théorie peut permettre par exemple de donner une version modèle théorique de la géométrie p-différentielle de Buium et de traiter certains problèmes de géométrie diophantienne (voir [Sca06]).

Pour revenir aux corps valués munis d'une dérivation, deux autres cas relativement distincts ont été étudiés. Le premier, l'étude au long cours du corps des transséries (voir par exemple [ADH13]), a amené à considérer des corps valués différentiels qui ne sont pas D-Henséliens et dont la dérivation, bien que ne préservant pas la valuation, est continue. Le second est celui de corps valués différentiels dans lesquels il n'y a aucune interaction entre la valuation et la dérivation. En particulier, cette dérivation est discontinue (voir par exemple [GP10]). Mais, mis à part les techniques liées au fait qu'on étudie des corps valués, ces deux autres cas ont en fait peu de choses en commun avec le cas étudié par Scanlon ou avec le cas d'un automorphisme de corps valué qui lui est étroitement lié.

Élimination des imaginaires

Lorsque l'on veut étudier une théorie donnée, une fois que la question de l'élimination des quantificateurs a été résolue, une autre question se pose, celle de la description des quotients définissables. En effet, pour de multiples raisons, on est naturellement amené à considérer des relations d'équivalence définissables et donc les quotients par ces relations d'équivalence. Mais ces quotients, qu'on appelle les ensembles interprétables, sortent de la catégorie des ensembles définissables et on perd alors l'essentiel de l'information qu'on avait pu établir jusqu'ici sur les ensembles définissables.

Des relations d'équivalence définissables apparaissent de la façon suivante. Soient $X \subseteq Y \times Z$ des ensembles $\varnothing$-définissables dans une structure M. On peut voir X comme la famille $(X_y)_{y \in Y}$ des fibres de X sur Y. On peut alors se demander s'il existe une autre paramétrisation $(U_w)_{w \in W}$ de cette famille telle que chaque ensemble de la famille X apparaisse une et une seule fois dans la famille U. Cette notion, proche de celle d'un espace de modules, est fondamentale car elle permet d'identifier l'ensemble des X_y avec l'ensemble définissable W. On dit alors que le point de W qui correspond à une fibre X_y de X est un *paramètre canonique* de X. Ce paramètre canonique est alors le « plus petit » ensemble au-dessus duquel X_y peut être défini ; un ensemble dont l'existence n'est pas assurée dans une théorie quelconque.
On peut construire W et U de manière « canonique » en considérant la relation d'équivalence $y_1 E y_2$ sur Y définie par $\forall z, z \in X_{y_1} \iff z \in X_{y_2}$ et en prenant $W = Y/E$ et $U = \{(z, \widehat{y}) : x \in X_y\}$ où $\widehat{y}$ est la E-classe de y. Mais les ensembles W et U sont interprétables et non définissables. Une telle construction n'est possible à l'intérieur de M que si les quotients interprétables dans M sont représentables par des ensembles définissables. C'est exactement cette propriété que l'on nomme l'*élimination des imaginaires* dans M et qui est équivalente à l'existence, dans M, d'un plus petit ensemble de définition pour tout ensemble définissable.
La notion d'élimination des imaginaires, introduite par Poizat dans [Poi83] est beaucoup plus récente que celle d'élimination des quantificateurs, qui a été fondamentale, pendant de longues années, dans toutes les applications de la théorie des modèles. Il n'est donc guère surprenant qu'il existe moins de résultats d'élimination des imaginaires. Parmi les exemples de théories éliminant les imaginaires, on trouve, tout d'abord, les deux théories qui ont motivé la définition de cette notion : ACF et DCF_0. L'élimination des imaginaires dans ces théories découle de l'existence, pour toute variété algébrique (respectivement différentielle), d'un plus petit corps (respectivement différentiel) de définition et de la définissabilité des polynômes symétriques. Parmi les autres exemples notables, on trouve aussi les groupes divisibles ordonnés, les corps réels clos (c'est-à-dire les corps élémentairement équivalents à $\mathbb{R}$) et plus généralement toutes les théories o-minimales de groupes ordonnés.
En général, pour une théorie qui n'élimine pas les imaginaires, la question est de savoir quels sont les paramètres canoniques d'ensembles définissables qu'il suffit de rajouter aux modèles de la théorie pour obtenir l'élimination des imaginaires. En rajoutant tous les paramètres canoniques de tous les ensembles définissables dans une théorie T, on obtient la théorie T^{eq} (dont les modèles seront notés M^{eq}) définie par Shelah qui élimine les imaginaires. Mais cette construction, si elle est très utile d'un point de vue abstrait, n'a que peu d'intérêt pour une théorie donnée car elle ne fournit aucune information sur les ensembles interprétables. Ce que l'on recherche dans les résultats d'élimination des imaginaires c'est une description « concrète » des ensembles interprétables. Par exemple, on peut vérifier que la théorie des ensembles infinis sans structure dans le langage contenant seulement l'égalité n'élimine pas les imaginaires ; en effet, à part les singletons, aucun ensemble fini n'a de paramètre canonique. Mais il suffit de rajouter, pour tout $n \in \mathbb{N}_{>0}$, une sorte S_n dont les éléments sont les sous-ensembles à n éléments de la sorte principale.
Considérons maintenant les corps valués algébriquement clos dans le langage avec une seule sorte pour le corps valué lui-même. Il découle de l'élimination des quantificateurs que

tout ensemble infini définissable est de même cardinalité que le corps valué. Or, il existe des corps valués algébriquement clos de cardinal $2^{\aleph_0}$ dont le groupe de valeur $\mathbf{\Gamma}$ est dénombrable. Il est donc impossible que l'ensemble interprétable $\mathbf{\Gamma} = \mathbf{K}^\star/\mathcal{O}^\star$ soit en bijection avec un sous-ensemble définissable du corps. Des contre-exemples plus élaborés permettent de montrer qu'il ne suffit pas de rajouter le groupe de valeur, le corps résiduel ni même l'ensemble des boules du corps valué pour obtenir l'élimination des imaginaires. Il en est de même pour les corps locaux de caractéristique nulle.

Le premier résultat d'élimination des imaginaires pour des corps valués a été démontré par Haskell, Hrushovski et Macpherson dans [HHM06]. Ils démontrent que dans les corps valués algébriquement clos, il suffit de rajouter les paramètres canoniques de certains ensembles, qui peuvent être vus comme des équivalents en dimension supérieure des boules. Ces ensembles sont, d'une part, les sous-$\mathcal{O}$-modules libres de rang n de $\mathbf{K}^n$, aussi connus sous le nom de réseaux et, d'autre part, pour tout réseau s, les ensembles de la forme $a + \mathfrak{M}s$ où $a \in s$. Le langage dans lequel on a rajouté des paramètres canoniques pour ces deux familles d'ensembles est connu sous le nom de *langage géométrique*.

Il a été démontré depuis par Mellor [Mel06] que les corps valués réels clos éliminent aussi les imaginaires dans le langage géométrique. Dans le Chapitre I de ce texte, on démontre de nouveaux résultats d'élimination des imaginaires dans le langage géométrique : d'une part pour les corps locaux de caractéristique nulle et d'autre part pour les corps pseudo-locaux (c'est-à-dire les ultraproduits de corps locaux) d'équicaractéristique nulle.

Pour ce qui est des corps valués enrichis, les résultats sont encore plus rares. Dans le cas des corps valués algébriquement clos avec structure analytique, il est démontré dans [HHM13] qu'ils n'éliminent pas les imaginaires dans le langage géométrique. Mais aucune description concrète des paramètres canoniques qu'il faudrait rajouter n'est connue. Dans ce texte, on prouve un résultat pour l'autre famille d'enrichissements décrite plus tôt, les corps valués avec opérateurs. Plus précisément, le Chapitre III contient des considérations générales sur les imaginaires dans les enrichissements d'ACVF qui sont appliqués dans le Chapitre IV pour démontrer que $\mathrm{VDF}_{\mathcal{EC}}$ élimine les imaginaires dans le langage géométrique augmenté d'un symbole pour la dérivation.

Nous avons remarqué précédemment que les résultats d'élimination des quantificateurs permettent de démontrer des résultats de modération. Pour ce qui est de l'élimination des imaginaires, c'est plutôt le contraire : la modération d'une théorie peut aider à comprendre ses imaginaires. Par exemple, la stabilité des théories ACF, DCF_0 et SCF joue un rôle non négligeable dans les preuves d'élimination des imaginaires. De même, la théorie des types stablement dominés et de la métastabilité a été développée pour étudier les imaginaires dans ACVF. Dans une preuve plus récente [Hru14 ; Joh] de l'élimination des imaginaires dans ACVF dans le langage géométrique , la densité des types définissables joue un rôle central. Enfin, dans la preuve de l'élimination des imaginaires pour $\mathrm{VDF}_{\mathcal{EC}}$ dans le langage géométrique présentée dans ce texte, les types définissables sont tout aussi importants, mais un autre ingrédient central de la preuve est le bon comportement, dans les théories NIP, des ensembles extérieurement définissables.

Après l'élimination

Une fois que l'on connait l'élimination des quantificateurs et des imaginaires pour une théorie donnée, tout un champ de possibilités s'ouvre à nous, et il n'y a plus vraiment d'étape suivante « canonique ». Par exemple, pour ce qui est d'ACVF, Hrushovski et Loeser [HL] étudient l'espace des types stablement dominés et montrent qu'il permet, non seulement, de redéfinir la notion d'espace de Berkovich mais aussi de prouver de nouveaux résultats sur leur topologie. Plus classiquement, on peut vouloir étudier les groupes et les corps définissables dans une théorie : les classifier mais aussi montrer comment certaines propriétés géométriques des ensembles définissables permettent d'en construire. Ces considérations sont centrales à la théorie dite géométrique de la stabilité et, appliquées à DCF_0 et SCF, ont permis à Hrushovski [Hru96] de démontrer certains résultats diophantiens dont la conjecture de Mordell-Lang pour les corps de fonctions.
Dans les corps algébriquement clos, d'après un résultat de Weil, ou plus précisément l'interprétation modèle théorique de ce résultat par Hrushovski (cf. [Poi87, Section 4.e]) ou van den Dries [Dri90], tout groupe définissable est définissablement isomorphe à un groupe algébrique. D'après un résultat de Bouscaren et Delon [BD01], les groupes définissables dans un corps K séparablement clos de degré d'imperfection fini sont définissablement isomorphes aux K-points d'un groupe algébrique. Dans les corps pseudo-finis, d'après Hrushovski et Pillay [HP94], la situation est similaire, à certains groupes finis près : tout groupe définissable dans un corps pseudo fini K contient un sous-groupe définissable d'indice fini, isogène aux K-points d'un groupe algébrique. Dans les corps différentiellement clos de caractéristique nulle, d'après un théorème de Pillay [Pil97], tout groupe définissable se plonge définissablement dans un groupe algébrique. Enfin, dans les corps algébriquement clos avec un automorphisme générique, Chatzidakis et Hrushovski [CH99], dans le cas de rang fini, puis Kowalski et Pillay [KP02], dans le cas général, montrent un résultat similaire à des groupes finis près. La Section IV.5 de ce texte contient les premiers pas d'une telle classification des groupes définissables dans les corps valués différentiels à la Scanlon.

Aperçu des résultats

Les principaux résultats de ce texte sont des résultats d'élimination des quantificateurs et des imaginaires dans les corps valués, des purs corps valués comme les corps locaux de caractéristique nulle ou les corps pseudo-locaux d'équicaractéristique nulle, et des corps valués enrichis par une dérivation, un automorphisme ou encore une structure analytique.
Le premier chapitre, écrit avec Ehud Hrushovski et Ben Martin, contient [HMR]. Ses principaux résultats modèles théoriques sont l'élimination des imaginaires dans le langage géométrique pour les corps locaux de caractéristique nulle (Théorème **A**) et pour les corps pseudo-locaux d'équicaractéristique nulle (Théorème **B**). On en déduit alors, dans le Corollaire **(I.2.7)** que l'élimination des imaginaires dans les corps locaux de caractéristique nulle est uniforme quand la caractéristique résiduelle est assez grande.
Ces résultats d'élimination sont ensuite appliqués pour compter des classes dans une famille de relations d'équivalence paramètrées par le groupe de valeur et définissables dans un corps local de caractéristique nulle. En utilisant des techniques, développées entre autres

par Denef, qui relient ces problèmes à des calculs d'intégrales p-adiques, on démontre le Théorème **C** selon lequel les fonctions zêta associées à ces problèmes de comptage sont uniformément rationnelles. Ce chapitre contient aussi les applications du Théorème **C** à l'étude de certaines fonctions zêta qui apparaissent en théorie de la croissance des sous-groupes et des représentations.

Dans le Chapitre II, on prouve un résultat (Théorème **D**) d'élimination des quantificateurs de corps dans des corps valués avec *à la fois* une structure analytique et un automorphisme. Bien que la motivation première ait été d'étudier le corps des vecteurs de Witt sur $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ muni de sa structure analytique et du relèvement du Frobenius (qui est une isométrie), le Théorème **D** est prouvé pour *tous* les automorphismes de corps valués. Ce chapitre tente aussi, dans les Sections II.A et II.B, de donner une approche plus systématique des preuves d'élimination des quantificateurs dans les corps valués enrichis, par le biais de certaines considérations plus abstraites.

Le sujet du Chapitre III est l'élimination des imaginaires dans certains enrichissements d'ACVF. La théorie $\mathrm{VDF}_{\mathcal{EC}}$ est le principal exemple motivant cette section. Plus précisément, on prouve un résultat de densité des types définissables (Théorème **E**) qui implique à la fois l'éliminations des imaginaires et l'existence d'extensions globales invariantes. Ce chapitre commence, dans la Section III.1, par un travail en commun avec Pierre Simon, que je remercie d'accepter que je présente ici ces résultats, sur des considérations plus abstraites à propos des ensembles extérieurement définissables dans les théories NIP. Ces considérations sont essentielles pour démontrer les autres résultats de ce chapitre.

Enfin, le Chapitre IV contient plusieurs résultats sur $\mathrm{VDF}_{\mathcal{EC}}$. Le principal (Théorème **F**) est l'élimination des imaginaires et la propriété d'extension invariante pour $\mathrm{VDF}_{\mathcal{EC}}$ dans le langage géométrique. C'est une application directe des résultats du chapitre précédent. On montre aussi que le corps des constantes est stablement plongé dans les modèles de $\mathrm{VDF}_{\mathcal{EC}}$, et on étudie les clôtures définissables et algébriques. Le lien entre les types dans $\mathrm{VDF}_{\mathcal{EC}}$ et ceux dans ACVF est formalisé par la définition d'un analogue des prolongations de la géométrie algébrique différentielle. Enfin, la dernière section de ce chapitre est consacrée à l'étude des groupes définissables dans $\mathrm{VDF}_{\mathcal{EC}}$ et leurs liens avec les groupes définissables dans ACVF.

Description détaillée des résultats

Chapitre I. Imaginaires dans les corps p-adiques : Outre leur intérêt intrinsèque pour les théoriciens des modèles, la principale motivation derrière les résultats de ce chapitre est l'étude de certaines fonctions de comptage en théorie des groupes. Commençons donc par un petit historique de ces questions de comptage et de leur lien à la théorie des modèles.
Commençons par les séries de Poincaré associées aux points p-adiques de variétés. Soit X une variété affine définie sur $\mathbb{Z}_p$. On appelle a_n le nombre de ces $\mathbb{Z}/p^n\mathbb{Z}$-points et $\widetilde{a}_n$ le nombre de points dans l'image de la fonction canonique $X(\mathbb{Z}_p) \to X(\mathbb{Z}/p^n\mathbb{Z})$. Pour étudier la croissance de ces deux suites, on définit les deux séries de Poincaré $P_X(t) :=$

$\sum_i a_i t^i$ et $\widetilde{P}_X(t) := \sum_i \widetilde{a}_i t^i$. La principale question concernant ces séries est celle de leur rationalité. Dans le cas de $P_X(t)$, cela est démontré par Igusa dans les années 1970. Pour ce qui est de $\widetilde{P}_X(t)$, la présence d'une image (et donc d'un quantificateur) rend les choses plus compliquées mais Denef démontre sa rationalité dans [Den84] en utilisant l'élimination des quantificateurs pour $\mathbb{Q}_p$ prouvée par Macintyre [Mac76].
Dans sa preuve, Denef considère les ensembles $A_n := \{x \in \mathbb{Z}_p : x \text{ est dans } X \text{ modulo } p^n\mathbb{Z}_p\}$ et $\widetilde{A}_n := \{x \in \mathbb{Z}_p : \text{il existe } y \in X(\mathbb{Z}_p) \text{ tel que } x \equiv y \mod p^n\mathbb{Z}_p\}$, qui sont définissables dans $\mathbb{Q}_p$, et prouve que $a_n = \mu(A_n)p^{nd}$ et $\widetilde{a}_n = \mu(\widetilde{A_n})p^{nd}$ où $d \in \mathbb{N}$ dépend de X et μ est la mesure de Haar sur $\mathbb{Q}_p$ telle que $\mu(\mathbb{Z}_p) = 1$. Il montre ensuite que la $\mathbb{Q}$-algèbre des fonctions engendrée par les fonctions $f : \mathbb{Q}_p \times \mathbb{Z} \to \mathbb{Z}$ et $x \mapsto p^{f(x)}$ où f est définissable dans $\mathbb{Q}_p$ est close par intégration. Le théorème de rationalité suit alors facilement. En utilisant un théorème d'élimination des quantificateurs pour $\mathbb{Q}_p$ uniforme, Pas [Pas91] et Macintyre [Mac90] démontrent que, quand X est définie sur $\mathbb{Z}$, la rationalité de ces séries est uniforme.
Dans [GSS88], Grunewald, Segal et Smith utilisent le résultat de Denef sur la définissabilité des intégrales p-adiques pour traiter un problème de comptage qui vient directement de la théorie des groupes. Soit G un groupe nilpotent, finiment engendré, sans torsion. Le nombre de sous-groupes de G d'indice n est fini ; on le note b_n. On trouve alors une famille, définissable dans $\mathbb{Q}_p$, de fonctions paramètrées par le groupe de valeur telle que l'intégrale de la fonction d'indice n est exactement b_{p^n}. En utilisant le résultat de Denef, on obtient alors la rationalité de la série $\zeta_{G,p}(t) := \sum_i b_{p^i} t^i$. Par des considérations similaires à celles de Pas et Macintyre, ou plus généralement la spécialisation p-adique de résultats d'intégration motivique, on obtient aussi des informations sur le comportement quand p varie. L'un des intérêts de ces résultats uniformes en p est que l'on peut définir la fonction zêta globale associée à G, $\zeta_G(s) = \sum_i b_i i^{-s}$ et qu'on a, comme pour la fonction zêta de Riemann qui n'est autre que $\zeta_{\mathbb{Z}}$, une décomposition d'Euler en fonctions zêta locales :

$$\zeta_G(s) = \prod_p \zeta_{G,p}(p^{-s}).$$

Une compréhension uniforme des fonctions zêta locales peut donc permettre de comprendre la fonction zêta globale.
La méthode pour associer à b_{p^n} une fonction définissable dans $\mathbb{Q}_p$ consiste à trouver, uniformément en n, un ensemble D_n/E_n interprétable dans $\mathbb{Q}_p$ qui soit en bijection avec l'ensemble des sous groupes de G d'indice p^n et à trouver une fonction f_n définissable dans $\mathbb{Q}_p$ telle que pour tout $x \in D_n$, si $\widehat{x}^{E_n}$ dénote la E_n-classe de x, $\mu(\widehat{x}^{E_n}) = p^{-\mathrm{val}(f_n(x))}$. Ainsi le résultat de Grunewald, Segal et Smith, tout comme celui de Denef sur les séries de Poincaré, peut se voir comme une instance d'un problème plus général qui consiste à compter les classes dans une famille $(E_n)_{n\in\mathbb{N}}$ de relations d'équivalence, définissable dans $\mathbb{Q}_p$. Dans le cas des résultats que l'on a déjà mentionnés, les relations d'équivalences en question sont suffisamment simples pour qu'on puisse trouver les fonctions f_n de façon explicite. Cependant, dans le cas général, tant qu'on ne sait pas exactement quelle est la forme des ensembles interprétables, il est impossible de trouver de tels f_n, d'où la nécessité de prouver un résultat d'élimination des imaginaires pour $\mathbb{Q}_p$, de préférence uniforme en p, pour pouvoir en déduire des informations sur les fonctions zêta globales.

Dans ce chapitre on prouve deux théorèmes d'élimination des imaginaires. Un théorème sur les corps locaux :

Théorème A :

Soit K un corps local de caractéristique mixte $(0,p)$. Si on rajoute une constante pour un générateur de $K \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$ sur $\mathbb{Q}_p \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$, alors la théorie de K dans le langage géométrique élimine les imaginaires.

Et un théorème sur leurs ultraproduits :

Théorème B :

La théorie des corps pseudo-locaux de caractéristique résiduelle nulle élimine les imaginaires dans le langage géométrique, si on rajoute une infinité de constantes.

Dans les deux cas, comme la valuation est discrète, les paramètres canoniques d'ensembles de la forme $a + \mathfrak{M}s$, où s est un réseau, ne sont, en fait, pas nécessaires. On déduit du second théorème, dans le Corollaire **(I.2.7)**, que l'élimination des imaginaires pour les corps locaux est uniforme quand la caractéristique résiduelle est grande.
Il s'en suit que tout ensemble interprétable dans un corps local K de caractéristique nulle peut être identifié avec un sous-ensemble définissable de $K^n \times (\mathrm{GL}_m(K)/\mathrm{GL}_m(\mathcal{O}))$ pour certains n et $m \in \mathbb{N}$, uniformément en p. On peut alors trouver assez facilement les fonctions f_n telles que précédemment, en utilisant le fait que la mesure de Haar sur $\mathrm{GL}_m(K)$ a une densité relative à la mesure de Haar sur K^{n^2} qui est une fonction définissable, et en déduire un résultat (Théorème **C**) de rationalité uniforme pour toute famille de relations d'équivalence uniformément définissables dans les corps locaux de caractéristique nulle.
Ce théorème permet, non seulement, de retrouver plusieurs résultats déjà connus de rationalité pour des fonctions zêta locales issues de la théorie des groupes, mais aussi d'en prouver de nouveaux pour lesquels les relations d'équivalence qui rentrent en jeu sont plus compliquées, comme par exemple le comptage des représentations à « isotwist » près.
Pour ce qui est des résultats d'élimination des imaginaires eux-mêmes, ils sont prouvés en utilisant des critères abstraits, Proposition **(I.2.11)** et Corollaire **(I.2.15)**, qui permettent de transférer un résultat d'élimination des imaginaires d'une certaine théorie, ici ACVF, à une autre. Dans ce chapitre, le transfert est rendu possible par le fait que les fonctions définissables dans les théories, pour lesquelles on veut éliminer les imaginaires, peuvent être recouvertes par des correspondances, définissables dans ACVF, qui, à chaque point, associent un nombre fini de points. L'existence d'extensions invariantes des types sur de petits ensembles de paramètres algébriquement clos ainsi qu'une description complète de tous les 1-types jouent aussi un rôle central.

Chapitre II. Corps analytiques de différence : Comme mentionné précédemment, ce chapitre a pour but l'étude des corps valués enrichis à la fois par une structure analytique et un automorphisme de corps valué. L'étude de ces structures n'est cependant pas seulement motivée par l'envie de comprendre les interactions entre deux types d'enrichissement de corps valués que l'on comprend bien et de donner un traitement modèle théorique de

structures de plus en plus complexes, mais aussi parce que c'est un cadre modèle théorique qui permet de comprendre certains problèmes de géométrie diophantienne et de théorie des nombres. Par exemple, c'est le cadre dans lequel traiter la géométrie p-différentielle de Buium. En effet, toute fonction p-différentielle définie sur $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ est une fonction définissable dans la structure $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ munie du relèvement du Frobenius, ainsi que des symboles pour toutes les fonctions analytiques p-adiques $\sum_I a_I X^I$ où $a_I \to 0$ quand $|I| \to \infty$. Dans [Sca06], Scanlon montre comment une bonne connaissance de la théorie des modèles des corps analytiques de différence peut permettre de montrer que certains résultats diophantiens sont uniformes. En particulier, il montre qu'on peut déduire une version en famille de la conjecture de Manin-Munford pour les variétés abéliennes sur $\mathrm{W}[k]$ à partir de la conjecture classique (c'est-à-dire du théorème de Raynaud), en utilisant, d'une part des résultats de Buium sur le lien entre ces questions et la géométrie p-différentielle et, d'autre part un théorème d'Ax-Kochen-Eršov pour les corps analytiques de différence.
Le résultat principal de ce chapitre est un résultat d'élimination des quantificateurs de corps pour une certaine classe de corps valués analytiques de différence :

Théorème D :

La théorie des corps valués σ-Henséliens de caractéristique nulle avec structure analytique élimine les quantificateurs de corps dans le langage avec des termes dominants

Dans le cas des corps valués munis de structure analytique, l'idée sur laquelle repose en grande partie les résultats de théorie des modèles est que, par le biais de la préparation de Weierstrass, la valuation, et même le terme dominant, d'une fonction analytique en une variable est en fait donnée par celle, ou celui, d'un polynôme. On peut alors démontrer que tout ensemble définissable unidimensionnel du corps valué est déjà définissable dans le langage des corps valués (avec terme dominant). Dans le cas de la preuve originale de Denef et van den Dries, l'une des complications est liée au fait qu'ils disposent de la préparation de Weierstrass uniquement pour les fonctions analytiques mais pas pour tous les termes du langage (qui font potentiellement intervenir l'inverse). On utilise ici une préparation de Weierstrass qui fonctionne pour tous les termes, prouvée par Cluckers et Lipshitz [CL11].
Par ailleurs, dans le cas des corps valués de différence sans structure analytique, le cœur des preuves consiste, tout d'abord, à adapter la notion d'Hensélianité au cas des polynômes de différence pour obtenir la notion de σ-Hensélianité, ainsi qu'à généraliser l'étude des 1-types utilisée pour démontrer l'élimination des quantificateurs dans le cas de corps valués sans opérateur, à l'étude de certains n-types spécifiques de corps valué qui correspondent au type de $(x, \sigma(x), \dots, \sigma^{n-1}(x))$ pour des éléments $x \in \mathbf{K}$ bien choisis. Ce dernier point apparaît relativement explicitement dans la Section II.6.
La preuve du Théorème **D** consiste à marier ces deux approches. Dans [Sca06], Scanlon avait déjà tenté de le faire dans le cas d'une isométrie mais la notion de σ-Hensélianité qu'il utilise est trop faible, bien que cela soit caché par des erreurs dans certains calculs. L'axiomatisation et les preuves ont du être entièrement refaites (et ont été généralisée au cas d'un automorphisme de corps valué quelconque) mais certaines idées de cet article restent fondamentales dans l'approche présentée ici.
Pour mener à bien ce mélange, il y a plusieurs difficultés. La première est que la notion

habituelle de σ-Hensélianité, telle que définie dans [BMS07] par exemple, n'est vraiment utilisable que si l'on considère des termes dont le développement de Taylor est fini, autrement dit des polynômes. L'autre difficulté est liée au fait que, comme expliqué ci-dessus, l'élimination des quantificateurs en présence d'un automorphisme nécessite d'étudier certains n-types de corps valués analytiques, mais que la technique principale utilisée dans le cadre analytique ne nous renseigne vraiment que sur les 1-types. L'ingrédient principal pour surmonter ces problèmes est de considérer les propriétés différentielles des termes, ce qui permet, entre autre, de définir une nouvelle notion de σ-Hensélianité adaptée au cadre analytique de différence.
Vu que c'est notre exemple principal, on donne aussi dans ce chapitre une axiomatisation de la théorie du corps $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ muni de sa structure analytique et du relèvement du Frobenius (dorénavant appelé W_p) ainsi qu'un résultat de modération :

0.5 (Proposition **(II.6.31)** et Corollaire **(II.7.5)**) **:**
La théorie de W_p est axiomatisée par le fait que c'est un corps valué σ-Hensélien de caractéristique mixte non ramifié avec structure analytique, que le corps résiduel est algébriquement clos, que l'automorphisme induit sur le corps résiduel est le Frobenius, que le groupe de valeur est élémentairement équivalent à $\mathbb{Z}$ et que l'automorphisme induit sur le groupe de valeur est l'identité. De plus la théorie de W_p est NIP.

Enfin, ce chapitre se veut aussi une approche systématique de certains faits bien connus autour de la question de l'élimination des quantificateurs dans les corps valués enrichis mais qui sont, en général, redémontrés dans chaque cas spécifique. Tout d'abord, pour montrer l'élimination en caractéristique mixte il suffit de le faire en équicaractéristique nulle. Ensuite, les résultats avec composantes angulaires découlent des résultats avec termes dominants. Enfin, les résultats d'élimination des quantificateurs de corps sont *resplendissants*, c'est-à-dire que l'on peut enrichir arbitrairement les sortes autres que le corps valué sans perdre l'élimination des quantificateurs de corps. Les Sections II.A et II.B contiennent donc des considérations plus abstraites qui permettent de prouver ces trois faits dans la plupart des enrichissements de corps valués que l'on pourrait vouloir étudier.

Chapitre III. Imaginaires dans certains enrichissements de ACVF **:** La motivation principale de ce chapitre est la résolution de la question de la propriété d'extension invariante dans la théorie $\mathrm{VDF}_{\mathcal{EC}}$. Comme nous l'avons déjà mentionné, lorsqu'ils étudiaient les imaginaires dans ACVF, Haskell, Hrushovski et Macpherson [HHM06] ont développé la notion de type stablement dominé, notion qui a ensuite été étudiée en détail dans [HHM08] et a amené à la définition de la métastabilité. L'exemple emblématique de théorie métastable est ACVF mais il serait intéressant de disposer d'autres exemples dans laquelle la partie stable est plus compliquée que ne l'est ACF. La théorie $\mathrm{VDF}_{\mathcal{EC}}$ est un bon candidat.
L'un des problèmes liés aux types stablement dominés est qu'on ne sait prouver qu'une forme de descente[10] relativement compliquée, qui suppose l'existence d'extensions globales invariantes, ce qu'on appelle communément la *propriété d'extension invariante*. C'est pour

[10] Le fait que tout type global stablement dominé sur B et invariant sur $C \subseteq B$ est stablement dominé sur C.

cette raison que la définition de la métastabilité donnée par Haskell, Hrushovski et Macpherson ne contient pas uniquement une propriété qui traduit la présence de nombreux types stablement dominés mais aussi la propriété d'extension invariante.
Pour prouver que $\mathrm{VDF}_{\mathcal{EC}}$ est aussi un exemple de métastabilité et pouvoir, par exemple, y étudier les groupes comme le fait Hrushovski pour ACVF dans [Hrub], il faut donc d'abord démontrer la propriété d'extension invariante dans $\mathrm{VDF}_{\mathcal{EC}}$. Cette propriété n'étant pas sans rapport avec les imaginaires, la question de leur élimination se pose donc aussi naturellement. Par analogie avec DCF_0, il est raisonnable de penser que $\mathrm{VDF}_{\mathcal{EC}}$ n'a pas plus d'imaginaires qu'ACVF et donc qu'elle élimine les imaginaires dans le langage géométrique.
Les techniques développées dans ce chapitre pour étudier $\mathrm{VDF}_{\mathcal{EC}}$ s'adaptent à un cadre plus général. L'un des résultats que l'on obtient est un critère abstrait pour l'élimination des imaginaires et la propriété d'extension invariante dans certains enrichissements de ACVF. On démontre au Chapitre IV, que ce critère s'applique à $\mathrm{VDF}_{\mathcal{EC}}$. La raison pour laquelle on se place dans un cadre aussi général est qu'il est vraisemblable que ces techniques s'avèreront utiles dans d'autres cas que $\mathrm{VDF}_{\mathcal{EC}}$, par exemple $\mathrm{VDF}_{\mathcal{EC}}$ avec une structure analytique ou bien les vecteurs de Witt sur $\overline{\mathbb{F}_p}^{\mathrm{alg}}$.
Comme c'est le cas dans la version de la preuve de l'élimination des imaginaires dans ACVF [Hru14 ; Joh], les types définissables jouent un rôle central dans l'élaboration de ce critère. En fait, le principal résultat de ce chapitre (Théorème **E**) est un résultat de densité des types définissables dont la base canonique est contrôlée. Pour la théorie $\mathrm{VDF}_{\mathcal{EC}}$ dans le langage géométrique, ce résultat peut s'énoncer ainsi :

Théorème 0.6 (Théorème **E** dans le cas de $\mathrm{VDF}_{\mathcal{EC}}$) **:**
Soient $M \models \mathrm{VDF}_{\mathcal{EC}}$, $A \subseteq M^{\mathrm{eq}}$ définissablement clos et X un ensemble A-définissable, alors il existe un type définissable p qui est $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant et consistant avec X, où par $\mathcal{G}(A)$ on entend l'ensemble des points de A qui sont dans les sortes du langage géométrique.

L'un des ingrédients qui joue un rôle déterminant dans le contrôle de la base canonique des types que l'on construit dans ce chapitre est le résultat abstrait prouvé dans la Section III.1 à propos des ensembles extérieurement définissables dans les théories NIP, et qui a pour corollaire l'énoncé suivant :

Théorème 0.7 (Théorème **(III.1.4)** dans le cas de $\mathrm{VDF}_{\mathcal{EC}}$) **:**
Soient $M \models \mathrm{VDF}_{\mathcal{EC}}$, $A \subseteq M$ définissablement clos. Si X est un ensemble A-définissable au sens de $\mathrm{VDF}_{\mathcal{EC}}$ qui est aussi extérieurement définissable au sens d'ACVF, alors il est $\mathcal{G}(A)$-définissable au sens de ACVF.

Cela peut se reformuler de la manière suivante : soit p un type de ACVF, supposons qu'il existe un schéma de définition pour p qui soit composé de formules de $\mathrm{VDF}_{\mathcal{EC}}$, alors il existe un schéma de définition pour p qui est donné par des formules d'ACVF.

Chapitre IV. Un peu de théorie des modèles des corps valués différentiels : Le dernier chapitre de ce texte contient plusieurs résultats sur la théorie $\mathrm{VDF}_{\mathcal{EC}}$. Le plus important d'entre eux est le théorème suivant :

Théorème F :

La théorie $\mathrm{VDF}_{\mathcal{EC}}$ élimine les imaginaires dans le langage géométrique et a la propriété d'extension invariante.

On étudie aussi dans ce chapitre certaines questions qui se posent quand on veut étudier $\mathrm{VDF}_{\mathcal{EC}}$. La première d'entres elles est de montrer que le corps des constantes est bien stablement plongé et est un pur modèle d'ACVF. La preuve de ce résultat s'adapte naturellement à la théorie de $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ muni du relèvement du Frobenius. La deuxième question est celle des clôtures définissables et algébriques. On démontre que la cloture définissable n'est pas aussi simple que l'on pourrait espérer. Elle contient, en général, strictement la clôture Hensélienne du corps différentiel engendré. On montre, tout de même, que :

0.8 (Corollaires **(IV.3.3)** et **(IV.3.4)**) **:**
Soient $M \models \mathrm{VDF}_{\mathcal{EC}}$ et $A \subseteq \mathbf{K}(M)$. Le corps $\mathbf{K}(\mathrm{dcl}(A))$ est une extension immédiate du corps différentiel engendré par A et le corps $\mathbf{K}(\mathrm{acl}(A))$ est une extension immédiate de la clôture algébrique du corps différentiel engendré par A.

On formalise aussi dans ce chapitre la relation entre les types dans $\mathrm{VDF}_{\mathcal{EC}}$ et les types dans ACVF en introduisant une notion de prolongation au niveau de l'espace des types qui est un analogue de celle définie dans DCF_0. La principale différence est que, dans DCF_0, la prolongation d'un ensemble définissable est un ensemble définissable dans ACF alors que, pour $\mathrm{VDF}_{\mathcal{EC}}$, c'est un type partiel de ACVF.
Enfin, dans la Section IV.5, on étudie les groupes définissables dans $\mathrm{VDF}_{\mathcal{EC}}$ et leurs liens avec les groupes définissables dans ACVF. La quasi-totalité de cette section consiste à montrer, en suivant [Hrub], que certains outils développés dans les théories stables pour étudier et construire des groupes se généralisent au contexte instable tant que les groupes que l'on considère ont des génériques définissables. On montre en particulier l'analogue du résultat de Hrushovski [Hru90] qu'un groupe $\star$-définissable dans une théorie stable est une pro-limite de groupes définissables. On donne aussi un théorème de construction de groupes à partir de « group chunks ». La notion de « group chunk » que nous utilisons ici est un peu plus générale que celle considérée habituellement en théorie des modèles, ce qui nous permet de traiter directement les groupes définissables non connexes.
À partir de ces considérations « néo-stables », on montre que la preuve du fait qu'un groupe définissable dans DCF_0 s'injecte définissablement dans un groupe définissable dans ACF (et donc dans un groupe algébrique), peut être reproduite dans un cadre abstrait et on en déduit que certains groupes définissables dans $\mathrm{VDF}_{\mathcal{EC}}$ s'injectent définissablement dans des groupes définissables dans ACVF.

CHAPTER 0

Preliminaries

Bérenger, au Logicien.
Cela me semble clair, mais cela ne résout pas la question.
Le Logicien, à Bérenger, en souriant d'un air compétent.
Évidemment, cher Monsieur, seulement, de cette façon, le problème est posé de façon correcte.
E. Ionesco, *Rhinocéros*, Acte I

First of all, let me recall some notations and conventions. When a is a (potentially infinite) tuple of elements from some set X, we will often write $a \in X$ when we should write $a \in X^I$ where I is the set of indices of the tuple.

When we assume that a structure M is κ-saturated (i.e. all types over set of parameters $A \subseteq M$ with $|A| < \kappa$ are realized in M) or strongly κ-homogenous (every elementary isomorphism between substructures of cardinal strictly less than κ extends to automorphism M) for some cardinal κ, whenever we consider $A \subseteq M$, it is understood that A is "small", i.e. $|A| < \kappa$. Similarly if we consider a tuple, a substructure or a submodel it will be assumed to be small. When we assume that a structure M is saturated or homogeneous enough, it means that M is κ-saturated or strongly κ-homogeneous for some cardinal κ greater than the cardinality of any tuple, subset, substructure or submodel of M we might have to consider.

When φ is an $\mathcal{L}$-formula with (tuple of) variables x and s, we will often write $\varphi(x;s)$ to specify that s is intended to be a tuple of parameters. Let M be some $\mathcal{L}$-structure, $A \subseteq M$ and $m \in M$ be a tuple, we will write $\varphi(A;m) := \{a \in A : M \vDash \varphi(a;m)\}$. Similarly if X is an $\mathcal{L}(M)$-definable set, we will write $X(M)$ for the M-points of X and $X(A) := X(M) \cap A$. Usually in this notation there is an implicit definable closure, but we want to avoid that because more often than not there will be multiple structures and languages around and hence multiple definable closures that could be implicit.

Let M be some $\mathcal{L}$-structure, $\Delta(x;s)$ be a set of $\mathcal{L}$-formulas, $A \subseteq B \subseteq M$ and $p \in \mathcal{S}_x^\Delta(B)$ be a Δ-type (i.e. a maximal consistent set of formulas of the form $\varphi(x;b)$ for $\varphi \in \Delta$ and $b \in B$). We will denote $p|_A \in \mathcal{S}_x^\Delta(A)$ the restriction of p to A: the set $\{\varphi(x;a) \in p : a \in A\}$. When p is definable (and the defining scheme also defines a global extension, cf. Remark **(0.3.10)**), and C[1] $\subseteq M$ contains B, we will also write $p|_C$ for the extension of p to C.

Finally, when we say that a set is $\mathcal{L}$-definable, we mean it is definable without parameters and when we say it is $\mathcal{L}(A)$-definable, this will obviously mean it is definable with parameters in A.

Most of the time, we will be working in multi-sorted languages.

[1]Because C is for Cookie, and that's good enough for me [Fat].

0.1. Imaginaries

Let us begin these preliminaries by recalling the notion of imaginaries whose elimination is quite central to this work. All of the definitions and results in this section are classical, but we will try to be very precise as terminology and notations tend to vary from one author to another.
In model theory, an imaginary is a point in a interpretable set or equivalently a class of a definable equivalence relation and, just as quantifier elimination results are about giving a simpler description of definable sets in term of boolean combination of certain atomic formulas, elimination of imaginaries is about giving a simpler description of interpretable sets in terms of definable sets (cf. Proposition **(0.1.7)**). Elimination of imaginaries is also linked to the question of finding canonical parameters for definable sets (cf. Definition **(0.1.4)**).

0.1.1. Codes

Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory and $M \vDash T$ be a saturated and homogeneous enough structure.

Definition 0.1.1 (Code)**:**
Let $M \vDash T$ and X be $\mathcal{L}(M)$-definable. We say that $A \subseteq M$ is a code for X if for all $\sigma \in \mathrm{Aut}(M)$:

σ stabilizes X globally if and only if σ fixes A pointwise.

We allow a code to be $\varnothing$ to be able to code $\mathcal{L}$-definable sets in theories without constants. Following [Hod93], we want to distinguish between uniform and non-uniform versions of codes. To do so we introduce the closely related (but more syntactic) notion of canonical parameter. These two notions are not usually distinguished but the definition of canonical parameter forces a canonical parameter to a be a finite tuple whereas we want to allow potentially infinite sets as codes.

Definition 0.1.2 (Canonical parameter)**:**
Let $M \vDash T$, X be $\mathcal{L}(M)$-definable and $\theta(x; s)$ be an $\mathcal{L}$-formula. We say that a tuple $a \in M$ is a canonical parameter for X via θ if for all $m \in M$ $\theta(M; m) = X$ if and only if $m = a$.
Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}$-definable family of sets, i.e. there exists an $\mathcal{L}$-formula $\varphi(x; s)$ and an $\mathcal{L}$-definable set Λ such that for all $\lambda \in \Lambda$, X_λ is defined by $\varphi(x; \lambda)$. We say that X admits uniform canonical parameters via θ if for all $\lambda \in \Lambda(M)$, there exists $a_\lambda \in M$ a canonical parameter for X via θ.

Proposition 0.1.3:
Let X be $\mathcal{L}(M)$-definable and $A \in M$ be a finite tuple, then the following are equivalent:

(i) *The set A is a code for X;*

(ii) *There exists a finite tuple $a \in A$ which is a canonical parameter for X via some $\mathcal{L}$-formula θ and such that $A \subseteq \mathrm{dcl}(a)$.*

Proof.

(i)⇒(ii) Let us assume that A is a code for X. Let $\varphi(x;m)$ be an $\mathcal{L}(M)$-formula such that $X = \varphi(M;m)$ and $p := \mathrm{tp}(m/A)$. For all $c \vDash p$, there exists $\sigma \in \mathrm{Aut}(M/A)$ such that $\sigma(m) = c$. As σ fixes A, it stabilizes X globally and it follows that $\varphi(M;c) = \varphi(M;\sigma(m)) = \sigma(\varphi(M;m)) = \sigma(X) = X = \varphi(M;m)$. We have just proved that:

$$p(s) \vdash \forall x\, \varphi(x;s) \iff \varphi(x;m).$$

By compactness, there is an $\mathcal{L}(A)$-formula $\psi(s;a)$ where a is a tuple from A such that $\psi(s;a) \Rightarrow (\forall x\, \varphi(x;s) \iff \varphi(x;m))$. It follows that X is defined by $\theta(x;a) := \forall s\, \psi(s;a) \Rightarrow \varphi(x;s)$.

Let now $q := \mathrm{tp}(a/\varnothing)$. Let $c \vDash q$ be such that $\theta(M;c) = \theta(M;a) = X$. There exists $\sigma \in \mathrm{Aut}(M)$ such that $\sigma(a) = c$. Moreover $\sigma(X) = \sigma(\theta(M;a)) = \theta(M;c) = X$. Hence σ must fix A pointwise and $c = \sigma(a) = a$. It follows that:

$$q(t) \vdash (\forall x\, \theta(x;t) \iff \theta(x;a)) \Rightarrow t = a.$$

By compactness, we can find an $\mathcal{L}$-formula $\xi(t)$ such that $(\xi(t) \wedge (\forall x\, \theta(x;t) \iff \theta(x;a))) \Rightarrow t = a$. Then a is a canonical parameter for X via $\xi(t) \wedge \theta(x;t)$.

(ii)⇒(i) Let $\theta(x;s)$ be an $\mathcal{L}$-formula such that $a \in A$ is a canonical parameter for X via θ. Then X is $\mathcal{L}(A)$-definable and hence any $\sigma \in \mathrm{Aut}(M/A)$ must stabilize X globally. Conversely, let $\sigma \in \mathrm{Aut}(M)$ be any automorphism which stabilizes X globally. Then $\theta(M;\sigma(a)) = \sigma(\theta(M;a)) = \sigma(X) = X = \theta(M;a)$ and hence $\sigma(a) = a$. ■

■

0.1.2. Elimination of imaginaries

Definition 0.1.4 (Elimination of imaginaries)**:**
We say that T eliminates imaginaries if every definable set in every model of T is coded (equivalently has a canonical parameter via some $\mathcal{L}$-formula θ).
We say that T uniformly eliminates imaginaries if every $\mathcal{L}$-definable family of sets admits uniform canonical parameters via some $\mathcal{L}$-formula θ.

If $\mathcal{S}$ is a set of sorts from $\mathcal{L}$, we will sometime say that T (uniformly) eliminates imaginaries up to $\mathcal{S}$ to mean that every set definable in any model of T has a (uniform) canonical parameter (via some $\mathcal{L}$-formula θ) which is a tuple of points from the sorts $\mathcal{S}$.
In [Poi83], where the notion of elimination of imaginaries was first introduced, it is shown that if there are enough constants in the theory, non uniform and uniform version of elimination of imaginaries are equivalent.

Proposition 0.1.5:
If $\mathcal{L}$ is such that there is a sort containing two constants and every sort contains at least one constant, then T eliminates imaginaries if and only if T eliminates imaginaries uniformly.

Proof. Let $M \vDash T$ be saturated enough and $\varphi(x;s)$ be an $\mathcal{L}$-formula. For all tuples $m \in M$ there exists a formula $\theta_m(x;t_m)$ and a tuple a_m such that a_m is a canonical parameter for

$\varphi(M;m)$ via θ_m. By compactness, there exists $(\theta_i(x;t_i))_{0\leqslant i<k}$ such that for all tuples $m\in M$ there exists $i_m < k$ and a_m such that a_m is a canonical parameter for $\varphi(M;m)$ via θ_{i_m}. Adding variables to t_i and specifying in θ_i that they must be equal to a given constant in the right sort, we may assume that all the t_i are equal to some t. Replacing $\theta_i(x;t)$ by $\theta_i(x;t) \wedge \bigwedge_{j<i} \neg(\forall x\, \theta_j(x;t) \iff \theta_i(x;t))$ for all $m\in M$ there is a unique i_m and a_m such that a_m is a canonical parameter for $\varphi(M;m)$ is via θ_{i_m}. Let c_1 and c_2 be two constants in the same sort and let $\theta(x;t,u) := \bigvee_i \theta_i(x;s) \wedge \bigwedge_{j\neq i} u_j = c_1 \wedge u_i = c_2$. Then for all $m\in M$, the tuple $a_m d_{i_m}$ is a canonical parameter for $\varphi(M;m)$ via θ where d_i is a tuple where every element is c_1 except the i-th which is c_2. ■

A theory T uniformly eliminates imaginaries if and only if the inclusion functor from the category of definable sets in T into the category of interpretable sets in T is an equivalence of categories. To reformulate that statement in a more model theoretic way, let me introduce the notion of a representable quotient.

Definition 0.1.6 (Representable equivalence relation)**:**
Let $M\vDash T$, D be $\mathcal{L}$-definable and $E\subseteq D^2$ be an $\mathcal{L}$-definable equivalence relation in M. We say that E is represented in M if there exists an $\mathcal{L}$-definable function f with domain D such that the fibers of f are exactly the E-classes, i.e. for all x, $y\in D(M)$, xEy if and only if $f(x)=f(y)$.

If an equivalence relation E on some set D is represented by the definable function f, then the interpretable quotient D/E is definably isomorphic to the definable set $\mathrm{Im}(f)$.

Proposition 0.1.7:
The theory T uniformly eliminates imaginaries if and only for all $M\vDash T$, every $\mathcal{L}$-definable equivalence relation in M is represented in M.

Proof. Assume T eliminates imaginaries uniformly and let E be an $\mathcal{L}$-definable equivalence relation on some D in some $M\vDash T$. Then for all $a\in D$, let E_a be the E-class of a. By uniform elimination of imaginaries, there exists a formula $\theta(x;s)$ such that for all $a\in D(M)$, E_a admits a canonical parameter via θ. Let $f(a)$ be the canonical parameter of E_a via θ. Then f is an $\mathcal{L}$-definable function and $f(a_1)=f(a_2)$ if and only if $E_{a_1}(M)=\theta(M;f(a_1))=\theta(M;f(a_2))=E_{a_2}(M)$, i.e. a_1Ea_2.
Conversely, let $\varphi(x;s)$ be an $\mathcal{L}$-formula. Let s_1Es_2 hold if $\forall x\,(\varphi(x;s_1) \iff \varphi(x;s_2))$. Then E is an $\mathcal{L}$-definable equivalence relation in any model of T and hence it must be represented by some $\mathcal{L}$-definable function f. Let $\theta(x;s) := \forall t\,(f(t)=s \Rightarrow \varphi(x;t))$. Then $\varphi(M;m)=\theta(M;f(m))$ and if $c\neq f(m)$, for any a such that $f(a)=c$, $\varphi(M;a)\neq\varphi(M;m)$ and so $\theta(M;c)\neq\varphi(M;m)$, i.e. $f(m)$ is a canonical parameter for $\varphi(M;m)$ via θ. ■

0.1.3. Shelah's eq construction

The notion of imaginaries first appeared in work by Shelah (e.g. [She78]) through the following construction which consists in adding new sorts so that all interpretable sets become definable. For the sake of simplicity, let us assume that T is complete until the end of Section 0.1.

Definition 0.1.8 (T^{eq})**:**
For every $\mathcal{L}$-definable $E \subseteq (\prod_{i<k} S_i)^2$, where S_i is an $\mathcal{L}$-sort, such that T implies that E is an equivalence relation, let S_E be a new sort and f_E be a new symbol $f_E : \prod_i S_i \to S_E$. Let $\mathcal{L}^{\mathrm{eq}}$ be the language $\mathcal{L}$ to which we add all the sorts S_E and all the function symbols f_E. Let T^{eq} be the $\mathcal{L}^{\mathrm{eq}}$-theory:

$$\begin{aligned} T \quad &\cup \quad \{f_E \textit{ is onto} : E \textit{ is an } \mathcal{L}\textit{-definable equivalence relation}\} \\ &\cup \quad \{\forall x \forall y\, f_E(x) = f_E(y) \iff xEy : E \textit{ is an } \mathcal{L}\textit{-definable equivalence relation}\} \end{aligned}$$

Every $M \models T$ can be extended in a unique way into a model M^{eq} of T^{eq} by interpreting S_E as $(\prod_i S_i(M))/E(M)$ and f_E as the canonical projection.

Note that T^{eq} is also complete.

Definition 0.1.9 (Dominant sorts)**:**
Let $\mathcal{S}$ be a set of $\mathcal{L}$-sorts. We say that the sorts in $\mathcal{R}$ are dominant in T if for every $\mathcal{L}$-sort S there exists a tuple of sorts $(R_i)_{0\leqslant i<k} \in \mathcal{R}$ and an $\mathcal{L}$-definable surjective function $f : \prod_i R_i \to S$.

Remark 0.1.10:

1. Let $\mathcal{R}$ be the set of $\mathcal{L}$-sorts. They are dominant in T^{eq}. We usually call them the real sorts while the new sorts S_E are usually called the imaginary sorts;
2. Let $M \models T^{\mathrm{eq}}$. The $\mathcal{L}^{\mathrm{eq}}(M^{\mathrm{eq}})$-structure induced on $\mathcal{R}$ is exactly the $\mathcal{L}(M)$-structure, i.e. for all $\mathcal{L}^{\mathrm{eq}}(M^{\mathrm{eq}})$-formula $\varphi(x)$ where all the variables in x are in the real sorts, there exists an $\mathcal{L}(M)$-formula θ such that $\forall x\, \varphi(x) \iff \theta(x)$;
3. The theory T^{eq} uniformly eliminates imaginaries;
4. If $N \preccurlyeq M$ then $N^{\mathrm{eq}} \preccurlyeq M^{\mathrm{eq}}$;
5. If T is model complete then so is T^{eq}. However, if T eliminates quantifiers, T^{eq} might not.

We will denote the definable closure in T^{eq} by $\mathrm{dcl}^{\mathrm{eq}}$ (similarly for the algebraic closure $\mathrm{acl}^{\mathrm{eq}}$).

Proposition 0.1.11:
Let $M \models T$, X be $\mathcal{L}(M)$-definable and A and $A' \in M^{\mathrm{eq}}$ be codes for X. Then $\mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A')$.

Proof. We may assume M is saturated and homogeneous enough. Let $\sigma \in \mathrm{Aut}(M^{\mathrm{eq}})$. Then, by definition, σ fixes A pointwise if and only if it stabilizes X globally, if and only if it fixes A' pointwise. It follows that A and A' are interdefinable. ∎

Notation 0.1.12 ($\ulcorner X \urcorner$)**:**
It follows that the set $\ulcorner X \urcorner := \mathrm{dcl}^{\mathrm{eq}}(A)$ does not depend on the actual choice of code for X but just on X itself. It is also the largest (with respect to the inclusion) code for X. We call it *the* code of X.

If we want to specify the theory (or the language) in which the code is considered, we will write $\ulcorner X \urcorner^{\mathcal{L}}$. Note that, by definition, a is a code for X if and only if $\mathrm{dcl}^{\mathrm{eq}}(a) = \ulcorner X \urcorner = \mathrm{dcl}^{\mathrm{eq}}(\ulcorner X \urcorner)$.

0.1.4. Weak elimination

Sometimes (see Example **0.1.18.3**) sets are not exactly coded, but are coded up to finite permutation. This is known as a weak code.

Definition 0.1.13 (Weak code):
Let $M \models T$ be saturated and homogeneous enough and let X be $\mathcal{L}(M)$-definable. We say that $A \subseteq M$ is a weak code for X if there exists a finite number of $(A_i)_{0 \leqslant i < k} \subseteq M$ such that $A_0 = A$ and for all $\sigma \in \mathrm{Aut}(M)$:

$$\sigma \text{ stabilizes } X \text{ globally if and only if } \sigma \text{ stabilizes } \{A_i : 0 \leqslant i < k\} \text{ globally.}$$

Definition 0.1.14 (Weak canonical parameter):
Let $M \models T$, X be $\mathcal{L}(M)$-definable and $\theta(x; s)$ be an $\mathcal{L}$-formula. We say that a tuple $a \in M$ is a weak canonical parameter for X via θ if there exists a finite number of tuples $(a_i)_{0 \leqslant i < k} \in M$ such that $a_0 = a$ and for all tuples $m \in M$, $\varphi(M; m) = X$ if and only if there exists i such that $m = a_i$.
We also define a uniform weak canonical parameter in the obvious way.

Proposition 0.1.15:
Let $M \models T$ be saturated and homogeneous enough. Let X be $\mathcal{L}(M)$-definable and $a \in M$ be a finite tuple. The following are equivalent:

(i) The tuple a is a weak canonical parameter for X via some $\mathcal{L}$-formula θ.

(ii) The tuple a (seen as a set) is a weak code of X;

(iii) We have $\ulcorner X \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$;

Proof. We obviously have (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follows from the fact that $a \in \mathrm{dcl}^{\mathrm{eq}}(c)$ if and only if a is fixed by $\mathrm{Aut}(M^{\mathrm{eq}}/c)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(c)$ if and only if the orbit under $\mathrm{Aut}(M^{\mathrm{eq}}/c)$ of a is finite. Now, if (iii) holds, by Proposition **(0.1.3)**, there exists $c \in \ulcorner X \urcorner$ a canonical parameter of X via some $\mathcal{L}^{\mathrm{eq}}$-formula θ. In particular, we have $c \in \mathrm{dcl}^{\mathrm{eq}}(a)$ and $a \in \mathrm{acl}^{\mathrm{eq}}(\mathrm{dcl}^{\mathrm{eq}}(c)) = \mathrm{acl}^{\mathrm{eq}}(c)$. Then, there is a finite to one $\mathcal{L}^{\mathrm{eq}}$-definable map f such that $f(a) = c$.
Let $\varphi(x; t) := \theta(x; f(t))$. By Remark **0.1.10.2**, we may assume that φ is an $\mathcal{L}$-formula. Then for all $m \in M$, $\varphi(M; m) = X$ if and only if $\theta(M; f(m)) = X$ if and only if $f(m) = c$, i.e. m is in the finite fiber of f above c. ∎

Definition 0.1.16 (Weak elimination of imaginaries):
We say that T weakly eliminates imaginaries if every definable set in every model of T admits a weak canonical parameter via some $\mathcal{L}$-formula θ.
We say that T uniformly weakly eliminates imaginaries if every $\mathcal{L}$-definable family of sets admits a uniform weak canonical parameter via some $\mathcal{L}$-formula θ.

Let us now show that the one difference between weak elimination of imaginaries and elimination of imaginaries is the ability to code finite sets.

Proposition 0.1.17:
Let T be a theory such that in every $M \vDash T$, any finite set (of finite tuples) is coded. Let X be $\mathcal{L}(M)$-definable. Then X is weakly coded in M if and only if it is coded in M.

Proof. Let a be a weak canonical parameter of X via some $\mathcal{L}$-formula θ. We have $a \in \mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$. Let $A = \{a_i : 0 \leqslant i < k\}$ be the orbit of a under the action of $\mathrm{Aut}(M^{\mathrm{eq}}/\ulcorner X \urcorner)$. For all i, there exists $\sigma \in \mathrm{Aut}(M^{\mathrm{eq}}/\ulcorner X \urcorner)$ such that $\sigma(a) = a_i$. Then $X = \sigma(X) = \theta(M; \sigma(a)) = \theta(M; a_i)$. It follows that any automorphism $\sigma \in \mathrm{Aut}(M)$ that stabilizes A globally, sends $X = \theta(M; a)$ to $X = \theta(M; a_i)$ for some i and thus stabilizes X globally. Therefore, σ stabilizes X globally if and only if σ stabilizes A globally, if and only if σ fixes e pointwise, where e is a code for A. Thus e is also a code for X. ∎

Finally, let us give some examples.

Example 0.1.18:

1. The theory ACF of algebraically closed fields in the language of rings $\mathcal{L}_{\mathrm{rg}}$ uniformly eliminates imaginaries and so does the theory DCF_0 of differentially closed fields of characteristic zero in $\mathcal{L}_{\mathrm{rg},\partial} := \mathcal{L}_{\mathrm{rg}} \cup \{\partial\}$. These are in fact the two examples that motivated the definition of the elimination of imaginaries in [Poi83].

2. The theory DOAG of divisible ordered abelian groups, i.e. the theory of $(\mathbb{Q}, +, 0, <)$, and the theory RCF of real closed fields both eliminate imaginaries, as they are o-minimal groups. But, in the first case, the elimination is not uniform. Indeed, let $\varphi(x; s, t) := s = t$ where $|x| = |s| = |t| = 1$. Then for all $M \equiv (\mathbb{Q}, +, 0, <)$ and $a, c \in M$, $\varphi(M; a, c) = M$ or $\varphi(M; a, c) = \varnothing$. If the elimination were uniform, there would exist a formula $\theta(x; u)$ and unique c_1, $c_2 \in M$ such that $\theta(M; c_1) = M$ and $\theta(M; c_2) = \varnothing$. In particular c_1 and c_2 are distinct elements in $\mathrm{dcl}(\varnothing) = \{0\}$, a contradiction. In fact, if the language is one sorted, we have just proved a converse to Proposition **(0.1.5)**.

3. The theory of infinite sets weakly eliminates imaginaries. It cannot eliminate imaginaries because finite sets are not coded. For reasons similar to those in the previous example, the weak elimination cannot be uniform either because $\mathrm{acl}(\varnothing) = \varnothing$. But infinite sets with two distinct constants uniformly weakly eliminate imaginaries.

0.2. Valued fields

We will now consider the other central subject in this work: valued fields. They have been studied by model theorists for more than half a century now, leading to a profusion of results and applications. In this section, we will describe the different languages used in model theory of valued fields and some of the most classical results of elimination. For more detail on various topics we will refer the reader to the preliminaries included in the relevant chapters of this text.

0.2.1. Definition and examples

Definition 0.2.1 (Valued Field)**:**
A valued field is a field K with a group morphism val *from* $\mathbf{K}^\star$ *to some ordered abelian group* Γ *such that for all $x, y \in K$ the ultrametric inequality holds:*

$$\mathrm{val}(x+y) \geqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\}.$$

Remark 0.2.2:

1. The valuation of 0 is usually defined to be some new point $+\infty$, bigger than any point in Γ.

2. The set $\mathcal{O} := \{x \in K : \mathrm{val}(x) \geqslant 0\}$ is a subring of K that is called the valuation ring. The ring $\mathcal{O}$ is a local ring whose unique maximal ideal is $\mathfrak{M} := \{x \in K : \mathrm{val}(x) > 0\}$ and such that for all $x \in \mathrm{Frac}(\mathcal{O}) = K$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. In fact, any local ring $\mathcal{O}$ with this property is called a valuation ring and is the valuation ring of a unique valuation on $K = \mathrm{Frac}(\mathcal{O})$: the canonical projection $K^\star \to K^\star/\mathcal{O}^\star$.

3. The field $k := \mathcal{O}/\mathfrak{M}$ is called the residue field and the canonical projection is usually denoted $\mathrm{res} : \mathcal{O} \to k$.

Let us now consider classical examples of valued fields.

Example 0.2.3:

1. Let K be any field. For every irreducible polynomial $P \in K[X]$, the ring $K[X]_{(P)} \subseteq K(X)$ is a valuation ring associated to the P-adic valuation $\mathrm{val}_P(P^n Q/R) := n \in \mathbb{Z}$ whenever $Q \wedge R = P \wedge Q = P \wedge R = 1$. For all $e \in \mathbb{R}$ strictly greater than one, $|x| = e^{-\mathrm{val}_P(x)}$ defines an ultrametric norm on $K(X)$ and all these norms are equivalent. Let us now consider $P = X$. The completion of $K[X]$ for the X-adic norm is the field $K((X))$ of Laurent series over K whose valuation ring is $K[[X]]$ the ring of power series. The value group of $K(X)$ and $K((X))$ is $\mathbb{Z}$ and their residue field is K.

2. The Laurent series construction is in fact a special case of a more general construction: Hahn fields. Let k be a field and Γ be an ordered abelian group. Then, let $k((t^\Gamma))$ be the field whose elements are the formal series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$ such that $\{\gamma : a_\gamma \neq 0\}$ is well-ordered. The valuation on $k((t^\Gamma))$ is defined by $\mathrm{val}(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma) := \min\{\gamma \in \Gamma : a_\gamma \neq 0\}$. The value group of $k((t^\Gamma))$ is Γ and its residue field is k. The Laurent series field over K is exactly $K((t^{\mathbb{Z}}))$.

3. Both of the previous examples have equicharacteristic, i.e. the characteristic of the residue field equals the characteristic of the field. It is also possible to have a characteristic zero valued field with a positive characteristic residue field. We then talk of mixed characteristic.

   Let p be a prime. The ring $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ is a valuation ring. It is associated to the p-adic valuation $\mathrm{val}_p(p^n a/b) := n \in \mathbb{Z}$ when $a \wedge b = a \wedge p = b \wedge p = 1$. The completion of $\mathbb{Q}$ with respect to the p-adic norm associated to the p-adic valuation is called $\mathbb{Q}_p$ the field of

p-adic numbers. Its valuation ring is $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ the ring of p-adic integers. The value group of both $\mathbb{Q}$ and $\mathbb{Q}_p$ for the p-adic valuation is $\mathbb{Z}$ and their residue fields is $\mathbb{F}_p$.

More generally, for any number field L whose ring of integers is R and any prime ideal ℓ in R, L can be endowed with an ℓ-adic valuation and the completion of L for the associated ℓ-adic norm is a finite extension of $\mathbb{Q}_p$ where p is such that $\ell \cap \mathbb{Z} = (p)$.

4. A mixed characteristic field is said to be unramified if $\mathrm{val}(p)$ has minimal positive valuation in the value group. There is a construction similar to the Hahn field construction in unramified mixed characteristic: the Witt vectors. Let k be a perfect field of positive characteristic. There is a unique, up to unique isomorphism, unramified mixed characteristic complete valued field with residue field k that we denote $\mathrm{W}(k)$. Its valuation ring is denoted $\mathrm{W}[k]$ and its value group is $\mathbb{Z}$. Note that $\mathrm{W}(\mathbb{F}_p)$ is exactly $\mathbb{Q}_p$.

0.2.2. Algebraically closed valued fields

The theory ACVF of (non trivially valued) algebraically closed valued fields is the first theory of valued fields to have been studied by model theorists. In the past fifteen years, various new aspects of this theory were studied extensively [HHM06; HHM08; HK06; HL]. There are many languages in which one can work depending on whether one wants a sort for the value group, the residue field or none of them.

Definition 0.2.4 (One sorted language)**:**
The language $\mathcal{L}_{\mathrm{div}}$ has one sort $\mathbf{K}$ and consists of the ring language $\mathcal{L}_{\mathrm{rg}} := \{+,-,\cdot,0,1\}$ enriched with a predicate $| \subseteq \mathbf{K}^2$ which is interpreted as $\mathrm{val}(x) \leqslant \mathrm{val}(y)$.

Definition 0.2.5 (Two sorted language)**:**
The language $\mathcal{L}_\Gamma$ has two sorts $\mathbf{K}$ and $\mathbf{\Gamma}$ and consists of the language of rings on $\mathbf{K}$, the language of ordered groups $\mathcal{L}_{\mathrm{og}} := \{+,-,0,<\}$ on $\mathbf{\Gamma}$ and a function $\mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}$.

Definition 0.2.6:
<Three sorted language) The language $\mathcal{L}_{\Gamma,\mathbf{k}}$ has three sorts $\mathbf{K}$, $\mathbf{\Gamma}$ and $\mathbf{k}$ and consists of the language of rings on $\mathbf{K}$ and $\mathbf{k}$, the language of ordered groups on $\mathbf{\Gamma}$, a function $\mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}$ and a function $\mathrm{res} : \mathbf{K}^2 \to \mathbf{k}$ interpreted as the residue of xy^{-1}.

Theorem 0.2.7:
The theory ACVF eliminates quantifiers in the one sorted, the two sorted and the three sorted languages. Its completions are given by fixing the characteristic of the valued field and of its residue field.

Proof. All these results follow from work by Abraham Robinson in [Rob77] although he only stated model completeness (in any of the three languages as they are equivalent). The elimination of quantifiers in the one sorted language follows immediately from model completeness and the existence of prime models (which are none other than the algebraic closures).

Quantifier elimination in the two sorted language is stated in [Wei84, Theorem 3.2] and quantifier elimination in the three sorted language follows from more general results in [Del82].
The description of the completions follows from quantifier elimination. ∎

Note that it also follows from the quantifier elimination results that $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded, that $\mathbf{k}$ is a pure algebraically closed field and $\mathbf{\Gamma}$ is a pure divisible ordered abelian group. Another consequence of these quantifier elimination results is that unary sets inside the sorts $\mathbf{K}$ have a very specific form.

Definition 0.2.8 (Swiss cheese)**:**
Let (K, val) *be a valued field. The open ball of center* $a \in K$ *and radius* $\gamma \in \mathrm{val}(K)$ *is the set* $\{x \in K : \mathrm{val}(x - a) > \gamma\}$. *The closed ball of center* $a \in K$ *and radius* $\gamma \in \mathrm{val}(K)$ *is the set* $\{x \in K : \mathrm{val}(x - a) \geqslant \gamma\}$. *We also consider the whole valued field* K *to be an open ball. Let* b *be a ball and* $(b_i)_{0 \leqslant i < k}$ *be subballs of* b. *Then the set* $b \setminus \bigcup_i b_i$ *is called a swiss cheese.*

Note that balls of radius $+\infty$ are singletons. In particular the outer ball or the holes of a swiss cheese can be singletons. We say that two swiss cheeses are trivially nested if the outer ball of one coincides with one of the holes of the other.

Theorem 0.2.9 ([Hol95])**:**
Any unary set $X \subseteq \mathbf{K}$ *definable in* ACVF *with parameters can be expressed in a unique way as a finite union of non trivially nested swiss cheeses.*

0.2.3. Henselian fields

A larger class of valued fields that is also widely considered in model theory and is the setting of a great variety of model theoretic results, is the class of characteristic zero Henselian fields. For the more algebraic considerations around Henselian fields, we refer the reader to [EP05].

Definition 0.2.10 (Henselianity)**:**
Let (K, val) *be a valued field. It is said to be Henselian if any of the following equivalent properties hold:*

(i) *For all polynomials* $P \in \mathcal{O}_X$ *and* $a \in \mathcal{O}$ *such that* $\mathrm{res}(P(a)) = 0 \neq \mathrm{res}(P'(a))$, *there exists* $c \in \mathcal{O}$ *such that* $P(c) = 0$ *and* $\mathrm{res}(c) = \mathrm{res}(a)$*;*

(ii) *For all polynomials* $P \in \mathcal{O}_X$ *and* $a \in \mathcal{O}$ *such that* $\mathrm{val}(P(a)) > 2\mathrm{val}(P'(a))$, *there exists* $c \in \mathcal{O}$ *sucht hat* $P(c) = 0$ *and* $\mathrm{val}(c - a) > \mathrm{val}(P(a)) - \mathrm{val}(P'(a))$*;*

(iii) *There is a unique extension of* val *to any finite extension of* K*;*

(iv) *There is a unique extension of* val *to any algebraic extension of* K*;*

Example 0.2.11:
All complete fields ($\mathbb{Q}_p$, $K((t))$ for example) are Henselian — this result is called Hensel's lemma and gave its name to the notion. Hahn fields are also Henselian.

Foremost among the classic results on the model theory of Henslian valued fields is the Ax-Kochen-Eršov principle (cf. [AK65; Erš65]) which was one of the first model theoretic results to be proved about Henselian valued fields, was essential in creating the great interest we witness today for the model theory of valued fields and still influences our vision of this subject.

Theorem 0.2.12 (Ax-Kochen-Eršov principle)**:**

Let K and L be two characteristic zero Henselian valued fields which are unramified if they have mixed characteristic. We have $K \equiv L$ as valued fields if and only if $\mathbf{\Gamma}(K) \equiv \mathbf{\Gamma}(L)$ as ordered groups and $\mathbf{k}(K) \equiv \mathbf{k}(L)$ as fields.

The Ax-Kochen-Eršov principle is an immediate consequence of some quantifier elimination theorems (e.g. **(0.2.17)**) but the original proof of the principle is 25 years older than these quantifier elimination results.
Before considering more general Henselian fields, let me state another fundamental result of Macintyre in the model theory of valued fields that is only concerned with a very specific Henselian field: $\mathbb{Q}_p$. For a generalization of this result to finite extensions of $\mathbb{Q}_p$, we refer the reader to [PR84].

Theorem 0.2.13 ([Mac76])**:**

Let $\mathcal{L}_{\mathrm{div},P} := \mathcal{L}_{\mathrm{div}} \cup \{P_n : n \in \mathbb{N}_{>0}\}$ where for all $n \in \mathbb{N}_{>0}$, P_n is interpreted as the set of n-th powers. The $\mathcal{L}_{\mathrm{div},P}$-theory of $\mathbb{Q}_p$ eliminates quantifiers.

Let us now define the leading term language, also known as amc-congruences in [Bas91; BK92; Kuh94] or as the $\mathbf{RV}$-language in [HK06] for example.

Definition 0.2.14 (Leading term language)**:**
Let $\mathcal{L}^{\mathbf{RV}^+}$ be a language with a sort $\mathbf{K}$, sorts $\mathbf{RV}_n$ for all $n \in \mathbb{N}_{>0}$, the ring language of $\mathbf{K}$, for all $n \in \mathbb{N}_{>0}$ a function symbol $\mathrm{rv}_n : \mathbf{K} \to \mathbf{RV}_n$ and for all $m|n$ a function symbol $\mathrm{rv}_{m,n} : \mathbf{RV}_n \to \mathbf{RV}_m$. The sort $\mathbf{RV}_n$ is interpreted as $\mathbf{K}^\star/(1 + n\mathfrak{M}) \cup \{0\}$ and the function symbols rv_n and $\mathrm{rv}_{m,n}$ are interpreted as the canonical projections. We also add to the language a constant symbol $0_n \in \mathbf{RV}_n$ interpreted as 0, a binary predicate $|_n$ interpreted as $\mathrm{val}_{\mathbf{RV},n}(x) \leqslant \mathrm{val}_{\mathbf{RV},n}(y)$ where $\mathrm{val}_{\mathbf{RV},n}$ is the function induced by val on $\mathbf{RV}_n$, and functions $+_{m,n} : \mathbf{RV}_n^2 \to \mathbf{RV}_m$ for all $m|n$ interpreted as the trace of the addition on $\mathbf{RV}_n^2 \times \mathbf{RV}_m$.

For more precisions on this language, the reader should refer to Section II.1.

Theorem 0.2.15 ([Bas91; BK92].)**:**

The $\mathcal{L}^{\mathbf{RV}^+}$-theory $\mathrm{T}_{\mathrm{Hen}}$ of characteristic zero Henselian fields eliminates field quantifiers resplendently.

Resplendent field quantifier elimination is essentially the property that whatever the enrichment on the $\mathbf{RV}_n$ sorts, any formula is equivalent to a formulas without field quantifiers. This notion is explained at length in Section II.A. In equicharacteristic zero, one can work in the restriction of $\mathcal{L}^{\mathbf{RV}^+}$ to $\mathbf{K}$ and $\mathbf{RV}_1$.

0.2.4. Angular components

Historically, characteristic zero Henselian valued fields have not been considered in the leading term language but first with a cross-section and then with angular components. The latter are still used nowadays because, although they add new definable sets, they separate the value group from the residue field.

Definition 0.2.16 (Angular component language)**:**
The angular component language $\mathcal{L}^{\mathrm{ac}}$ has a sort $\mathbf{K}$, a sort $\mathbf{\Gamma}$ and sorts $\mathbf{R}_n$ for all $n \in \mathbb{N}_{>0}$, the ring language on $\mathbf{K}$ and each of the $\mathbf{R}_n$, the ordered group language on $\mathbf{\Gamma}$, for all n a function symbol $\mathrm{ac}_n : \mathbf{K} \to \mathbf{R}_n$ and for all $m|n$ a function symbol $\mathrm{res}_{m,n} : \mathbf{R}_n \to \mathbf{R}_m$. For all $n \in \mathbb{N}_{>0}$, $\mathbf{R}_n$ is interpreted as the ring $\mathcal{O}/(n\mathfrak{M})$. The maps $\mathrm{res}_{m,n}$ are interpreted as the canonical projections and the maps ac_n as angular components, i.e. group morphisms $\mathbf{K}^\star \to \mathbf{R}_n^\star$ that send 0 to 0 and such that $\mathrm{ac}_n|_{\mathcal{O}^\star} = \mathrm{res}_n|_{\mathcal{O}^\star}$.

This language is usual known as the Denef-Pas language. Note that $\mathbf{R}_0$ is the residue field $\mathbf{k}$. For more precisions and the relation between this language and leading terms, one may look at Section II.1. As with leading terms, in equicharacteristic zero, one may work with the restriction of $\mathcal{L}^{\mathrm{ac}}$ to $\mathbf{K}$, $\mathbf{\Gamma}$ and $\mathbf{k} = \mathbf{R}_0$.
Let $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ be the $\mathcal{L}^{\mathrm{ac}}$-theory of Henselian valued fields with angular components, $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen},0}$ be the $\mathcal{L}^{\mathrm{ac}}$-theory of equicharacteristic zero Henselian valued fields with angular components and for all p and e the $\mathcal{L}^{\mathrm{ac}}$-theory $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathrm{Hen},p}$ of characteristic $(0,p)$ Henselian valued fields with ramification index at most e, i.e. $\mathrm{val}(p)$ is at most e times bigger than the smallest positive element of $\mathbf{\Gamma}$. In the latter case we add a constant symbol for a uniformizer π, i.e. $\mathrm{val}(\pi)$ is the smallest positive element of $\mathbf{\Gamma}$ and we assume that, for all $n \in \mathbb{N}_{>0}$, $\mathrm{ac}_n(\pi) = 1$.

Theorem 0.2.17 ([Pas89])**:**
The theories $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen},0}$ and $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathrm{Hen},p}$ eliminate field quantifiers resplendently.

Note that, although angular components sometime enlarge the class of definable sets, any saturated enough valued field can be equipped with angular components (cf. [Pas90, Corollary 1.6]).

0.2.5. Geometric sorts

Let me introduce one last language, the geometric language of [HHM06] that was introduced in order to prove elimination of imaginaries in ACVF.

Definition 0.2.18 (Geometric language)**:**
The geometric language $\mathcal{L}^{\mathcal{G}}$ consists of the sort $\mathbf{K}$, for all $n \in \mathbb{N}_{>0}$, sorts $\mathbf{S}_n$ and $\mathbf{T}_n$, the ring language on $\mathbf{K}$ and functions $s_n : \mathbf{K}^{n^2} \to \mathbf{S}_n$ and $t_n : \mathbf{S}_n \times \mathbf{K}^n \to \mathbf{T}_n$. The sort $\mathbf{S}_n$ is interpreted as the sort of all $\mathcal{O}$-lattices in $\mathbf{K}^n$, i.e. rank n free $\mathcal{O}$-modules in $\mathbf{K}^n$. The sort $\mathbf{T}_n$ is interpreted as $\bigcup_{s\in\mathbf{S}_n} s/\mathfrak{M}s$, the function s_n sends a $\mathbf{K}$-linearly independent tuple $(b_i)_{0\leqslant i<n} \in \mathbf{K}^n$ to the $\mathcal{O}$-modules generated by the b_i and any other tuple to, for example, $\mathcal{O}^n$ and t_n sends a lattice s and a point $a \in s$ to the coset $a + \mathfrak{M}s$. If $a \notin s$, $t_n(s,a) = \mathfrak{M}s$, for example.

The sort $\mathbf{S}_1$ is in fact $\mathbf{\Gamma}$ and s_0 is exactly the valuation. There is also an $\mathcal{L}^{\mathcal{G}}$-definable map $\tau_n : \mathbf{T}_n \to \mathbf{S}_n$ sending $a + \mathfrak{M}s$ to s and the fiber of τ_1 above $\mathcal{O}$ is $\mathbf{k}$. A more detailed account of those geometric sorts can be found in Section I.2.2. Let $\mathrm{ACVF}^{\mathcal{G}}$ be the $\mathcal{L}^{\mathcal{G}}$-theory of algebraically closed valued fields.

Theorem 0.2.19 ([HHM06, Theorem 1.0.1])**:**
The theory $\mathrm{ACVF}^{\mathcal{G}}$ *eliminates imaginaries.*

0.3. The independence property

Because the value group is ordered, valued fields cannot be stable or even simple. But provided the residue field is tame enough, valued fields are still reasonably tame. In particular, many of the interesting theories of valued fields are NIP, a tameness assumption that appeared independently in work of Shelah in [She71] and in learning theory as finite VC-dimension in [VČ71].

0.3.1. Definitions and examples

Definition 0.3.1 (Independence property)**:**
Let M be an $\mathcal{L}$-structure. An $\mathcal{L}$-formula $\varphi(x,y)$ has the independence property in M if there exist tuples $(a_i)_{i\in\mathbb{N}} \in M$ and $(b_j)_{j\in\mathcal{P}(\mathbb{N})} \in M$ such that for all i and j, $M \vDash \varphi(a_i, b_j)$ if and only if $i \in j$.
An $\mathcal{L}$-theory T is said to be NIP *(not the independence property) or dependent if no $\mathcal{L}$-formula has the independence property in any model of T.*

Definition 0.3.2 (VC-dimension)**:**
Let X be a set and $S \subseteq \mathcal{P}(X)$. The VC-dimension of S is defined as the greatest $n \in \mathbb{N}$ such that there exists $A \subseteq X$ of cardinal at most n and $\{A \cap Y : Y \in S\} = \mathcal{P}(A)$.

As announced earlier, these two notions are closely linked.

Proposition 0.3.3:
Let M be $\aleph_0$-saturated. An $\mathcal{L}$-formula $\varphi(x,y)$ has the independence property in M if and only if the family $\{\varphi(M,m) : m \in M$ is a tuple $\}$ has infinite VC-dimension.

Example 0.3.4:

1. Let X be a set and S be the set of all sets of cardinal at most n, then S has VC-dimension n.

2. The theory ACF is NIP (as are all strongly minimal theories, cf. Definition **(0.3.19)**).

3. Let $(X,<)$ be a totally ordered set and S be the set of all intervals. Then S has VC-dimension 2.

4. Real closed fields are NIP (as are all o-minimal theories, cf. Definition **(0.3.20)**).

5. Let (K, val) be a valued field and S be the set of all balls. Then S has VC-dimension 2.

6. The theory ACVF is NIP (as are all C-minimal theories, cf. Definition **(0.3.22)**).

7. By a theorem of [GS84], the theory of ordered abelian groups is NIP.

Let us now give a theorem that formalizes the principle, mentioned earlier, that a Henselian valued field is tame if its residue field is tame (which, because the value groups is always NIP, is a form of Ax-Kochen-Eršov principle for NIP). This theorem was first proved in equicharacteristic zero by Delon in [Del81].

Theorem 0.3.5 ([Bél99])**:**
Let T be any completion of $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ or $\mathrm{T}_{\mathrm{Hen}}$ that implies either equicharacteristic zero or non ramification in mixed characteristic. Then T is NIP *if and only if* **k***, as a field, is* NIP.

0.3.2. Some properties of NIP theories

First let us give a combinatorial result about families of finite VC-dimension known as the (p, q)-theorem that will be used in Section III.1. To be precise, the statement we give here is a special case of the dual version of the (p, q)-theorem (see [Sim, Corollary 6.13]). For the full statement, one can look at [Mat04] or [Sim, Theorem 6.10].

Theorem 0.3.6:
For all $k \in \mathbb{N}$, there exists q and $n \in \mathbb{N}$ such that for any set X, any finite $A \subseteq X$ and any $S \subseteq \mathcal{P}(X)$ of VC-dimension at most k, if for all $A_0 \subseteq A$ of size at most q there exists $Y \in S$ containing A_0, then there exists $Y_1 \dots Y_n \in \mathcal{S}$ such that $A \subseteq \bigcup_i Y_i$.

The other result about NIP theories that we will be using is the existence of uniform honest definitions.

Definition 0.3.7 (Externally definable set)**:**
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $X \subseteq A$ is externally definable if there exists an $\mathcal{L}$-formula $\varphi(x; s)$ and a tuple $m \in M$ such that $X = \varphi(A; m)$.

Definition 0.3.8 (Stable embdedness)**:**
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that A is stably embedded if every externally definable subset of A is in fact of the form $X(A)$ where X is $\mathcal{L}(A)$-definable.

Before we go any further, let us recall a notion central to stable theories: definable types. Not only will it allow to fix some notations, it will also shed some light on some later conventions.

Definition 0.3.9 (Definable type)**:**
Let M be an $\mathcal{L}$-structure, $A \subseteq M$, $\Delta(x; s)$ be a set of $\mathcal{L}$-formulas and $p \in \mathcal{S}^{\Delta}_x(A)$. We say that the Δ-type p is $\mathcal{L}(A)$-definable if for all $\varphi(x; s) \in \Delta(x; s)$, there exists an $\mathcal{L}(A)$-formula $\theta(s)$ such that for all tuples $m \in M$, $\varphi(x; m) \in p$ if and only if $M \vDash \theta(m)$.
We will denote $\theta(s)$ by $d_p x\, \varphi(x; s)$.

Remark 0.3.10:

1. Assume $A \preccurlyeq M$, and Δ is closed under boolean combinations. For all $\varphi \in \Delta$, we have

$$M \vDash \forall s\, (d_p x\, (\varphi(x;s) \wedge \psi(x;s)) \iff (d_p x\, \varphi(x;s) \wedge d_p x\, \psi(x;s))),$$

$$M \vDash \forall s\, (d_p x\, \neg\varphi(x;s) \iff \neg d_p x\, \varphi(x;s))$$

   and

$$M \vDash \forall s\, d_p x\, \varphi(x;s) \Rightarrow \exists x\, \varphi(x;s).$$

   It follows that p has a (unique) global definable extension and this extension has the same defining scheme.

2. If A is not a model, the situation is more complicated. There is no reason for the defining scheme of p to be compatible with $\neg$ and $\wedge$ or to be consistent, i.e. $M \vDash \forall s\, d_p x\, \varphi(x;s) \Rightarrow \exists x\, \varphi(x;s)$. Therefore, we have to distinguish two notions of definability: those definable types that have a global definable extension given by the defining scheme and those that do not. Most of the time in this work, we will be considering global definable types, but the rare time types are not over a model, we will specify which notion we are considering.

One can check that a set A is stably embedded if every type over A is definable (by a defining scheme that might not define a global type). Hence, in a stable theory T, every set $A \subseteq M \vDash T$ is stably embedded (and that is equivalent to stability). The existence of honest definitions in NIP theories tells us that this fact remains "finitely" true in a uniform way:

Theorem 0.3.11 ([CS13])**:**

Let T be an NIP *$\mathcal{L}$-theory and $\varphi(x;s)$ be an $\mathcal{L}$-formula, $M \vDash T$, $A \subseteq M$ and $b \in M$. There exists a formula $\psi(x;t)$ such that for any finite $A_0 \subseteq \varphi(A;b)$, there exists a tuple $d \in A$ such that $A_0 \subseteq \psi(A;d) \subseteq \varphi(A;b)$.*

In fact, the (p,q)-theorem can be used to show that these honest definitions are even more uniform:

Theorem 0.3.12 ([CS])**:**

Let T be an NIP *$\mathcal{L}$-theory and $\varphi(x;s)$ be an $\mathcal{L}$-formula. There exists a formula $\psi(x;t)$ such that for any $M \vDash T$, any $A \subseteq M$, any tuple $b \in N$ and any $A_0 \subseteq \varphi(A;b)$ finite, there exists a tuple $d \in A$ such that $A_0 \subseteq \psi(A;d) \subseteq \varphi(A;b)$.*

We now want to define types that behave generically as if one were in a stable theory. But first, we need some more definitions.

Definition 0.3.13 ($p \otimes q$)**:**
Let M be a saturated enough $\mathcal{L}$-structure. Let p and $q \in \mathcal{S}(M)$ be such that q is $\mathrm{Aut}(M/A)$-invariant for some $A \subseteq M$. The type $p \otimes q \in \mathcal{S}(M)$ is the type such that for all $C \subseteq M$ containing A, $p \otimes q|_C = \mathrm{tp}(ab/C)$ for any $a \vDash p|_C$ and $b \vDash q|_{Ca}$.

One can check that because q is $\mathrm{Aut}(M/A)$-invariant, this is well-defined. Note that if p is also $\mathrm{Aut}(M/A)$-invariant, $p \otimes q$ is $\mathrm{Aut}(M/A)$-invariant and that if p and q are both definable, $p \otimes q$ is defined by $d_p x\, d_q y \varphi(x, y; s)$ where $d_p x \varphi(x, y; s)$ is the formula in the defining scheme of p corresponding to φ and similarly for q.
Let T be an NIP $\mathcal{L}$-theory.

Proposition 0.3.14:
Let $(a_i)_{i\in I}$ be an indiscernible sequence and $\varphi(x)$ be an $\mathcal{L}(M)$-formula. Then there exists an end segment $I_0 \subseteq I$ such that either for all $i \in I_0$, $M \vDash \varphi(a_i)$ or for all $i \in I_0$, $M \vDash \neg\varphi(a_i)$

Definition 0.3.15 ($\lim(\mathcal{I})$)**:**
*Let $\mathcal{I} = (a_i)_{\in I}$ be an indiscernible sequence. We define $\lim(\mathcal{I})$ to be the global type such that for all $\mathcal{L}$-formula $\varphi(x; s)$ and $m \in M$, $\varphi(x; m) \in \lim(\mathcal{I})$ if and only if $\varphi(x; m)$ holds on an end segment of $\mathcal{I}$. By Proposition **(0.3.14)**, this is a complete type.*

For any $\mathrm{Aut}(M/A)$-invariant type p, we define $p^{\otimes 1} := p$, $p^{\otimes n+1} := p^{\otimes n} \otimes p$ and $p^{\otimes\omega}(x_{i<\omega}) := \bigcup_n p^{\otimes n}(x_{i<n})$. Note that because $p^{\otimes n}(x_{i<n}) \subseteq p^{\otimes n+1}(x_{i<n+1})$ this is a well-defined type. We call its realisations Morley sequences of p.

Definition 0.3.16 (Generically stable type, [She04; HP11])**:**
Let $M \vDash T$ be a saturated enough and let $A \subseteq M$ and $p \in \mathcal{S}(M)$ be $\mathrm{Aut}(M/A)$-invariant, then p is generically stable if any of the following equivalent properties hold:

(i) p is $\mathcal{L}(M)$-definable and finitely satisfiable in some (small) $N \preccurlyeq M$;

(ii) For all $q \in \mathcal{S}(M)$ which is $\mathrm{Aut}(M/C)$-invariant for some small $C \subseteq M$, $p \otimes q = q \otimes p$;

(iii) $p^{\otimes 2}(x, y) = p^{\otimes 2}(y, x)$;

(iv) Any Morley sequence of p is totally indiscernible;

(v) For all $\mathcal{I} \vDash p^{\otimes\omega}|_A$, $\lim(\mathcal{I}) = p$.

Definition 0.3.17 ($f_\star p$)**:**
Let $p \in \mathcal{S}(C)$ and f be an $\mathcal{L}(C)$-definable function defined on p. The type $f_\star p$ is $\mathrm{tp}(f(a)/C)$ for any $a \vDash p$.

One can check that if $p \in \mathcal{S}(M)$ is $\mathrm{Aut}(M/C)$-invariant and f is $\mathcal{L}(C)$-definable, then $f_\star p$ is also $\mathrm{Aut}(M/C)$-invariant. Similarly if $p \in \mathcal{S}(M)$ is $\mathcal{L}(C)$-definable and f is $\mathcal{L}(C)$-definable, $f_\star p$ is $\mathcal{L}(C)$-definable and we can take $d_{f_\star p} y \varphi(y; s) = d_p x \varphi(f(x); s)$. Similarly, when $(f_i)_{i\in I}$ is a tuple of $\mathcal{L}(C)$-definable functions (what is usually called a $\star$-definable function cf. Definition **(IV.5.2)**), $f_\star p$ also makes sense.

Lemma 0.3.18:
Let $M \vDash T$, $A \subseteq M$, $p \in \mathcal{S}(M)$ be $\mathrm{Aut}(M/A)$-invariant and generically stable and f be $\mathcal{L}(A)$-definable and defined on p, then $f_\star p$ is generically stable.

Proof. Let $C \supseteq A$, $a \vDash p|_C$ and $b \vDash p|_{Ca}$. Then $(a, b) \vDash p^{\otimes 2}|_C$. As p is generically stable, we also have $(b, a) \vDash p^{\otimes 2}$. It follows that $\mathrm{tp}(f(a), f(b)/C) = \mathrm{tp}(f(b), f(a)/C)$. But

$f(a) \models f_\star p|_C$ and $f(b) \models f_\star p|_{Cf(a)}$. It follows that both $(f(a), f(b))$ and $(f(b), f(a))$ are realisations of $(f_\star p)^{\otimes 2}|_C$. ■

0.3.3. C-minimality

This section is not exactly about NIP, but about subclasses of NIP, among them the class of C-minimal theories, which are all defined by asking that the unary sets definable in models of a given theory are all quantifier free definable in a particularly simple sublanguage. The most simple of these minimality notions (and by far the oldest, it is already central to the proof of the Baldwin-Lachlan theorem, cf. [BL71]) is minimality with respect to the language of equality:

Definition 0.3.19 (Strong minimality)**:**
An $\mathcal{L}$-theory T is strongly minimal if every unary set definable in models of T is either finite or cofinite.

Strongly minimal theories are the tamest theories possible (they are, in some sense, ω-stable theories of dimension one). Amongst them, we find infinite sets, infinite k-vector spaces for some fixed field k and algebraically closed fields.
The next notion of minimality we will consider is minimality with respect to an order. It was first studied in [PS86], based on previous work by van den Dries in [Dri84b].

Definition 0.3.20 (o-minimality)**:**
Let $\mathcal{L}$ be a language containing a binary relation symbol $<$ and T be an $\mathcal{L}$-theory which implies that $<$ is a linear order. The theory T is o-minimal if every unary set definable in models of T is a finite union of intervals.

Among o-minimal theories, one can find the theory of discrete orders with or without end points, dense orders with or without end points, the theories DOAG (divisible ordered abelian groups) and RCF (real closed fields), the theory of the field $\mathbb{R}$ with the exponential function and even the theory of the field $\mathbb{R}$ with the exponential function and analytic functions on compact domains.
The last class we want to consider is somehow a mix of the two previous ones that was introduced in [MS96].

Definition 0.3.21 (C-relation)**:**
Let X be a set, a C-relation on X is a relation $C \subseteq X^3$ that verifies the following axioms:

(i) $\forall x \forall y \forall z\, (C(x;y;z) \Rightarrow C(x;z;y))$;

(ii) $\forall x \forall y \forall z\, (C(x;y;z) \Rightarrow \neg C(y;x;z))$;

(iii) $\forall x \forall y \forall z \forall w\, (C(x;y;z) \Rightarrow (C(w;y;z) \vee C(x;w;z)))$;

(iv) $\forall x \forall y\, (x \neq y \Rightarrow \exists z\, z \neq y \wedge C(x;y;z))$.

A C-relation is essentially linked to the presence of a tree with meets. Indeed for any C-relation on a set X there exists a tree $(T, \leqslant)$ with meets (where the root is at the bottom) such that the leaves of T are exactly the points in X and $C(x,y,z)$ holds if and only if $x \wedge z < y \wedge z$. If (K, val) is a valued field, the relation $\mathrm{val}(x-z) < \mathrm{val}(y-z)$ is a C-relation and the associated tree is the tree of closed balls. By analogy with valued fields, in a set X with a C-relation, an open ball is a set of the form $\{x \in X : C(a,x,b)\}$ (we also consider the whole set X to be an open ball), a closed ball in a set of the form $\{x \in X : \neg C(x,a,b)\}$ and a swiss cheese is a ball from which a finite number of balls have been removed.

Definition 0.3.22 (C-minimality)**:**
Let $\mathcal{L}$ be a language containing a ternary relation symbol C and T be an $\mathcal{L}$-theory which implies that C is a C-relation. The theory T is C-minimal if every unary set definable in models of T is a finite union of swiss cheeses.

It follows from Theorem **(0.2.9)**, that ACVF is a C-minimal theory for the C-relation described above. Algebraically closed valued fields with analytic structure (cf. Section II.3) are also C-minimal. To finish, let us give a formal statement, in the case of valued fields, of the previous somewhat vague statement that C-minimality is a mix of strong minimality and o-minimality.

Proposition 0.3.23:
Let T be a C-minimal $\mathcal{L}$-theory of valued fields, where C is defined as above. Then $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded, the induced structure on $\mathbf{k}$ is strongly minimal and the induced structure on $\mathbf{\Gamma}$ is o-minimal.

Proof. Let $M \vDash T$ and $X \subseteq \mathbf{\Gamma}$ be a unary $\mathcal{L}$-definable set. The set $\mathrm{val}^{-1}(X)$ is both a (potentially infinite) union of annuli around 0 and a finite union of swiss cheeses, hence it is a finite union of annuli around 0 and X must be a finite union of intervals. Therefore, $\mathbf{\Gamma}$ is o-minimal in T and by [HO10], $\mathbf{\Gamma}$ is stably embedded in models of T.
Similarly, let $X \subseteq \mathbf{k}$ be a unary $\mathcal{L}$-definable set. The set $\mathrm{val}^{-1}(X)$ is both a (potentially infinite) union of disjoint open balls of radius 0 inside $\mathcal{O}$ and a finite union of swiss cheeses, hence it must either be a finite union of disjoint open balls of radius 0 or $\mathcal{O}$ minus a finite number of open balls of radius 0. In the first case, X is finite and in the other X is cofinite. Thus $\mathbf{k}$ is strongly minimal in T and that easily implies that $\mathbf{k}$ is stably embedded. ■

0.4. Stable domination

In ACVF, the value group, being ordered, is obviously unstable and the residue field, being strongly minimal, is stable. To build on this idea that a valued field is controlled by its residue field and value group, one would like to show that in some sense the theory is stable-like over the value group. Let T be an $\mathcal{L}$-theory.

0.4.1. Definition and properties

Let us first begin with some equivalent definitions of stable embeddedness for partial types (cf. the appendix of [CH99]). Let M be a sufficiently saturated and homogeneous $\mathcal{L}$-structure.

Proposition 0.4.1:
Let P be a partial $\mathcal{L}$-type. The following are equivalent:

(i) $P(M)$ *is stably embedded in* M*;*

(ii) *For all tuples* $a \in M$, $\mathrm{tp}(a/\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(P(M))) \vdash \mathrm{tp}(a/P(M))$*;*

(iii) *For all tuples* $a \in M$, $\mathrm{tp}(a/P_0) \vdash \mathrm{tp}(a/P(M))$ *for some (small)* $P_0 \subseteq P(M)$*;*

(iv) *For all tuples* $a \in M$, $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(P_0)) \vdash \mathrm{tp}(a/P(M))$ *for some (small)* $P_0 \subseteq P(M)$*;*

(v) *For all tuples* $a \in M$, $\mathrm{tp}(a/P(M))$ *is* $\mathcal{L}(P(M))$*-definable.*

Definition 0.4.2 (Stable stably embedded sets)**:**
Let $C \subseteq M$ and P be a partial $\mathcal{L}(C)$-type. We say that P is stable, stably embedded if any of the following statements hold:

(i) *There is no* $\mathcal{L}(M)$*-formula* $\varphi(x, y)$*, tuples* $(a_i)_{i<\omega} \in P(M)$ *and tuples* $(b_i)_{i<\omega} \in M$ *such that* $M \vDash \varphi(a_i, b_j)$ *if and only if* $i \leqslant j$*;*

(ii) P *is stably embedded and the* $\mathcal{L}(C)$*-induced structure on* P *is stable;*

(iii) *For all tuples* $a \in P(M)$ *and* $A \subseteq M$, $\mathrm{tp}(a/A)$ *is definable;*

(iv) *Let* $\kappa > |\mathcal{L}| + |C|$ *with* $\kappa = \kappa^{\aleph_0}$ *and* $B \subseteq M$ *which contains* C *and has cardinality* κ*, then there are at most* κ 1*-types over* B *realized in* $P(M)$*.*

In $\mathrm{ACVF}^{\mathcal{G}}$, stable, stably embedded definable sets can all be described, and they are closely linked to the residue field:

Proposition 0.4.3 ([HHM06, Lemma 2.6.2 and Remark 2.6.3])**:**
Let $M \vDash \mathrm{ACVF}^{\mathcal{G}}$, $C \subseteq M$ and P be an $\mathcal{L}^{\mathcal{G}}(C)$-definable set. The following are equivalent:

(i) P *is stable, stably embedded;*

(ii) P *is* **k***-internal, i.e. there exists a finite* $A \subseteq M$ *such that* $P \subseteq \mathrm{dcl}(A \cup k)$*;*

(iii) $P \subseteq \mathrm{dcl}(E \cup k)$ *for some finite* $E \subseteq P$*;*

(iv) *There is no* $\mathcal{L}^{\mathcal{G}}(M)$*-definable map from* P^l *for some* $l \in \mathbb{N}$ *onto an infinite interval in* $\mathbf{\Gamma}$*.*

Let us go back to a more general setting and let $\mathcal{L}$ be a language, M be a saturated and homogeneous enough $\mathcal{L}$-structure that eliminates imaginaries and $C \subseteq M$.

Definition 0.4.4 (St_C)**:**
The stable part over C, denoted St_C, is a structure whose sorts are all the $\mathcal{L}(C)$-definable stable, stably embedded sets with their $\mathcal{L}(C)$-induced structure.

Remark 0.4.5:

1. Finite sets are stable, stably embedded, hence $\mathrm{acl}(C) \subseteq \mathrm{St}_C$.

2. The structure St_C is stably embedded in M and its theory is stable. We will denote by $\downarrow$ the independence relation in St_C.

3. At some point we might have two or more languages around so we will specify the language by writing $\mathrm{St}_C^{\mathcal{L}}$.

Definition 0.4.6 (Stable domination):
Let $p \in \mathcal{S}(C)$, we say that p is stably dominated if for every $a \models p$ and $B \subseteq M$ such that $\mathrm{St}_C(\mathrm{dcl}(CB)) \downarrow_C \mathrm{St}_C(\mathrm{dcl}(Ca))$,

$$\mathrm{tp}(B/\mathrm{St}_C(\mathrm{dcl}(Ca))) \vdash \mathrm{tp}(B/Ca).$$

Let me now give some properties of stably dominated types.

Lemma 0.4.7:
Let $\mathcal{R}$ be a set of dominant sorts in T and $p \in \mathcal{S}(C)$. Assume that for all $B \subseteq \mathcal{R}(M)$ such that $\mathrm{St}_C(\mathrm{dcl}(CB)) \downarrow_C \mathrm{St}_C(\mathrm{dcl}(Ca))$, we have $\mathrm{tp}(B/\mathrm{St}_C(\mathrm{dcl}(Ca))) \vdash \mathrm{tp}(B/Ca)$. Then p is stably dominated.

Proof. Let $a \models p$ and $B \subseteq M$ be such that $\mathrm{St}_C(\mathrm{dcl}(CB)) \downarrow_C \mathrm{St}_C(\mathrm{dcl}(Ca))$. By domination of $\mathcal{R}$, there exists $D \subseteq \mathcal{R}(M)$ such that $B \subseteq \mathrm{dcl}(D)$. As St_C is stably embedded, we may assume that the $\mathrm{St}_C(\mathrm{dcl}(CD)) \downarrow_C \mathrm{St}_C(\mathrm{dcl}(Ca))$ and hence $\mathrm{tp}(D/\mathrm{St}_C(\mathrm{dcl}(Ca))) \vdash \mathrm{tp}(D/Ca)$. Let B' be such that $B' \equiv_{\mathrm{St}_C(\mathrm{dcl}(Ca))} B$. There exists $\sigma \in \mathrm{Aut}(M/\mathrm{St}_C(\mathrm{dcl}(Ca)))$ such that $\sigma(B) = B'$. We have $\sigma(D) \equiv_{\mathrm{St}_C(\mathrm{dcl}(Ca))} D$ and hence $\sigma(D) \equiv_{Ca} D$. Because $B \subseteq \mathrm{dcl}(D)$, it follows that $B' = \sigma(B) \equiv_{Ca} B$. ■

Proposition 0.4.8 ([HHM08, Proposition 3.13]):
Let $p \in \mathcal{S}(C)$ be stably dominated, then p has a global $\mathcal{L}(\mathrm{acl}(C))$-definable extension.

Proposition 0.4.9 ([HHM08, Proposition 3.31.(iii)]):
Let $a \in M$ be a tuple. The type $\mathrm{tp}(a/C)$ is stably dominated if and only if $\mathrm{tp}(a/\mathrm{acl}(C))$ is.

Proposition 0.4.10 ([HHM08, Proposition 3.32.(iii)]):
Let $c \in M$ be a tuple and f be an $\mathcal{L}(C)$-definable function. If $\mathrm{tp}(c/C)$ is stably dominated then so is $\mathrm{tp}(f(c)/C)$.

Proposition 0.4.11 ([HHM08, Proposition 4.1]):
Let $p \in \mathcal{S}(M)$ be $\mathrm{Aut}(M/C)$-invariant and let $B \subseteq M$ contain C. If $p|_C$ is stably dominated, so is $p|_B$.

The other direction (i.e. a positive answer to the question whether a global C-invariant type p is stably dominated over $B \supseteq C$ is stably dominated over C) is called descent and is much more complicated. In fact, we only know how to prove it under the assumption that the type of B over C has a global invariant extension:

Proposition 0.4.12 ([HHM08, Theorem 4.9]):
Let p and $q \in \mathcal{S}(M)$ be $\mathrm{Aut}(M/C)$-invariant. Assume that for all $b \models q|_C$, $p|_{Cb}$ is stably dominated. Then $p|_C$ is also stably dominated.

Let us now define the invariant extension property:

Definition 0.4.13 (Invariant extension property)**:**
Let T be an $\mathcal{L}$ theory that eliminates imaginaries, $A \subseteq M$ for some $M \vDash T$. We say that T has the invariant extension property over A if, for all $N \vDash T$, every type $p \in \mathcal{S}(A)$ can be extended to an $\mathrm{Aut}(N/A)$-invariant type.
We say that T has the invariant extension property if T has invariant extensions over any $A = \mathrm{acl}(A) \subseteq M \vDash T$.

Lemma 0.4.14:
Let T be an $\mathcal{L}$ theory that eliminates imaginaries. Assume that T has the invariant extension property. Let p and $q \in \mathcal{S}(M)$ be $\mathrm{Aut}(M/C)$-invariant and $B \subseteq M$, $C \subseteq B$, be such that $p|_B$ is stably dominated. Then $p|_C$ is also stably dominated.

Proof. Let q be an $\mathrm{Aut}(M/\mathrm{acl}(C))$-invariant extension of $\mathrm{tp}(B/\mathrm{acl}(C))$. Let $B' \vDash q|_{\mathrm{acl}(C)}$. Then there is an automorphism $\tau \in \mathrm{Aut}(M/\mathrm{acl}(C))$ such that $\tau(B') = B$. Because p is $\mathrm{Aut}(M/C)$-invariant $p|_{B'} = \tau(p|_B)$. It follows that $p|_{B'}$ is stably dominated. Hence, by Proposition **(0.4.12)**, $p|_{\mathrm{acl}(C)}$ is stably dominated. We conclude by Proposition **(0.4.9)**. ■

In [HHM08], it is shown that there are many stably dominated types in ACVF. In fact, ACVF is what is called a metastable theory. Let T be an NIP $\mathcal{L}$-theory eliminating imaginaries and let $\mathbf{\Gamma}$ be a stably embedded $\mathcal{L}$-definable set with an $\mathcal{L}$-definable linear order.

Definition 0.4.15 (Maximally complete field)**:**
A valued field K is said to be maximally complete if for every chain of balls $(b_i)_{i\in I}$ in K, the intersection $\bigcap_{i\in I} b_i(K)$ is non empty.

This is also often referred to as spherical completeness. Equivalently, K is maximally complete if every pseudo-convergent sequence from K has a pseudo-limit in K (see Definition **(II.4.12)**). Note that a maximally complete field is in particular Henselian and that Hahn fields are maximally complete.

Definition 0.4.16 (Metastability)**:**
We say that T is metastable over $\mathbf{\Gamma}$ if:

(i) *The theory T has the invariant extension property.*

(ii) Existence of metastability bases: *For all $A \subseteq M$, there exists $C \subseteq M$ containing B such that for all tuples $a \in M$, $\mathrm{tp}(a/C\mathbf{\Gamma}(\mathrm{dcl}(Ca)))$ is stably dominated.*

Theorem 0.4.17 ([HHM08, Theorem 12.18.(ii) and Corollary 8.16])**:**
The theory ACVF is metastable over $\mathbf{\Gamma}$. In fact, the metastability bases are exactly the maximally complete models of ACVF.

0.4.2. Orthogonality to Γ

The following definition of orthogonality is only equivalent to classical notions if $\mathbf{\Gamma}$ is stably embedded and eliminates imaginaries for the $\mathcal{L}$-induced structure.

Definition 0.4.18 (Orthogonality to $\mathbf{\Gamma}$)**:**
Let $\mathbf{\Gamma}$ be an $\mathcal{L}$-definable set and $p \in \mathcal{S}(M)$ be an $\mathrm{Aut}(M/C)$-invariant type. It is orthogonal to $\mathbf{\Gamma}$ if any of the following equivalent statements hold:

(i) For all $B \subseteq M$ containing A and $a \models p|_B$, $\mathbf{\Gamma}(\mathrm{dcl}(Ba)) = \mathbf{\Gamma}(\mathrm{dcl}(B))$.

(ii) For all $\mathcal{L}(M)$-definable functions f whose domain contains p and whose codomain is $\mathbf{\Gamma}$, the type $f_\star p$ is realized in M.

The three notions we have defined: stable domination, generic stability and orthogonality to $\mathbf{\Gamma}$; are related.

Proposition 0.4.19:
Let T be an NIP *$\mathcal{L}$-theory eliminating imaginaries and let $\mathbf{\Gamma}$ be a stably embedded $\mathcal{L}$-definable set with an $\mathcal{L}$-definable linear order. Let $p \in \mathcal{S}(M)$ be an $\mathrm{Aut}(M/C)$-invariant type. We have:*

(i) If p is stably dominated then it is generically stable;

(ii) If p is generically stable it is orthogonal to $\mathbf{\Gamma}$.

Proof.

(i) By Definition **(0.3.16)**, we have to show that $p \otimes p(x, y) = p \otimes p(y, x)$. Let $A = \mathrm{acl}(A) \subseteq M$ contain C, $a \models p|_A$ and $b \models p|_{Aa}$. We have to show that $a \models p|_{Ab}$. We have $\mathrm{St}_A(\mathrm{dcl}(Ab)) \models (\mathrm{St}_{A\star} p)|_{Aa}$ where St_A is considered as a $\star$-definable function sending x to $\mathrm{St}_A(\mathrm{dcl}(Ax))$. As p is $\mathrm{Aut}(M/C)$ invariant and $C \subseteq A$, $(\mathrm{St}_{A\star} p)|_{Aa}$ does not fork over A. In particular $\mathrm{St}_A(\mathrm{dcl}(Ab)) \downarrow_A \mathrm{St}_A(\mathrm{dcl}(Aa))$. By, Lemma **(0.4.14)**, $p|_A$ is stably dominated and hence $\mathrm{tp}(a/A\mathrm{St}_A(\mathrm{dcl}(Ab))) \vdash \mathrm{tp}(a/Ab)$.

Let $a' \models p|_{Ab}$. Then, as above $\mathrm{St}_A(\mathrm{dcl}(Ab)) \downarrow_A \mathrm{St}_A(\mathrm{dcl}(Aa'))$ and hence

$$\mathrm{St}_A(\mathrm{dcl}(Aa')) \equiv_{\mathcal{L}(A\mathrm{St}_A(\mathrm{dcl}(Ab)))} \mathrm{St}_A(\mathrm{dcl}(Aa)).$$

Because St_A is stably embedded, we have, in fact, $a' \equiv_{\mathcal{L}(A\mathrm{St}_A(\mathrm{dcl}(Ab)))} a$ and hence $\mathrm{tp}(a/Ab) = \mathrm{tp}(a'/Ab) = p|_{Ab}$.

(ii) Let f be any $\mathcal{L}(M)$-definable function into Γ defined on p. By Lemma **(0.3.18)** $f_\star p$ is generically stable. But because $\mathbf{\Gamma}$ has an $\mathcal{L}$-definable order, the only generically stable types in $\mathbf{\Gamma}$ are realized. ■

■

In ACVF, the instability is essentially contained in $\mathbf{\Gamma}$ and the three notions coincide. The equivalence of (i) and (iii) is already proved in [HHM08, Section 10].

Proposition 0.4.20 ([HL, Proposition 2.8.1])**:**
Let $M \models \mathrm{ACVF}^{\mathcal{G}}$, $C \subseteq M$ and $p \in \mathcal{S}(M)$ be an $\mathrm{Aut}(M/C)$-invariant type. The following are equivalent;

(i) p is stably dominated;

(ii) p *is generically stable;*

(iii) p *is orthogonal to* $\mathbf{\Gamma}$.

Proof. By Proposition **(0.4.19)**, it suffices to prove that (iii)⇒(i). Let $C_0 \preccurlyeq M$ contain C and be maximally complete. Let $a \vDash p|_{C_0}$. By Theorem **(0.4.17)**, $\mathrm{tp}(a/C_0\mathbf{\Gamma}(\mathrm{dcl}(C_0a)))$ is stably dominated. But, because p is orthogonal to $\mathbf{\Gamma}$, $\mathbf{\Gamma}(\mathrm{dcl}(C_0a)) = \mathbf{\Gamma}(C_0)$ and hence $\mathrm{tp}(a/C_0) = p|_{C_0}$ is stably dominated. We can now conclude by Proposition **(0.4.11)**. ■

CHAPTER I

Imaginaries in p-adic fields

Le Logicien
La logique mène au calcul mental.
E. Ionesco, *Rhinocéros*, Acte I

This chapter contains [HMR], joint with Ehud Hrushovski and Ben Martin. In the paper, one can also find an appendix by Raf Cluckers that is not included here. The Appendix consists of a complete proof of the rationality results in Section I.6 for fixed p, which also extends to the analytic case, while avoiding an explicit elimination of imaginaries.
Although I was not directly involved in the elaboration of sections I.7 and I.8 either, they are included here both to keep the paper as a whole and to show an application of the model theoretic results.

I.1. Introduction

This paper concerns the model theory of the p-adic numbers $\mathbb{Q}_p$ and applications to certain counting problems arising in group theory. Recall that a theory (in the model-theoretic sense of the word) is said to have *elimination of imaginaries* (EI) if the following holds: for every model M of the theory, for every definable subset D of some M^n and for every definable equivalence relation R on D, there exists a definable function $f: D \to M^m$ for some m such that the fibers of f over $f(D)$ are precisely the equivalence classes of R. In other words, elimination of imaginaries states that every pair (D, E) (consisting of a definable set D and a definable equivalence relation E on it) reduces to a pair (D', E') where E' is the equality—where, as in descriptive set theory, we say that (D, E) *reduces to* (D, E) if there exists a $\varnothing$-definable map $f: D' \to D$ with $xE'y \iff f(x)Ef(y)$.
The theory of $\mathbb{Q}_p$ (in the language of rings with a predicate for $v(x) \geqslant v(y)$) does not admit EI [SM93]: for example, no such f exists for the definable equivalence relation R on $\mathbb{Q}_p$ given by xRy if $\mathrm{val}(x-y) \geqslant 1$, because $\mathbb{Q}_p/R$ is countably infinite but any definable subset of $\mathbb{Q}_p^m$ is either finite or uncountable. We show that the theory of $\mathbb{Q}_p$ (and even of any finite extension) with some extra sorts for $\mathbb{Z}_p$-lattices admits EI (cf. Theorem **A**). In Theorem **B**, we show that the theory of ultraproducts of $\mathbb{Q}_p$ also eliminates imaginaries if we add similar sorts. It follows that the elimination in $\mathbb{Q}_p$ is uniform in p (see Corollary **(I.2.7)** for a precise statement of the uniformity). In fact, we prove a more general result (Corollary **(I.2.15)**): given two theories $\widetilde{T}, T$ satisfying certain hypotheses, $\widetilde{T}$ has EI if T does. In our application, T is the theory of algebraically closed valued fields of mixed characteristic ($\mathrm{ACVF}_{0,p}$) or equicharacteristic zero ($\mathrm{ACVF}_{0,0}$) and $\widetilde{T}$ is either the theory of a finite extension of $\mathbb{Q}_p$ or the theory of an ultraproduct of $\mathbb{Q}_p$ where p varies, with appropriate extra constants in each

cases (in fact in the latter case Corollary **(I.2.15)** does not apply immediately but a variant does).

The notion of an invariant extension of a type plays a key part in our proof. If T is a theory, $M \models T$, $A \subseteq M$ and p is a type over A then an *invariant extension* of p is a type q over M such that $q|_A = p$ and q is $\mathrm{Aut}(M/A)$-invariant. The theory ACVF is not stable; in [HHM06; HHM08], Haskell, the first author and Macpherson used invariant extensions of types to study the stability properties of ACVF and to define notions of forking and independence. They proved that ACVF plus some extra sorts admits EI.

Let us now come back to the meaning of our elimination of imaginaries result. It shows that any (D', E') can be reduced to a (D, E) of a special kind—namely, the equivalence relation on $\mathrm{GL}_N(\mathbb{Q}_p)$ for some N whose equivalence classes are the left $\mathrm{GL}_N(\mathbb{Z}_p)$-cosets. The quotients D/E have a specific geometric meaning; but can one explain abstractly in what way they are special? One useful statement is that an arbitrary equivalence relation is reduced to a quotient by a definable group action. Another property concerns volumes; the E-classes have volumes that are *motivically invertible*; indeed, they are equivalent to a polydisk of an appropriate dimension and size.

Indeed, it is only this property of the geometric imaginaries that is actually used in the application in Section I.6 where we show an abstract uniform rationality result, Theorem **C**, for zeta functions counting the number of classes in some definable family of equivalence relations. The proof of this result relies on representing the number of classes of some definable relation E on some definable set D as an integral. The idea, going back to Denef and Igusa, is simple: the number of classes of E on D equals the volume of D, for any measure such that each E-class has measure one. The question is how to come up (definably) with such a measure. The setting is that we already have the Haar measure μ on $\mathbb{Q}_p$ (normalized so that $\mu(\mathbb{Z}_p) = 1$), and for simplicity (one can easily reduce to this case) let us assume each E-class $[x]_E \in D/E$ has finite, nonzero measure. The question then is to show that there exists a definable function $f\colon D \to \mathbb{Q}_p$ such that the measure of each E-class $[x]_E$ is of the form:

$$\mu([x]_E) = |f(x)| \tag{I.1}$$

and then replace μ by $f^{-1}\mu$. In practice, f is usually given explicitly (cf. [GSS88, Section 2]). This is straightforward for the case of p-power index subgroups of Γ coded in the usual way. For more complicated equivalence relations, however, it is not clear *a priori* that such an f can be found, even in principle.

Given EI to the geometric sorts, we can represent E as the coset equivalence relation of $\mathrm{GL}_n(\mathbb{Z}_p)$. In this case we can take Haar measure on $\mathrm{GL}_n(\mathbb{Z}_p)$ where automatically each class has measure one. Equivalently, with respect to the additive Haar measure, we use the explicit factor $1/|\det(M)|$. In other words for this canonical E, the reciprocal of the (additive) Haar measure of any E-class *is* represented.

We illustrate the power of Theorem **C** by using it to prove rationality results for certain zeta functions of finitely generated nilpotent groups. Grunewald, Segal and Smith [GSS88] showed that subgroups of p-power index of such a group Γ can be parametrized p-adically. More precisely, these subgroups can be *coded*: that is, placed in bijective correspondence with the set of equivalence classes of some definable equivalence relation on a definable subset D of some $\mathbb{Q}_p^N$. Let $a_n < \infty$ denote the number of subgroups of Γ of index n. Using p-adic integration over D and results of Denef and Macintyre, Grunewald, Segal and Smith showed that the p-local subgroup zeta function $\sum_{n=0}^{\infty} a_{p^n} t^n$ is a rational function of t, and that the degrees of the numerator and denominator of this rational function are bounded independently of p. Du Sautoy and others have calculated subgroup zeta functions explicitly in many cases [Sau01; SW08; Vol04] and studied uniformity questions (the behavior of the p-local subgroup zeta function as the prime p varies). For instance, du Sautoy and Grunewald proved a uniformity result by showing that the p-adic integrals that arise in the calculation of subgroup zeta functions fall into a special class they call *cone integrals* [SG00]. See the start of Section I.7 for further discussion of uniformity in the context of subgroup zeta functions.

We also consider situations where it is not clear how to construct suitable definable p-adic integrals. The main one, and the original motivation for our results, is in the area of representation growth. This is analogous to subgroup growth: one counts not the number $a_{p,n}$ of index p^n subgroups of a group Γ, but the number $b_{p,n}$ of irreducible p^n-dimensional complex characters of Γ (modulo tensoring by one-dimensional characters if Γ is nilpotent). Jaikin [Jai06] proved, under mild technical restrictions, that the p-local representation zeta functions of semi-simple compact p-adic analytic groups are rational (the second author is grateful to him for explaining his work). Representation growth of finitely generated pro-p groups was studied by Jaikin [Jai06]; Lubotzky and the second author gave a partial criterion for an arithmetic group to have the congruence subgroup property in terms of its representation growth [LM04, Theorem 1.2]. We prove that the p-local representation zeta function $\zeta_{\Gamma,p}(s) := \sum_{n=0}^{\infty} b_{p,n} p^{-ns}$ of a finitely generated nilpotent group Γ is a rational function of p^{-s} and has good uniformity properties as p varies (Theorem I.8).

The results in Sections I.7 and I.8 both follow the same idea: we show how to interpret (definably) in $\mathbb{Q}_p$ the sets we want count. More precisely, in Section I.7 we show how to interpret in $\mathbb{Q}_p$ the set of finite-index subgroups H of Γ and we show that the equivalence relations that arise are uniformly definable in p. This allows us to apply Theorem **C**. The same idea is used in Section I.8, but the details are more complicated. We show how to interpret in $\mathrm{Th}(\mathbb{Q}_p)$ the set of pairs (N, σ), where N is a finite-index normal subgroup of Γ and σ is an irreducible character of Γ/N, up to twisting by one-dimensional characters. The key idea is first to code triples (H, N, χ), where H is a finite-index subgroup of Γ, N is a finite-index normal subgroup of H and χ is a one-dimensional character of H/N—the point is that finite nilpotent groups are monomial, so any irreducible character is induced from a one-dimensional character of a subgroup. The equivalence relation of giving the same induced character can be formulated in terms of restriction, and shown to be definable. Inspecting these constructions shows that they are all uniform in p, so again Theorem **C** applies.

Since the first draft of this paper [HM08] was circulated, there has been considerable activity in the field of representation growth. Jaikin (*loc. cit.*) used the coadjoint orbit formalism of Howe and Kirillov to parametrize irreducible characters of p-adic analytic groups; rationality of the representation zeta function then follows from the usual arguments of semi-simple compact p-adic integration. Voll used similar ideas to parametrize irreducible characters of finitely generated torsion-free nilpotent groups, and showed that representation zeta functions are rational and satisfy a local functional equation [Vol10] (in fact, he proved this for a very general class of zeta functions that includes representation zeta functions and subgroup zeta functions as special cases). Stasinski and Voll [SV14] proved a uniformity result for representation zeta functions and calculated these zeta functions for some families of nilpotent groups. Ezzat [Ezz12] and Snocken [Sno13] have calculated further examples of representation zeta functions of nilpotent groups. For work on representation growth for other kinds of group, see [LL08; Avn+12; Avn+13; Avn11].

The Kirillov orbit method has the advantage that it linearises the problem of parametrizing irreducible representations and simplifies the form of the imaginaries that appear. The disadvantage is that the proof of rationality only applies to $\zeta_{\Gamma,p}(s)$ for almost all p—one must discard a finite set of primes. *We stress that our result Theorem I.8 is the only known proof of rationality of $\zeta_{\Gamma,p}(s)$ that works for every p.*

This paper falls naturally into two parts. The first part is model-theoretic: in Section I.2 we establish an abstract criterion, Proposition **(I.2.11)**, for elimination of imaginaries and apply it in Sections I.4 and I.5 to prove Theorems **A** and **B**. Section I.3 consists of a study of unary types in henselian valued fields, which is used extensively in Sections I.4 and I.5. In Section I.6 we establish the general rationality result Theorem **C**.

In the second part (Sections I.7 and I.8), we apply Theorem **C** to prove rationality of some group-theoretic zeta functions, including the representation zeta functions of finitely generated nilpotent groups (Theorem I.8). The main tools are results from profinite groups; no ideas from model theory are used in a significant way beyond the notion of definability.

I.2. Elimination of imaginaries

I.2.1. Definition and first properties

We denote by $\mathbb{N}$ (respectively $\mathbb{N}_{>0}$) the nonnegative (respectively positive) integers, respectively. For standard model-theoretic concepts and notation such as dcl (definable closure) or acl (algebraic closure) we refer the reader to any introduction to model theory, e.g. [Poi00].

Notation I.2.1:
If X is a definable (possibly ∞-definable) set in some structure M and $A \subseteq M$, we will write $X(A) \coloneqq \{a \in A : M \vDash X(a)\}$. If we want to make the parameters of X explicit, we will write $X(A; b)$.

We say that the definable set X is *coded* (in M) if it can be written as $R(M; b)$, where b is a tuple of elements of M, and where $b \neq b'$ implies that $R(M; b) \neq R(M; b')$. In this situation $\mathrm{dcl}(b)$ only depends on X and is called a code for X. It is denoted $\ulcorner X \urcorner$. We say T *eliminates imaginaries* (EI) if every definable relation in every model of T is coded. Equivalently, if there are at least two constants, T eliminates imaginaries if for any equivalence relation E there is a definable function whose fibers are exactly the equivalence classes of E (cf. [Poi83, Lemme 2]).

For any theory T, by adding sorts for every definable quotient we obtain a theory T^{eq} that has elimination of imaginaries. These new sorts are called *imaginary sorts* and the old sorts

from T are called the *real sorts*. Similarly, to any model M of T we can associate a (unique) model M^{eq} of T^{eq} that has the same real sorts as M. In general, we use the notation $\ulcorner X \urcorner$ to refer to the code of X with respect to T^{eq}. We will denote by $\mathrm{dcl}^{\mathrm{eq}}$ the definable closure in M^{eq} and similarly for $\mathrm{acl}^{\mathrm{eq}}$.
We will consider many-sorted theories with a distinguished collection $\mathcal{S}$ of sorts, referred to as the *dominant* sorts; we assume that for any sort S, there exists a $\varnothing$-definable partial function from a finite product of dominant sorts onto S (and this function is viewed as part of the presentation of the theory). The set of elements of dominant sorts in a model M is denoted $\mathrm{dom}(M)$.
The following lemma and remark—which reduce elimination of imaginaries to coding certain functions—will not be used explicitly in the p-adic case, but they are essential guidelines as unary functions of the kind described in the remark are central to the proof of Proposition **(I.2.11)**.

Lemma I.2.2 (cf. [HHM06, Remark 3.2.2])**:**
A theory T admits elimination of imaginaries if every function definable (with parameters) whose domain is contained in a dominant sort is coded in any model of T.

Proof. It suffices to show that every definable function f is coded. Indeed, to code a set, it suffices to code the identity on this set. Pulling back by the given $\varnothing$-definable functions, it suffices to show that every definable function whose domain is contained in a product $M_1 \times \ldots \times M_n$ of dominant sorts is coded. For $n = 1$, this is our assumption. For larger n, we use induction, considering a definable function $f : M_1 \times \ldots \times M_n \to M^k$ as the function f' mapping $c \in M_1$ to the code of the function $y \mapsto f(c, y)$. By compactness there are finitely many definable functions h_i covering f'. The codes of these h_i allow us to code f. ∎

Remark I.2.3:
In Lemma **(I.2.2)**, it suffices to consider definable functions $f = f_e$ on a definable subset D of a dominant sort, defined with a parameter e from an imaginary sort Y, such that *either* f is the identity on its domain (unary EI), *or* there exists an $A \subseteq M$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(A, f_e(c))$ for any $c \in D$—i.e. f_e is determined by any one of its values—and $\mathrm{tp}(e/A)$ implies the type of e over Ac for any $c \in D$.
Indeed, let e be imaginary. There exist $c_1, \ldots, c_n \in \mathrm{dom}(M)$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(c_1, \ldots, c_n)$. For all l, $0 \leqslant l \leqslant n$, let $A_l := \mathrm{dcl}^{\mathrm{eq}}(e, c_1, \ldots, c_l) \cap M$. We know that $e \in \mathrm{dcl}^{\mathrm{eq}}(A_n)$ and we want to show that $e \in \mathrm{dcl}^{\mathrm{eq}}(A_0) = \mathrm{dcl}^{\mathrm{eq}}(\mathrm{dcl}^{\mathrm{eq}}(e) \cap M)$; i.e. e is interdefinable with a tuple of real elements.
Let us proceed by reverse induction. Suppose $e \in \mathrm{dcl}^{\mathrm{eq}}(A_{l+1})$, let $A := A_l$ and let $c := c_{l+1}$. Then over Ac, e is interdefinable with a real tuple d; so $d = f(e, c)$ and $e = h(d)$ for some A-definable functions f, h. By unary EI, any Ae-definable subset of a dominant sort has a code in $\mathrm{dcl}^{\mathrm{eq}}(Ae) \cap M = \mathrm{dcl}^{\mathrm{eq}}(A) \cap M = A$ and hence is A-definable. Thus, $\mathrm{tp}(c/A) \vdash \mathrm{tp}(c/Ae)$ and, for any $c' \vDash \mathrm{tp}(c/A)$, $e = h(f(e, c'))$. Let D be an A-definable set with $c \in D$ and such that $e = h(f(e, c'))$ for any $c' \in D$. Then $f_e(y) = f(e, y)$ has a code in $A = A_l$ and we have $e \in \mathrm{dcl}^{\mathrm{eq}}(A_l)$.

Definition I.2.4:
We will say that a theory T eliminates imaginaries up to uniform finite imaginaries (EI/UFI) *if for all $M \vDash T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $d \in M$ such that $e \in \operatorname{acl}^{\mathrm{eq}}(d)$ and $d \in \operatorname{dcl}^{\mathrm{eq}}(e)$. The theory T is said to* eliminate finite imaginaries (*EFI) if any $e \in \operatorname{acl}^{\mathrm{eq}}(\varnothing)$ is interdefinable with a tuple from M.*

Let us now give a criterion for elimination of imaginaries from [Hru09].

Lemma I.2.5:
A theory T eliminates imaginaries if it eliminates imaginaries up to uniform finite imaginaries and for every set of parameters A, T_A eliminates finite imaginaries.

Proof. Let $e \in M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$. Then by EI/UFI, there exists $d \in M$ such that $e \in \operatorname{acl}^{\mathrm{eq}}(d)$ and $d \in \operatorname{dcl}^{\mathrm{eq}}(e)$. Hence e is a finite imaginary in T_d and there exists $d' \in M$ such that $e \in \operatorname{dcl}^{\mathrm{eq}}(dd')$ and $dd' \in \operatorname{dcl}^{\mathrm{eq}}(ed) = \operatorname{dcl}^{\mathrm{eq}}(e)$, i.e. e is coded by dd'. ∎

I.2.2. Valued fields

Let L be a valued field, with valuation ring $\mathcal{O}(L)$, maximal ideal $\mathcal{M}(L)$ and residue field $\mathbf{k}(L)$. We will find it convenient to consider the value group $\mathbf{\Gamma}(L)$ in both an additive notation (with valuation $\mathrm{val} : L \to \mathbf{\Gamma}(L) \cup \{\infty\}$) and a multiplicative notation (with reverse order and absolute value $|.|$), depending on the setting. In fact the latter notation will only be used when L is a finite extension of $\mathbb{Q}_p$ and in that case we will take $|x|$ to be $q^{-\mathrm{val}(x)}$ where $q = |\mathrm{res}(L)|$.
We will consider valued fields in the geometric language whose sorts (later referred to as the geometric sorts) are as follows. We take a single dominant sort $\mathbf{K}$, for the field L itself. The additional sorts $\mathbf{S}_n, \mathbf{T}_n$ for $n \in \mathbb{N}$ are given by

$$\mathbf{S}_n := \mathrm{GL}_n(\mathbf{K})/\mathrm{GL}_n(\mathcal{O}) \simeq \mathrm{B}_n(\mathbf{K})/\mathrm{B}_n(\mathcal{O}),$$

the set of lattices in $\mathbf{K}^n$, and

$$\mathbf{T}_n := \mathrm{GL}_n(\mathbf{K})/\mathrm{GL}_{n,n}(\mathcal{O}) \simeq \bigcup_{m \leqslant n} \mathrm{B}_n(\mathbf{K})/\mathrm{B}_{n,m}(\mathcal{O}) \simeq \bigcup_{e \in \mathbf{S}_n} e/\mathcal{M}e.$$

Here a *lattice* is a free $\mathcal{O}$-submodule of $\mathbf{K}^n$ of rank n, B_n is the group of invertible upper triangular matrices, $\mathrm{GL}_{n,m}(\mathcal{O})$ is the group of matrices in $\mathrm{GL}_n(\mathcal{O})$ whose mth column reduces mod $\mathcal{M}$ to the column vector of $\mathbf{k}$ having a one in the mth entry and zeroes elsewhere, and $\mathrm{B}_{n,m}(\mathcal{O}) := \mathrm{B}_n(\mathcal{O}) \cap \mathrm{GL}_{n,m}(\mathcal{O})$. There is a canonical map from $\mathbf{T}_n$ to $\mathbf{S}_n$ taking $f \in e/\mathcal{M}e$ to the lattice e.
It is easy to see, using elementary matrices, that $\mathrm{GL}_n(\mathbf{K}) = \mathrm{B}_n(\mathbf{K})\mathrm{GL}_n(\mathcal{O})$, justifying the equivalence of the first two definitions of $\mathbf{S}_n$. Equivalently, it is shown in [HHM06, Lemma 2.4.8] that every lattice has a basis in triangular form.
Note that there is an obvious injective $\varnothing$-definable function $\mathbf{S}_m \times \mathbf{S}_{m'} \to \mathbf{S}_{m+m'}$, namely $(\lambda, \lambda') \mapsto \lambda \times \lambda'$, so we can identify any subset of a product of $\mathbf{S}_{n_i}$ with a subset of $\mathbf{S}_n$, where $n = \sum_i n_i$.

Note also that $\mathbf{S}_1$ can be identified with $\mathbf{\Gamma}$ by sending the coset $c\mathcal{O}^\star$ to $v(c)$. Then $\mathbf{k}$ can be identified with the fiber of $T_1 \to \mathbf{S}_1$ above the zero element of $\mathbf{S}_1 = \mathbf{\Gamma}$. More generally, let $\overline{\mathbf{B}} := \{\{x : \mathrm{val}(x-a) \geqslant \mathrm{val}(b)\} : a, b \in \mathbf{K}\}$ and $\dot{\mathbf{B}} := \{\{x : \mathrm{val}(x-a) > \mathrm{val}(b)\} : a, b \in \mathbf{K}\}$ be the sets of *closed* (respectively *open*) balls with center in $\mathbf{K}$ and radius in $v(\mathbf{K})$. Then $\overline{\mathbf{B}}$ embeds into $\mathbf{S}_2 \cup \mathbf{K}$ and $\dot{\mathbf{B}}$ into T_2. Indeed, the set of closed balls of radius $+\infty$ is identified with $\mathbf{K}$. The group $G(\mathbf{K})$ of affine transformations of the line acts transitively on the closed balls of nonzero radius; the stabilizer of $\mathcal{O} \in \overline{\mathbf{B}}$ is $G(\mathcal{O})$, so $\overline{\mathbf{B}} \smallsetminus \mathbf{K} \cong G(\mathbf{K})/G(\mathcal{O}) \subseteq \mathrm{GL}_2(\mathbf{K})/\mathrm{GL}_2(\mathcal{O})$. The group $G(\mathbf{K})$ also acts transitively on $\dot{\mathbf{B}}$ and the stabilizer of $\mathcal{M} \in \dot{\mathbf{B}}$ is $G(\mathbf{K}) \cap \mathrm{GL}_{2,2}(\mathcal{O})$. We will write $\mathbf{B} := \overline{\mathbf{B}} \cup \dot{\mathbf{B}}$ for the set of all balls. Note, however, that if $\mathbf{\Gamma}$ has a smallest positive element, the open balls are also closed balls.

In Sections I.3 and I.5, we will also consider the sort $\mathbf{RV} := \mathbf{K}^\star/(1+\mathcal{M})$ and the canonical projection $\mathrm{rv} : \mathbf{K}^\star \to \mathbf{RV}$. The structure on $\mathbf{RV}$ is given by its group structure and the structure induced by the exact sequence $\mathbf{k}^\star \to \mathbf{RV} \to \mathbf{\Gamma}$ where the second map is denoted $\mathrm{val}_{\mathrm{rv}}$—i.e. we have a binary predicate interpreted as $\mathrm{val}_{\mathrm{rv}}(x) \leqslant \mathrm{val}_{\mathrm{rv}}(y)$, a unary predicate interpreted as $\mathbf{k}^\star$ and the ring structure on $\mathbf{k}$ (adding a zero to $\mathbf{k}^\star$). This exact sequence induces on each fiber of $\mathrm{val}_{\mathrm{rv}}$ the structure of a $\mathbf{k}$-vector space (if we add a zero to the fiber). When $T \supseteq \mathrm{T}_{\mathrm{Hen},0}$ (the theory of henselian valued fields with a residue field of characteristic zero), $\mathbf{RV}$ is stably embedded and the structure induced on $\mathbf{RV}$ is exactly the one described above. Note that $\mathbf{RV}$ and T_1 can be identified (if we add a zero to each fiber of $\mathrm{val}_{\mathrm{rv}}$).

The theory of a structure is determined by the theory of the dominant sorts; so, for any field L we can speak of $\mathrm{Th}(L)$ in the geometric sorts. We take the geometric language $\mathcal{L}^{\mathcal{G}}$ to include the ring structure on the sort $\mathbf{K}$, the natural maps $\mathrm{GL}_n(\mathbf{K}) \to \mathbf{S}_n(\mathbf{K})$, $\mathrm{GL}_n(\mathbf{K}) \times \mathbf{K}^n \to \mathbf{T}_n(\mathbf{K})$.

In [HHM06], it is shown that ACVF eliminates imaginaries in $\mathcal{L}^{\mathcal{G}}$. Let us now give the counterpart of this theorem for p-adic fields.

By $\mathcal{L}^{\mathcal{G}^-}$, we denote the restriction of $\mathcal{L}^{\mathcal{G}}$ to the sorts $\mathbf{K}$ and $\mathbf{S}_n$. For all sets $N \subseteq \mathbb{N}_{>0}$, we will also consider an expansion $\mathcal{L}^{\mathcal{G}}_N$ of $\mathcal{L}^{\mathcal{G}^-}$ by a constant a and for all $n \in N$ a tuple of constants c_n of length n in the field sort.

By an *uniformizer* of a valued field we mean an element $a \in \mathcal{O}$ whose valuation generates the value group. By an *(unramified) n-Galois uniformizer* we mean a tuple c of elements of the valued field whose residue generate $T(\mathbf{k})/(T(\mathbf{k}))^{\cdot n}$ where T is the restriction of scalar of $\mathbb{G}_m$ from $\mathbf{k}[\omega_n]$ to $\mathbf{k}$ for ω_n some primitive nth root of unity and $T(\mathbf{k})^{\cdot n}$ denotes the nth powers in $T(\mathbf{k})$. Note that because $\mathbf{k}[\omega_n]$ is of degree at most n over $\mathbf{k}$, c is a tuple of length at most n. Adding some zeroes at the end of the tuple, we may assume it is a tuple of length exactly n.

Let PL_0 be the theory of pseudo-local fields of residue characteristic 0, i.e. henselian fields with value group a $\mathbb{Z}$-group (i.e. an ordered group elementarily equivalent to $(\mathbb{Z}, 0, <)$) and residue field a pseudo-finite field of characteristic 0. So PL_0 is the theory of ultraproducts $\prod \mathbb{Q}_p/\mathcal{U}$ of p-adics over non-principal ultrafilters on the set of primes. In fact, let $\mathfrak{L}_p$ be a set of finite extensions of $\mathbb{Q}_p$ and $\mathfrak{L} := \bigcup_p \mathfrak{L}_p$. Any ultraproduct $\prod_{L \in \mathfrak{L}} L/\mathcal{U}$ of residue characteristic zero— i.e. the ultrafilter $\mathcal{U}$ does not contains any set included in some $\mathfrak{L}_{p_0}$— is a model of PL_0. Note that if $\mathfrak{L}_p$ is nonempty for infinitely many p there actually is an ultrafilter on $\mathfrak{L}$ such that $\prod_{L \in \mathfrak{L}} L/\mathcal{U}$ has residue characteristic zero.

Let $L \vDash \mathrm{PL}_0$ be an $\mathcal{L}^{\mathcal{G}^-}$-structure. A *proper expansion* of L to $\mathcal{L}^{\mathcal{G}}_N$ is a choice of a and c_n

for each $n \in N$ such that a is a uniformizer and c_n is an unramified n-Galois uniformizer. Note that because there are only finitely many possibilities for the minimal polynomial of ω_n over $\mathbf{k}$, the class of proper expansions to $\mathcal{L}^{\mathcal{G}}_N$ of models of PL_0 is elementary. Let us call it ${\mathrm{PL}_0}^N$.

A proper expansion of $L \in \mathfrak{L}_p$ to $\mathcal{L}^{\mathcal{G}}_N$ is a choice of a and c_n for each $n \in N$ such that a is a uniformizer and if n is prime with p, c_n is an unramified n-Galois uniformizer. If n is not prime to p take all of the other constants to be zero except for one, in c_p for example, which is a generator of L over $\mathbb{Q}_p$. Note that a residue characteristic zero ultraproduct of proper expansions to $\mathcal{L}^{\mathcal{G}}_N$ of $L \in \mathfrak{L}$ is a model of ${\mathrm{PL}_0}^N$.

The two main elimination of imaginaries results in this paper are the following. The first is for finite extensions of $\mathbb{Q}_p$:

Theorem A:

The theory of $\mathbb{Q}_p$ eliminates imaginaries in $\mathcal{L}^{\mathcal{G}^-}$. The same is true for any finite extension L of $\mathbb{Q}_p$, provided one adds a constant symbol for a generator of $L \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$ over $\mathbb{Q}_p \cap \overline{\mathbb{Q}}^{\mathrm{alg}}$.

The second is for their ultraproducts of residue characteristic zero:

Theorem B:

${\mathrm{PL}_0}^{\mathbb{N}}$ *eliminates imaginaries in $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$.*

Remark I.2.6:

1. Although the T_n are needed to obtain EI in algebraically closed valued fields, they are not needed here. Indeed, if a valued field L has a discrete valuation (i.e. the value group has a smallest positive element $\mathrm{val}(\lambda_0)$), then for any lattice e, $\lambda_0 e$ is itself a lattice, and a coset h of $\lambda_0 e$— a typical element of $\mathbf{T}_n$—can be coded by the lattice in $\mathbf{K}^{n+1}$ generated by $h \times \{1\}$. Hence all elements of $T_n(L)$ are coded in $\mathbf{S}_{n+1}(L)$.

2. In Theorem **A**, we need to add constants for elements of a subfield F with a certain number of properties:

   a. F contains a uniformizer;

   b. $\mathrm{res}(F) = \mathbf{k}$;

   c. $\overline{\mathbf{K}}^{\mathrm{alg}} = \overline{F}^{\mathrm{alg}}\mathbf{K}$ (in fact, we need that for every finite extension L of $\mathbf{K}$ there is a generator of $\mathcal{O}(L)$ whose minimal polynomial is over F).

   It suffices to take $F = \mathbb{Q}[c]$, where c generates L over $\mathbb{Q}_p$. Note that we can choose such a $c \in \overline{\mathbb{Q}}^{\mathrm{alg}}$.

   Note also that a proper expansion of some finite extension L of $\mathbb{Q}_p$ to $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$ contains a generator of L over $\mathbb{Q}_p$. Hence proper extensions of L eliminate imaginaries in $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$.

3. In Theorem **B**, we need to name elements of a subfield F satisfying (a), (c) as above, and:

   d. $\mathrm{res}(F)(\mathbf{k}^\star)^{\cdot n} = \mathbf{k}$ for all n;

e. $\mathbf{k}$ admits EI in the language of rings augmented by constants for elements of $\operatorname{res}(F)$.

We can choose F to be generated by a uniformizer a and unramified n-Galois uniformizers c_n for all n. It is clear that such an F satisfies (a). Furthermore, $\overline{\mathbf{k}}^{\mathrm{alg}} = \overline{\operatorname{res}(F)}^{\mathrm{alg}}\mathbf{k}$. Indeed, let ω_n be a primitive nth root of unity, and let $d_n := \sum_i c_{n,i}\omega_n^i$. The degree n extension of $\mathbf{k}[\omega_n]$ is contained in $\mathbf{k}[\omega_n, \sqrt[n]{d_n}]$ by Kummer theory and it contains the degree n extension of $\mathbf{k}$.

Now (c) is a consequence of (a) and the above statement and (e) also follows as any extension of degree n is generated by an element in $\overline{\operatorname{res}(F)}^{\mathrm{alg}}$, so there is an irreducible polynomial of degree n with $\operatorname{res}(F)$-definable parameters; this is the hypothesis of [CH99, Proposition B.(3)]. Finally for any n, there is a d such that $\{x \in \mathbf{k} : x^n = 1\} = \{x \in \mathbf{k} : x^d = 1\}$ and $\mathbf{k}$ contains primitive dth roots of unity. Then $c_d \in \mathbf{k}$ generates $\mathbf{k}^\star/(\mathbf{k}^\star)^{\cdot d} = \mathbf{k}^\star/(\mathbf{k}^\star)^{\cdot n}$, so (d) holds.

4. It would be nice to find a more precise description of the imaginaries if no constants are named. For finite extensions of $\mathbb{Q}_p$, this is done in Remark **(I.4.6)**.

Before going any further, let us show that Theorem **B** allows us to prove a uniform version of Theorem **A**.

Corollary I.2.7:
Let $\mathfrak{L}_p$ be any set of finite extensions of $\mathbb{Q}_p$ and let $\mathfrak{L} := \bigcup_p \mathfrak{L}_p$. Let $E(x,y)$ be an $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$-formula (where x, y range over definable sets X, Y). Then there exist integers m, l, a set of integers N, a prime p_0 and some $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$-formula $\varphi(x,w)$ such that the following uniform statement of elimination of imaginaries holds. For all $p \geqslant p_0$ and all proper expansions to $\mathcal{L}^{\mathcal{G}}_N$ of $L_p \in \mathfrak{L}_p$, $\varphi(x,w)$ defines a function

$$f_{L_p} : X \to \mathbf{S}_m(L_p) \times \mathbf{K}(L_p)^l$$

and

$$L_p \models (\forall x, x')(f_{L_p}(x) = f_{L_p}(x') \iff (\forall y) E(x,y) \equiv E(x',y))$$

Proof. Assume $\mathfrak{L}_p$ is nonempty for infinitely many p, otherwise the statement is trivial. The formula $\forall y\, E(x,y) \equiv E(x',y)$ defines an equivalence relation in any ultraproduct L of fields in $\mathfrak{L}$. By Theorem **B**, there is a formula $\varphi(x,w)$ (which works for any proper expansion to $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$ of any such ultraproduct of residue characteristic zero) such that $\varphi(x,w)$ defines a function f and $f(x) = f(x')$ if and only if $\forall y\, E(x,y) \equiv E(x',y)$. By compactness, for some N, this equivalence is valid in proper expansions to $\mathcal{L}^{\mathcal{G}}_N$.
Let us now assume there is an infinite set $I \subseteq \mathfrak{L}$ such that I has a nonempty intersection with infinitely many $\mathfrak{L}_p$ and for every $L \in I$, there is a proper expansion of L to $\mathcal{L}^{\mathcal{G}}_{\mathbb{N}_{>0}}$ such that we do not have $f(x) = f(x')$ if and only if $\forall y\, E(x,y) \equiv E(x',y)$ in L. Then there exists an ultrafilter on $\mathfrak{L}$ containing I but containing no set included in some $\mathfrak{L}_{p_0}$ and $\prod_{L\in\mathfrak{L}} L/\mathcal{U} \models \mathrm{PL_0}^{\mathbb{N}}$; but we do not have $f(x) = f(x')$ if and only if $\forall y\, E(x,y) \equiv E(x',y)$ in this ultraproduct, a contradiction. ∎

Remark I.2.8:

1. In particular, whenever $E(x,y)$ is interpreted in L_p as an equivalence relation, $f_p(x)$ codes the E-class of X.

2. Assume $\mathfrak{L}_p$ is finite for all p. Then $\bigcup_{p<p_0} \mathfrak{L}_p$ is finite and, using Theorem **A** (and Remark **I.2.6.2**), we can find a φ and an N that work for all $L \in \bigcup_p \mathfrak{L}_p$ and not just for p big enough.

The proof of Theorems **A** and **B** uses elimination of imaginaries and the existence of invariant extensions in the theory of algebraically closed valued fields. Recall that a theory T has the invariant extension property if whenever $A = \mathrm{acl}(A) \subseteq M \vDash T$ and $c \in M$, $\mathrm{tp}(c/A)$ extends to an $\mathrm{Aut}(M/A)$-invariant type over M. This holds trivially for any finite field, and by inspection, for $\mathrm{Th}(\mathbb{Z},+,<)$ and, although we will only use a weaker version of the extension property (Corollary **(I.3.10)**) in the proof of Theorem **A**, we will show that the theory of a finite extension of $\mathbb{Q}_p$ (with the geometric sorts) enjoys the stronger version (Remark **(I.4.7)**).

I.2.3. Real elimination of imaginaries

To illustrate the idea of transferring imaginaries from one theory to the other, consider the following way of deducing EI for RCF (the theory of real closed fields) from EI for ACF (the theory of algebraically closed fields).

Example I.2.9:
Let F be a field considered in a language extending the language of rings. Assume for all $M \vDash \mathrm{Th}(F)$:

(i) (Algebraic boundedness): Let $A \subseteq M$; then $\mathrm{acl}(A) \subseteq \overline{A}^{\mathrm{alg}}$ (where $\overline{A}^{\mathrm{alg}}$ denotes the ACF algebraic closure);

(ii) (Rigidity of finite sets): No automorphism of M can have a finite cycle of size > 1. Equivalently, for each n, there exists $\varnothing$-definable functions $r_{i,n}(x_1,...,x_n)$ that are symmetric in the x_i, such that $S = \{r_{1,n}(S),\ldots,r_{n,n}(S)\}$. (Here $r_{i,n}(S)$ denotes $r_{i,n}(x_1,...,x_n)$ when $S = \{x_1,\ldots,x_n\}$, possibly with repetitions.);

(iii) (Unary EI): Every definable subset of M is coded.

Then $\mathrm{Th}(F)$ eliminates imaginaries (in the single sort of field elements).

Proof. Let $f : M \to M$ be a definable function. By Lemma **(I.2.2)**, it suffices to prove that f is coded. Let H be the Zariski closure of the graph of f. Since the theory is algebraically bounded, the set $H(x) := \{y : (x,y) \in H\}$ is finite for any x, of size bounded by some n. Let $U_{n,i}$ be the set of x such that $f(x) = r_{i,n}(H(x))$. Then, by elimination of imaginaries in ACF, H—being a Zariski closed set—is coded in $\overline{M}^{\mathrm{alg}}$. But the code is definable over M and hence is in the perfect closure of M. Replacing this code by some p^nth power in the characteristic p case, we can suppose it belongs to M. Moreover, each U_i (being unary) is coded; these codes together give a code for f. ■

Note that RCF satisfies the hypotheses of Example **(I.2.9)**, but $\mathrm{Th}(\mathbb{Q}_p)$ (in the field sort alone) does not. More precisely, as shown in the introduction, the value group cannot be be definably embedded into $\mathbb{Q}_p^n$. Hence hypothesis (iii) fails for $\mathrm{Th}(\mathbb{Q}_p)$ in the field sort alone.

Remark I.2.10:
If F is a field satisfying (i), (iii), then F has EI/UFI. This is an immediate consequence of Proposition **(I.2.11)** as hypotheses (ii) and (iv) of Proposition **(I.2.11)** are true if T is the theory of algebraically closed fields in the language of rings.

I.2.4. Criterion for elimination of imaginaries

Let T be a complete theory in a language $\mathcal{L}$. Assume T eliminates quantifiers and imaginaries. Let $\widetilde{T}$ be a complete theory in a language $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$; assume $\widetilde{T}$ contains the universal part of T.
In a model $\widetilde{M}$ of the theory $\widetilde{T}$, three kinds of definable closure can be considered: the usual definable closure $\mathrm{dcl}_{\widetilde{\mathcal{L}}}$, the definable closure in $\widetilde{M}^{\mathrm{eq}}$, denoted $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}$ and the imaginary definable closure restricted to real points $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}} \cap \widetilde{M}$. As $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}} \cap \widetilde{M}$ and $\mathrm{dcl}_{\widetilde{\mathcal{L}}}$ take the same value on sets of real points, we will denote both of them $\mathrm{dcl}_{\widetilde{\mathcal{L}}}$. One must take care however that if A contains imaginary elements, $A \notin \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)$.
As T eliminates imaginaries, these distinctions are not necessary in models of T and we will only need $\mathrm{dcl}_{\mathcal{L}}$. One should note that, as T eliminates quantifiers, $\mathrm{dcl}_{\mathcal{L}}$ is the closure under quantifier-free $\mathcal{L}$-definable functions and hence that, for any $A \subseteq \widetilde{M}$, $\mathrm{dcl}_{\mathcal{L}}(A) \cap \widetilde{M} \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)$.
Analogous statements hold for $\mathrm{acl}_{\mathcal{L}}$, $\mathrm{acl}_{\widetilde{\mathcal{L}}}$, $\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}$, $\mathrm{tp}_{\mathcal{L}}$, $\mathrm{tp}_{\widetilde{\mathcal{L}}}$, etc.
One should also be careful that if $\widetilde{M} \vDash \widetilde{T}$ is contained in some $M \vDash T$, there is no reason in general that $\widetilde{M}^{\mathrm{eq}}$ should be contained in M. In fact, the whole purpose of the following proof is to show that under certain hypotheses every element of $\widetilde{M}^{\mathrm{eq}}$ is interdefinable with a tuple in $\widetilde{M}$.
Since we are describing notation, let us also point out that we will write interchangeably $\mathrm{dcl}(A, b) = \mathrm{dcl}(Ab) = \mathrm{dcl}(A \cup \{b\})$.

Proposition I.2.11:
*Assume that T and $\widetilde{T}$ are as above. Let $\widetilde{M}$ be an $|L|^+$-saturated and $|L|^+$-homogeneous model of $\widetilde{T}$ and let $M \vDash T$ be such that $\widetilde{M}\big|_{\mathcal{L}} \leqslant_{\mathcal{L}} M$ and such that any automorphism of $\widetilde{M}$ extends to an automorphism of M. If conditions (i)–(iv) below hold for any $A = \mathrm{acl}_{\widetilde{\mathcal{L}}}(A) \subseteq \widetilde{M}$, then $\widetilde{T}$ admits elimination of imaginaries up to uniform finite imaginaries (see Definition **(I.2.4)**).*

(i) *(Relative algebraic boundedness) For all elements $c \in \mathrm{dom}(\widetilde{M})$ and $\widetilde{M}' \prec \widetilde{M}$, $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\widetilde{M}'c) \subseteq \mathrm{acl}_{\mathcal{L}}(\widetilde{M}'c)$.*

(ii) *(Internalizing $\mathcal{L}$-codes) Let $e \in \mathrm{dcl}_{\mathcal{L}}(\widetilde{M})$. Then there exists a tuple e' of elements of $\widetilde{M}$ such that an automorphism of M stabilizing $\widetilde{M}$ globally fixes e if and only if it fixes e'.*

(iii) *(Unary EI) Every $\widetilde{\mathcal{L}}(\widetilde{M})$-definable unary subset of* $\operatorname{dom}(\widetilde{M})$ *is coded in* $\widetilde{M}$.

(iv) *(Invariant types) For all* $c \in \operatorname{dom}(\widetilde{M})$, *there exists an* $\operatorname{Aut}(M/A)$*-invariant type* p *over* M *such that* $p|_{\widetilde{M}}$ *is consistent with* $\operatorname{tp}_{\widetilde{\mathcal{L}}}(c/A)$.

Moreover, for any $\mathcal{L}(M)$*-definable function* r *whose domain contains* p, *let* $\partial_p r$ *be the* p*-germ of* r *(where two* $\mathcal{L}(M)$*-definable functions* r, r' *have the same* p*-germ if they agree on a realization of* p *over* M*). Then:*

$(*)$ *There exists a directed order* I *and a sequence* $(\varepsilon_i)_{i\in I}$, *with* $\varepsilon_i \in \operatorname{dcl}_{\mathcal{L}}(A, \ulcorner r \urcorner)$ *such that* $\sigma \in \operatorname{Aut}(M/A)$ *fixes* $\partial_p r$ *if and only if* σ *fixes almost every* ε_i*—i.e.* σ *fixes all* $i \geqslant i_0$, *for some* $i_0 \in I$.

Some comments on the proposition:

1. There are two ways to ensure that automorphisms of $\widetilde{M}$ extend to automorphisms of M. The first is to take M homogeneous enough. The other is to take M atomic over $\widetilde{M}$; in the case of valued fields, we could take M to be the algebraic closure of $\widetilde{M}$.

2. In fact, we will only need (iv) for $|A| \leqslant |\widetilde{\mathcal{L}}|$.

3. If p is definable then, for a uniformly defined family of functions r_b, $\partial_p r_b$ is an imaginary (and we could take ε_i to be that imaginary). Nevertheless, if p is not definable and say (ε_i) is countable then Condition (iv) implies that the germ is a Σ_2^0-hyperimaginary, i.e. an equivalence class of sequences indexed by I where the equivalence relation is given by a countable union of countable intersections of definable sets (although each definable set will involve only a finite number of indices, the countable union of countable intersections can involve them all). In the case of ACVF one also has that σ fixes $\partial_p r$ if and only if σ fixes cofinally many ε_i; in this case the equivalence relation is also a countable intersection of countable unions of definable sets, so it is Δ_2^0.

4. Hypotheses (ii) and (iii) are special cases of elimination of imaginaries. It would be nice to move (iii) from the hypotheses to the conclusion, i.e. assuming only (i), (ii) and (iv), to show that every imaginary is "equivalent" to an imaginary of M definable over $\widetilde{M}$.

First let us clarify how $\operatorname{Aut}(\widetilde{M})$ acts on $\operatorname{dcl}_{\mathcal{L}}(\widetilde{M})$ as this action will be used implicitly throughout the proof. Any $\widetilde{\sigma} \in \operatorname{Aut}(\widetilde{M})$ can be extended to an automorphism $\sigma \in \operatorname{Aut}(M)$ and all these extensions are equal on $\operatorname{dcl}_{\mathcal{L}}(\widetilde{M})$, hence we have a well-defined action of $\widetilde{\sigma}$ on $\operatorname{dcl}_{\mathcal{L}}(\widetilde{M})$ and the notation $\operatorname{Aut}(\widetilde{M}/B)$ makes sense even if $B \subseteq \operatorname{dcl}_{\mathcal{L}}(\widetilde{M})$. Similarly, if p is an $\operatorname{Aut}(M/B)$-invariant type, $\operatorname{Aut}(\widetilde{M}/B)$ acts on p-germs of $\mathcal{L}(\widetilde{M})$-definable functions. We begin our proof with elimination of finite sets:

Lemma I.2.12:
Assume (ii) holds in Proposition **(I.2.11)**. *Then every finite set* $E \subseteq \widetilde{M}$ *is coded.*

Proof. By EI for T, the finite set E is coded by a tuple $e' \in M$; e' may consist of elements in $\mathrm{dcl}_{\mathcal{L}}(\widetilde{M})$ but outside $\widetilde{M}$. By (ii), there exists a tuple e of elements of $\widetilde{M}$ such that an automorphism of $\mathrm{dcl}_{\mathcal{L}}(\widetilde{M})$ leaving $\widetilde{M}$ invariant fixes E if and only if it fixes e' if and only if it fixes e. Thus E and $e \in \widetilde{M}$ are interdefinable. ∎

Proof (Proposition **(I.2.11)**). Let $e \in \widetilde{M}^{\mathrm{eq}}$. There exists $c_1, \ldots, c_n \in \mathrm{dom}(\widetilde{M})$, we have $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(c_1, \ldots, c_n)$. Let $A_i := \mathrm{dcl}_{\widetilde{\mathcal{L}}}(e, c_1, \ldots, c_i) \subseteq \widetilde{M}$. The claim is that $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_0)$. We have $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_n)$ and show by reverse induction on $l \leqslant n$ that $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_l)$. Assume inductively that $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_{l+1})$. Let $A := A_l$, $c := c_{l+1}$. It is easy to check that

$$A = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ae).$$

As $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_{l+1})$, for some tuple $d \in A_{l+1} = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ace)$, some $\widetilde{\mathcal{L}}(A)$-definable function f and some $\widetilde{\mathcal{L}}(A)$-definable, finite-set-valued function g, we have

$$e \in g(d), d = f(c, e).$$

Let $f_e(x) := f(x, e)$. Let $\bar{A} := \mathrm{acl}_{\widetilde{\mathcal{L}}}(A)$ and $\widetilde{p} := \mathrm{tp}_{\widetilde{\mathcal{L}}}(c/\bar{A})$.
Let $\widetilde{M_0} \prec \widetilde{M}$ such that $\widetilde{M_0}^{\mathrm{eq}}$ contains Ae. Note that for all c' in the domain of f_e, $f_e(c') \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(\widetilde{M_0}c)$. By (i), there exists an $\mathcal{L}(\widetilde{M_0})$-definable finite-set-valued function $\varphi_{c'}$ such that $f_e(c') \in \varphi_{c'}(c')$. By compactness, for some finite set I_0 and $\mathcal{L}(\widetilde{M_0})$-definable finite-set-valued functions $(\varphi_i)_{i \in I_0}$, the following holds: for any c' in the domain of f_e, $f_e(c') \in \varphi_i(c')$ for some $i \in I_0$. Let $\varphi(x) := \bigcup_{i \in I_0} \varphi_i(x)$; so $f_e(c') \in \varphi(c')$ for all c' in the domain of f_e. Hence if Φ is the set of all $\mathcal{L}(\widetilde{M})$-definable, finite-set-valued functions ψ with a domain containing that of f_e and such that for all c' in the domain of f_e, $f_e(c) \in \psi(c)$, then Φ is nonempty.
Let p be an $\mathrm{Aut}(M/\bar{A})$-invariant type over M extending $\widetilde{p}$, as in (iv). For $m \in \mathbb{N}$, let Φ_m be the set of all $\mathcal{L}(\widetilde{M})$-definable functions $\varphi \in \Phi$ such that for $c \models p$, $\varphi(c)$ is an m-element set. Note that as p is a complete type, m does not depend on c. Let m be minimal such that Φ_m is nonempty. Clearly all $\varphi \in \Phi_m$ share the same p-germ: if φ, φ' do not have the same p-germ, then $\varphi''(x) := \varphi(x) \cap \varphi'(x)$ lies in $\Phi_{m'}$ for some $m' < m$. Pick $F_E \in \Phi_m$, defined over $E \subseteq \widetilde{M}$. So F_E covers f_e, F_E is $\mathcal{L}(E)$-definable, and the p-germ of F_E is invariant under $\mathrm{Aut}(\widetilde{M}/\bar{A}e)$.

Claim I.2.13: *The p-germ of F_E is invariant under* $\mathrm{Aut}(\widetilde{M}/\bar{A})$.

Proof. Let (ε_i) be a sequence as in (iv), coding the germ of F_E on p. Note that $\varepsilon_i \in \mathrm{dcl}_{\mathcal{L}}(\widetilde{M})$ (since F_E is $\mathcal{L}(E)$-definable and $E \subseteq \widetilde{M}$). By (ii), we may replace ε_i by a tuple of $\widetilde{M}$, without changing $\mathrm{Aut}(\widetilde{M}/\varepsilon_i)$; we do so.
Now, almost all ε_i must be in $\mathrm{acl}_{\widetilde{\mathcal{L}}}(\bar{A}e)$. For otherwise, by moving to a subsequence we may assume all ε_i are outside $\mathrm{acl}_{\widetilde{\mathcal{L}}}(\bar{A}e)$. So $\mathrm{Aut}(\widetilde{M}/\bar{A}e\varepsilon_i)$ has infinite index in $\mathrm{Aut}(\widetilde{M}/\bar{A}e)$. By Neumann's Lemma, for any finite set X of indices, there exists $\tau \in \mathrm{Aut}(\widetilde{M}/\bar{A}e)$ with $\tau(\varepsilon_i) \neq \varepsilon_i$, for all $i \in X$. By compactness (and homogeneity of $\widetilde{M}$), there exists $\tau \in \mathrm{Aut}(\widetilde{M}/\bar{A}e)$ with $\tau(\varepsilon_i) \neq \varepsilon_i$ for all i. But then τ fails to fix the p-germ of F, contradicting the $\mathrm{Aut}(\widetilde{M}/\bar{A}e)$-invariance of this germ.
So for almost all i, some finite set $\mathcal{E}_i$ containing ε_i is defined over Ae. By Lemma **(I.2.12)**, the finite set $\mathcal{E}_i$ is coded in $\widetilde{M}$. But $A = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ae)$, so $\mathcal{E}_i$ is defined over A. Hence $\varepsilon_i \in \bar{A}$, i.e.

ε_i is fixed by $\mathrm{Aut}(\widetilde{M}/\bar{A})$. This being the case for almost all i, the p-germ of F_E is invariant under $\mathrm{Aut}(\widetilde{M}/\bar{A})$. ♦

Claim I.2.14: $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A)$.

Proof. It suffices to show that if $((e_i, E_i) : i \in \mathbb{N})$ is an indiscernible sequence over $\bar{A}$ with $e_0 = e, E_0 = E$, then $e_i = e_j$ for some $i \neq j$. Let $c \models p|_{A(E_i)_{i\in\mathbb{N}}}$ such that $c \models \widetilde{p}$. By (iii) and because $A = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ae)$, $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$ implies $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/Ae)$; hence $\mathrm{tp}_{\widetilde{\mathcal{L}}}(e/A)$ implies $\mathrm{tp}_{\widetilde{\mathcal{L}}}(e/Ac)$. So $\mathrm{tp}_{\widetilde{\mathcal{L}}}(e_i/Ac) = \mathrm{tp}_{\widetilde{\mathcal{L}}}(e/Ac)$.
By Claim **(I.2.13)**, the p-germs of the F_{E_i} are equal; so $F_{E_i}(c)$ is a finite set F that does not depend on i. But $f(c, e_i) \in F$, so $f(c, e_i)$ takes the same value on some infinite set I' of indices i. Hence so does the finite set $g(f(c, e_i))$. As $e \in g(f(c, e))$, it follows that $e_i \in g(f(c, e_i))$, so infinitely many e_i lie in the same finite set and $e_i = e_j$ for some $i \neq j$. ♦

We have just shown that e lies in $\mathrm{acl}^{\mathrm{eq}}(A) = \mathrm{acl}^{\mathrm{eq}}(\mathrm{dcl}^{\mathrm{eq}}(e) \cap \widetilde{M})$. This concludes the proof of Proposition **(I.2.11)**. ■

Let us now show that this first criterion can be turned into a criterion for elimination of imaginaries.

Corollary I.2.15:
*Let $\widetilde{T}$ and T be as in Proposition **(I.2.11)** and let us suppose moreover that:*

(v) *(Weak rigidity) For all $A = \mathrm{acl}_{\widetilde{\mathcal{L}}}(A)$ and $c \in \mathrm{dom}(\widetilde{M})$, $\mathrm{acl}_{\widetilde{\mathcal{L}}}(Ac) = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ac)$.*

Then $\widetilde{T}$ eliminates imaginaries.

Proof. Let $e \in \widetilde{M}^{\mathrm{eq}}$ be an imaginary element. There exists $c_1, \ldots, c_n \in \mathrm{dom}(\widetilde{M})$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(c_1, \ldots, c_n)$. For all $l \leqslant n$, let $A_l := \mathrm{acl}_{\widetilde{\mathcal{L}}}(e, c_1, \ldots, c_l) \subseteq \widetilde{M}$. Then $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_n)$; we show by reverse induction on $l \leqslant n$ that $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_l)$. We assume inductively that $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_{l+1})$. Let $A := A_l, c := c_{l+1} \in \mathrm{dom}(\widetilde{M})$. It is easy to check that

$$A = \mathrm{acl}_{\widetilde{\mathcal{L}}}(Ae)$$

and that, for some tuple d,

$$d \in A_{l+1} = \mathrm{acl}_{\widetilde{\mathcal{L}}}(Ace), e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(Ad).$$

By Proposition **(I.2.11)**, $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_0)$, so $d \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(Ac)$. By weak rigidity (v), $d \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ac)$. Thus $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(Ac)$.
Say $e = h(c)$, where h is an $\widetilde{\mathcal{L}}^{\mathrm{eq}}(A)$-definable function. Then $h^{-1}(e)$ is an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable subset of $\mathrm{dom}(\widetilde{M})$, hence by (iii) it has a code $e' \in \widetilde{M}$. Clearly e and e' are interdefinable over A. As $e' \in \widetilde{M}$, we have $e' \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ae) = A$. So $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A) = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_l)$. This finishes the induction and shows that $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A_0)$.
Let a be a tuple from A_0 such that e is $\widetilde{\mathcal{L}}^{\mathrm{eq}}(a)$-definable. Let a' be the (finite) set of conjugates of a over e. Then $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(e) = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(a')$ and, by Lemma **(I.2.12)**, a' is coded, hence e is interdefinable with some sequence from $\widetilde{M}$. ■

Keeping (v) out of Proposition **(I.2.11)** makes the proof of the EI criterion messier than strictly necessary. Nonetheless, distinguishing the case without (v) is important for ultraproducts of the p-adics where (v) fails.
The following lemma will be used to prove (v) in the p-adic case.

Lemma I.2.16:
Assume that for any $a \in \widetilde{M}$, there exists a tuple c from $\mathrm{dom}(\widetilde{M})$ with $a \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(c)$ and such that $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a))$ extends to an $\mathrm{Aut}(\widetilde{M}/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a))$-invariant type over $\widetilde{M}$. Then (v) follows from:

(v′) *If $B \subseteq \mathrm{dom}(\widetilde{M})$ then $\mathrm{acl}_{\widetilde{\mathcal{L}}}(B) \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(B)$.*

Proof. Let $A = \{a_i : i < \kappa\}$. For each i, pick a tuple c_i of elements of $\mathrm{dom}(\widetilde{M})$ with $a_i \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(c_i)$, and extend $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c_i/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a_i))$ to an $\mathrm{Aut}(\widetilde{M}/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a_i))$-invariant type $\widetilde{p}_i$. Let $A_0 :=$ A, and recursively let $A_{i+1} := A_i \cup \{b_i\}$, where $b_i \vDash \widetilde{p}_i|_{\mathrm{acl}_{\widetilde{\mathcal{L}}}(A_i c)}$, and $A_\lambda := \bigcup_{i<\lambda} A_i$ for limit λ.

Claim I.2.17: $\mathrm{acl}_{\widetilde{\mathcal{L}}}(Ac) \cap \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A_i c) \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ac)$.

Proof. We proceed by induction on i. The limit case is trivial. To move from i to $i+1$, let $d \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(Ac) \cap \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A_{i+1}c)$ and let $\sigma \in \mathrm{Aut}(\widetilde{M}/A_i c)$. As $\mathrm{tp}_{\widetilde{\mathcal{L}}}(b_i/\mathrm{acl}_{\widetilde{\mathcal{L}}}(A_i c))$ is invariant under σ, $d \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(A_i c)$ and d is definable over $A_i c b_i$, we have $\sigma(d) = d$, i.e. d is definable over $A_i c$ and hence $d \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ac)$ by induction. ♦

Now $A_\kappa \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A_\kappa \cap \mathrm{dom}(\widetilde{M}))$ and so $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A_\kappa c) = \mathrm{dcl}_{\widetilde{\mathcal{L}}}((A_\kappa c) \cap \mathrm{dom}(\widetilde{M}))$. By (v′) this set contains $\mathrm{acl}_{\widetilde{\mathcal{L}}}(A_\kappa c)$ and hence $\mathrm{acl}_{\widetilde{\mathcal{L}}}(Ac)$. Applying Claim **(I.2.17)** with $i = \kappa$, we obtain (v).
■

I.3. Extensible 1-types in intersections of balls

The goal of this section is to establish some results about unary types in henselian fields (specifically, finite extensions of $\mathbb{Q}_p$ and ultraproducts of such fields), which will be useful to prove that Proposition **(I.2.11)** can be applied to these fields.
In this section, we will not be considering valued fields in the geometric language as we need quantifier elimination and not elimination of imaginaries. Let $\mathcal{R}$ be a set of symbols; we will be working in the countable language $\mathcal{L} := \{\mathbf{K}, +, \cdot, ^{-1}, \mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}, r : \mathbf{K} \to \mathbf{K}_r, \dots\}_{r\in\mathcal{R}}$ where the $\mathbf{K}_r$ are new sorts, each r is such that $r|_{\mathbf{K}^\star}$ is a surjective group homomorphism $\mathbf{K}^\star \to \mathbf{K}_r$ that vanishes on $1 + \mathcal{M}^{\cdot\nu}$ for some $\nu = \nu(r) \in \mathbb{N}$ and the $\dots$ refer to additional constants on $\mathbf{K}$ and additional relations on the sorts $\mathbf{K}_r$ and $\mathbf{\Gamma}$. Let T be some theory of discretely valued fields in this language that eliminates quantifiers. Throughout this section M will be a sufficiently saturated model of T and $\lambda_0 \in \mathbf{K}(M)$ a uniformizer. We will write $\overline{r}$ for the (possibly infinite) tuple of all $r \in \mathcal{R}$.
Assume $\mathbf{\Gamma}$ is definably well-ordered (every nonempty definable subset with a lower bound has a least element). Let

$$Q_{\mathcal{R}} := \{(u, v_r, \dots)_{r\in\mathcal{R}} : (\exists x \in \mathbf{K})\mathrm{val}(x) = u, \forall r \in \mathcal{R}, r(x) = v_r\}.$$

We write $\mathrm{val}(x) \gg \mathrm{val}(y)$ if $\mathrm{val}(x) > \mathrm{val}(y) + m\mathrm{val}(\lambda_0)$ for all $m \in \mathbb{N}$.

Finite extensions of the p-adics fit in this setting, by Prestel-Roquette [PR84], if we take the r_n to be the canonical projections $\mathbf{K}^\star \to \mathbf{K}^\star/(\mathbf{K}^\star)^{\cdot n}$. Note that every element of these finite groups is in $\mathrm{dcl}^{\mathrm{eq}}(\varnothing)$. In the case of ultraproducts of p-adic fields of residue characteristic zero and more generally of henselian valued fields with residue characteristic zero (denoted as $\mathrm{T}_{\mathrm{Hen},0}$), one map r suffices: the canonical projection $\mathrm{rv} : \mathbf{K}^\star \to \mathbf{RV}$.
Observe that $\mathrm{val}(x-y) \gg \mathrm{val}(x-z)$ implies $r(x-z) = r(y-z)$ for all $r \in \mathcal{R}$. Indeed $(y-z)/(x-z) = 1 + (y-x)/(x-z) \in 1 + \mathcal{M}^{\cdot \nu(r)}$.

Notation I.3.1:
If $b \in \mathbf{B}(M)$, $x \in \mathrm{dom}(M)$ and $x \notin b$, the valuation $\mathrm{val}(x-y)$ takes the same value for all $y \in b$. We denote it $\mathrm{val}(x-b)$. By $\mathrm{rad}(b)$ we denote the infimum of $\mathrm{val}(y-y')$, $y, y' \in b$. Moreover for all $r \in \mathcal{R}$, if $\mathrm{val}(x-b) + \nu(r)\mathrm{val}(\lambda_0) \leqslant \mathrm{rad}(b)$, then $r(x-y) = r(x-y')$ for all $y, y' \in b$. We write $r(x-b) = r(x-y)$ in this case.

Definition I.3.2:
Let $f = (f_i)_{i\in I}$ be a family of A-definable functions for some $A \subseteq M^{\mathrm{eq}}$. A partial type p over A is complete over A relative to f *if the map* $\mathrm{tp}(c/A) \mapsto \mathrm{tp}(f(c)/A)$ *is injective on the set of extensions of p to $S(A)$.*

Remark I.3.3:
The partial type $p(x)$ is complete over A relative to f if and only if for every formula $\varphi(x)$ over A, there exists a formula $\theta(u)$ over A such that $p \vdash (\varphi(x) \iff \theta(f(x)))$.

For the rest of the section we are going to study generic types of intersections of balls. Let $\bar{b} = \{b_i : i \in I\}$ be a descending sequence of balls in $\mathbf{B}(M)$. Let $P := \bigcap_{i\in I} b_i$. Let $P_{\mathbf{\Gamma}} := \{\gamma \in \mathbf{\Gamma} : \forall i \in I\, \gamma > \mathrm{rad}(b_i)\}$. For any A with $b_i \in \mathrm{dcl}^{\mathrm{eq}}(A)$, we define the generic type of P over $A \subseteq M^{\mathrm{eq}}$ to be

$$q_P|_A := P(x) \cup \{x \notin b : b \in \mathbf{B}(\mathrm{acl}^{\mathrm{eq}}(A)), b \,\text{strictly included in}\, P\}.$$

In Section I.4, we will also be considering the ACVF-generic of such an intersection P, i.e. the same notion of genericity but considered in algebraically valued fields. Note that if L is a valued field, $A \subseteq L$ and P is an intersection of balls in $\mathbf{B}(A)$, then the difference between the generic type of P over A in L and in $\overline{L}^{\mathrm{alg}}$ is that the latter must also avoid balls that do not have a center or a radius in L but in $\overline{L}^{\mathrm{alg}}$.

Remark I.3.4:
If P is a strict intersection, i.e. P is not equal to a ball or equivalently $\bar{b}$ does not have a minimal element, then for an element to be generic in P over A it suffices to check that x is not contained in any ball $b \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ contained in P. Indeed, if $b \in \mathbf{B}(\mathrm{acl}^{\mathrm{eq}}(A))$, then the smallest ball containing all A-conjugates of b is strictly included in P and is definable over A.

In what follows, we will consider $A \subseteq M^{\mathrm{eq}}$ containing all constants in $\mathbf{K}$, and $\bar{b}$ a decreasing sequence of balls in $\mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ (indexed by some ordinal). Unless otherwise mentioned, until Proposition **(I.3.9)**, we will suppose that $P = \bigcap_i b_i$ is strict.

Lemma I.3.5:
Suppose $A \subseteq \mathbf{K}(M)$. Fix $a \in A$ with $a \in b_i$ for each i. Then $q_P|_A$ is complete relative to $\mathrm{val}(x-a)$ and to $\overline{r}(x-a)$. Moreover, if $P(\mathrm{dcl}^{\mathrm{eq}}(A)) = \varnothing$ then $q_P|_A$ is complete.

Proof. Taking into account quantifier elimination, we must show the following: let $c, c' \in M$ be two realizations of $q := q_P|_A$ such that $(\mathrm{val}(c-a), \overline{r}(c-a))$ has the same type over A as $(\mathrm{val}(c'-a), \overline{r}(c'-a))$; then the substructures $A(c)$, $A(c')$ generated by c, c' over A (which are simply the fields generated by c, c' over A) are isomorphic over A.

Extend the valuation from $\mathbf{K}(M)$ to $L := \overline{\mathbf{K}(M)}^{\mathrm{alg}}$—the algebraic closure of $\mathbf{K}(M)$. The intersection of $\mathbf{K}^\star(M)$ with $\ker(r) \cdot (1 + \lambda_0^{\nu(r)}\mathcal{O}(L))$ is $\ker(r)$: indeed $\ker(r) \subseteq \mathbf{K}^\star(M)$ and $(1 + \lambda_0^{\nu(r)}\mathcal{O}(L)) \cap \mathbf{K}^\star(M) = (1 + \lambda_0^{\nu(r)}\mathcal{O}(K)) \subseteq \ker(r)$. It follows that the natural map $\mathbf{K}^\star(L) \to \mathbf{K}^\star(L)/\ker(r) \cdot (1 + \lambda_0^{\nu(r)}\mathcal{O}(L))$ extends r. We also call the extended map r. By construction, the following still holds: for all $x, y, z \in L$, $\mathrm{val}(x-y) \gg \mathrm{val}(x-z)$ implies $r(x-z) = r(y-z)$.

Then it suffices to show that $\overline{A}^{\mathrm{alg}}(c), \overline{A}^{\mathrm{alg}}(c')$ are $\overline{A}^{\mathrm{alg}}$-isomorphic, by an isomorphism commuting with the extensions of the maps r (one can then restrict the isomorphism to $A(c)$). As $(\mathrm{val}(c-a), \overline{r}(c-a))$ and $(\mathrm{val}(c'-a), \overline{r}(c'-a))$ realize the same type over A, by taking a conjugate of c' over A, we may assume the tuples are equal.

Take any $d \in \overline{A}^{\mathrm{alg}}$. If $d \notin b_i$ for some i, then $\mathrm{val}(c-d) = \mathrm{val}(c'-d)$. Moreover, for any $k \in \mathbb{N}$, $\mathrm{val}(c-c') \geqslant \mathrm{rad}(b_{i+k}) \geqslant \mathrm{rad}(b_i) + k\mathrm{val}(\lambda_0) > \mathrm{val}(c-d) + k\mathrm{val}(\lambda_0)$; and it follows that $\overline{r}(c-d) = \overline{r}(c'-d)$.

If $d \in b_i$ for each i, then the smallest ball $b \in \mathbf{B}(L)$ containing a and all the conjugates of d over A is (quantifier-free) A-definable in L. As $\mathbf{\Gamma}$ is definably well-ordered, the $\mathbf{K}(M)$-points of b form a ball $b' \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ which is included in P. Hence c and c' are not in b' nor, in fact, in any of the balls centered around b' with radius $\mathrm{rad}(b') - k\mathrm{val}(\lambda_0)$. It follows that $\mathrm{val}(c-d) = \mathrm{val}(c-a)$ and $\overline{r}(c-d) = \overline{r}(c-a)$, and similarly for c'. Thus $\mathrm{val}(c'-d) = \mathrm{val}(c-d)$ and $\overline{r}(c-d) = \overline{r}(c'-d)$.

As any rational function g over A is a ratio of products of constant or linear polynomials, it follows that $\mathrm{val}(g(c)) = \mathrm{val}(g(c')), \overline{r}(g(c)) = \overline{r}(g(c'))$. This proves the first part of the lemma.

If P does not contain any point in A, then there cannot be any $d \in \overline{A}^{\mathrm{alg}}$ such that $d \in b_i$ for each i. Indeed, let $d_{j \leqslant n}$ be the $\mathcal{L}$-conjugates of d over A; then $e := 1/n \sum_j d_j \in \mathrm{dcl}^{\mathrm{eq}}(A)$ and for all i, $d_j \in b_{i+k}$ where k is such that $k\mathrm{val}(\lambda_0) \geqslant \mathrm{val}(n)$ and $\mathrm{val}(e-d) \geqslant \mathrm{val}(1/n) + \mathrm{rad}(b_{i+k}) \geqslant \mathrm{rad}(b_i)$. It follows that $e \in P(\mathrm{dcl}^{\mathrm{eq}}(A))$, a contradiction. But the hypothesis about $(\mathrm{val}(x-a), r(x-a))$ is only used when $d \in P$. Thus the second assertion follows. ■

Remark I.3.6:
Suppose T extends $\mathrm{T}_{\mathrm{Hen},0}$ and $A \subseteq \mathbf{K}(M)$. Without any assumption on P (it can be strict, a closed ball or an open ball), if $P(A) = \varnothing$ then P is a complete type. The exact same proof works as balls are convex in residue characteristic zero and the unique $r = \mathrm{rv}$ we need has kernel $1 + \mathcal{M}$, i.e. $\mathrm{val}(x-y) > \mathrm{val}(x-z)$ alone implies $r(x-z) = r(y-z)$.

We now want to prove (in Proposition **(I.3.9)**) that Lemma **(I.3.5)** is true without the assumption that $A \subseteq \mathbf{K}(M)$.

Lemma I.3.7:
Suppose $A \subseteq M^{\mathrm{eq}}$ is such that P contains no $b \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$. Then $q_P|_A$ is a complete type.

Proof. Let us suppose A is countable. Then the partial type $P = \bigcap_{n=1,2,\dots} b_n$ is not isolated over A; for if the formula $\theta(x)$ with parameters in A implies $x \in b_i$ for all i, then, as $\mathbf{\Gamma}$ is definably well-ordered, there is a smallest ball b containing θ. This ball is strictly contained in P and is A-definable, a contradiction. Then by the omitting types theorem, there exists a model M_0 such that $A \subseteq M_0^{\mathrm{eq}}$ and $P(M_0) = \emptyset$. By Lemma **(I.3.5)**, $q_P|_{\mathbf{K}(M_0)}$ is a complete type, and, as $\mathbf{K}$ is dominant in M_0^{eq}, P is a complete type over M_0^{eq} and hence over A.
If A is not countable, let c and c' be generic in P over A and let $(M_0^{\mathrm{eq}}, A_0) \prec (M^{\mathrm{eq}}, A)$ be countable (in the language where we add a predicate for A) and contain c and c'. Let Q be the intersection of all A_0-definable balls in M_0 that contain c; then Q is strict, it contains no A_0-definable ball and also contains c' (all of this is expressed in the type of c, c' in the language with the new predicate). By the countable case, c and c' have the same type over A_0 in M_0^{eq} and hence, they have the same type over A in M^{eq}. ■

Lemma I.3.8:
Let $q_{\mathcal{R}}$ be a complete type over A extending $Q_{\mathcal{R}}$. Suppose $q_{\mathcal{R}}$ implies both that $u \in P_{\mathbf{\Gamma}}$, and that, for any $\gamma \in P_{\mathbf{\Gamma}}(\mathrm{dcl}^{\mathrm{eq}}(A))$, $\gamma > u$. Then

$$q_P|_M \cup \bigcup_{a \in P(M)} q_{\mathcal{R}}(\mathrm{val}(x-a), \overline{r}(x-a))$$

is consistent.

Proof. We may assume M has an element a' with $a' \in b_i$ for each i. Note that $q_{\mathcal{R}}$ is consistent with $\{\gamma > u : \gamma \in P_{\mathbf{\Gamma}}(M)\}$. Indeed, for any $\gamma \in P_{\mathbf{\Gamma}}(M)$, if $q_r \vdash u \geqslant \gamma$, then some $\psi \in q_r$ is bounded below by γ; but then the minimum $\gamma' \leqslant \gamma$ of ψ in M exists as $\mathbf{\Gamma}$ is definably well-ordered, γ' is in $P_{\mathbf{\Gamma}}(\mathrm{dcl}^{\mathrm{eq}}(A))$ and $q_{\mathcal{R}} \vdash \gamma' \leqslant u$, contradicting our hypothesis.
Let c' be such that $(\mathrm{val}(c'), \overline{r}(c')) \vDash q_{\mathcal{R}} \cup \{\gamma < u : \gamma \in P_{\mathbf{\Gamma}}(M)\}$ and $d := a' + c'$. Clearly $d \vDash q_P|_M$; indeed $\mathrm{val}(d - a') = \mathrm{val}(c') \in P_{\mathbf{\Gamma}}$ and thus $d \in b_i$ for all i. Now, let us assume there exist $b \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(M))$ included in P and containing d. Taking a bigger ball, we can suppose that $a' \in b$, too; but then $\mathrm{val}(d - a') = \mathrm{val}(c') > \mathrm{rad}(b) \in P_{\mathbf{\Gamma}}(M)$ contradicting the choice of c'. Moreover for any $a \in P(M)$, $\mathrm{val}(d - a') = \mathrm{val}(c') \ll \mathrm{val}(a - a')$. Thus $\mathrm{val}(d - a) = \mathrm{val}(d - a') = \mathrm{val}(c')$ and $\overline{r}(d - a) = \overline{r}(d - a') = \overline{r}(c')$, and d realizes the given type. ■

Proposition I.3.9:
Assume P is strict and fix $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ with $a \subseteq b_i$ for each i. Then $q_P|_A$ is complete relative to $\mathrm{val}(x-a)$ and to $\overline{r}(x-a)$. Moreover, if P does not contain any ball in $\mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ then $q_P|_A$ is complete.

Proof. The second case is tackled in Lemma **(I.3.7)**. So we can suppose that such an $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ exists. Let $c, c' \vDash q_P|_A$ such that $q_{\mathcal{R}} := \mathrm{tp}(\mathrm{val}(c-a)\overline{r}(c-a)/A) = \mathrm{tp}(\mathrm{val}(c'-a)\overline{r}(c'-a)/A)$. Let $M_0 \prec M$ such that $A \subseteq M_0^{\mathrm{eq}}$. It follows from Lemma **(I.3.8)** that there exists $c_0 \vDash q_P|_{M_0^{\mathrm{eq}}} \cup q_{\mathcal{R}}(\mathrm{val}(x-a)\overline{r}(x-a))$. Taking conjugates of c and c' over A, we can

suppose that $(\mathrm{val}(c-a)\overline{r}(c-a)) = (\mathrm{val}(c_0-a)\overline{r}(c_0-a)) = (\mathrm{val}(c'-a)\overline{r}(c'-a))$ as these three tuples have the same type over A.
Then, as shown in the proof of Lemma **(I.3.8)**, $c, c' \models q_P|_{M_0^{\mathrm{eq}}}$. By Lemma **(I.3.5)**, c and c' have the type over M_0^{eq} and hence over A. ∎

Corollary I.3.10:
Let L be a finite extension of $\mathbb{Q}_p$, $M \models \mathrm{Th}(L)$ and $A \subseteq M$ such that $\mathbf{B}(\mathrm{acl}^{\mathrm{eq}}(A)) \subseteq A$. Let $c \in \mathrm{dom}(M)$. Then $\mathrm{tp}(c/A)$ extends to an $\mathrm{Aut}(M/A)$-invariant type.

Proof. Let $W(c;A) := \{b \in \mathbf{B}(A) : c \in b\}$, $P := \bigcap_{b\in W(c;A)} b$. As the residue field of M is finite, P cannot reduce to a single ball (that ball would be the union of finitely many proper subballs, each in $\mathbf{B}(\mathrm{acl}^{\mathrm{eq}}(A))$, hence in A and one of them would contain c). Note that $c \models q_P|_A$.
If there is no ball $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ contained in P, then let $q_{\mathcal{R}}$ be any $\mathrm{Aut}(M/A)$-invariant type extending $Q_{\mathcal{R}}$ which implies $u \in P_{\Gamma}$ and $\alpha > u$ for all $\alpha \in P_{\Gamma}(M)$. If such a ball a exists, we suppose $q_{\mathcal{R}}$ also extends $\mathrm{tp}(v(c-a), \overline{r}(c-a)/A)$. By Lemma **(I.3.8)**, $q^* := q_P|_{M(x)} \cup \bigcup_{a\in P(M)} q_{\mathcal{R}}(\mathrm{val}(x-a), \overline{r}(x,a))$ is consistent. Clearly q^* is $\mathrm{Aut}(M/A)$-invariant. It follows from Proposition **(I.3.9)** that q^* is complete and that it extends $\mathrm{tp}(c/A)$. ∎

Let N_n be the group of matrices of the form I_n+b, where I_n is the identity matrix in GL_n, and b is an upper triangular matrix with all entries having valuation $\gg 0$. Thus $N_n = \mathrm{B}_n(\mathcal{O}) \cap \bigcap_m (I_n + \lambda_0^m \mathrm{B}_n(\mathcal{O}))$.

Lemma I.3.11:
There exists an $\mathrm{Aut}(M)$-invariant type $p|_M$ of matrices $a \in N_n$, invariant under right multiplication: for all $A \subseteq M^{\mathrm{eq}}$ and $b \in N_n(A)$, if $c \models p|_A$, then $cb \models p|_A$. The type p is complete relative to the norms and $\overline{r}$-values of the entries. Moreover, if $Q_{\mathcal{R}}$ has an $\mathrm{Aut}(M)$-invariant 1-type t extending the cut defined by $(\mathrm{val}(\lambda_0), 2\mathrm{val}(\lambda_0), \dots)$ on the left, then p can be taken to be complete.

Proof. Let $P := \bigcap_i (\lambda_0^i \mathcal{O})$ and $q := q_P|_M$; then q is $\mathrm{Aut}(M)$-invariant and complete relative to val and $\overline{r}$ by Proposition **(I.3.9)** (as P contains 0). If t as above exists, then take $q := q_P|_M \cup t(\mathrm{val}(x), \overline{r}(x))$ which is consistent by Lemma **(I.3.8)**, complete and $\mathrm{Aut}(M)$-invariant.
Let p be the type of upper-triangular matrices obtained by taking the $n(n+1)/2$th tensor power of q (where by tensor product, here we mean the tensor product of types; see just below for a more explicit statement), using the lexicographic order on the matrix entries, and adding 1 on the diagonal: thus for all $A \subseteq M^{\mathrm{eq}}$, if $a \in M_n$, then $I_n + a \models p|_A$ if and only if $a_{11} \models q|_A, a_{12} \models q|_{\mathrm{dcl}^{\mathrm{eq}}(Aa_{11})}, \dots a_{22} \models q|_{\mathrm{dcl}^{\mathrm{eq}}(Aa_{11}\dots a_{1,n})}, \dots, a_{n,n} \models q|_{\mathrm{dcl}^{\mathrm{eq}}(Aa_{11},\dots a_{n,n-1})}$, while $a_{ij} = 0$ for $i > j$.
The fact that p is an $\mathrm{Aut}(M)$-invariant (partial) type of elements of N_n is clear. As for the right translation invariance, let $I_n + b \in N_n(A)$ and $I_n + a \models p|_A$; we have to show that $(I_n + a)(I_n + b) = I_n + a + b + ab \models p|_A$. Let $d := a + b + ab$. Then $d_{11} = a_{11} + b_{11} + a_{11}b_{11}$. We have

$$\mathrm{val}(a_{11}b_{11}) > \mathrm{val}(b_{11}) \gg \mathrm{val}(a_{11}) \gg 0.$$

So $\mathrm{val}(d_{11}) = \mathrm{val}(a_{11})$ and hence d_{11} also realizes $q_P|_A$. Furthermore, we also have $\overline{r}(d_{11}) = \overline{r}(a_{11})$; it follows that $d_{11} \models q|_A$. Similarly

$$d_{12} = a_{12} + b_{12} + a_{11}b_{12} + a_{12}b_{22};$$

here a_{12} has strictly bigger valuation than any of the other summands, so again $\mathrm{val}(d_{12}) = \mathrm{val}(a_{12})$ and $\overline{r}(d_{12}) = \overline{r}(a_{12})$, thus $d_{12} \models q|_{\mathrm{dcl}^{\mathrm{eq}}(Aa_{11})}$. But since $b \in \mathrm{dcl}^{\mathrm{eq}}(A)$, we have $d_{11} \in \mathrm{dcl}^{\mathrm{eq}}(Aa_{11})$, so $d_{12} \models q|_{\mathrm{dcl}^{\mathrm{eq}}(Ad_{11})}$. Continuing in this way we see that $I_n + d \models p|_A$. ∎

Corollary I.3.12:
Let R be a left coset of N_n in $\mathrm{B}_n(\mathbf{K})$. There exists an $\mathrm{Aut}(M/\ulcorner R \urcorner)$-invariant type of elements of R.

Proof. Pick $g \in R$, let p be the right-N_n-invariant type of Lemma **(I.3.11)**, and for all $A \subseteq M^{\mathrm{eq}}$, let $p^g|_A := \mathrm{tp}(cg/Ag)$, where $c \models p|_{\mathrm{dcl}^{\mathrm{eq}}(Ag)}$. Then $p^g|_A = p^{hg}|_A$ for $h \in N_n(\mathrm{dcl}^{\mathrm{eq}}(A))$, since p is right-N_n-invariant. Thus any automorphism fixing $\ulcorner R \urcorner$ must fix the global type $p^g|_M$. ∎

Corollary I.3.13:
Let L be a finite extension of $\mathbb{Q}_p$ and let $M \models \mathrm{Th}(L)$, $e \in \mathbf{S}_n(M)$, $E := \mathrm{acl}^{\mathrm{eq}}(e)$. Then there exists an $\mathrm{Aut}(M/E)$-invariant type of bases for e.

Proof. It was noted in Section I.2.2 that any lattice e has a triangular basis; this basis can be viewed as the set of columns of a matrix in $\mathrm{B}_n(\mathbf{K})$. Let b, b' be two such bases, and suppose $b' = \sigma(b)$, $\sigma \in \mathrm{Aut}(M/E)$. Then as $e/\lambda_0^m e$ is finite for all m, the columns of b, b' must be in the same coset of $\lambda_0^m e$ for each m. Thus if we write $b' = ab$ with $a \in \mathrm{B}_n(\mathcal{O})$, then $a = I_n$ modulo $\lambda_0^m \mathcal{O}$ for each m, so $a \in N_n$ and $\mathrm{Aut}(M/E)$ preserves the coset $N_n b$. So it suffices to take the $\mathrm{Aut}(M/\ulcorner N_n b \urcorner)$-invariant type of elements of $N_n b$ guaranteed by Corollary **(I.3.12)**. ∎

Let us now suppose that T extends $\mathrm{T}_{\mathrm{Hen},0}$. Using similar techniques, we can extend the previous results to the case when P is a closed ball (this case is only relevant to Section I.5). For the last result, though, we will also need the residue field to be pseudo-finite.
Let b be a closed ball. We will write res_b for the map that sends $x \in b$ to $x + \mathrm{rad}(b)\mathcal{M}$, the maximal open subball of b containing x.

Lemma I.3.14:
Let $b \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ a closed ball and q a complete type over A extending $\mathrm{res}_b(b)$ such that $q \vdash x \neq b'$ for all $b' \in \mathrm{res}_b(b)(\mathrm{acl}^{\mathrm{eq}}(A))$. Then

$$q_b|_M \cup q(\mathrm{res}_b(x))$$

is consistent.

Proof. Let us first show that q is consistent with $\{x \neq b' : b' \in \mathrm{res}_b(b)(M^{\mathrm{eq}})\}$. If not there is a finite number of balls $b_i \in \mathrm{res}_b(b)(M^{\mathrm{eq}})$ such that $q \vdash \bigvee_i x = b_i$. If we take a minimal number of such balls, each of them must realize q and hence be algebraic over A, a contradiction.
Now, let c be such that $\mathrm{res}_b(c) \models q \cup \{x \neq b' : b' \in \mathrm{res}_b(b)(M^{\mathrm{eq}})\}$; then $c \models q_b|_M$. Indeed $c \in b$ and if c is in $b' \in \mathbf{B}(M^{\mathrm{eq}})$ such that $b' \subseteq b$, then $c \in \mathrm{res}_b(b') \in \mathrm{res}_b(b)(M^{\mathrm{eq}})$, contradicting the choice of c. ∎

Lemma I.3.15:
Suppose $P = b$ is a closed ball. Then $q_b|_A$, the generic type of b, is complete relative to res_b.

Proof. If $A \subseteq \mathbf{K}(M)$ then, by the same considerations as in Lemma **(I.3.5)** (and, as $\mathrm{T}_{\mathrm{Hen},0} \subseteq T$, taking $\overline{r} = \mathrm{rv}$ is enough), it suffices to show that if c and c' are realizations of $q_b|_A$ such that $\mathrm{res}_b(c) = \mathrm{res}_b(c')$ then for all $d \in \overline{A}^{\mathrm{alg}}$, $\mathrm{rv}(c-d) = \mathrm{rv}(c'-d)$. If $d \in \mathrm{res}_b(c)$, then $c \in \mathrm{res}_b(c) = \mathrm{res}_b(d) \in \mathbf{B}(\mathrm{acl}^{\mathrm{eq}}(A))$ as $d \in \overline{A}^{\mathrm{alg}}$. This contradicts the fact that $c \models q_b|_A$. Hence $d \notin \mathrm{res}_b(c)$. As c and $c' \in \mathrm{res}_b(c) = \mathrm{res}_b(c')$, $\mathrm{val}(c-c') > \mathrm{val}(c-d)$ and $\mathrm{rv}(c-d) = \mathrm{rv}(c'-d)$.
If A is not contained in $\mathbf{K}$, let $c, c' \models q_b|_A$ such that $q := \mathrm{tp}(\mathrm{res}_b(c)/A) = \mathrm{tp}(\mathrm{res}_b(c')/A)$. By Lemma **(I.3.14)**, there exists $c_0 \models q_b|_M \cup q$. Taking A-conjugates of c and c', we can suppose that $\mathrm{res}_b(c) = \mathrm{res}_b(c_0) = \mathrm{res}_b(c')$. Then, as seen in the proof of Lemma **(I.3.14)**, $c, c' \models q_b|_M$. By the previous paragraph c and c' have the same type over M and hence over A. ∎

Corollary I.3.16:
Suppose $P = b$ is a closed ball and let $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ be contained in b. Then $q_b|_A$ is complete relative to $\mathrm{rv}(x-a)$.

Proof. If $c, c' \models q_b|_A$, then $\mathrm{val}(c-a) = \mathrm{val}(c'-a) = \mathrm{rad}(b)$ and hence $\mathrm{res}_b(c) = \mathrm{res}_b(c')$ if and only if $\mathrm{rv}(c-a) = \mathrm{rv}(c'-a)$. Thus the corollary follows immediately from Lemma **(I.3.15)**.
∎

Corollary I.3.17:
Suppose $\mathbf{k}$ is pseudo-finite, $\mathbf{k}(A)$ contains the constants needed for $\mathbf{k}$ to have EI and $P = b$ is a closed ball that contains no ball $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$. Then $x \in b$ generates a complete type over A.

Proof. By Lemma **(I.3.15)**, it suffices to show that $\mathrm{res}_b(b)$ is a complete type over A. But $\mathrm{res}_b(b)$ is a definable 1-dimensional affine space over $\mathbf{k}$—i.e. a $V := \gamma\mathcal{O}/\gamma\mathcal{M}$-torsor where $\gamma := \mathrm{rad}(b)$. Hence $H := \mathrm{Aut}(\mathrm{res}_b(b)/\mathbf{k}, A)$ is a subgroup of a semi-direct product of V and the multiplicative group $\mathbb{G}_m(\mathbf{k})$. The subgroup $H \cap V$ (i.e. the group of translations of $\mathrm{res}_b(b)$ that also are automorphisms over A and $\mathbf{k}$) is ∞-definable over A. Indeed, it is the set $\{u \in V : \forall \overline{y}\, \forall x\, (x \in \mathrm{res}_b(b) \wedge \overline{y} \in \mathbf{k}) \implies (\varphi(x,\overline{y}) \iff \varphi(x+u,\overline{y}))$ for all A-formulas $\varphi(x,\overline{y})\}$.
As $\mathbf{k}$ is a pseudo-finite field, V has no nontrivial proper definable subgroup. And because in a pseudo-finite field any ∞-definable group is an intersection of definable groups, V has no nontrivial proper ∞-definable subgroup either. If $H \cap V = V$ then H acts transitively on $\mathrm{res}_b(b)$ (by translation) and, as $H \leqslant \mathrm{Aut}(\mathrm{res}_b(b)/A)$, we are done. On the contrary, if $H \cap V = \{1\}$, then H contains no translations and must either have exactly one fixed point or be the trivial group and hence fix all points in $\mathrm{res}_b(b)$.
Suppose H has only one fixed point $a \in \mathrm{res}_b(b)$ and let $\theta \in \mathrm{Aut}(\mathrm{res}_b(b)/A)$. For any $\sigma \in H$, $\theta^{-1} \circ \sigma \circ \theta \in H$ and hence $(\theta^{-1} \circ \sigma \circ \theta)(a) = a$, i.e. $\theta(a)$ is fixed by σ. As a is the only point fixed by H, $\theta(a) = a$ and $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$: but this is a contradiction. It follows that H fixes every point in $\mathrm{res}_b(b)$ and hence, because $\mathbf{k}$ is stably embedded, $\mathrm{res}_b(b) \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{k}, A)$. But then we must also have $V \subseteq \mathrm{dcl}^{\mathrm{eq}}(\mathbf{k}, A)$. Hence $(V, \mathrm{res}_b(b))$ is A-definably isomorphic to a definable (regular) homogeneous space (G, R) of $\mathbf{k}^{\mathrm{eq}} = \mathbf{k}$. As $\mathbf{k}$ is stably embedded, (G, R) is definable over $A' := \mathbf{k}^{\mathrm{eq}}(\mathrm{dcl}^{\mathrm{eq}}(A)) = \mathbf{k}(\mathrm{dcl}^{\mathrm{eq}}(A))$.

Hence we only have to show that any A'-definable 1-dimensional affine space in a pseudo-finite field has an A'-point to obtain a contradiction. Let us consider $\mathbf{k}$ elementarily embedded in the fixed field of $L \models ACFA$ and let $\overline{A'}$ be the algebraic closure of A' in L. Note that A is algebraically closed in $ACFA$ and is a model of ACF. By usual arguments (e.g., [KP02]) there exists an ACF $\overline{A'}$-definable homogeneous space (G', S') and interalgebraic group configurations in (G, R) and (G', S'). Replacing G' with its identity component G'_0 and S' with the G'_0-orbit of any $\overline{A'}$-point in S' (there is such a point because $\overline{A'} \models ACF$), we can suppose that G' is connected. By some additional classical arguments (although the literature mainly concerns itself with groups and not homogeneous spaces at this point: see [KP02] again), there is an $\overline{A'}$-definable subgroup H of $G \times G'$, such that $H_0 := \{x \in G : (x, 0) \in H\}$ and $H'_0 := \{x \in G' : (0, x) \in H\}$ are finite central subgroups and the left and right projections of H must have finite index in G (respectively G'). But as G and G' are connected, these projections must be the groups themselves. As G has no torsion (we are in characteristic 0), H_0 is trivial. Taking the quotient of (G', S') by H'_0—i.e. considering the group G'/H'_0 acting on the H'_0 orbits of S'—the group H is in fact (the graph of) an isomorphism. In particular, as G has no proper definable subgroup, this implies that the action of G' on S' is also regular, i.e. S' is a G'-torsor.
Let (a, a') be generic in $R \times S'$, let X be the H-orbit of (a, a') and let $P := \mathrm{tp}(aa'/\overline{A'})$. As P and X have the same dimension (equal to 1), P cannot be covered with infinitely many H-orbits (pseudo-finite fields have the (E) property of [HP94]) and as $\overline{A'}$ is algebraically closed (including imaginaries), X must contain P and hence is $\overline{A'}$-definable. Moreover, it is quite easy to see that X is (the graph of) an isomorphism between R and S'. As S' contains $\overline{A'}$-points, so does R. Let d be one of these points, and let $(d_i)_{i=1\dots n}$ be its A'-conjugates. Then $1/n \sum_i d_i \in R(A')$, and we have the A'-point we have been looking for. ∎

To conclude this section, let me summarize the classification of unary types in PL_0.

Proposition I.3.18:
Suppose T extends PL_0 and $\mathbf{k}(A)$ contains the constants needed for $\mathbf{k}$ to have EI. Let $a \in \mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ with $a \subseteq b_i$ for each i. Then $q_P|_A$ is complete relative to $\mathrm{val}(x-a)$ and to $\overline{r}(x-a)$. Moreover, if P does not contain any ball in $\mathbf{B}(\mathrm{dcl}^{\mathrm{eq}}(A))$ then $q_P|_A$ is complete.

Proof. If P is strict we can apply Proposition **(I.3.9)**. If not apply Corollary **(I.3.16)** or Corollary **(I.3.17)**. ∎

I.4. The p-adic case

Let L be a finite extension of $\mathbb{Q}_p$. As stated in Remark **I.2.6.2**, it can be shown that there exists a number field $F \subseteq L$ that contains a uniformizer λ_0 of L, such that $\mathrm{res}(L) = \mathrm{res}(F)$ and such that every finite extension L' of L is generated by an element α whose minimal polynomial is defined over F and such that α also generates the valuation ring $\mathcal{O}(L')$ over $\mathcal{O}(L)$. Let T_L denote the theory of L in $\mathcal{L}^{\mathcal{G}} \cup \{P_n : n \in \mathbb{N}_{>0}\} \cup \{c\}$, where the predicates P_n stand for the nonzero nth powers (in the sort $\mathbf{K}$) and c generates F over $\mathbb{Q}$. Then T_L is model complete (cf. [PR84]) and it is axiomatized by the fact that $\mathbf{K}$ is a henselian valued

field with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$, by the isomorphism type of F and by the definition of the P_n predicates.
We now check the hypotheses of Corollary **(I.2.15)** for $\widetilde{T} = T_L$ and $T = \mathrm{ACVF}^{\mathcal{G}}_{0,p,F}$ (the theory of algebraically closed valued fields of mixed characteristic in the geometric language with a constant for c; the F in the subscript is there to recall that we added a constant for a generator of F to the theory). We use the same notation as in Proposition **(I.2.11)**.

(i) Relative algebraic boundedness: By model completeness and the nature of the axioms (the only axioms that are not universal are the fact that the field is henselian and the definition of the predicates P_n; but both state the existence of algebraic points) $\mathrm{acl}_{\mathcal{L}}(\widetilde{M}'c) \cap \widetilde{M}$ is an elementary submodel of $\widetilde{M}$, hence certainly $\widetilde{\mathcal{L}}$-definably closed.

(ii) Internalizing $\mathcal{L}$-codes: As $\mathbf{K}(\widetilde{M})$ is henselian, $\mathbf{K}(\mathrm{dcl}_{\mathcal{L}}(\widetilde{M})) = \mathbf{K}(\widetilde{M})$, hence if $e \in \mathbf{K}$, there is nothing to do. For any element e of $\mathbf{S}_n(M)$ let us write $\Lambda(e) \subseteq \mathbf{K}^n$ for the lattice represented by e. If $e \in \mathbf{S}_n(\mathrm{dcl}_{\mathcal{L}}(\widetilde{M}))$, $\Lambda(e)$ has a basis in some finite extension L_0 of $L := \mathbf{K}(\widetilde{M})$. Say $[L_0 : L] = m_0$; let L' be the join of all field extensions of L of degree m_0. Then L' is a finite extension of L such that any $\sigma \in \mathrm{Aut}(M)$ stabilizing $\widetilde{M}$ globally, stabilizes L' globally; let $[L' : L] = m$. By hypothesis, there is a generator a of L' over L whose characteristic polynomial over L is defined over F. One has an a-definable isomorphism $f_a : L' \to L^m$ (as vector spaces over L), with $f_a(\mathcal{O}(L')) = \mathcal{O}(L)^m$ (i.e. $\mathcal{O}(L')$ is a free $\mathcal{O}(L)$-module of rank m). The morphism f_a further induces an isomorphism of the lattice $\Lambda(e)(L')$ with a lattice $f_a(\Lambda(e)(L')) = \Lambda(e')(L)$ for some $e' \in \mathbf{S}_{nm}(\widetilde{M})$. As any a' of the (finitely many) that are $\mathrm{Aut}(L'/F)$-conjugate to a is also $\mathrm{Aut}(L'/L)$-conjugate to a, we see that $\Lambda(e')(L) = f_{a'}(\Lambda(e)(L'))$ as well. Thus e and e' are interdefinable in the sense required in (ii).
Similarly for $\mathbf{T}_n$ (alternatively, for finite extensions L' of L, the value group also has a least element, hence we can apply Remark **I.2.6.1**).

Remark I.4.1:
We have proved something slightly stronger than (ii): we also have $e \in \mathrm{dcl}_{\mathcal{L}}(e')$. The inverse of f_a is a linear map $L^m \to L'$, say $g_a(\alpha_1, \dots, \alpha_m) := \sum \alpha_i a^i$. From the viewpoint of M, g_a is an a-definable linear map with $g_a(\Lambda(e')) = \Lambda(e)$ (as g_a is $\mathbf{K}$-linear, this remains true for the lattices generated by the L- or L'- points of $\Lambda(e)$ and $\Lambda(e')$). Moreover this is also true for any of the finitely many conjugates of a. Thus $e \in \mathrm{dcl}_{\mathcal{L}}(e')$.

The following corollary of this stronger version of (ii) is not needed for what follows but it sheds some light on the interaction between automorphisms of $\widetilde{M}$ and $\mathcal{L}(M)$-definable sets.

Corollary I.4.2:
Let $A = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A) \subseteq \widetilde{M}$. Let G be the group of automorphisms of M that stabilizes $\widetilde{M}$ globally and fixes A point-wise. Let $e \in M$, and assume $g(e) = e$ for all $g \in G$. Then $e \in \mathrm{dcl}_{\mathcal{L}}(A)$.

Proof. We have $e \in \mathrm{dcl}_{\mathcal{L}}(\widetilde{M})$, since $\mathrm{Aut}(M/\widetilde{M})$ fixes e. Let e' be as in (ii). Then G fixes e'; since G maps surjectively to $\mathrm{Aut}(\widetilde{M}/A)$, we have $e' \in A$. By the above Remark **(I.4.1)**, $e \in \mathrm{dcl}_{\mathcal{L}}(e')$. So $e \in \mathrm{dcl}_{\mathcal{L}}(A)$. ■

(iii) Unary EI: In [Sco97] P. Scowcroft has proved a weak version of this, where the sets are classes of equivalence relations in two variables. R. Cluckers has suggested that a strong version may be true: every unary subset can be coded in $\mathbf{B}$. This is what we prove here. Let e be an imaginary code for a unary subset $D \subseteq \mathbf{K}(\widetilde{M})$. Let $A := \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(e)$ and $B := \mathbf{B}(A)$.

Claim I.4.3: *For all* $c \in \mathbf{K}(\widetilde{M})$, $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/B) \vdash \mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$.

Proof. Following the notation of Corollary **(I.3.10)**, recall that $W(c;A) = \{b \in \mathbf{B}(A) : c \in b\}$. Let $P := \bigcap W(c;A) = \bigcap W(c;B)$, a strict intersection. Then $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/B) \vdash q_P|_B = q_P|_A$. By Proposition **(I.3.9)**, either $q_P|_A$ is a complete type and we are done, or there is some $a \in B$ such that $a \subseteq P$ and $q_P|_A$ is complete relative to $\overline{r}(x-a)$ and $\mathrm{val}(x-a)$. As $\mathbf{K}^\star = F(\mathbf{K}^\star)^{\cdot n}$ for all n, $\mathrm{tp}_{\widetilde{\mathcal{L}}}(\overline{r}(c-a)/A)$ follows from its type over F, i.e. over $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\varnothing)$. Moreover $\mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(B)) = \mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A))$ (as elements of $\mathbf{\Gamma}$ are coded by balls). Thus, as $\mathbf{\Gamma}$ is stably embedded and has unary EI, $\mathrm{tp}_{\widetilde{\mathcal{L}}}(\mathrm{val}(c-a)/B) \vdash \mathrm{tp}_{\widetilde{\mathcal{L}}}(\mathrm{val}(c-a)/A)$ and we have the expected result. ■

As D is $\widetilde{\mathcal{L}}(A)$-definable, D is also $\mathrm{Aut}(\widetilde{M}/B)$-invariant, so that by compactness D is definable over B. Hence $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(B)$. We conclude as in Corollary **(I.2.15)**: there is a tuple a from $\mathbf{B}$ with $a \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(e)$ and $e \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(a)$; so $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(e) = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(a')$, where a' is the finite set of $\widetilde{\mathcal{L}}(e)$-conjugates of a. We already know that finite sets are coded (e.g., by (ii) and Lemma **(I.2.12)**).

(iv) Invariant types and germs: The main ingredient for this proof is the C-minimality of ACVF.
Let $A = \mathrm{acl}_{\widetilde{\mathcal{L}}}(A)$, $c \in \mathbf{K}(\widetilde{M})$, $W(c;A) = \{b_i : i \in I\}$ and $P = \bigcap_i b_i$. The balls b_i are linearly ordered by inclusion, and we order I correspondingly: $i \leqslant j$ holds if $b_j \subseteq b_i$. As seen previously, P is a strict intersection. Let p be the ACVF generic of P.
If $r(x,b)$ is an $\mathcal{L}$-definable function, let $X(b',b'') := \{x : r(x,b') \neq r(x,b'')\}$. Then $X(b',b'')$ is a finite Boolean combination of balls and there exists $i = i(b',b'')$ such that $X(b',b'') \cap P$ is contained in a proper subball of P if and only if for each $j \geqslant i$, $X(b',b'') \cap b_j$ is contained in a proper subball of b_j.
Define an equivalence relation E_i by $b' E_i b''$ if and only if $X(b',b'') \cap b_i$ is contained in a proper subball of b_i (i.e. $r(x,b')$ and $r(x,b'')$ have the same germ on the ACVF-generic of b_i). Let $e_i := b/E_i$. Then:

$$\begin{aligned}
&\sigma \in \mathrm{Aut}(M/A) \text{ fixes the germ of } r(x,b)\\
&\quad\iff r(x,b) \text{ and } r(x,\sigma b) \text{ have the same } p\text{-germ}\\
&\quad\iff X(b,\sigma b) \cap P \text{ is contained in a proper subball of } P\\
&\quad\iff \text{for some } i, \text{ for all } j \geqslant i,\ bE_j\sigma(b)\\
&\quad\iff \text{for some } i, \text{ for all } j \geqslant i,\ \sigma \text{ fixes } e_j.
\end{aligned}$$

As for the consistency of $p|_{\widetilde{M}}$ with $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$: by definition of the ACVF generic, $p|_{\widetilde{M}}$ is generated by P along with all formulas $x \notin b$, where $b \in \mathbf{B}(\mathrm{dcl}_{\mathcal{L}}(\widetilde{M}))$ is a proper subball of P. As P is part of $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$, it suffices to show that $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$ does not imply any formula

$x \in d$ with d an $\mathcal{L}(\widetilde{M})$-definable finite union of balls $d_j \in \mathbf{B}(\mathrm{dcl}_{\mathcal{L}}(\widetilde{M}))$ strictly included in P.

Claim I.4.4: *For all $b \in \mathbf{B}(\mathrm{dcl}_{\mathcal{L}}(\widetilde{M}))$, such that $b(\widetilde{M}) \neq \varnothing$, there exists $b' \in \mathbf{B}(\widetilde{M})$ such that $b(\widetilde{M}) = b'(\widetilde{M})$.*

Proof. As $\mathbf{\Gamma}$ is definably well-ordered, $\inf\{\mathrm{val}(a-c) : a, c \in b(\widetilde{M})\} = \gamma \in \mathbf{\Gamma}(\widetilde{M})$. We can now take b' to be the ball of radius γ around any point in $b(\widetilde{M})$. ■

It follows that $d(\widetilde{M})$ is equal to a finite union d' of balls in $\mathbf{B}(\widetilde{M})$ and $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$ implies $x \in d' \subseteq P$. But this would contradict Lemma **(I.3.8)**.

(v) Weak rigidity: We use Lemma **(I.2.16)**. The hypothesis that for all $a \in \widetilde{M}$ there is a tuple $c \in \mathbf{K}(\widetilde{M})$ such that $a \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(c)$ and $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a))$ extends to an $\mathrm{Aut}(\widetilde{M}/\mathrm{acl}_{\widetilde{\mathcal{L}}}(a))$-invariant type, holds trivially when $a \in \mathbf{K}(\widetilde{M})$ and follows from Corollary **(I.3.13)** when $a \in \Lambda(\widetilde{M})$. For $a \in \mathbf{T}(\widetilde{M})$, as the value group has a least element, a is coded by an element of $S(\widetilde{M})$ (see Remark **I.2.6.1**) and hence, applying Corollary **(I.3.13)** to the code in $S(\widetilde{M})$, we are done.
The assumption (v′) of Lemma **(I.2.16)** is proved for $\mathbb{Q}_p$ by van den Dries in [Dri84a]. Let us briefly recall his proof to check that it adapts to the finite extension of $\mathbb{Q}_p$ case.
Let $B \subseteq \mathbf{K}(\widetilde{M})$ (we can assume that $B = \mathrm{dcl}_{\mathcal{L}}(B) \cap \mathbf{K}(\widetilde{M})$ is a field and contains F). Let $\sigma \in \mathrm{Aut}(\widetilde{M}/B)$ and let $B' = \{c \in \mathbf{K}(\mathrm{acl}_{\mathcal{L}}(B) \cap \widetilde{M}) : \sigma(c) = c\}$. It suffices to show that $B' \models T_L$. Indeed, by model completeness, $B' \prec \widetilde{M}$ will then contain $\mathrm{acl}_{\widetilde{\mathcal{L}}}(B)$, hence $\mathrm{acl}_{\widetilde{\mathcal{L}}}(B)$ is rigid over B.
As noted in the proof of (i), in order to show that $B' \models T_L$, we only have to show that B' is henselian and that the definition of the P_n is preserved.
By the universal property of the henselization, B' is an algebraic extension of B^h and hence it is henselian. Moreover, let $x \in B' \cap P_n(\widetilde{M})$ and let $y \in \mathbf{K}(\widetilde{M})$ such that $x = y^n$. Note first that $(y/\sigma(y))^n = x/\sigma(x) = 1$ and thus that $y/\sigma(y) \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(\varnothing)$. Furthermore, for all $m \in \mathbb{N}$, there exists $q \in F$ such that $yq \in P_m(\widetilde{M})$ (because $\mathbf{K}(\widetilde{M})$ is henselian, $\mathbf{\Gamma} = \mathrm{val}(F^\star) + m\mathbf{\Gamma}$ and $\mathcal{O} = \mathcal{O}(F) + \lambda_0^{m'}\mathcal{O}$). But then $y/\sigma(y) = yq/\sigma(yq) \in P_m(\widetilde{M})$. As $\bigcap_m P_m(\mathrm{acl}_{\widetilde{\mathcal{L}}}(\varnothing)) = \{1\}$, it follows that $y = \sigma(y)$, i.e. $y \in B'$.

Remark I.4.5:
As in [Dri84a], it follows from this proof that the restriction of T_L to the sort $\mathbf{K}$ has definable Skolem functions.

Proof of Theorem A: By Corollary **(I.2.15)**, we have EI to the sorts $\mathbf{K}, \mathbf{S}_n, \mathbf{T}_n$. But as is explained in Remark **I.2.6.1**, the sorts $\mathbf{T}_n$ are not actually needed.

We finish the section with some additional remarks.

Remark I.4.6:
If we do not want to add a constant c to the language, then it suffices to add "Galois-twisted $\mathbf{S}_n$", interpreted as $\mathbf{S}_n(K')$ for K' ranging over the finite extensions of $\mathbf{K}$.
Indeed, by Theorem **A**, any imaginary e is interdefinable over c with some tuple of real elements e'. So we have an e-definable function f_e with $f_e(c) = e'$ and a $\varnothing$-definable function

h with $h(c, f_e(c)) = e$. As c is algebraic over $\mathbb{Q}$, restricting to e-conjugates of c, we can take the graph of f_e (a finite set) to be a complete type over e.
With the new sorts, it is clear that (ii) holds without adding a constant and f_e is coded by some tuple $d \in \widetilde{M}$. Let us now show that d is a code for e. If e' is $\widetilde{\mathcal{L}}(d)$-conjugate to e there is some $\sigma \in \mathrm{Aut}(\widetilde{M}/d)$ such that $\sigma(e) = e'$. As σ fixes d, $\sigma(c)$ is also in the domain of f_e and hence $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c'/e) = \mathrm{tp}_{\widetilde{\mathcal{L}}}(c/e)$, i.e. $e' = \sigma(e) = \sigma(h(c, f_e(c))) = h(\sigma(c), f_e(\sigma(c))) = e$. This implies that d is a code for e.

Remark I.4.7:
Let $A = \mathrm{acl}_{\widetilde{\mathcal{L}}}(A) \subseteq \widetilde{M} \models T_L$. Then every type over A extends to an $\mathrm{Aut}(\widetilde{M}/A)$-invariant type.

Proof. Let $c \in \widetilde{M}$; then $c = f(a_1, \ldots, a_n)$, where $a_i \in \mathrm{dom}(\widetilde{M})$, and f is $\varnothing$-definable. It suffices to extend $\mathrm{tp}_{\widetilde{\mathcal{L}}}(a_1, \ldots, a_n/A)$ to an $\mathrm{Aut}(\widetilde{M}/A)$-invariant type. If $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/\widetilde{M})$ and $\mathrm{tp}_{\widetilde{\mathcal{L}}}(d/\widetilde{M}c)$ are $\mathrm{Aut}(\widetilde{M}/A)$-invariant, then so is $\mathrm{tp}_{\widetilde{\mathcal{L}}}(cd/\widetilde{M})$; so it suffices to show that $\mathrm{tp}_{\widetilde{\mathcal{L}}}(a_i/A_i)$ extends to an $\mathrm{Aut}(\widetilde{M}/A_i)$-invariant type for each i, where $A_i := \mathrm{dcl}_{\widetilde{\mathcal{L}}}(A_{i-1}a_{i-1})$. But (by hypothesis (v) of Corollary **(I.2.15)**) we have that $A_i = \mathrm{acl}_{\widetilde{\mathcal{L}}}(A_i)$, so Corollary **(I.3.10)** applies. ∎

Remark I.4.8:
Rigidity of finite sets fails for the theory of the p-adics in the geometric language, i.e. $\mathrm{acl}_{\widetilde{\mathcal{L}}} \neq \mathrm{dcl}_{\widetilde{\mathcal{L}}}$.

Proof. As the value group is stably embedded, one can find a non-trivial automorphism σ fixing the value group in a sufficiently saturated model. By definability of the angular component function ac, it follows that x and $\sigma(x)$ have the same angular coefficient. Take $a \in \mathcal{O}$ with $\sigma(a) \neq a$. Let $\gamma := \mathrm{val}(\sigma(a) - a)$, $\mathrm{ac}(\sigma(a) - a) =: \alpha$. Then $\mathrm{val}(\sigma^2(a) - \sigma(a)) = \gamma$, $\mathrm{ac}(\sigma^2(a) - \sigma(a)) = \alpha$, etc. As $p \cdot \alpha = 0$ in the residue field, $(\sigma^p(a) - a) = \sum_{i=0}^{p-1}(\sigma^{i+1}(a) - \sigma^i(a))$ has valuation $\delta > \gamma$. Thus in the ring $\mathcal{O}/\delta\mathcal{O}$, the image of a is not a fixed point, but has an orbit of size p under σ. This set of size p is not rigid. ∎

Remark I.4.9:
The same techniques developed here to prove elimination of imaginaries in $\mathbb{Q}_p$ can also be used to give an alternative proof for elimination of imaginaries in real closed valued fields (see [Mel06]). Hypothesis (i) of Corollary **(I.2.15)** also follows from the fact that the algebraic closure is a model, (ii) follows as in the p-adic case, (iii) follows from the description of 1-types given in [Mel06, Proposition 4.8]; and so does the existence of the type in (iv). The rest of (iv) is proved exactly as here and so is (v).

I.5. The asymptotic case

Recall that $\mathrm{T}_{\mathrm{Hen},0}$ denotes the theory of henselian fields of residue characteristic 0 and PL_0 is the theory of henselian fields with value group a $\mathbb{Z}$-group and residue field a pseudo-finite field of characteristic 0. Our goal is now to prove that any completion T_F of PL_0 in the language $\mathcal{L}^{\mathcal{G}}$ with constants added for some subfield $F \subseteq \mathbf{K}$ (see Remark **I.2.6.2**) eliminates

imaginaries. We will be using Proposition **(I.2.11)** with $\widetilde{T} = T_F$ and $T = \mathrm{ACVF}^{\mathcal{G}}_{0,0,F}$. We still follow the notation of this proposition.
It is worth noting that we will not, in general, be able to use Corollary **(I.2.15)** as there are some ultraproducts of p-adics where (v) is false. Indeed, it is shown in [BH12, Theorem 7] that there exist a characteristic zero pseudo-finite field L, $A \subseteq L$, and $b \in L$ such that b has a finite orbit over A. Then A can be identified with the set $A = \{at^0 : a \in A\} \subseteq L((t)) \models \mathrm{PL}_0$ and b is algebraic but not definable over A'. It is easy to build a counter-example to (v) using A' and b.

(i) Relative algebraic boundedness: The proof is not as simple as in the p-adic case and needs some preliminary lemmas and definitions.

Definition I.5.1:
We will say that that $\widetilde{T}$ is algebraically bounded (with respect to T) within the sort S *if for all $\widetilde{M} \models \widetilde{T}$ and $A \subseteq \mathrm{dom}(\widetilde{M})$, $S(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq S(\mathrm{acl}_{\mathcal{L}}(A))$.*
Even if S is stably embedded, one must beware that this is, in general, slightly different from saying that $\mathrm{Th}_{\widetilde{\mathcal{L}}}(S)$ (the theory induced by $\widetilde{T}$ on the sort S) is algebraically bounded (with respect to $\mathrm{Th}_{\mathcal{L}}(S)$), as in the latter case, one requires that $S(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq S(\mathrm{acl}_{\mathcal{L}}(A))$ holds for all $A \subseteq S$.

Lemma I.5.2:
Let $T_F \supseteq \mathrm{T}_{\mathrm{Hen},0}$ be such that $\mathbf{k}^\star/(\mathbf{k}^\star)^{\cdot n}$ is finite and $\mathbf{k}^\star = (\mathbf{k}^\star)^{\cdot n}\mathrm{res}(F)$. Then:

(i) If $A = \mathrm{acl}_{\mathcal{L}}(\mathbf{K}(A)) \cap \widetilde{M}$, then $\mathbf{\Gamma}(A) = \mathrm{val}(\mathbf{K}(A))$;

(ii) If $\mathrm{Th}_{\widetilde{\mathcal{L}}}(\mathbf{k})$ and $\mathrm{Th}_{\widetilde{\mathcal{L}}}(\mathbf{\Gamma})$ are algebraically bounded, then $\widetilde{T}$ is algebraically bounded within $\mathbf{k}$ and $\mathbf{\Gamma}$.

Proof.

(i) For any $a \in \mathbf{K}(\widetilde{M})^\star$ and $\gamma \in \mathbf{\Gamma}(\widetilde{M})$ such that $n\gamma = \mathrm{val}(a)$ for some $n \in \mathbb{N}$, there exist $x \in \mathbf{K}(\widetilde{M})$ such that $\mathrm{val}(ax^{-n}) = 0$ and $c \in F$ such that $\mathrm{res}(ax^{-n}c^{-1}) \in \mathbf{k}^{\cdot n}$. As $\widetilde{M}$ is an equicharacteristic zero henselian field, $ax^{-n}c^{-1} \in (\mathbf{K}(\widetilde{M})^\star)^{\cdot n}$ and hence $ac^{-1} \in (\mathbf{K}(\widetilde{M})^\star)^{\cdot n}$, i.e. there exists $a' \in \overline{F(a)}^{\mathrm{alg}} \cap \widetilde{M} \subseteq \mathrm{acl}_{\mathcal{L}}(a) \cap \widetilde{M}$ such that $\mathrm{val}(a') = \gamma$. As $\mathbf{\Gamma}(\mathrm{acl}_{\mathcal{L}}(\mathbf{K}(A))) = \mathbb{Q} \otimes \langle \mathrm{val}(\mathbf{K}(A))\rangle$, the statement follows.

(ii) Delon shows in [Del82, Theorem 2.1] that in the three-sorted language $(\mathbf{K}, \mathbf{k}, \mathbf{\Gamma})$ with val and res, field quantifiers can be eliminated up to formulas of the form

$$\varphi^\star(x, r) = \exists y \in \mathbf{K} \bigwedge_i y_i x_i \in (\mathbf{K}^\star)^{\cdot n_i} \wedge \mathrm{val}(y_i) = 0 \wedge \varphi(r, \mathrm{res}(y)),$$

where r is a tuple of variables from $\mathbf{k}$. It follows immediately that if $A \subseteq \mathbf{K}(\widetilde{M})$ then $\mathbf{\Gamma}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathrm{acl}_{\widetilde{\mathcal{L}}}(\mathrm{val}(A)) \subseteq \mathrm{acl}_{\mathcal{L}}(\mathrm{val}(A)) \subseteq \mathrm{acl}_{\mathcal{L}}(A)$, where the first inclusion follows from field quantifier elimination and the second from algebraic boundedness of $\mathrm{Th}_{\widetilde{\mathcal{L}}}(\mathbf{\Gamma})$.

The presence of the φ^* makes it a little more complicated for $\mathbf{k}$, but $\varphi^*(a,r)$ implies that $a_i y_i \in (\mathbf{K}^\star)^{\cdot n_i}$ for some y_i such that $\mathrm{val}(y_i) = 0$ and hence that $n_i | \mathrm{val}(a_i)$. By first statement, there exists $b_i \in \mathrm{acl}_{\mathcal{L}}(A) \cap \widetilde{M}$ such that $n\mathrm{val}(b_i) = \mathrm{val}(a_i)$ and thus that $\varphi^*(a,r) \iff \exists y \in \mathbf{k} \bigwedge_i y_i \mathrm{res}(a_i b_i^{-n}) \in (\mathbf{k}^\star)^{\cdot n_i} \wedge \varphi(r,y)$. It is now clear that $\mathbf{k}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subset \mathrm{acl}_{\widetilde{\mathcal{L}}}(\mathrm{res}(\mathrm{acl}_{\mathcal{L}}(A) \cap \widetilde{M}))$ and we can conclude as for $\mathbf{\Gamma}$. ∎

∎

In the next three lemmas, we will suppose that the hypotheses of the previous lemma apply to $\widetilde{T}$.

Lemma I.5.3:
For all $A \subseteq \mathbf{K}(\widetilde{M})$, $\mathbf{RV}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathbf{RV}(\mathrm{acl}_{\mathcal{L}}(A))$, i.e. $\widetilde{T}$ is algebraically bounded within $\mathbf{RV}$.

Proof. Let $c \in \mathbf{RV}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A))$ and let $\gamma := \mathrm{val}_{\mathrm{rv}}(c)$. Then by Lemma **(I.5.2)**, $\gamma \in \mathbb{Q} \otimes \mathrm{val}(A)$. It follows that there exist $c' \in \mathbf{K}(\mathrm{acl}_{\mathcal{L}}(A) \cap \widetilde{M})$ and $n \in \mathbb{N}$ such that $\mathrm{val}(c') = n\gamma$. Then $c^n/\mathrm{rv}(c') \in \mathbf{k}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathbf{k}(\mathrm{acl}_{\mathcal{L}}(A))$—also by Lemma **(I.5.2)**—and hence $c \in \mathrm{acl}_{\mathcal{L}}(A)$. ∎

Lemma I.5.4:
For any $A = \mathrm{acl}_{\mathcal{L}}(\mathbf{K}(A)) \cap \widetilde{M}$, $\mathbf{B}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) = \mathbf{B}(A)$. Moreover, any ball $b \in \mathbf{B}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A))$ contains a point in A.

Proof. Let $b \in \mathbf{B}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A))$ and let Q be the intersection of all balls in $\mathbf{B}(A)$ that contain b. As Q is $\mathrm{Aut}(M/A)$-invariant, it suffices to show that b contains Q (and hence is equal to Q) to show it is $\mathrm{Aut}(M/A)$-invariant and thus in $\mathrm{dcl}_{\mathcal{L}}(A) \cap \widetilde{M} = A$.
If $Q(A) = \varnothing$, it follows from Remark **(I.3.6)** that Q is a complete type over A in $\widetilde{M}$, so Q is contained in b. Hence we can assume that we have a point $a \in Q(A)$. We can suppose $a \notin b$, or, because $\mathrm{rad}(b) \in \mathbf{\Gamma}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathbf{\Gamma}(\mathrm{acl}_{\mathcal{L}}(A) \cap \widetilde{M}) = \mathbf{\Gamma}(A)$, we would be done.
If Q is a closed ball that strictly contains b, then b is contained in a unique maximal open subball b' of Q. This b' is interdefinable over A (in M) with $\mathrm{rv}(b - a) \in \mathbf{RV}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathbf{RV}(\mathrm{acl}_{\mathcal{L}}(A) \cap \widetilde{M}) = \mathbf{RV}(A)$, where the first inequality follows from Lemma **(I.5.3)**. Hence b' is in A, contains b and is strictly contained in Q, contradicting the definition of Q.
Finally, if Q is a strict intersection or an open ball, then $\mathrm{val}(b - a) \in \mathbf{\Gamma}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(A)) = \mathbf{\Gamma}(A)$, thus the closed ball of radius $\mathrm{val}(b-a)$ around a would be in A, would contain b and would be strictly contained in Q, a contradiction.
As for the second point, once we know that $b \in \mathbf{B}(A)$, since $\mathrm{acl}_{\mathcal{L}}(A)$ is a model of ACVF, b contains a point c in $\mathbf{K}(\mathrm{acl}_{\mathcal{L}}(A)) = \overline{\mathbf{K}(A)}^{\mathrm{alg}}$ and, as balls are convex in residue characteristic zero, the average of the $\mathrm{Aut}(M,A)$-conjugates of c is in $b(\mathrm{dcl}_{\mathcal{L}}(A) \cap \widetilde{M}) = b(A)$. ∎

Lemma I.5.5:
For any $A \subseteq \mathrm{dcl}_{\mathcal{L}}(\mathbf{K}(A)) \cap \widetilde{M}$, $\mathrm{acl}_{\widetilde{\mathcal{L}}}(A) \subseteq \mathrm{acl}_{\mathcal{L}}(A)$. In particular, for any $\widetilde{M}' \prec \widetilde{M}$ and $c \in \mathbf{K}(\widetilde{M})$, $\mathrm{acl}_{\widetilde{\mathcal{L}}}(\widetilde{M}'c) \subseteq \mathrm{acl}_{\mathcal{L}}(\widetilde{M}'c)$.

Proof. Let $C = \mathrm{acl}_{\mathcal{L}}(A) \cap M$, then $C = \mathrm{acl}_{\mathcal{L}}(K(C)) \cap M$, and let $e \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(A)$. If $e \in \mathbf{K} \subseteq \mathbf{B}$, then Lemma **(I.5.4)** applies to e—viewed as a ball with an infinite radius—and we have $e \in C \subseteq \mathrm{acl}_{\mathcal{L}}(A)$.

The remaining sorts $\mathbf{S}_n$ and $\mathbf{T}_n$ can be viewed as $\mathrm{B}_n(\mathbf{K})/H$ (or a union of such in the case of $\mathbf{T}_n$) where H is an $\mathcal{L}$-definable subgroup. Note that there exists an increasing sequence of $\mathcal{L}$-definable subgroups $(G_i)_{i=1\ldots m}$ of $\mathrm{B}_n(\mathbf{K})$ with $G_0 = \{1\}$ and $G_m = \mathrm{B}_n(\mathbf{K})$ such that for every i, there exists an $\mathcal{L}$-definable morphism $\varphi_i : G_i \to G$ with kernel G_{i-1}, where G is either the additive group $\mathbb{G}_a(\mathbf{K})$, or the multiplicative group $\mathbb{G}_m(\mathbf{K})$, and such that for every point $a \in G(C)$, $\varphi^{-1}(a)$ contains a point in $G_i(C)$. It suffices to show by induction on i that if $H_i = G_i \cap H$ is an $\mathcal{L}$-definable subgroup of G_i and $e \in (G_i/H_i)(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(C))$ then e is $\mathcal{L}(C)$-definable.
Let $\varphi : G_i \to G$, where $G = \mathbb{G}_a(\mathbf{K})$ or $G = \mathbb{G}_m(\mathbf{K})$, be a group homomorphism with kernel G_{i-1}. Then $e \in (G_i/H_i)(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(C))$ can be viewed as an almost $\widetilde{\mathcal{L}}(C)$-definable coset $eH_i \subseteq G_i$—i.e. a finite union of these cosets is $\widetilde{\mathcal{L}}(C)$-definable—and $\varphi(eH_i)$ is an almost $\widetilde{\mathcal{L}}(C)$-definable coset of $\varphi(H_i)$. Moreover, the group $H := \varphi(H_i)$ is an $\mathcal{L}$-defined subgroup of G. If $G = \mathbb{G}_a$, H has the form $y\mathcal{O}$ or $y\widetilde{\mathcal{M}}$ and if $G = \mathbb{G}_m$, $H = \mathcal{O}^\star$ or $H = 1 + I$ where I is some proper ideal of $\mathcal{O}$. Thus the cosets of H are either balls or annuli of the form $y\mathcal{O}^\star$. In both cases, $\varphi(eH_i)$ has a point $a \in C$ (in the ball case, apply Lemma **(I.5.4)**, and in the other case, this is because such an annulus is equal to some $\mathrm{val}^{-1}(\gamma)$ where $\gamma \in \mathbf{\Gamma}(\mathrm{acl}_{\widetilde{\mathcal{L}}}(C)) = \mathbf{\Gamma}(C) = \mathrm{val}(\mathbf{K}(C))$, by Lemma **(I.5.2)**).
Let $a' \in \varphi^{-1}(a) \cap G_i(C) = (a'G_{i-1}) \cap G_i(C)$; then $a'^{-1}(eH_i \cap a'G_{i-1})$ is a coset of $H_{i-1} = H_i \cap G_{i-1}$ in G_{i-1} that is almost $\widetilde{\mathcal{L}}(C)$-definable. By induction, $a'^{-1}(eH_i \cap a'G_{i-1})$ is $\mathcal{L}(C)$-definable, but then $(eH_i \cap a'G_{i-1})$ is also $\mathcal{L}(C)$-definable and hence eH_i—the only coset of H_i that contains $eH_i \cap a'G_{i-1}$—is $\mathcal{L}(C)$-definable. ∎

(ii) Internalizing $\mathcal{L}$-codes: Let $L = \prod \mathbb{Q}_p/\mathcal{U}$ be an non principal ultraproduct. Provided we have a subfield of constants F such that every finite extension of L is generated by an element whose minimal polynomial is over F and which also generates the valuation ring over $\mathcal{O}(L)$, the proof for finite extensions of $\mathbb{Q}_p$ goes through for $\mathrm{Th}(L)$.

(iii) Unary EI: In the following lemmas, we will consider a theory T_F extending PL_0 where we have added constants F containing a uniformizer λ_0, such that $\mathrm{res}(F)$ contains the necessary constants for $\mathbf{k}$ to have EI and for all $n \in \mathbb{N}_{>0}$, $\mathbf{k}^\star = (\mathbf{k}^\star)^{\cdot n}\mathrm{res}(F)$. Let $\widetilde{M} \models T_F$ be saturated and homogeneous enough.
We will first study the imaginaries in $\mathbf{RV}$. For all $\gamma \in \mathbf{\Gamma}(\widetilde{M})$, let us write $\mathbf{RV}_\gamma := \mathrm{val}_{\mathrm{rv}}^{-1}(\gamma)$. Let H be a (small[1]) subgroup of $\mathbf{\Gamma}(\widetilde{M})$ containing $1 := \mathrm{val}(\lambda_0)$, and let $\mathbf{RV}_H := \bigcup_{\gamma\in H} \mathbf{RV}_\gamma$ where a point 0_γ is added to every $\mathbf{RV}_\gamma$. The structure induced by $T_{F,H}$ on $\mathbf{RV}_H$ is that of an enriched family of (1-dimensional) $\mathbf{k}$-vector spaces and we view it as a structure with one sort for each $\mathbf{RV}_\gamma \cup \{0_\gamma\}$. As H is a group, $\mathbf{RV}_H$ is closed under tensor products and duals. These $\mathbf{k}$-linear structures are studied in [Hru12]. Let us recall some of the definitions there.

Definition I.5.6:
Let $A = (V_i)_{i\in I}$ be a $\mathbf{k}$-linear structure.

(i) We say that A has flags *if for any vector space V_i in A with $\dim(V_i) > 1$, there are vector spaces V_j and V_l in A with $\dim(V_j) = \dim(V_i) - 1$, $\dim(V_l) = 1$ and a $\varnothing$-definable exact*

[1] With respect to the saturation and homogeneity of $\widetilde{M}$.

sequence $0 \to V_l \to V_i \to V_j \to 0$.

(ii) *We say that* A has roots *if for any* 1*-dimensional* V_i *and any* $m \geqslant 2$, *there exist* V_j *and* V_l *in* A *and* $\varnothing$*-definable* $\mathbf{k}$*-linear embeddings* $f : V_j^{\otimes m} \to V_l$ *and* $g : V_i \to V_l$ *such that* $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$.

Lemma I.5.7:
The theory of $\mathbf{RV}_H$ *with the structure induced by* $T_{F,H}$ *eliminates imaginaries.*

Proof. It follows from [Hru12, Proposition 5.10] that it suffices to show that $\mathbf{RV}_H$ has flags and roots. As every $\mathbf{RV}_a$ is 1-dimensional, the structure trivially has flags. But it does not have roots. Let us extend H to some H' such that $\mathbf{RV}_{H'}$ has roots.
Let $R := \{r \in \mathbb{N}_{>0} : \mathbf{k}(\widetilde{M})$ contains nontrivial rth roots of unity$\}$, $L := \mathbf{K}(\widetilde{M})[\lambda_0^{1/r} : r \in R]$ and $H' := \langle H, 1/r : r \in R\rangle \subseteq \mathrm{val}(L)$. Note that L is a ramified extension of $\mathbf{K}(\widetilde{M})$ and that $\mathrm{res}(L) = \mathbf{k}(\widetilde{M})$, hence $\mathbf{RV}_H(\widetilde{M}) = \mathbf{RV}_H(L)$. Now $\mathbf{RV}_1$ has rth roots in $\mathbf{RV}_{H'}$ for any r. Indeed, if $r \in R$ then $\mathbf{RV}_{1/r}$ is an rth root and if $r \notin R$, then as the map $\mathbf{RV} \to \mathbf{RV} : x \mapsto x^r$ is injective, V_1 is its own rth root.
Let us show that for any $\gamma \in H'$ and any $r \geqslant 2$, $\mathbf{RV}_\gamma$ has an rth root. As $\gamma \in H'$, there exists $n \in \mathbb{N}$ such that $n\gamma \in H \subseteq \mathbf{\Gamma}(\widetilde{M})$, a $\mathbb{Z}$-group. Hence there exist $\alpha \in H$ and $m \in \mathbb{N}$ such that $n\gamma = rn\alpha + m$. Let $\mathbf{RV}_\beta$ be an nrth root of $\mathbf{RV}_1$; then $\mathbf{RV}_\alpha \otimes \mathbf{RV}_\beta{}^{\otimes m}$ is an rth root of $\mathbf{RV}_\gamma$.
By [Hru12, Proposition 5.10], $\mathbf{RV}_{H'}$ has elimination of imaginaries.
Any automorphism $\widetilde{\sigma}$ of $\mathbf{RV}_H$ can be extended to an automorphism of $\mathbf{RV}_{H'}$. Indeed, if $h \in \mathbf{RV}_{H'}$ then $\mathrm{val}_{\mathrm{rv}}(h) = \gamma + n/r$ where $\gamma \in H$, $n \in \mathbb{Z}$ and $r \in R$, and $h\mathrm{rv}(\lambda_0)^{-n/r} \in \mathbf{RV}_H$. Taking $\sigma(h) := \widetilde{\sigma}(h\mathrm{rv}(\lambda_0)^{-n/r})\mathrm{rv}(\lambda_0)^{n/r}$ will work. Moreover, we can find an automorphism of $\mathbf{RV}_{H'}$ fixing only $\mathbf{RV}_H$. Consider the homomorphism $\varphi : H' \to \mathbf{k}(\widetilde{M})$ sending $\gamma + n/r$ to d_r^n where $(d_r)_{r\in\mathbb{N}} \in \mathbf{k}(\widetilde{M})$ is such that for all r and l, we have $d_r^r = 1$, $d_r \neq 1$ if $r \in R$ and $d_{lr}^l = d_r$. Then $\theta : h \mapsto h\varphi(\mathrm{val}_{\mathrm{rv}}(h))$ is a group automorphism of $\mathbf{RV}_{H'}$ inducing the identity on both $\mathbf{k}$ and H' hence an automorphism of the full structure of $\mathbf{RV}_{H'}$. It is easy to see that θ fixes only $\mathbf{RV}_H$.
Note that because each fiber is a sort, if $X \subseteq RV_H^l$ for some $l \in \mathbb{N}$ and X is definable in $\mathbf{RV}_H$, then it is defined by the same formula in $\mathbf{RV}_{H'}$. Hence it is coded by some $x \in \mathbf{RV}_{H'}$. But as there are isomorphisms of $\mathbf{RV}_{H'}$ fixing only $\mathbf{RV}_H$, we must have $x \in \mathbf{RV}_H$, and as automorphisms of $\mathbf{RV}_H$ extend to $\mathbf{RV}_{H'}$, x is also a code for X in $\mathbf{RV}_H$. ∎

Proposition I.5.8:
The theory induced by T_F *on the sort* $\mathbf{RV}$ *(see Section I.2.2) eliminates imaginaries to the sorts* $\mathbf{RV}$ *and* $\mathbf{\Gamma}$.

Proof. First let us show that for all $n \in \mathbb{N}_{>0}$, $\mathbf{RV}/\mathbf{RV}^{\cdot n}$ is finite and $\mathbf{RV} = \mathbf{RV}^{\cdot n}\mathrm{rv}(F)$. Let $a \in \mathbf{RV}$. As $\mathbf{\Gamma}$ is a $\mathbb{Z}$-group, there exist $y \in \mathbf{RV}$ and $r \in \mathbb{N}$ such that $r < n$ and $\mathrm{val}_{\mathrm{rv}}(a) = \mathrm{val}_{\mathrm{rv}}(y^n) + \mathrm{val}(\lambda_0^r)$. Hence $\mathrm{val}_{\mathrm{rv}}(ay^{-n}\mathrm{rv}(\lambda_0)^{-r}) = 0$, i.e. $ay^{-n}\mathrm{rv}(\lambda_0)^{-r} \in \mathbf{k}^\star$. As $\mathbf{k}^\star = (\mathbf{k}^\star)^{\cdot n}\mathrm{res}(F)$, there exists $m \in \mathrm{res}(F)$ such that $ay^{-n}m^{-1}\mathrm{rv}(\lambda_0^{-r}) \in (\mathbf{k}^\star)^{\cdot n}$, i.e. $a \in m\mathrm{rv}(\lambda_0^r)\mathbf{RV}^{\cdot n}$.
Moreover, for any $A \subseteq \mathbf{RV}(\widetilde{M})$, $\mathrm{val}_{rv}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \mathbb{Q} \otimes \mathrm{val}_{\mathrm{rv}}(A)$. Indeed, let $\gamma \in \mathbf{\Gamma}(\widetilde{M}) \smallsetminus \mathbb{Q} \otimes \mathrm{val}_{\mathrm{rv}}(A)$ and $d \in \bigcap(\mathbf{k}(\widetilde{M})^\star)^{\cdot n} \smallsetminus \{1\}$, then there exists a group homomorphism $\varphi_d : \mathbf{\Gamma}(\widetilde{M}) \to \mathbf{k}^\star(\widetilde{M})$ such that $\varphi_d(\mathrm{val}_{\mathrm{rv}}(A)) = \{1\}$, $\varphi_d(\gamma) = d$ and $\psi_d : t \mapsto t\varphi_d(\mathrm{val}_{\mathrm{rv}}(t))$

defines an automorphism of $\mathbf{RV}(\widetilde{M})$ fixing A, $\mathbf{k}$ and $\mathbf{\Gamma}$, which sends any $x \in \mathrm{val}_{\mathrm{rv}}^{-1}(\gamma)$ to $dx \neq x$. Hence $\mathrm{val}_{\mathrm{rv}}^{-1}(\gamma)$ cannot contain any point definable over A.
Let us now code finite sets. For any tuple $\gamma \in \mathbf{\Gamma}$, let $\mathbf{RV}_\gamma$ denote $\prod_i \mathbf{RV}_{\gamma_i}$.

Claim I.5.9: *In the theory induced by T_F on the sorts $\mathbf{RV} \cup \mathbf{\Gamma}$, finite sets are coded.*

Proof. Let $X \subseteq \mathbf{RV}^i \times \mathbf{\Gamma}^j$ be finite. As $\mathbf{\Gamma}$ is ordered, we can suppose that there are tuples γ and $\gamma' \in \mathbf{\Gamma}$ such that $X \subseteq \mathbf{RV}_\gamma \times \{\gamma'\}$. By Lemma **(I.5.7)**, the projection of X on $\mathbf{RV}_\gamma$ is coded (over γ) by some $x \in \mathbf{RV}_{\langle 1,\gamma\rangle}$. It is easy to see that $x\gamma\gamma'$ is a code for X. ♦

To prove elimination of imaginaries in $\mathbf{RV}$ to the sorts $\mathbf{RV}$ and $\mathbf{\Gamma}$, by Lemma **(I.2.2)**, it suffices to code $\widetilde{\mathcal{L}}(A)$-definable functions $f : \mathbf{RV} \to R$, where R is either $\mathbf{RV}$ or $\mathbf{\Gamma}$, for any $A \subseteq \mathbf{RV}(\widetilde{M})$. Let us first consider the case $R = \mathbf{RV}$. Let D be the domain of f and X its graph.

Lemma I.5.10:
If there exist n and $m \in \mathbb{Z}$ such that for all $x \in D$, $n\mathrm{val}_{\mathrm{rv}}(f(x)) - m\mathrm{val}_{\mathrm{rv}}(x)$ is constant, then f is coded.

Proof. Let $\gamma_f := n\mathrm{val}_{\mathrm{rv}}(f(x)) - m\mathrm{val}_{\mathrm{rv}}(x) \in \mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner))$. For all $y \in \mathbf{k}^\star$ and $x, z \in \mathbf{RV}$, let $y \cdot (x, z) := (y^n x, y^m z)$. This defines an action of $\mathbf{k}^\star$ on any $\mathbf{RV}_\gamma$ where γ is a 2-tuple. Let $y \in \bigcap_n (\mathbf{k}^\star)^{\cdot n}$ and $\gamma \in \mathbf{\Gamma}(\widetilde{M})^2$ be such that $n\gamma_2 - m\gamma_1 = \gamma_f$ and $\gamma_1 \notin \mathbb{Q} \otimes \langle \mathrm{val}_{\mathrm{rv}}(A)\rangle$. By a similar automorphism construction as above, there is $\psi \in \mathrm{Aut}(\mathbf{RV}(\widetilde{M})/A)$ such that for all $x \in \mathbf{RV}_\gamma$, $\psi(x) = y \cdot x$ and hence $x \in X$ implies $y \cdot x \in X$. By compactness, there exists $N \in \mathbb{N}_{>0}$ such that for any $x \in \mathbf{RV}$ with $\mathrm{val}_{\mathrm{rv}}(x) \notin \mathbb{Q} \otimes \langle \mathrm{val}_{\mathrm{rv}}(A)\rangle$ and for any $y \in (\mathbf{k}^\star)^{\cdot N}$, if $x \in X$ then $y \cdot x \in X$. Let $X' := \{x \in X : \forall y \in (\mathbf{k}^\star)^{\cdot N}, y \cdot x \in X\}$. Then it suffices to code X' and $X \smallsetminus X'$. Note that $(x, y) \in X \smallsetminus X'$ implies $\mathrm{val}_{\mathrm{rv}}(x) \in \mathbb{Q} \otimes \langle \mathrm{val}_{\mathrm{rv}}(A)\rangle$.

Claim I.5.11: *Suppose that X is stable under the action of $(\mathbf{k}^\star)^{\cdot N}$. Then f is coded.*

Proof. Let $E \subseteq \mathrm{rv}(F)$ intersect all the classes of $\mathbf{RV}$ modulo $\mathbf{RV}^{\cdot(Nn)}$. Fix $\gamma \in \mathbf{\Gamma}$. For any $x \in D_\gamma := D \cap \mathbf{RV}_\gamma$, there are $y \in \mathbf{RV}^{\cdot N}$ and $e \in E$ such that $x = y^n e$. As X is $(\mathbf{k}^\star)^{\cdot N}$-stable, one can check that $g_\gamma(e) := y^{-m} f(x)$ depends only on e and γ. One can also check that $\mathrm{val}_{\mathrm{rv}}(g_\gamma(e)) = 1/n(\gamma_f + m\mathrm{val}_{rv}(e)) \in \mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner)) =: H$ and g_γ is in fact a function (with a finite graph G_γ) definable in $\mathbf{RV}_H$. By Lemma **(I.5.7)** and compactness, there is a definable function $g : \mathbf{\Gamma} \to {\mathbf{RV}_H}^l$ for some $l \in \mathbb{N}$ such that $g(\gamma)$ codes g_γ (over H). It is quite clear that g is $\widetilde{\mathcal{L}}(\ulcorner f \urcorner)$-definable, but as $X = \bigcup_{\gamma \in \mathbf{\Gamma}} (\mathbf{k}^\star)^{\cdot N} G_\gamma$, f is also $\widetilde{\mathcal{L}}(H\ulcorner g \urcorner)$-definable.
Now, as $\mathbf{\Gamma}$ has Skolem functions, we can definably order $\mathrm{Im}(g)$, and, because ${\mathbf{RV}_H}^l$ is internal to $\mathbf{k}$ and the induced theory on $\mathbf{k}$ is simple, $\mathrm{Im}(g)$ must be finite (a simple theory cannot have the strict order property). Thus $\mathrm{Im}(g) \subseteq \mathrm{acl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner)$. For any $e \in \mathrm{Im}(g)$, $g^{-1}(e) \subseteq \mathbf{\Gamma}$ is coded. Let d be the tuple of all codes of fibers and corresponding images, then $d \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner)$ and $\ulcorner f \urcorner \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\gamma d)$ for some $\gamma \in H = \mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner))$. We can conclude by coding the finite set of $\ulcorner f \urcorner$-conjugates of γd (by Lemma **(I.5.9)**). ▲

Claim I.5.12: *Suppose that for all $x \in D$, $\mathrm{val}_{\mathrm{rv}}(x) \in \mathbb{Q} \otimes \langle \mathrm{val}_{\mathrm{rv}}(A)\rangle$. Then f is coded.*

Proof. By compactness, D must be contained in only finitely many $\mathbf{RV}_{\gamma_i}$. All of these γ_i are $\widetilde{\mathcal{L}}(\ulcorner f \urcorner)$-definable and hence f lies inside $\mathbf{RV}_H$, where $H := \mathbf{\Gamma}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner))$. By Lemma **(I.5.7)**, f is coded by some d over H, hence there is some tuple $\gamma \in H$ such that $d\gamma$ codes f. ▲

Now, Claim **(I.5.11)** allows us to code X' and Claim **(I.5.12)** allows us to code $X \smallsetminus X'$. This concludes the proof of Lemma **(I.5.10)**. ♦

Let us now show that we can reduce to Lemma **(I.5.10)**. As $f(x) \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(Ax)$, we have $\mathrm{val}_{\mathrm{rv}}(f(x)) \in \mathbb{Q} \otimes \langle \mathrm{val}_{\mathrm{rv}}(Ax)\rangle$. By compactness, for all i in some finite set I, there exist $n_i, m_i \in \mathbb{Z}$ and $\gamma_i \in \mathbb{Q} \otimes \mathrm{val}_{rv}(A) \cap \mathbf{\Gamma}(\widetilde{M})$ such that for all $x \in D$, there exists $i \in I$ with $g_i(x) := n_i \mathrm{val}_{\mathrm{rv}}(f(x)) - m_i \mathrm{val}_{\mathrm{rv}}(x) = \gamma_i$. Define $E_{i,\gamma}$ to be the fiber of g_i above γ. Then $D \subseteq \bigcup_{i\in I} E_{i,\gamma_i}$. Let us assume that $|I|$ is minimal such that this inclusion holds.

Claim I.5.13: *The set $X := \{(\gamma_i)_{i\in I} \in \mathbf{\Gamma} : D \subseteq \bigcup_{i\in I} E_{i,\gamma_i}\}$ is finite.*

Proof. We proceed by induction on $|I|$. Let us assume X is infinite, and pick any $x \in D$. By the pigeon hole principle, there exists $i_0 \in I$ and an infinite set $Y \subseteq X$ such that for all $(\gamma_i)_{i\in I} \in Y$, $x \in E_{i_0,\gamma_{i_0}}$, i.e. $g_{i_0}(x) = \gamma_{i_0}$. It follows that for all $(\gamma_i)_{i\in I}$ and $(\delta_i)_{i\in I} \in Y$, $\gamma_{i_0} = \delta_{i_0}$ and $E_{i_0,\gamma_{i_0}} = E_{i_0,\delta_{i_0}} =: E$. By minimality of $|I|$, $D \smallsetminus E$ is nonempty and the set $\{(\gamma_i)_{i\in I\smallsetminus\{i_0\}} \in \mathbf{\Gamma} : D \smallsetminus E \subseteq \bigcup_{i\in I\smallsetminus\{i_0\}} E_{i,\gamma_i}\}$ is finite by induction, but it contains $\{(\gamma_i)_{i\in I\smallsetminus\{i_0\}} : (\gamma_i)_{i\in I} \in Y\}$ which is infinite, a contradiction. ♦

Then any $(\gamma_i)_{i\in I} \in X$ is in $\mathrm{acl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner)$, $f_i := f|_{E_{i,\gamma_i}}$ satisfies the conditions of Lemma **(I.5.10)** and it suffices to code each f_i. Indeed let d be the tuple of the codes for those functions; then $d \in \mathrm{acl}_{\widetilde{\mathcal{L}}}(\ulcorner f \urcorner)$ and, as $f = \bigcup_{i\in I} f_i$, $\ulcorner f \urcorner \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(d)$. The code of the finite set of $\ulcorner f \urcorner$-conjugates of d—which exists by Claim **(I.5.9)**—is a code for f.
Finally, if $R = \mathbf{\Gamma}$, then for all $\gamma \in \mathbf{\Gamma}(\widetilde{M})$, $f^{-1}(\gamma) \subseteq \mathbf{RV}$ is coded by the case $R = \mathbf{RV}$. Hence f is interdefinable with a function from $\mathbf{\Gamma}$ to $\mathbf{RV}^l \times \mathbf{\Gamma}^m$ for some l and m. So we have to code functions from $\mathbf{\Gamma}$ to $\mathbf{\Gamma}$ (which we already know how to code) and from $\mathbf{\Gamma}$ to $\mathbf{RV}$. Let $g : \mathbf{\Gamma} \to \mathbf{RV}$ be a definable function and let $h := g \circ \mathrm{val}_{\mathrm{rv}}$. Then $h : \mathbf{RV} \to \mathbf{RV}$ is coded as we have just shown and, as for all $\gamma \in \mathbf{\Gamma}$, $h(\mathrm{val}_{\mathrm{rv}}^{-1}(\gamma)) = \{g(\gamma)\}$, a code for h is also a code for g. This concludes the proof of Proposition **(I.5.8)**. ■

Remark I.5.14:

1. Let $B_m := RV/(\mathbf{k}^\star)^{\cdot m}$. We have a homomorphism $B_m \to \mathbf{\Gamma}$ whose finite kernel is $\mathbf{k}^\star/(\mathbf{k}^\star)^{\cdot m}$. Hence $B_m^{\cdot m}$ maps injectively into $\mathbf{\Gamma}$, and our assumptions on constants imply that there is a set of $\varnothing$-definable representatives for the cosets of $B_m^{\cdot m}$ in B_m. Thus the theory (and imaginaries) of B_m reduce to those of $\mathbf{\Gamma}$.

2. On the other hand, it can be shown that every unary definable subset D of $\mathbf{RV}$ is a finite union of pullbacks from B_m for some m and subsets of $\mathrm{val}_{\mathrm{rv}}^{-1}(a)$ for a lying in some finite subset F_D of $\mathbf{\Gamma}$. This m is uniform in families, and F_D can be defined canonically as the set of $a \in \mathbf{\Gamma}$ such that $\mathrm{val}_{\mathrm{rv}}^{-1}(a)$ is not a pullback from B_m. This gives another proof of unary EI in $\mathbf{RV}$ (with the stated constants), given EI in any $\mathbf{RV}_H$.

   A similar (but slightly more complicated) decomposition is also true in higher dimension (e.g., adapt [HK06, Lemma 3.25] to our case by replacing $\mathbf{\Gamma}$ by a suitable B_m). Moreover, EI in $\mathbf{RV}$ also follows from this decomposition.

Let us come back to unary EI in T_F (in fact, the proof given here would work in any theory $T \supseteq \mathrm{T}_{\mathrm{Hen},0}$ such that $\mathbf{\Gamma}$ is definably well-ordered and $\mathbf{RV}$ has unary EI). We will proceed as in the case of finite extensions of $\mathbb{Q}_p$. First let us show that the analogue of Claim **(I.4.3)** is still true in this case.

Claim I.5.15: *Let* $A = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A)$*,* $B := \mathbf{B}(A)$ *and* $c \in \mathbf{K}(\widetilde{M})$*. Then* $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/B) \vdash \mathrm{tp}_{\widetilde{\mathcal{L}}}(c/A)$*.*

Proof. As any element in $\mathbf{RV}$ is coded by a ball, RV is stably embedded and has unary EI, the claim is true if $c \in \mathbf{RV}(\widetilde{M})$. Recall that $W(c;A) := \{b \in \mathbf{B}(A) : c \in b\}$. If $P := \bigcap W(c;A) = \bigcap W(c;B)$ does not contain any ball in B then P is a complete type over A and B (by Proposition **(I.3.18)**) and we are done. If P does contain a ball $b \in B$, then, by Proposition **(I.3.18)**, P is complete relative to $\mathrm{rv}(x-b)$. But $\mathrm{tp}_{\widetilde{\mathcal{L}}}(\mathrm{rv}(x-b)/B) \vdash \mathrm{tp}_{\widetilde{\mathcal{L}}}(\mathrm{rv}(x-b)/A)$ and we are also done. ■

Unary EI in T_F follows as for finite extensions of $\mathbb{Q}_p$.

(iv) Invariant types and germs: The same proof as for finite extensions of $\mathbb{Q}_p$ (nearly) works as we only used there that $\mathbf{\Gamma}$ is definably well-ordered. The one difference is that P can be a closed ball. But in that case p, the ACVF generic of P, is definable, thus the p-germ of any r is an imaginary element e, and one may take $I = \{0\}$ and $e_0 = e$. Moreover, the inconsistency of $\mathrm{tp}(c/A)$ and $p|_{\widetilde{M}}$ would—by Claim **(I.4.4)**—contradict Lemma **(I.3.14)**.

Corollary I.5.16:
Let $T_F \supseteq \mathrm{T}_{\mathrm{Hen},0}$ *be an* $\widetilde{\mathcal{L}}$*-theory such that* $\mathrm{Th}_{\widetilde{\mathcal{L}}}(\mathbf{k})$ *and* $\mathrm{Th}_{\widetilde{\mathcal{L}}}(\mathbf{\Gamma})$ *are algebraically bounded,* $\mathbf{\Gamma}$ *is definably well-ordered,* $\mathbf{RV}$ *has unary EI,* $\mathbf{K}$ *has a finite number of extensions of any given degree and* $\mathbf{k}^\star/(\mathbf{k}^\star)^{\cdot n}$ *is finite. Suppose also that we have added constants for a field* $F \subseteq \mathbf{K}$ *such that* $\mathbf{k}^\star = (\mathbf{k}^\star)^{\cdot n}\mathrm{res}(F)$ *and any finite extension of* $\mathbf{K}$ *is generated by an element whose minimal polynomial is over* F *and which also generates the valuation ring over* $\mathcal{O}(\mathbf{K})$*. Then* T_F *has EI/UFI in the sorts* $\mathbf{K}$ *and* $\mathbf{S}_n$*.*
In particular this is true of ultraproducts of the p*-adics (if we add some constants as in Remark* ***I.2.6.2****).*

Proof. By Proposition **(I.2.11)** we have EI/UFI in the sorts $\mathbf{K}$, $\mathbf{S}_n$ and T_n but as noted earlier the sorts T_n are not needed when the value group has a smallest positive element. ■

Elimination of finite imaginaries: As we already know that $\mathbf{RV}$ eliminates imaginaries, it suffices to show that every finite imaginary in PL_0 (over arbitrary parameters) can be coded in $\mathbf{RV}$ (the proof is adapted from [Hru09, Lemma 2.10]).

Definition I.5.17:
If $C \subseteq C'$*, we say that* C' *is* stationary *over* C *if* $\mathrm{dcl}^{\mathrm{eq}}(C') \cap \mathrm{acl}^{\mathrm{eq}}(C) = \mathrm{dcl}^{\mathrm{eq}}(C)$*. A type* $p = \mathrm{tp}(c/C)$ *is* stationary *if* cC *is stationary over* C*.*

Remark I.5.18:

1. It is clear that is C'' is stationary over C' and C' over C, then so is C'' over C.
2. If $\mathrm{tp}(c/C)$ generates a complete type over $\mathrm{acl}^{\mathrm{eq}}(C)$, then $\mathrm{tp}(c/C)$ is stationary. Indeed, let $x \in \mathrm{dcl}^{\mathrm{eq}}(Cc) \cap \mathrm{acl}^{\mathrm{eq}}(C)$; then there is a C-definable function f such that $f(c) = x$. As $\mathrm{tp}(c/C)$ generates a complete type over $\mathrm{acl}^{\mathrm{eq}}(C)$, there is a C-definable

set D such that for all $c' \in D$, $f(c') = x$, hence $x \in \mathrm{dcl}^{\mathrm{eq}}(C)$.

Lemma I.5.19:
Let $\widetilde{T}$ be a theory extending PL_0 *(in the geometric language with possibly new constants). For all $\widetilde{M} \vDash \widetilde{T}$ and $A \subseteq \widetilde{M}$, there exists $C \preceq \widetilde{M}$ containing $\mathbf{RV}(\widetilde{M}) \cup A$ and stationary over $\mathbf{RV}(\widetilde{M}) \cup A$.*

Proof. Let us first prove the following claim.

Claim I.5.20: *Let $B = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(B) \subseteq \widetilde{M}$ such that $\mathbf{RV}(\widetilde{M}) \subseteq B$ and $b \in \mathbf{B}(\widetilde{M})$. Then there exists a tuple $c \in \mathbf{K}(\widetilde{M})$ with $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/B)$ stationary, $b \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(c)$ and $b(\widetilde{M}) \cap c \neq \varnothing$.*

Proof. Let us first suppose that $b \in \mathbf{RV}(\widetilde{M})$, i.e. that b is of the form $c(1+\widetilde{\mathcal{M}})$. Let $P \subseteq b$ be a minimal (for the inclusion) intersection of balls in $\mathbf{B}(B)$. For any $c \vDash P$ we have $b = \mathrm{rv}(c)$, hence it suffices to show that P is a complete stationary type over B.
As P does not strictly contain any ball in $\mathbf{B}(B)$ by definition, it cannot contain balls in $\mathbf{B}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(B))$ (we are in residue characteristic zero) and by Proposition **(I.3.18)**, P is a complete type over $\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(B)$. By Remark **I.5.18.2**, P is stationary over B.
Now if $b \in \mathbf{B}(\widetilde{M})$, pick any $r \in \mathbf{RV}(\widetilde{M})$ such that $\mathrm{val}_{\mathrm{rv}}(r) = \mathrm{rad}(b)$. Applying the claim to r, we find $c \in \mathbf{K}(\widetilde{M})$ such that $\mathrm{tp}_{\widetilde{\mathcal{L}}}(c/B)$ is stationary and $\mathrm{rad}(b) \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(c)$. It now suffices to find a point $d \in b$ whose type is stationary over $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(Bc)$, but we can proceed as in the first case. Then $b \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(cd)$ and $\mathrm{tp}(cd/B)$ is stationary. ♦

Starting with $B := \mathrm{dcl}_{\widetilde{\mathcal{L}}}(\mathbf{RV}(\widetilde{M}) \cup A)$, and applying the claim iteratively, we find $C \supseteq A \cup \mathbf{RV}(\widetilde{M})$ such that C is stationary over $A \cup \mathbf{RV}(\widetilde{M})$, $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(C) = C$, $\mathbf{B}(\widetilde{M}) \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(\mathbf{K}(C))$ and every ball in $\mathbf{B}(\widetilde{M})$ has a point in C.

Claim I.5.21: *We have $C \subseteq \mathrm{dcl}_{\widetilde{\mathcal{L}}}(\mathbf{K}(C))$.*

Proof. Let $e \in C$. If $e \in \mathbf{K}$ then the result is trivial, thus we only have to consider $e \in \mathbf{S}_n$ or $e \in \mathbf{T}_n$. Let us consider the same decomposition of $\mathbf{S}_n$ and $\mathbf{T}_n$ as in the proof of Lemma **(I.5.5)** and show by induction on i that for all $e \in (G_i/H_i)(\widetilde{M})$, e is $\widetilde{\mathcal{L}}(\mathbf{K}(C))$-definable.
If we write e as eH_i then, as proved in Lemma **(I.5.5)**, $\varphi_i(eH_i)$ is either a ball or a set of the form $y\mathcal{O}^\star$ and hence is definable over $\mathbf{B}(\widetilde{M})$ and has a point $a' \in \mathbf{K}(C)$. Let $a \in \varphi_i^{-1}(a')(C)$. Then $a^{-1}eH_i \cap G_{i-1}$ is a coset of H_{i-1} in G_{i-1} which is $\widetilde{\mathcal{L}}(\mathbf{K}(C))$-definable by induction, and hence so is eH_i. ♦

As $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(C) = C$, $\mathbf{K}(C) = \mathbf{K}(C)^h \vDash \mathrm{T}_{\mathrm{Hen},0}$ and, as $\mathrm{rv}(\mathbf{K}(C)) = \mathbf{RV}(\widetilde{M})$, it follows from field quantifier elimination in $\mathrm{T}_{\mathrm{Hen},0}$ in the language with sorts the $\mathbf{K}$ and $\mathbf{RV}$, that $\mathbf{K}(C) \preceq \mathbf{K}(\widetilde{M})$. But this implies that $C = \mathrm{dcl}_{\widetilde{\mathcal{L}}}(\mathbf{K}(C)) \preceq \widetilde{M}$. ■

Lemma I.5.22:
Let $\widetilde{T}$ be a theory that extends PL_0 *(in the geometric language) and $A \subseteq \widetilde{M} \vDash \widetilde{T}$. Then every finite imaginary sort of $\widetilde{T}_A$ is in definable bijection with a finite imaginary sort of* $\mathbf{RV}$ *(with the structure induced by $\widetilde{T}_A$).*

Proof. Let $Y = D/E$ be a finite imaginary sort (in $\widetilde{T}_A$) and $\pi : D \to Y$ be the canonical surjection. As the field sort is dominant, we can assume that D is a definable subset of

$\mathbf{K}^n$ for some n. Let $C \supseteq A$ be as in Lemma **(I.5.19)**. As Y is finite and $C \prec \widetilde{M}$, $Y(C) = Y(\widetilde{M})$ and there exists a finite set $H \subseteq \mathbf{K}^n(C)$ meeting every E-class. Let W be some finite set in $\mathbf{RV}(C)$, of bigger cardinality than H, and $h : W \to H$ any surjection. Note that any such surjection is $\widetilde{\mathcal{L}}(C)$-definable. Composing, we have an $\widetilde{\mathcal{L}}(C)$-definable surjection $\psi : W \to Y$. But there are finitely many maps $W \to Y$, hence they are all algebraic over $\mathbf{RV}(C) \cup A = \mathbf{RV}(\widetilde{M}) \cup A$ and by stationarity of C over $\mathbf{RV}(\widetilde{M}) \cup A$, ψ is $\widetilde{\mathcal{L}}(\mathbf{RV}(\widetilde{M}) \cup A)$-definable. Let $e \in \mathbf{RV}(\widetilde{M})$ be such that ψ and W are $\widetilde{\mathcal{L}}(Ae)$-definable.
Let W be defined by the $\widetilde{\mathcal{L}}(Ae)$-formula $\varphi(x, e)$ and ψ by the $\widetilde{\mathcal{L}}(Ae)$-formula $\psi(x, y, e)$ (which implies that for any e', $\psi(\widetilde{M}, \widetilde{M}, e')$ is the graph of a function whose domain is $\varphi(\widetilde{M}, e')$). Then the formulas $\varphi(x, z)$ and $\psi(x, y, z)$ define, respectively, a subset D' of $\mathbf{RV}^{|e|+1}$ and a surjection $\psi : D' \to Y$. Let E' be defined by $E'((x, z), (x', z')) \iff \psi(x, z) = \psi(x', z')$. Then we have an $\widetilde{\mathcal{L}}(A)$-definable bijection $D'/E' \to Y$ and, as $\mathbf{RV}$ is considered with the structure induced by $\widetilde{T}_A$, D'/E' is a finite imaginary sort of $\mathbf{RV}$. ∎

Proof of Theorem B: Let $K \models \mathrm{PL}_0$ and $\widetilde{T} := \mathrm{Th}(K)$ (with constants added as in Corollary **(I.5.16)**). As we have already proved EI/UFI in Corollary **(I.5.16)**, by Lemma **(I.2.5)** it is enough to show that for any A, T_A eliminates finite imaginaries in the sorts $\mathbf{K}$, $\mathbf{S}_n$. Let $e \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A)$; then, by Lemma **(I.5.22)**, there exists an $\mathbf{RV}$-imaginary e' interdefinable over A with e. By EI in $\mathbf{RV}$ to the sorts $\mathbf{RV}$ and $\mathbf{\Gamma}$ (Proposition **(I.5.8)**), there exists a tuple $d \in \mathbf{RV} \cup \mathbf{\Gamma}$ such that e' is interdefinable with d, hence e is interdefinable with d over A. We have shown that any finite imaginary of T_A is coded (over A) in $\mathbf{RV} \cup \mathbf{\Gamma} = \mathbf{S}_1 \cup \mathbf{T}_1$ which are themselves coded in $\mathbf{S}_1 \cup \mathbf{S}_2$.

For a more canonical treatment of the parameters F in the pseudo-finite case, see [CH99]—it would be interesting to adapt it to the pseudo-local setting.

I.6. Rationality

Let $r \in \mathbb{N}$. For all tuples $l \in \mathbb{N}^r$, when $t = (t_i)_{1 \leqslant i \leqslant r}$, we write t^l for $\prod_{i \leqslant r} t_i^{l_i}$. We say a power series $\sum_{l \in \mathbb{N}^r} a_l t^l \in \mathbb{Q}[[t_1, \ldots, t_r]]$ with each $a_l \in \mathbb{N}$ is *rational* if it is equal to a rational function in $t_1, \ldots, t_r$ with coefficients from $\mathbb{Q}$. In this section we prove that certain zeta functions that come from counting the equivalence classes of definable equivalence relations are rational.
Let L_p be a finite extension of $\mathbb{Q}_p$. By a *definable family* $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{Z}^r}$ of subsets of L_p^N we mean a definable subset R_{L_p} of $L_p^N \times \mathbb{Z}^r$—where $\mathrm{val}(L_p^\star)$ is identified with $\mathbb{Z}$—and we write $R_{L_p,l}$ for the fiber above l of the projection from R_{L_p} to $\mathbb{Z}^r$. By a *definable family* $E_{L_p} = (E_{L_p,l})_{l \in \mathbb{Z}^r}$ of equivalence relations on R_{L_p} we mean a definable equivalence relation E_{L_p} on R_{L_p} such that for every $x, y \in R_{L_p}$, if $x E_{L_p} y$ then there exists $l \in \mathbb{Z}^r$ such that $x, y \in R_{L_p,l}$. We then have a definable equivalence relation $E_{L_p,l}$ on $R_{L_p,l}$ for every l, and by a slight abuse of notation we can regard $(E_{L_p,l})_{l \in \mathbb{Z}^r}$ as a definable family of subsets of $\mathbb{Q}_p^{2N}$. The set $\mathbb{N}^r$ is a definable subset of $\mathbb{Z}^r$, so it makes sense to talk of definable families $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{N}^r}$, etc.
Let $\mathfrak{L}_p$ be a set of finite extensions of $\mathbb{Q}_p$ and $\mathfrak{L} := \bigcup_p \mathfrak{L}_p$. We will say that $(R_{L_p})_{L_p \in \mathfrak{L}}$ and

$(E_{L_p})_{L_p \in \mathfrak{L}}$ are uniformly $\varnothing$-definable in L_p if there exists two $\mathcal{L}^{\mathcal{G}}$-formulas φ and θ independent of L_p such that for all $L_p \in \mathfrak{L}$, $R_{L_p} = \varphi(L_p)$ and $E_{L_p} = \theta(L_p)$.
Now we come to the main result of this section.

Theorem C:

Let $\mathfrak{L}_p$ and $\mathfrak{L}$ be as above (note that we do not assume $\mathfrak{L}_p$ is nonempty for infinitely many p). For all $L_p \in \mathfrak{L}$, let $R_{L_p} = (R_{L_p,l})_{l \in \mathbb{N}^r}$ be a family of subsets of L_p^N and let $E_{L_p} = (E_{L_p,l})_{l \in \mathbb{N}^r}$ be a family of equivalence relations on $(R_{L_p,l})_{l \in \mathbb{N}^r}$ uniformly $\varnothing$-definable in L_p. Suppose that for each $l \in \mathbb{N}^r$ and L_p, $a_{L_p,l} = |R_{L_p,l}/E_{L_p,l}| < \infty$. Then, for every $L_p \in \mathfrak{L}$, the power series

$$S_{L_p} := \sum_{l \in \mathbb{N}^r} a_{L_p,l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]] \text{ is rational.}$$

Moreover, there exist $k, n, d \in \mathbb{N}$, there exist tuples $(a_j)_{j \leqslant k}$ of integers and $(b_j)_{j \leqslant k}$ of elements of $\mathbb{N}^r$, and for all tuples $l \in \mathbb{N}^r$ with $|l| := \sum_{i \leqslant r} l_i \leqslant d$ there exist $q_l \in \mathbb{Q}$ and varieties X_l over $\mathbb{Z}$, such that the following holds:

(1) for all j, a_j and b_j are not both 0;

(2) for all $p \gg 0$ and all $L_p \in \mathfrak{L}_p$, we have

$$S_{L_p} = \frac{\sum_{|l| \leqslant d} q_l |X_l(\mathrm{res}(L_p))| t^l}{|\mathrm{res}(L_p)|^n \prod_{j=1}^k (1 - |\mathrm{res}(L_p)|^{a_j} t^{b_j})}. \tag{I.2}$$

We say that a family of power series $\sum_{l \in \mathbb{N}^r} a_{p,l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$ for each prime p is *uniformly rational* if it is of the form given in (I.2).

Remark I.6.1:

1. Assume $\mathfrak{L}_p$ is finite for all p. Let $\mathcal{L}_{\mathrm{rg}}$ be the language of rings. At the cost of replacing X_l by quantifier-free $\mathcal{L}_{\mathrm{rg}}$-definable sets, we can make (I.2) hold for all L_p. In particular, suppose we are given definable R_{p_0} and E_{p_0} as above, but just for a single prime p_0 and a single L_{p_0}. Then taking $\mathfrak{L} = \{L_{p_0}\}$, we obtain that the power series
$$S_{L_{p_0}} := \sum_{l \in \mathbb{N}^r} a_{L_{p_0},l} t^l \in \mathbb{Q}[[t_1, \dots, t_r]]$$
and is of the form (I.2) where X_l can be assume to be a single point.
2. Note that the level of uniformity we obtain in Theorem **C** is very similar to the one obtained in [SV14, Theorem A].
3. Usually, in this kind of rationality theorem, we can take $q_l = 1$ for all l. There are two reasons why more complicated rational coefficients appear here. The first reason is to turn the X_l into varieties instead of definable sets and the other reason is to get rid of the residual constant symbols that appear due to elimination of imaginaries.

For any finite extension L_p of $\mathbb{Q}_p$, it is natural here to consider the invariant Haar measure μ_{L_p} on $\mathrm{GL}_N(L_p)$. In terms of the additive Haar measure $\mu_{L_p,+}^{N^2}$ on $L_p^{N^2}$, μ_{L_p} can be de-

fined thus: for any continuous $f\colon \mathrm{GL}_N(L_p) \to \mathbb{C}$ with compact support, $\int f(x)d\mu_{L_p}(x) = \int f(x)|\det(x)|^{-N}d\mu^{N^2}_{L_p,+}(x)$. As $\det(x)$ is definable (uniformly in p), Denef's results on definability of p-adic integration [Den00] extend immediately to $d\mu_{L_p}$ and the motivic counterpart of this result—see [DL02], although the result we will be needing is already implicit in older work by Denef and Pas (see, e.g., [Pas89])—also extends to $d\mu_{L_p}$.
By left invariance, $\mu_{L_p}(A\cdot \mathrm{GL}_n(\mathcal{O}(L_p))) = \mu_{L_p}(\mathrm{GL}_n(\mathcal{O}(L_p)))$, a number that depends only on the normalization. We choose a normalization for $\mu_{L_p,+}$ and μ_{L_p} such that for any $A \in \mathrm{GL}_N(L_p)$, we have

$$\mu_{L_p}(A.\mathrm{GL}_N(\mathcal{O}(L_p))) = 1. \tag{I.3}$$

Proof (Theorem **C**). By uniform EI (Corollary **(I.2.7)**) there exist integers m_1 and m_2, some $N \subseteq \mathbb{N}_{>0}$ and some $\mathcal{L}^{\mathcal{G}}_N$-formula $\varphi(x, w)$ such that for all $p \gg 0$, for all $L_p \in \mathfrak{L}_p$, for any choice of uniformizer $a_{L_p} \in L_p$ and, for all $n \in N$, an unramified n-Galois uniformizer $b_{n,L_p} \in L_p$, φ defines a function $f'_{L_p}\colon R_{L_p} \to L_p^{m_2} \times \mathbf{S}_{m_1}(L_p)$ such that for every $x, y \in R_{L_p}, xE_{L_p}y \iff f'_{L_p}(x) = f'_{L_p}(y)$. Let $f'_{L_p} = (f''_{L_p}, f_{L_p})$ where $f''_{L_p} : R_{L_p} \to L_p^{m_2}$ and $f_{L_p} : R_{L_p} \to \mathbf{S}_{m_1}(L_p)$. For $l \in \mathbb{N}^r$, let $\mathcal{E}_{L_p,l} = \{f'_{L_p}(x) : x \in R_{L_p,l}\}$ and $\mathcal{E}_{L_p} = \bigcup_l \mathcal{E}_{L_p,l}$; so $\mathcal{E}_{L_p,l} \subseteq L_p^{m_2} \times \mathbf{S}_{m_1}(L_p)$ is finite, and it is the series $\sum_l |\mathcal{E}_{L_p,l}|t^l$ we wish to understand. Let $\pi_{L_p} : \mathcal{E}_{L_p} \to \mathbf{S}_{m_1}(L_p)$ be the projection, and let $F_{L_p,l} = \pi_{L_p}(\mathcal{E}_{L_p,l})$.
It follows from Lemma **(I.5.5)** and the fact that on the valued field sort the model-theoretic algebraic closure in ACVF coincides with the field-theoretic algebraic closure, that the size of the fiber $e_{L_p,l}(x) = |\pi^{-1}_{L_p}(x) \cap \mathcal{E}_{L_p,l}|$ is bounded by some positive integer N uniformly in $p \gg 0$. We may thus partition $F_{L_p,l}$ into finitely many pieces $F^{\nu}_{L_p,l} = \{x \in F_{L_p,l} : e_{L_p,l}(x) = \nu\}$; then

$$\sum_l |\mathcal{E}_{L_p,l}|t^l = \sum_{\nu \leqslant N} \nu \sum_l |F^{\nu}_{L_p,l}|t^l,$$

so it suffices to prove that the series for $F^{\nu}_{p,l}$ has the form (I.2).
Fix ν and let $F_{L_p,l} = F^{\nu}_{L_p,l}$; we need to retain only the information that $F_{L_p,l}$ is a family of finite subsets of $\mathbf{S}_m(L_p)$, uniformly $\varnothing$-definable in L_p. We can identify each element of $\mathbf{S}_m(L_p)$ with an element of $\mathrm{GL}_m(L_p)/\mathrm{GL}_m(\mathcal{O}(L_p))$, i.e. with a left coset of $\mathrm{GL}_m(\mathcal{O}(L_p))$; let $G_{L_p,l}$ be the union of these cosets. By Equation (I.3), we have

$$\mu_{L_p}(G_{L_p,l}) = |F_{L_p,l}|.$$

Thus

$$\sum_l |F_{L_p,l}|t^l = \sum_l \mu_{L_p}(G_{L_p,l})t^l \in \mathbb{Q}[[t_1, \dots, t_r]].$$

Uniform rationality now follows by Theorem [DL02, Theorem 1.1 and Theorem 3.1], up to the fact that because the sets $G_{L_p,l}$ are $\mathcal{L}^{\mathcal{G}}_N$-definable, the varieties X_l are over $\mathbb{Z}[\overline{T}]$ where $\overline{T}$ is specialized in $\mathrm{res}(L_p)$ to any tuple ($k_n : n \in N$ and k_n is the residue of an unramified n-Galois uniformizer).
Let $C_n(L_p) \coloneqq \{k_n \in \mathrm{res}(L_p) : k_n$ is the residue of an unramified n-Galois uniformizer$\}$. If $\mathrm{res}(L_p)[\omega_n]$ is of degree $d = d_{n,L_p}$ over $\mathrm{res}(L_p)$, then

$$|C_n(L_p)| = \frac{\varphi(n)(|\mathrm{res}(L_p)|^d - 1)}{n}$$

where φ is the Euler totient function. Let $C = \prod_{n \in N} C_n$ and for all $c \in C(L_p)$, let $X_{c,l}(L_p)$ be the L_p-points of the specialization of X_l to c. Then $Y_l := \coprod_{c \in C} X_{c,l}$ is an $\mathcal{L}_{\mathrm{rg}}$-definable set such that

$$|X_l(\mathrm{res}(L_p))| = \sum_{d|n} 1_{d_{n,L_p}=d} \frac{-n}{\varphi(n)} \frac{|Y_l(\mathrm{res}(L_p))|}{1 - |\mathrm{res}(L_p)|^d}$$

where $1_{d_{n,L_p}=d} = 1$ if $d_{n,L_p} = d$ and 0 otherwise. Replacing $|X_l(\mathrm{res}(L_p))|$ by this expression, we obtain a rational function of the right form where the X_l are $\mathcal{L}_{\mathrm{rg}}$-definable, but, by [DL02, Theorem 2.1], X_l may be assumed to be a $\mathbb{Z}$-variety for $p \gg 0$.
For L_p such that p is too small, we can still prove the rationality of S_{L_p} by the same argument using results for finite extensions of p-adic fields instead of those for ultraproducts: replace Corollary **(I.2.7)** by Theorem **A**, Lemma **(I.5.5)** by the proof of (i) in Section I.4 and [DL02, Theorem 1.1] by [Den00, Theorem 1.5 and Theorem 1.6.1]. ■

It follows from the uniform formula I.2 we gave for S_{L_p} in Theorem **C**, that there exists a tuple of polynomials $(P_i)_{1 \leqslant i \leqslant r}$ from $\mathbb{Q}[X]$ and a polynomial $Q \in \mathbb{Q}[X]$ such that we have the following uniform growth estimate on $a_{L_p,l}$: for all $p \gg 0$ and all $L_p \in \mathfrak{L}$,

$$a_{L_p,l} \leqslant Q(q) P_1(q)^{n_1} \cdots P_r(q)^{n_r}, \tag{I.4}$$

where $q = |\mathrm{res}(L_p)|$.
Below we consider uniformly $\varnothing$-definable (in p) families that arise in the following way. Take $\mathcal{L}_p$ to be $\{\mathbb{Q}_p\}$ for all p. Let $\mathcal{D}_p \subseteq \mathbb{Q}_p^N$ be uniformly $\varnothing$-definable and let $\mathcal{E}_p$ be a uniformly $\varnothing$-definable equivalence relation on $\mathcal{D}_p$. Suppose that $f_{p,1}, \ldots, f_{p,r} \colon \mathcal{D}_p \to \mathbb{Q}_p \setminus \{0\}$ are uniformly $\varnothing$-definable functions such that for every $l \in \mathbb{Z}^r$, the subset $\{x \in \mathcal{D}_p : \mathrm{val}(f_{p,i}(x)) = l_i\}$ is a union of $\mathcal{E}_p$-equivalence classes. Set $D_p = \{(x, \mathrm{val}(f_{p,1}(x)), \ldots, \mathrm{val}(f_{p,r}(x))) : x \in \mathcal{D}_p\} \subseteq \mathbb{Q}_p^N \times \mathbb{Z}^r$ and define $E_p \subseteq D_p \times D_p$ by $(x, s_1, \ldots, s_r) E_p (x', s_1', \ldots, s_r')$ if $x \mathcal{E}_p x'$ and $s_i = s_i'$ for all i. Then we can regard D_p as a uniformly $\varnothing$-definable family of subsets of $\mathbb{Q}_p^N$ and E_p as a uniformly $\varnothing$-definable family of equivalence relations on D_p.

I.7. Zeta functions of groups

We now consider some applications to some zeta functions that arise in group theory. Most of the examples in this section come from the theory of subgroup growth of finitely generated nilpotent groups. In Section I.8 we consider the representation zeta function of finitely generated nilpotent groups. We use Theorem **C** to prove uniform rationality of these zeta functions. In the subgroup case this gives alternative proofs of results of [GSS88] and [SG00].
Throughout this section Γ is a finitely generated nilpotent group. For any $n \in \mathbb{N}$, the number a_n of index n subgroups of Γ is finite (for background on subgroup growth, see [LS03]). The (global) subgroup zeta function of Γ is defined by $\xi_\Gamma(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ and the p-local subgroup zeta function by $\xi_{\Gamma,p}(s) := \sum_{n=0}^{\infty} a_{p^n} p^{-ns}$ (the symbol ζ is commonly used to denote the subgroup zeta function but we reserve this for the representation zeta function in Section I.8). These expressions converge if $\mathrm{Re}(s)$ is large enough. Grunewald, Segal and

Smith observed in [GSS88] that Euler factorization holds: we have

$$\xi_\Gamma(s) = \prod_p \xi_{\Gamma,p}(s),$$

where p ranges over all primes. Theorem I.7 below (and [GSS88, Theorem 1]) says that $\xi_{\Gamma,p}(s)$ is a rational function of p^{-s}. Hence $\xi_\Gamma(s)$ enjoys many of the properties of the Riemann zeta function.

To understand the behavior of the global subgroup zeta function, one needs to study the behaviour of the rational function $\xi_{\Gamma,p}(s)$ as p varies (cf. [Avn11]). Du Sautoy and Grunewald introduced a class of p-adic integrals they called *cone integrals*. They showed [SG00, Theorem 1.3] that if $\tau_p(s) := \sum_{n=0}^{\infty} b_{p,n} p^{-ns}$ is the zeta function arising from an Euler product of suitable cone integrals then $\tau_p(s)$ is uniformly rational (in the variable $t := p^{-s}$) in the sense of Section I.6. In fact, they proved a considerably stronger result [SG00, Theorem 1.4] and deduced various analytic properties of $\tau(s)$ [SG00, Theorem 1.5]: for instance, they showed that $\tau(s)$ can be meromorphically continued a short distance to the left of its abscissa of convergence [SG00, Theorem 1.1]. It follows from these results on cone integrals that $\xi_{\Gamma,p}(s)$ is uniformly rational [SG00, Section 5]. For Γ a finitely generated free nilpotent group of class 2, a stronger uniformity result holds: there is a polynomial $W(X,Y) \in \mathbb{Q}[X,Y]$ such that $\xi_{\Gamma,p}(s) = W(p,p^{-s})$ for every prime p [GSS88, Theorem 2]. Du Sautoy, however, has given an example showing that this stronger result does not hold in general [Sau01].

Theorem I.7 below deals with some variations on the subgroup zeta function. In order to formulate the problem in terms of definable equivalence relations, we need to recall some facts about nilpotent pro-p groups, including the notion of a good basis for a subgroup of a torsion-free nilpotent group [GSS88, Section 2]; we will need these ideas in Section I.8 as well. We write $\widehat{G}_p$ for the pro-p completion of a group G. Let $j\colon \Gamma \to \widehat{\Gamma}_p$ be the canonical map. Then $\widehat{\Gamma}_p$ is finitely generated as a pro-p group, so every finite-index subgroup of $\widehat{\Gamma}_p$ is open (cf. [Dix+99, Theorem 1.17]) and has p-power index (cf. [Dix+99, Lemma 1.18]). Since Γ is finitely generated nilpotent, every subgroup of p-power index is open in the pro-p topology on Γ; in particular, there is a bijection $H \mapsto \overline{j(H)}$ between index p^n subgroups of Γ and index p^n subgroups of $\widehat{\Gamma}_p$, and $\overline{j(H)} \cong \widehat{H}_p$ (see [GSS88, Proposition 1.2]). For any $H \trianglelefteq \Gamma$ of index p^n, we have $\Gamma/H \cong \widehat{\Gamma}_p/\overline{j(H)}$.

Let Δ be a finitely generated torsion-free nilpotent group. A tuple $a_1, \dots, a_R$ of elements of Δ is a *Mal'cev basis* if any element of Δ can be written uniquely in the form $a_1^{\lambda_1} \cdots a_R^{\lambda_R}$, where the $\lambda_i \in \mathbb{Z}$. We call the λ_i *Mal'cev coordinates*. Moreover, we require that group multiplication and inversion in Δ are given by polynomials in the λ_i with coefficients in $\mathbb{Q}$, and likewise for the map $\Delta \times \mathbb{Z} \to \Delta$, $(g,\lambda) \mapsto g^\lambda$. We may regard the a_i as elements of the pro-p completion $\widehat{\Delta}_p$, and analogous statements hold, except that λ and the Mal'cev coordinates λ_i now belong to $\mathbb{Z}_p$ (see [GSS88, Section 2]). In particular, the map $j\colon \Delta \to \widehat{\Delta}_p$ is injective and we may identify $\widehat{\Delta}_p$ with $\mathbb{Z}_p^R$.

Now let H be a finite-index subgroup of $\widehat{\Delta}_p$, of index p^n, say. In [GSS88], a *good basis for H* is defined as an R-tuple $h_1, \dots, h_R \in H$ such that every element of H can be written uniquely in the form $h_1^{\lambda_1} \cdots h_R^{\lambda_R}$ ($\lambda_i \in \mathbb{Z}_p$), and satisfying an extra property which does not concern us here. We say that $h_1, \ldots, h_R \in \widehat{\Delta}_p$ is a *good basis* if it is a good basis for some finite-index subgroup H of $\widehat{\Delta}_p$. For each i, we can write

$$h_i = a_1^{\lambda_{i1}} \cdots a_R^{\lambda_{iR}} \tag{I.5}$$

and we recover $\left|\widehat{\Delta}_p : H\right| = p^n$ from the formula

$$|\lambda_{11}\lambda_{22}\cdots\lambda_{RR}| = p^{-n}. \tag{I.6}$$

Any finite-index subgroup of $\widehat{\Delta}_p$ admits a good basis. Often we will identify a good basis $h_1, \ldots, h_R$ with the R^2-tuple of coordinates (λ_{ij}).

Proposition I.7.1:
The set $\mathcal{D}_p$ of good bases $(\lambda_{ij}) \subseteq \mathbb{Z}_p^{R^2}$ of $\widehat{\Delta}_p$ is uniformly $\varnothing$-definable in p.

Proof. This follows from the proof of [GSS88, Lemma 2.3]. ■

For each nonnegative n consider the following:
(a) the number of index p^n subgroups of Δ;
(b) the number of normal index p^n subgroups of Δ;
(c) the number of index p^n subgroups A of Δ such that $\widehat{A}_p \cong \widehat{\Delta}_p$;
(d) the number of conjugacy classes of index p^n subgroups of Δ;
(e) the number of equivalence classes of index p^n subgroups of Δ, where we define $A \sim B$ if $\widehat{A}_p \cong \widehat{B}_p$.
The rationality statement in (a)–(d) of the following result are due to Grunewald, Segal, and Smith [GSS88, Theorem 1]; for uniformity statements in (a)–(d), see [SG00, Section 1] and the start of this section. Here we give a different proof.

Theorem I.7.2:
Let $b_{p,n}$ be as described in any of (a)–(e) above. Then the power series $\sum_{n=0}^{\infty} b_{p,n} t^n$ is uniformly rational.

Proof. Consider case (a). Let $\mathcal{D}_p$ be as in Proposition I.7. Define $f_p : \mathcal{D}_p \to \mathbb{Z}_p$ by $f_p(\lambda_{ij}) = \lambda_{11} \cdots \lambda_{RR}$; note that f_p is uniformly $\varnothing$-definable in p. Define an equivalence relation $\mathcal{E}_p$ on $\mathcal{D}_p$ as follows: two R-tuples (λ_{ij}), (μ_{ij}), representing good bases $h_1, \ldots, h_R$ and $k_1, \ldots, k_R$ for subgroups H, K respectively, are equivalent if and only if $H = K$.
Now $\mathcal{E}_p$ is uniformly $\varnothing$-definable in p: it is the subset of $\mathcal{D}_p \times \mathcal{D}_p$ given by the conjunction for $1 \leqslant i, j \leqslant R$ of the formulae

$$(\exists \sigma_1^{(i)}, \ldots, \sigma_R^{(i)} \in \mathbb{Z}_p)\ k_i = h_1^{\sigma_1^{(i)}} \cdots h_R^{\sigma_R^{(i)}}$$

and

$$(\exists \tau_1^{(j)}, \ldots, \tau_R^{(j)} \in \mathbb{Z}_p)\ h_j = k_1^{\tau_1^{(j)}} \cdots k_R^{\tau_R^{(j)}},$$

and these become polynomial equations independent of p over $\mathbb{Q}$ in the λ_{ij}, the μ_{ij}, the σ_i and the τ_j when we write the h_i and k_j in terms of their Mal'cev coordinates (Equation (I.5)).

Construct D_p and E_p from $\mathcal{E}_p$ and $\mathcal{D}_p$ as at the end of Section I.6. Using Equation (I.6), we see that for each $n \in \mathbb{N}$, $D_{p,n}/E_{p,n}$ consists of precisely $b_{p,n}$ equivalence classes. We deduce from Theorem **C** (taking $\mathfrak{L}_p = \{\mathbb{Q}_p\}$) that $\sum_{n=0}^{\infty} b_{p,n}t^n$ is uniformly rational.
The proofs in cases (b)–(e) are similar, modifying the definitions of $\mathcal{D}_p$ and $\mathcal{E}_p$ appropriately. For example, in (b) we replace $\mathcal{D}_p$ by the set $\mathcal{D}_p^{\trianglelefteq}$ of tuples (λ_{ij}) that define a normal finite-index subgroup H; a tuple (λ_{ij}) corresponding to a finite-index subgroup H belongs to $\mathcal{D}_p^{\trianglelefteq}$ if and only if it satisfies the formula

$$(\forall g \in \widehat{\Delta}_p)(\forall h \in H)(\exists \nu_1, \dots, \nu_R \in \mathbb{Z}_p)\; ghg^{-1} = h_1^{\nu_1} \cdots h_R^{\nu_R},$$

which is made up of polynomial equations independent of p over $\mathbb{Q}$ in the ν_i, the λ_{ij} and the Mal'cev coordinates of g and h. In case (d), the equivalence relation is the subset of $\mathcal{D}_p \times \mathcal{D}_p$ given by the formula:
there exists $g \in \widehat{\Delta}_p$, there exist $\sigma_i^{(j)}, \tau_i^{(j)} \in \mathbb{Z}_p$ for $1 \leqslant j \leqslant R$ such that $gh_jg^{-1} = k_1^{\sigma_1^{(j)}} \cdots k_R^{\sigma_R^{(j)}}$ and $g^{-1}k_jg = h_1^{\tau_1^{(j)}} \cdots h_R^{\tau_R^{(j)}}$ for $1 \leqslant j \leqslant R$.
This is made up of polynomial equations independent of p over $\mathbb{Q}$ in the Mal'cev coordinates of g and of the h_i and the k_i. In cases (c) and (e), we can express the isomorphism condition in terms of polynomials in the Mal'cev coordinates; compare the proof of Proposition I.7 below. ∎

Remark I.7.3:
Du Sautoy and Grunewald prove that Theorem I.7 (a) and (b) actually hold for an arbitrary finitely generated nilpotent group Γ, possibly with torsion. To prove this in our setting, write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ. Theorem I.7 now follow for cases (a)–(e) from our arguments above with suitable modifications: for example, for case (a), we count not all index p^n subgroups of Δ, but only the ones that contain Θ. For details, compare the argument of the last two paragraphs of Lemma I.8.

The proof for case (d) of Theorem I.7 is not given explicitly in [GSS88], but the appropriate definable integral can be constructed using the methods in the proof of [Sau05, Theorem 1.2]; what makes this work is that the equivalence classes are the orbits of a group action. The language of [Sau05] contains symbols for analytic functions, but our methods still apply there because we can use the results of Cluckers from the Appendix, which do hold in the analytic setting.
Observe that Theorem I.7 for case (e) is new; here the equivalence relation does not arise from any obvious group action, and Theorem **C** gives a genuinely new way of proving uniform rationality.
Here is another application, to the problem of counting finite p-groups.

Proposition I.7.4:
Fix positive integers c, d. Let $c_{p,n}$ be the number of finite p-groups of order p^n and nilpotency class at most c, generated by at most d elements. Then the power series $\sum_{n=0}^{\infty} c_{p,n}t^n$ is uniformly rational.

Proof. Let Δ be the free nilpotent group of class c on d generators (note that Δ is torsion-free). Any finite p-group of order p^n and nilpotency class at most c and generated by at

most d elements is a quotient of $\widehat{\Delta}_p$ by some normal subgroup of index p^n. Let $\mathcal{D}_p^{\trianglelefteq}$ and f_p be as in the proof of Theorem I.7. Define an equivalence relation $\mathcal{E}_p$ on $\mathcal{D}_p^{\trianglelefteq}$ as follows: two R-tuples (λ_{ij}), (μ_{ij}), representing good bases $h_1, \ldots, h_R$ and $k_1, \ldots, k_R$ for subgroups H, K respectively, are equivalent if and only if $\widehat{\Delta}_p/H \cong \widehat{\Delta}_p/K$.
The result will follow as in Theorem I.7 if we can show that $\mathcal{E}_p$ is uniformly $\varnothing$-definable in p. Let $a_1, \ldots, a_R$ be the Mal'cev basis of $\widehat{\Delta}_p$, as before. We claim that $\mathcal{E}_p \subseteq \mathcal{D}_p^{\trianglelefteq} \times \mathcal{D}_p^{\trianglelefteq}$ is given by the following conditions:

$$|f_p(\lambda_{ij})| = |f_p(\mu_{ij})|, \tag{I.7}$$

$$(\exists b_1, \ldots, b_r \in \widehat{\Delta}_p)(\forall \nu_1, \ldots, \nu_r \in \mathbb{Z}_p)\ a_1^{\nu_1} \cdots a_R^{\nu_R} \in H \iff b_1^{\nu_1} \cdots b_R^{\nu_R} \in K \tag{I.8}$$

and

$$\begin{gathered}(\forall \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_r \in \mathbb{Z}_p)(\exists \nu_1, \ldots, \nu_r \in \mathbb{Z}_p) \\ (a_1^{\sigma_1} \cdots a_R^{\sigma_R} a_1^{\tau_1} \cdots a_R^{\tau_R} = a_1^{\nu_1} \cdots a_R^{\nu_R}) \wedge (b_1^{\sigma_1} \cdots b_R^{\sigma_R} b_1^{\tau_1} \cdots b_R^{\tau_R} = b_1^{\nu_1} \cdots b_R^{\nu_R}).\end{gathered} \tag{I.9}$$

To prove this, suppose Equations (I.7), (I.8) and (I.9) hold. Then $\left|\widehat{\Delta}_p : H\right| = \left|\widehat{\Delta}_p : K\right|$ and the map $a_i H \mapsto b_i K$ defines an isomorphism from $\widehat{\Delta}_p/H$ onto $\widehat{\Delta}_p/K$. Conversely, if g is an isomorphism from $\widehat{\Delta}_p/H$ onto $\widehat{\Delta}_p/K$ then $\left|\widehat{\Delta}_p : H\right| = \left|\widehat{\Delta}_p : K\right|$, so $|f_p(\lambda_{ij})| = |f_p(\mu_{ij})|$. Moreover, we can choose $b_i \in \widehat{\Delta}_p$ such that $g(a_i H) = b_i K$ for $1 \leqslant i \leqslant R$. Then for all $\nu_1, \ldots, \nu_r \in \mathbb{Z}$ we have

$$(*) \quad a_1^{\nu_1} \cdots a_R^{\nu_R} \in H \iff b_1^{\nu_1} \cdots b_R^{\nu_R} \in K$$

and for all $\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_r \in \mathbb{Z}$ there exist $\nu_1, \ldots, \nu_r \in \mathbb{Z}$ such that

$$(**) \quad (a_1^{\sigma_1} \cdots a_R^{\sigma_R} a_1^{\tau_1} \cdots a_R^{\tau_R} = a_1^{\nu_1} \cdots a_R^{\nu_R}) \wedge (b_1^{\sigma_1} \cdots b_R^{\sigma_R} b_1^{\tau_1} \cdots b_R^{\tau_R} = b_1^{\nu_1} \cdots b_R^{\nu_R});$$

since H, K are closed and the group operations are continuous, $(*)$ and $(**)$ hold with $\mathbb{Z}$ replaced by $\mathbb{Z}_p$. This proves the claim. The formulae above involve only the function f—which is uniformly $\varnothing$-definable in p—and polynomials independent of p over $\mathbb{Q}$ in the Mal'cev coordinates, so $\mathcal{E}$ is uniformly $\varnothing$-definable in p, as required. ∎

Du Sautoy's proof [Sau99, Theorem 2.2], [Sau00, Theorems 1.6 and 1.8] uses the fact that an isomorphism $\widehat{\Delta}_p/H \to \widehat{\Delta}_p/K$ lifts to an automorphism of $\widehat{\Delta}_p$, which implies that the equivalence relation $\mathcal{E}_p$ arises from the action of the group $\mathrm{Aut}(\widehat{\Delta}_p)$, a compact p-adic analytic group. This allows one to express the power series $\sum_{n=0}^{\infty} c_{p,n} t^n$ as a cone integral, from which uniform rationality follows (see the start of this section). Our proof is simpler in its algebraic input, as elimination of imaginaries allows us to use less information about $\mathcal{E}_p$.

Remark I.7.5:
Let Γ be a finitely generated nilpotent group and let $c_{p,n}$ be the number of isomorphism classes of quotients of Γ of order p^n. Then the power series $\sum_{n=0}^{\infty} c_{p,n} t^n$ is uniformly rational. If Γ is torsion-free then this follows immediately from the proof of Proposition I.7. If Γ has torsion then we write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ and modify the proof of Proposition I.7 accordingly (cf. Remark I.7).

I.8. Twist isoclasses of characters of nilpotent groups

By a *representation* of a group G we shall mean a finite-dimensional complex representation, and by a *character* of G we shall mean the character of such a representation. A character is said to be *linear* if its degree is one. We write $\langle\ ,\ \rangle_G$ for the usual inner product of characters of G. If χ is linear then we have

$$\langle \chi\sigma_1, \chi\sigma_2\rangle_G = \langle \sigma_1, \sigma_2\rangle_G \tag{I.10}$$

for all characters σ_1 and σ_2. If $G' \leqslant G$ has finite index then we write $\mathrm{Ind}_{G'}^{G}\cdot$ and $\mathrm{Res}_{G'}^{G}\cdot$ for the induced character and restriction of a character respectively. For background on representation theory, see [CR81]. Below when we apply results from the representation theory of finite groups to representations of an infinite group, the representations concerned always factor through finite quotients.

We denote the set of irreducible n-dimensional characters of G by $\mathrm{R}_n(G)$. If $N \trianglelefteq G$ then we say the character χ of an irreducible representation ρ factors through G/N if ρ factors through G/N (this depends only on χ, not on ρ).

Notation I.8.1:
We say a character σ of G is *admissible* if σ factors through a finite quotient of G. If p is prime then we say σ is *p-admissible* if σ factors through a finite p-group quotient of G. We write $\mathrm{R}_n^{\mathrm{ad}}(G)$ ($\mathrm{R}_n^{(p)}(G)$) for the set of admissible (p-admissible) characters in $\mathrm{R}_n(G)$. Note that $\mathrm{R}_n^{(p)}(G)$ is empty if n is not a p-power [CR81, (9.3.2) Proposition].

Given $\sigma_1, \sigma_2 \in \mathrm{R}_n(G)$, we follow [LM85] and say that σ_1 and σ_2 are *twist-equivalent* if $\sigma_1 = \chi\sigma_2$ for some linear character χ of G. Clearly this defines an equivalence relation on $\mathrm{R}_n(G)$; we call the equivalence classes *twist isoclasses*.

Observation I.8.2:
Let σ_1, σ_2 be two irreducible degree n characters of G that are twist-equivalent: say $\sigma_2 = \chi\sigma_1$. If $N \trianglelefteq G$ such that σ_1, σ_2 both factor through G/N, then χ also factors through G/N.

If $N_1, N_2 \trianglelefteq G$ have finite (p-power) index then $N_1 \cap N_2$ also has finite (p-power) index. This implies that when we are working with twist isoclasses in $\mathrm{R}_n^{\mathrm{ad}}(G)$ ($\mathrm{R}_n^{(p)}(G)$), we need only consider twisting by admissible (p-admissible) linear characters.

Fix a finitely generated nilpotent group Γ. The set $\mathrm{R}_n(\Gamma)$ can be given the structure of a quasi-affine complex algebraic variety. Lubotzky and Magid analyzed the geometry of this variety and proved the following result [LM85, Theorem 6.6].

Theorem I.8.3:

There exists a finite quotient $\Gamma(n)$ of Γ such that every irreducible n-dimensional representation of Γ factors through $\Gamma(n)$ up to twisting. In particular, there are only finitely many twist isoclasses of irreducible n-dimensional characters.

Thus the number of degree n twist isoclasses is a finite number a_n.

Definition I.8.4:
We define the (global) representation zeta function $\zeta_\Gamma(s)$ by $\zeta_\Gamma(s) := \sum_{n=1}^{\infty} a_n n^{-s}$ and the p-local representation zeta function $\zeta_{\Gamma,p}(s)$ by $\zeta_{\Gamma,p}(s) := \sum_{n=0}^{\infty} a_{p^n} p^{-ns}$.

It is shown in [SV14, Lemma 2.1] that $\zeta_\Gamma(s)$ converges on some right-half plane. Voll noted [Vol11, Section 3.2.1] that $\zeta_\Gamma(s)$ has an Euler factorization

$$\zeta_\Gamma(s) = \prod_p \zeta_{\Gamma,p}(s)$$

for any finitely generated nilpotent group (cf. the proof of Lemma I.8).

Theorem I.8.5:
The power series $\sum_{n=0}^{\infty} a_{p^n} t^n$ is uniformly rational.

Note that we do not have to assume that Γ is torsion-free.
We prove Theorem I.8 by showing how to parametrize twist isoclasses in a definable way. The equivalence relation in the parametrization is not simply the twist-equivalence relation, which arises from the action of a group—the group of linear characters of Γ—but a more complicated equivalence relation.
The correspondence between index p^n subgroups of Γ and index p^n subgroups of $\widehat{\Gamma}_p$ gives a canonical bijection between $\mathrm{R}_{p^n}^{(p)}(\Gamma)$ and $\mathrm{R}_{p^n}^{(p)}(\widehat{\Gamma}_p)$, and it is clear that this respects twisting by p-admissible characters.

Lemma I.8.6:
For every $n \in \mathbb{N}$, there is a bijective correspondence between the sets $\mathrm{R}_{p^n}(\Gamma)/(\text{twisting})$ and $\mathrm{R}_{p^n}^{(p)}(\widehat{\Gamma}_p)/(\text{twisting})$.

Proof. It suffices to show that given any $\sigma \in \mathrm{R}_{p^n}(\Gamma)$, some twist of σ factors through a finite p-group quotient of Γ. By Theorem I.8, we can assume that σ factors through some finite quotient F of Γ. Then F, being a finite nilpotent group, is the direct product of its Sylow l-subgroups F_l, where l ranges over all the primes dividing $|F|$. Moreover [CR81, Theorem 10.33], σ is a product of irreducible characters σ_l, where each σ_l is a character of F_l. Since the degree of an irreducible character of a finite group divides the order of the group [CR81, Proposition 9.3.2], all of the σ_l for $l \neq p$ are linear. We may therefore twist σ by a linear character of F to obtain a character that kills F_l for $l \neq p$, and this linear character is admissible by Observation I.8. The new character factors through F_p, and we are done. ∎

The key idea is that finite p-groups are monomial: that is, every irreducible character is induced from a linear character of some subgroup. We parametrize p-admissible irreducible characters of $\widehat{\Gamma}_p$ by certain pairs (H, χ), where H is a finite-index subgroup of $\widehat{\Gamma}_p$ and χ is a p-admissible linear character of H: to a pair we associate the induced character $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$. We can parametrize these pairs using the theory of good bases for subgroups of $\widehat{\Gamma}_p$, and this description is well-behaved with respect to twisting. Two distinct pairs (H, χ) and (H', χ') may give the same induced character; this gives rise to a definable equivalence relation on the set of pairs.

If ψ is a character of H then we denote by $g.\psi$ the character of $g.H := gHg^{-1}$ defined by $(g.\psi)(ghg^{-1}) = \psi(h)$.

Lemma I.8.7:
(a) Let $\sigma \in \mathrm{R}^{(p)}_{p^n}(\widehat{\Gamma}_p)$. Then there exists $H \leqslant \widehat{\Gamma}_p$ such that $\left|\widehat{\Gamma}_p\colon H\right| = p^n$, together with a p-admissible linear character χ of H such that $\sigma = \mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$.
(b) Let H be a p-power index subgroup of $\widehat{\Gamma}_p$ and let χ be a p-admissible linear character of H. Then $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is a p-admissible character of $\widehat{\Gamma}_p$, and $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is irreducible if and only if for all $g \in \widehat{\Gamma}_p \smallsetminus H$, $\mathrm{Res}^{g.H}_{g.H\cap H} g.\chi \neq \mathrm{Res}^{H}_{g.H\cap H}\chi$. Moreover, if ψ is a p-admissible linear character of $\widehat{\Gamma}_p$ and $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is irreducible then $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\left(\left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi\right) = \psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$.
(c) Let $H, H' \leqslant \widehat{\Gamma}_p$ have index p^n, and let χ, χ' be p-admissible linear characters of H, H' respectively such that $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ and $\mathrm{Ind}_{H'}^{\widehat{\Gamma}_p}\chi'$ are irreducible. Then $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi = \mathrm{Ind}_{H'}^{\widehat{\Gamma}_p}\chi'$ if and only if there exists $g \in \widehat{\Gamma}_p$ such that $\mathrm{Res}^{g.H}_{g.H\cap H'} g.\chi = \mathrm{Res}^{H'}_{g.H\cap H'}\chi'$.

Proof. (a) Since σ is p-admissible, it factors through some finite p-group F. Since finite p-groups are monomial [CR81, Theorem 11.3], there exist $L \leqslant F$ of index p^n and a linear character χ of L such that σ—regarded as a character of F—equals $\mathrm{Ind}_L^F\chi$. Let H be the pre-image of L under the canonical projection $\widehat{\Gamma}_p \to F$. Regarding χ as a character of H, it is easily checked that $\left|\widehat{\Gamma}_p\colon H\right| = p^n$ and $\sigma = \mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$.
(b) Since χ is p-admissible, the kernel K of χ has p-power index in $\widehat{\Gamma}_p$, so K contains a p-power index subgroup N such that $N \trianglelefteq \widehat{\Gamma}_p$. Clearly $N \leqslant \ker(\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi)$, so $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is p-admissible. The irreducibility criterion follows immediately from [CR81, Theorem 10.25]. By Frobenius reciprocity,

$$\begin{aligned}
&\left\langle \mathrm{Ind}_H^{\widehat{\Gamma}_p}\left(\left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi\right), \psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right\rangle_{\widehat{\Gamma}_p}\\
= &\left\langle \left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi, \mathrm{Res}_H^{\widehat{\Gamma}_p}\left(\psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right)\right\rangle_H\\
= &\left\langle \left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi, \left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\mathrm{Res}_H^{\widehat{\Gamma}_p}\left(\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right)\right\rangle_H\\
= &\left\langle \chi, \mathrm{Res}_H^{\widehat{\Gamma}_p}\left(\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right)\right\rangle_H \text{ by Equation (I.10)}\\
= &\left\langle \mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi, \mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right\rangle_{\widehat{\Gamma}_p}\\
= &\ 1.
\end{aligned}$$

Now $\psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is irreducible, because $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ is, and the degrees of $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\left(\left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi\right)$ and $\psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ are equal. We deduce that $\psi\,\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi = \mathrm{Ind}_H^{\widehat{\Gamma}_p}\left(\left(\mathrm{Res}_H^{\widehat{\Gamma}_p}\psi\right)\chi\right)$.
(c) The Mackey Subgroup Theorem [CR81, Theorem 10.13] gives

$$\mathrm{Res}_{H'}^{\widehat{\Gamma}_p}\left(\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi\right) = \sum_{\overline{g}\in H'\backslash\widehat{\Gamma}_p/H} \mathrm{Ind}^{H'}_{g.H\cap H'}\left(\mathrm{Res}^{g.H}_{g.H\cap H'}\, g.\chi\right). \tag{I.11}$$

Here the sum is over a set of double coset representatives g for $H'\backslash\widehat{\Gamma}_p/H$ (the characters on the RHS of the formula are independent of choice of representative). Since $\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi$ and

$\mathrm{Ind}_{H'}^{\widehat{\Gamma}_p}\chi'$ are irreducible, they are distinct if and only if their inner product is zero. We have

$$\begin{aligned}
&\left\langle \mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi, \mathrm{Ind}_{H'}^{\widehat{\Gamma}_p}\chi'\right\rangle_{\widehat{\Gamma}_p} \\
= &\left\langle \mathrm{Res}_{H'}^{\widehat{\Gamma}_p}(\mathrm{Ind}_H^{\widehat{\Gamma}_p}\chi), \chi'\right\rangle_{H'} \text{ by Frobenius reciprocity} \\
= &\sum_{\overline{g}\in H'\backslash\widehat{\Gamma}_p/H} \left\langle \mathrm{Ind}_{g.H\cap H'}^{H'}\left(\mathrm{Res}_{g.H\cap H'}^{g.H}\, g.\chi\right), \chi'\right\rangle_{H'} \text{ by the Mackey Subgroup Theorem} \\
= &\sum_{\overline{g}\in H'\backslash\widehat{\Gamma}_p/H} \left\langle \mathrm{Res}_{g.H\cap H'}^{g.H} g.\chi, \mathrm{Res}_{g.H\cap H'}^{H'}\chi'\right\rangle_{g.H\cap H'} \text{ by Frobenius reciprocity.}
\end{aligned}$$

This vanishes if and only if each of the summands vanishes, which happens if and only if $\mathrm{Res}_{g.H\cap H'}^{g.H} g.\chi \neq \mathrm{Res}_{g.H\cap H'}^{H'}\chi'$ for every g, since the characters concerned are linear. The result follows. ∎

Write Γ as a quotient Δ/Θ of a finitely generated torsion-free nilpotent group Δ: for example, we may take Δ to be the free class c nilpotent group on N generators for appropriate N and c. Let $\pi\colon\Delta \to \Gamma$ be the canonical projection, and let $i\colon\Theta \to \Delta$ be inclusion. Let $\widehat{\Delta}_p$, $\widehat{\Theta}_p$ be the pro-p completions of Δ, Θ respectively. Then π (respectively i) extends to a continuous homomorphism $\widehat{\pi}_p\colon\widehat{\Delta}_p \to \widehat{\Gamma}_p$ (respectively $\widehat{i}_p\colon\widehat{\Theta}_p \to \widehat{\Delta}_p$), and the three groups $\widehat{i}_p(\widehat{\Theta}_p)$, $\ker\widehat{\pi}_p$, and the closure of Θ in $\widehat{\Delta}_p$ all coincide (compare [Dix+99, Chapter 1, Ex. 21]; because Δ is finitely generated nilpotent, it can in fact be shown that $\widehat{i}_p$ is injective, and hence an isomorphism onto its image). Clearly p-admissible representations of $\widehat{\Gamma}_p$ correspond bijectively to p-admissible representations of $\widehat{\Delta}_p$ that kill $\ker\widehat{\pi}_p$. Now Θ is finitely generated (see, e.g., [Wag97, Lemma 1.2.2]), so we can choose a Mal'cev basis $\theta_1,\dots,\theta_s$ for Θ. We identify the θ_i with their images in $\widehat{\Delta}_p$.

Let μ_{p^n} be the group of all complex p^nth roots of unity, and let μ_{p^∞} be the group of all complex p-power roots of unity.

Lemma I.8.8:
The groups μ_{p^∞} and $\mathbb{Q}_p/\mathbb{Z}_p$ are isomorphic.

Proof. Let $p^{-\infty}\mathbb{Z} \leqslant \mathbb{Q}$ be the group of rational numbers of the form np^{-r} for $n \in \mathbb{Z}$ and r a nonnegative integer. Then $p^{-\infty}\mathbb{Z}\cap\mathbb{Z}_p = \mathbb{Z}$ and $\mathbb{Z}_p p^{-\infty}\mathbb{Z} = \mathbb{Q}_p$, so $\mathbb{Q}_p/\mathbb{Z}_p \cong p^{-\infty}\mathbb{Z}/\mathbb{Z}$, by one of the standard group isomorphism theorems. The map $q \mapsto e^{2\pi i q}$ gives an isomorphism from $p^{-\infty}\mathbb{Z}/\mathbb{Z}$ to μ_{p^∞}. ∎

Let $\Phi\colon\mu_{p^\infty} \to \mathbb{Q}_p/\mathbb{Z}_p$ be the isomorphism described above. Any p-admissible linear character of a pro-p group takes its values in μ_{p^∞}, so we use Φ to identify p-admissible linear characters with p-admissible homomorphisms to $\mathbb{Q}_p/\mathbb{Z}_p$.

Lemma I.8.9:
Let $\mathcal{D}_p \subseteq \mathbb{Z}_p^{R^2}\times\mathbb{Q}_p^R$ be the set of tuples (λ_{ij}, y_k), where $1\leqslant i,j\leqslant R$ and $1\leqslant k\leqslant R$, satisfying the following conditions:
(a) the λ_{ij} form a good basis $h_1,\dots,h_R$ for some finite-index subgroup H of $\widehat{\Delta}_p$ such that $\ker\widehat{\pi}_p \leqslant H$;
(b) the prescription $h_i \mapsto y_i \bmod \mathbb{Z}_p$ gives a well-defined p-admissible homomorphism $\chi\colon H \to$

$\mathbb{Q}_p/\mathbb{Z}_p$ that kills $\ker\widehat{\pi}_p$;

(c) the induced character $\mathrm{Ind}_H^{\widehat{\Delta}_p}\chi$ is irreducible.

Then $\mathcal{D}_p$ is uniformly $\varnothing$-definable in p. Moreover, $\mathrm{Ind}_H^{\widehat{\Delta}_p}\chi$ is a p-admissible character of $\widehat{\Delta}_p$ that kills $\ker\widehat{\pi}_p$ and hence induces a p-admissible character of $\widehat{\Gamma}_p$, and every p-admissible irreducible character of $\widehat{\Gamma}_p$ arises in this way.

Notation I.8.10:
Given $(\lambda_{ij}, y_k) \in \mathcal{D}_p$, we write $\Psi(\lambda_{ij}, y_k)$ for the pair (H, χ). Since the h_i generate H topologically, the p-admissible homomorphism χ defined by the y_i is unique.

Proof. Condition (a) is uniformly $\varnothing$-definable in p, by Proposition I.7 (to the formulae that define the set of good bases we add the formulae $(\exists \nu_{1j}, \dots, \nu_{rj} \in \mathbb{Z}_p)\ \theta_j = h_1^{\nu_{1j}} \cdots h_R^{\nu_{Rj}}$ for $1 \leqslant j \leqslant s$). Given that (a) holds, we claim that (b) holds if and only if there exists an R^2-tuple (μ_{ij}) such that:
(i) (μ_{ij}) defines a good basis $k_1, \dots, k_R$ for a finite-index subgroup K of $\widehat{\Delta}_p$;
(ii) $K \trianglelefteq H$;
(iii) $\ker\widehat{\pi}_p \subseteq K$;
(iv) there exist $y \in \mathbb{Q}_p$, $r_1, \dots, r_R \in \mathbb{Z}_p$, $h \in H$ such that $|y| = |H/K|$ and for every i we have $\overline{h^{r_i}} = \overline{h_i}$ and $r_i y = y_i \bmod \mathbb{Z}_p$. (Here $\overline{x}$ denotes the image of $x \in H$ under the canonical projection $H \to H/K$.)

To see this, note that if (b) holds then $\ker\chi$ is a finite-index subgroup of H which satisfies (ii) and (iii). Take (μ_{ij}) to be any tuple defining a good basis for K. Then H/K, being isomorphic to a finite subgroup of $\mathbb{Q}_p/\mathbb{Z}_p$, is cyclic, so choose $h \in H$ that generates H/K and choose $y \in \mathbb{Q}_p$ such that $\chi(h) = y \bmod \mathbb{Z}_p$. We can choose $r_1, \dots, r_R \in \mathbb{Z}$ such that $\overline{h_i} = \overline{h}^{r_i}$ for each i, and it is easily checked that (iv) holds.
Conversely, suppose there exists a tuple (μ_{ij}) satisfying (i)–(iv). The map $\mathbb{Z}_p \to H, \lambda \mapsto h^\lambda$ is continuous because it is polynomial with respect to the Mal'cev coordinates, so there exists an open neighborhood U of 0 in $\mathbb{Z}_p$ such that $h^\lambda \in K$ for all $\lambda \in U$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, we may therefore find $n_1, \dots, n_R \in \mathbb{Z}$ such that $\overline{h_i} = \overline{h^{n_i}}$ for each i. Hence H/K is cyclic with generator $\overline{h}$.
We have a monomorphism $\beta\colon H/K \to \mathbb{Q}_p/\mathbb{Z}_p$ given by $\beta(\overline{h}^n) = ny \bmod \mathbb{Z}_p$. Let χ be the composition $H \to H/K \xrightarrow{\beta} \mathbb{Q}_p/\mathbb{Z}_p$. The canonical projection $H \to H/K$ is continuous [Dix+99, 1.2 Proposition], so we have $\chi(h^\lambda) = \lambda y \bmod \mathbb{Z}_p$ for every $\lambda \in \mathbb{Z}_p$. Condition (iv) implies that $\chi(h_i) = y_i \bmod \mathbb{Z}_p$ for every i, as required.
Now condition (i) is uniformly $\varnothing$-definable in p, by Proposition I.7. Condition (iii) can be expressed as

$$\begin{aligned}(\forall \nu_1, \dots, \nu_s \in \mathbb{Z}_p)\, &[(\exists \sigma_1, \dots, \sigma_s \in \mathbb{Z}_p)\theta_1^{\nu_1} \cdots \theta_s^{\nu_s} = h_1^{\sigma_1} \cdots h_R^{\nu_R}] \\ &\Rightarrow (\exists \tau_1, \dots, \tau_s \in \mathbb{Z}_p)\theta_1^{\nu_1} \cdots \theta_s^{\nu_s} = k_1^{\tau_1} \cdots k_R^{\tau_R}.\end{aligned} \tag{I.12}$$

Equation (I.12) can be expressed in terms of polynomials independent of p over $\mathbb{Q}$ in the λ_{ij}, the μ_{ij}, the ν_k, the σ_k and the τ_k, so condition (iii) is uniformly $\varnothing$-definable in p. (Note that the θ_k are fixed elements of Δ, so their Mal'cev coordinates are not just elements of $\mathbb{Z}_p$ but elements of $\mathbb{Z}$.)

Similar arguments show that conditions (ii) and (iv) are also uniformly $\varnothing$-definable conditions in p. In (iv), note that the conditions $\overline{h^{r_i}} = \overline{h_i}$ imply by the argument above that h is a generator for H/K, so the condition $|y| = |H/K|$ can be expressed as

$$\left(h^{y^{-1}} \in K\right) \wedge \left(\left(\forall z \in \mathbb{Q}_p\right) |z| < |y| \implies h^{z^{-1}} \notin K\right).$$

This shows that (condition (a))$\wedge$(condition (b)) is uniformly $\varnothing$-definable in p.

Condition (iii) implies that χ kills $\ker \widehat{\pi}_p$. Hence $\mathrm{Ind}_H^{\widehat{\Delta}_p} \chi$ kills $\ker \widehat{\pi}_p$, so $\mathrm{Ind}_H^{\widehat{\Delta}_p} \chi$ gives rise to an irreducible p-admissible character of $\widehat{\Gamma}_p$. By Lemma I.8 (b), irreducibility of the induced character can be written as

$$(\forall g \in \widehat{\Delta}_p \smallsetminus H)(\exists h \in H)\ ghg^{-1} \in H \text{ and } \chi(ghg^{-1}) \neq \chi(h).$$

Writing this in terms of the Mal'cev coordinates, we see that condition (c) is uniformly $\varnothing$-definable in p.

By Lemma I.8 (a), any p-admissible irreducible character σ of $\widehat{\Gamma}_p$ is of the form $\mathrm{Ind}_L^{\widehat{\Gamma}_p} \chi$ for some finite-index subgroup L of $\widehat{\Gamma}_p$ and some p-admissible linear character χ of L. Let H be the pre-image of L under the canonical projection $\widehat{\Delta}_p \to \widehat{\Gamma}_p$. Regarding σ, χ as representations of $\widehat{\Delta}_p, H$ respectively, it is easily checked that $\sigma = \mathrm{Ind}_H^{\widehat{\Delta}_p} \chi$. Choose (λ_{ij}) defining a good basis $h_1, \dots, h_R$ for H, and choose y_k such that $\chi(h_k) = y_k \bmod \mathbb{Z}_p$ for all k. The above argument shows that $(\lambda_{ij}, y_k) \in \mathcal{D}_p$. This completes the proof. ■

Define $f_p : \mathcal{D}_p \to \mathbb{Z}_p$ by $f_p(\lambda_{ij}, y_k) = \lambda_{11} \cdots \lambda_{RR}$. Define an equivalence relation $\mathcal{E}_p$ on $\mathcal{D}_p$ by $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ if $\mathrm{Ind}_H^{\widehat{\Delta}_p} \chi$ and $\mathrm{Ind}_{H'}^{\widehat{\Delta}_p} \chi'$ are twist-equivalent, where $(H, \chi) = \Psi(\lambda_{ij}, y_k)$ and $(H', \chi') = \Psi(\lambda'_{ij}, y'_k)$. The degree of $\mathrm{Ind}_H^{\widehat{\Delta}_p} \chi$ equals $|f_p(\lambda_{ij}, y_k)|_p^{-1}$, and likewise for (λ'_{ij}, y'_k), so if $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ then $f_p(\lambda_{ij}, y_k) = f_p(\lambda'_{ij}, y'_k)$.

Construct D_p and E_p from $\mathcal{E}_p$ and $\mathcal{D}_p$ as at the end of Section I.6. It follows from Lemma I.8 and the definition of $\mathcal{E}_p$ that $D_{p,n}$ is the union of precisely a_{p^n} $E_{p,n}$-equivalence classes (note that if one representation of $\widehat{\Gamma}_p$ is the twist of another by some linear character ψ of $\widehat{\Delta}_p$ then ψ is automatically a character of $\widehat{\Gamma}_p$, by Observation I.8). To complete the proof of Theorem I.8, it suffices by Theorem **C** to show that E_p is a family of equivalence relations on D_p that is uniformly definable over p. Hence it is enough to prove the following result.

Proposition I.8.11:
The equivalence relation $\mathcal{E}_p$ is uniformly $\varnothing$-definable in p.

Proof. Let $\mathcal{D}'_p \subseteq \mathbb{Q}_p^R$ be the set of R-tuples $(z_1, \dots, z_R)$ such that the prescription $a_i \mapsto z_i \bmod \mathbb{Z}_p$ gives a well-defined p-admissible linear character of $\widehat{\Delta}_p$ that kills $\ker \widehat{\pi}_p$. We denote this character by $\Xi(z_1, \dots, z_R)$ (or just $\Xi(z_k)$). Similar arguments to those in the proof of Lemma I.8 show that $\mathcal{D}'_p$ is definable. Let $(z_1, \dots, z_R) \in \mathcal{D}'_p$, let $(H, \chi) = \Psi(\lambda_{ij}, y_k)$ and let $h_1, \dots, h_R$ be corresponding the good basis for H. Then $h_k = a_1^{\lambda_{k1}} \cdots a_R^{\lambda_{kR}}$, so $\Xi(z_k) = \lambda_{k1} z_1 + \cdots + \lambda_{kR} z_R$. Hence $\mathrm{Res}_H^{\widehat{\Delta}_p} \Xi(z_k)\, \chi = \Psi(\lambda_{ij}, y_k + \lambda_{k1} z_1 + \cdots + \lambda_{kR} z_R)$. Applying Lemma I.8 (b) and (c), we see that if $(H', \chi') = \Psi(\lambda'_{ij}, y'_k)$ then $(\lambda_{ij}, y_k) \sim (\lambda'_{ij}, y'_k)$ if and only if

$$\begin{aligned}(\exists (z_1, \dots, z_R) \in \mathcal{D}'_p)\ (\exists g \in \widehat{\Delta}_p)\ (\forall h \in H)\ & ghg^{-1} \in H' \\ & \Rightarrow \left(\mathrm{Res}_H^{\widehat{\Delta}_p} \Xi(z_k)\, \chi\right)(h) = \chi'(ghg^{-1}).\end{aligned}$$

Writing this in terms of the Mal'cev coordinates, we obtain an equation independent of p involving $\mathcal{D}_p'$ and p-adic norms of polynomials over $\mathbb{Q}$ in the λ_{ij}, the y_k, the λ_{ij}', the y_k', the z_k, and the Mal'cev coordinates of g and h. We deduce that $\mathcal{E}_p$ is uniformly $\varnothing$-definable in p, as required. ■

We give a simple example. Let $\mathcal{H}$ be the Heisenberg group $\langle a, b, c : [a, b] = c, [a, c] = [b, c] = 1\rangle$. Nunley and Magid [NM89] explicitly calculated the twist isoclasses of $\mathcal{H}$ and showed that

$$\zeta_{\mathcal{H},p}(s) = 1 + \sum_{n=1}^{\infty}(p-1)p^{n-1}p^{-ns}.$$

The formula for the sum of a geometric progression gives

$$\zeta_{\mathcal{H},p}(s) = \frac{1-p^{-s}}{1-p^{1-s}}.$$

Hence

$$\zeta_{\mathcal{H}}(s) = \prod_p \zeta_{\mathcal{H},p}(s) = \prod_p \frac{1-p^{-s}}{1-p^{1-s}} = \frac{\zeta(s-1)}{\zeta(s)},$$

where $\zeta(s)$ denotes the Riemann zeta function. Ezzat [Ezz14, Theorem 1.1] and Stasinski and Voll [SV14, Theorem B] calculated $\zeta_\Gamma(s)$ for various generalizations Γ of $\mathcal{H}$.
The subgroup zeta function of the Heisenberg group is given by

$$\frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}$$

[SG00, Section 1]. This and other calculations of [SV14] suggests that the representation zeta function is better behaved than the subgroup zeta function. The same is true for semisimple arithmetic groups [LM04].

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CHAPTER II

Analytic difference fields

Le Logicien, au Vieux Monsieur.
La logique n'a pas de limites !
E. Ionesco, *Rhinocéros*, Acte I

This chapter contains [Rid].

Introduction

Since the work of Ax, Kochen and Eršov on valued fields (e.g. [AK65]) and their proof that the theory of a Henselian valued field is essentially controlled (in equicharacteristic zero) by the theory of the residue field and the value group, model theory of Henselian valued fields has been a very active and productive field. Among later developments one may note Macintyre's result in [Mac76] of elimination of quantifiers for p-adic fields and the proof by Pas of valued fields quantifier elimination for equicharacteristic zero Henselian fields with angular components in [Pas89], which implies the Ax-Kochen-Eršov principle. Another notable result is the one by Basarab and Kuhlmann (see [Bas91; BK92; Kuh94]) of valued field quantifier elimination for Henselian valued fields with amc-congruences, a language that does not make the class of definable sets grow (as angular components do). Another result in the Ax-Kochen-Eršov spirit is the proof by Delon in [Del81] — extended by Bélair in [Bél99] — that Henselian valued fields do not have the independence property if and only if their residue field does not have it (their value group never has the independence property by [GS84].)
But model theorists have not limited themselves to giving an increasingly refined description of the model theory of Henselian valued fields, there have also been attempts at extending those results to valued fields with more structure. The two most notable enrichments that have been studied are, on the one hand, analytic structures as initiated by [DD88] and studied thereafter by a great number of people (among many others [Dri92; DHM99; LR00; LR05; CLR06; CL11]) and, on the other hand, D-structures (a generalization of both difference and differential structures), first for differentials and certain isometries in [Sca00] but also for greater classes of isometries in [Sca03; BMS07; AD10] and then for automorphisms that might not be isometries [Azg10; Pal12; Hrua; GP10; DO]. The model theory of valued differential fields is also quite central to the model theoretic study of transseries (see for example [ADH13]) but the techniques and results in this last field seem quite orthogonal to those in other references given above and to our work here.
The goal of the present paper is to study valued fields with *both* an analytic structure and an automorphism. The main result of this paper is Theorem **D**, which states that σ-Henselian

(cf. Definition **(II.4.10)**) valued fields with analytic structure and any automorphism σ eliminate field quantifiers resplendently in the leading term language (cf. Definition **(II.1.1)**). We then deduce various Ax-Kochen-Eršov type results for analytic difference fields (both with respect to the theory and the independence property). We also try to give a systematic and comprehensive approach to quantifier elimination in (enriched) valued fields through some more abstract considerations (mainly in the appendix).

In [Sca06], Scanlon already attempted to study analytic difference fields in the case of an isometry, but the definition of σ-Henselianity given there is too weak to actually work, although some incorrect computations hide this fact. The axiomatization and all the proofs had to be redone entirely but, as stated earlier, this paper does not only contain a corrected version of the results in [Sca06], it also generalizes these results from the isometric case to the case of any valued field automorphism.

Some ideas from [Sca06] could be salvaged though, among them the fact that Weierstrass preparation (see Definition **(II.3.22)**) allows us to be close enough to the polynomial case to adapt the proofs from the purely valued difference setting. Nevertheless this adaptation is not as straightforward as one would hope, essentially because Weierstrass preparation only holds in one variable, but one variable in the difference world actually gives rise to many variables in the non difference world. The main ingredient to overcome this obstacle is a careful study of differentiability of terms in many variables (see Definition **(II.4.4)**) that allows us to give a new definition of σ-Henselianity in **(II.4.10)**. These techniques can probably be used to prove results in greater generality, for example: valued fields with both analytic structure and D-structure, a notion, defined in [Sca00], that encompasses both the differential and the difference case.

As explained in [Sca06], our interest in the model theory of valued fields with both analytic structure and difference structure is not simply a wish to see Ax-Kochen-Eršov type of results extended to more and more complicated structures and in particular to the combination of two structures where things are known to work well. It is also motivated by the fact that this is the right model-theoretic setting in which to understand Buium's p-differential geometry. More precisely any p-differential function over $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ can be defined in $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped with the lifting of the Frobenius and symbols for all p-adic analytic functions $\sum a_I x^I$ where $\mathrm{val}(a_I) \to \infty$ as $|I| \to \infty$. See [Sca06, Section 4] for an example of how a good model theoretic understanding of this structure can help to show uniformity of certain diophantine results.

The organization of this text is as follows. Section II.1 is a description of the languages, with either angular components or **RV**-structure, that we will be using. In Section II.2, we show that it is possible to transfer elimination of quantifier results from equicharacteristic zero to mixed characteristic (using the theoretical framework of Appendix II.B). Sections II.3 and II.4 describe the class of analytic difference fields we will be studying. Section II.5 is concerned with purely analytical matters, it describes the link between analytic 1-types and the underlying algebraic 1-type. In Section II.6 we prove the main result of this paper, Theorem **D**: a field quantifier elimination result for σ-Henselian analytic difference fields. We also prove an Ax-Kochen-Eršov principle for these fields. Finally Section II.7 shows how this quantifier elimination result also allows us to give conditions (on the residue field and the value group) for such fields to have (or not have) the independence property. The appendix contains an account of the more abstract model theory at work in the rest of the paper to help smooth out the arguments. Appendix II.B, in particular, sets up a general setting for transfer of elimination of quantifier results.

I would like to thank Élisabeth Bouscaren and Tom Scanlon for our numerous discussions. Without them none of the mathematics presented here would be understandable, correct or even exist. I also want to thank Raf Cluckers for having so readily answered all my questions about analytic structures as I was discovering them. Finally, I would like to thank Koushik Pal for taking the time to discuss the non-isometric case with me. Our discussions led to the generalization of the proofs to the non-isometric case.

II.1. Languages of valued fields

We will be considering valued fields of *characteristic zero*. They will mainly be considered in two kinds of languages. On the one hand, the language with leading terms, also known in the work of Basarab and Kuhlmann (cf. [Bas91; BK92; Kuh94]) as amc-congruences and in later work as $\mathbf{RV}$-sorts (e.g. [HK06]) and on the other hand the language with angular components, also known as the Denef-Pas language.

Definition II.1.1 (Leading term language)**:**
The language $\mathcal{L}^{\mathbf{RV}}$ has the following sorts: a sort $\mathbf{K}$ and a family of sorts $(\mathbf{RV}_n)_{n\in\mathbb{N}_{>0}}$. On the sort $\mathbf{K}$, the language consists of the ring language. The language also contains functions $\mathrm{rv}_n : \mathbf{K} \to \mathbf{RV}_n$ for all $n \in \mathbb{N}_{>0}$ and $\mathrm{rv}_{m,n} : \mathbf{RV}_n \to \mathbf{RV}_m$ for all $m|n$.

Any valued field can be considered as an $\mathcal{L}^{\mathbf{RV}}$-structure by interpreting $\mathbf{K}$ as the field and $\mathbf{RV}_n$ as $(\mathbf{K}^\star/1 + n\mathfrak{M}) \cup \{0\}$ where $\mathfrak{M}$ is the maximal ideal of the valuation ring $\mathcal{O}$. We will write $\mathbf{RV}_n^\star$ for $(\mathbf{K}^\star/1 + n\mathfrak{M}) = \mathbf{RV}_n \smallsetminus \{0\}$. Then rv_n is interpreted as the canonical surjection $\mathbf{K}^\star \to \mathbf{RV}_n^\star$ and it sends 0 to 0; $\mathrm{rv}_{n,m}$ is interpreted likewise. We will denote the $\mathcal{L}^{\mathbf{RV}}$-theory of characteristic zero valued fields by T_{vf}. If we need to specify the residual characteristic, we will write $\mathrm{T}_{\mathrm{vf},0,0}$ or $\mathrm{T}_{\mathrm{vf},0,p}$.
We will be denoting $\mathbf{RV} := \bigcup_n \mathbf{RV}_n$. These sorts are closed in $\mathcal{L}^{\mathbf{RV}}$ (see Definition **(II.A.7)**). The sorts in $\mathbf{RV}$ have a lot of structure given by the following commutative diagram (where $\mathbf{R}_n := \mathcal{O}/n\mathfrak{M}$):

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}^\star \arrow[r] \arrow[dd, "\mathrm{res}_n"] \arrow[ddd, bend right, "\mathrm{res}_m"'] & \mathbf{K}^\star \arrow[dr, "\mathrm{val}"] \arrow[dd, "\mathrm{rv}_n"] \arrow[ddd, bend right, "\mathrm{rv}_m"'] & & \\
 & & & \mathbf{\Gamma} \arrow[r] & 0 \\
1 \arrow[r] & \mathbf{R}_n^\star \arrow[r] \arrow[d, "\mathrm{res}_{m,n}"] & \mathbf{RV}_n^\star \arrow[ur, "\mathrm{val}_{\mathbf{RV},n}"'] \arrow[d, "\mathrm{rv}_{m,n}"] & & \\
1 \arrow[r] & \mathbf{R}_m^\star \arrow[r] & \mathbf{RV}_m^\star \arrow[uur, bend right, "\mathrm{val}_{\mathbf{RV},m}"'] & &
\end{tikzcd}$$

and all of this structure is definable in $\mathcal{L}^{\mathbf{RV}}$, although not without quantifiers. In order to eliminate $\mathbf{K}$-quantifiers, we will have to add some structure on the $\mathbf{RV}$ sorts.

Definition II.1.2:
The language $\mathcal{L}^{\mathbf{RV}^+}$ is the enrichment of $\mathcal{L}^{\mathbf{RV}}$ with, on each $\mathbf{RV}_n$, the language of (multiplicative) groups $\{1_n, \cdot_n\}$, a symbol 0_n and a binary predicate $|_n$, and functions $+_{m,n} : \mathbf{RV}_n^2 \to \mathbf{RV}_m$ for all $m|n$.

The multiplicative structure on $\mathbf{RV}_n$ is interpreted as its multiplicative (semi-)group structure, i.e. the group structure of $\mathbf{RV}_n^\star$ and $0_n \cdot_n x = x \cdot_n 0_n = 0_n$, $x|_n y$ is interpreted as $\mathrm{val}_{\mathbf{RV},n}(x) \leqslant \mathrm{val}_{\mathbf{RV},n}(y)$ and for all $x, y \in \mathbf{K}$ such that $\mathrm{val}(x+y) \leqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\} + \mathrm{val}(n) - \mathrm{val}(m)$, $\mathrm{rv}_n(x) +_{m,n} \mathrm{rv}_n(y) = \mathrm{rv}_m(x+y)$ and 0_n otherwise. This is well defined. We will denote by $\mathrm{T}_{\mathrm{Hen}}$ the theory of characteristic zero Henselian valued fields in $\mathcal{L}^{\mathbf{RV}^+}$.

Remark II.1.3:

1. If K has equicharacteristic zero, then for all $m|n$, $\mathrm{rv}_{m,n}$ is an isomorphism. Hence if we are working in equicharacteristic zero, we will only need to consider $\mathbf{RV}_1$. In that case we also have that $\mathbf{R}_1 = \mathbf{R}_1^\star \cup \{0\} \subseteq \mathbf{RV}_1^\star \cup \{0\} = \mathbf{RV}_1$. The additive structure is also simpler: we only need to consider the $+_{1,1}$ function on $\mathbf{RV}_1$. It extends the additive structure of $\mathbf{R}_1$ and makes every fiber of $\mathrm{val}_{\mathbf{RV},1}$ into an $\mathbf{R}_1$-vector space of dimension 1 (if we consider 0_1 to be the zero of every fiber).

2. If K has mixed characteristic p, then whenever $m|n$ and $\mathrm{val}(n) = \mathrm{val}(m)$ — i.e. when p does not divide n/m — $\mathrm{rv}_{m,n}$ is an isomorphism. In particular for all $n \in \mathbb{N}_{>0}$, $\mathrm{rv}_{n,p^{\mathrm{val}(n)}}$ is an isomorphism (where we identify $\mathrm{val}(p)$ and 1).

3. One could wonder then why consider all the $\mathbf{RV}_n$ when the only relevant ones are the $\mathbf{RV}_{p^n}$ in mixed characteristic p and $\mathbf{RV}_1$ in equicharacteristic zero. The main reason is that we want enough uniformity to be able to talk of T_{vf} without specifying the residual characteristic or adding a constant for the characteristic exponent (in particular if one wishes to consider ultraproducts of valued fields with growing residual characteristic, although we will not do so here).

The use of this language is mainly motivated by the following result that originates in [Bas91; BK92], although the phrasing in terms of resplendence first appears in [Sca97]. By resplendent quantifier elimination relative to $\mathbf{RV}$, we mean that quantifiers on the sorts other than those in $\mathbf{RV}$ can be elimated (namely the field quantifiers here) and that this result is true whatever the enrichment on the $\mathbf{RV}$-sorts (see Appendix II.A for precise definitions).

Theorem II.1.4:
The theory $\mathrm{T}_{\mathrm{Hen}}$ eliminates $\mathbf{K}$-quantifiers resplendently relative to $\mathbf{RV}$.

Later, we will add analytic and difference structures, hence we will consider an enrichment of $\mathcal{L}^{\mathbf{RV}}$ by new terms on $\mathbf{K}$ and predicates and terms on $\mathbf{RV}$ (although none on both $\mathbf{K}$ and $\mathbf{RV}$; this is what we call in Appendix II.A an $\mathbf{RV}$-enrichment of a $\mathbf{K}$-term enrichment of $\mathcal{L}^{\mathbf{RV}}$). Let $\mathcal{L}$ be such a language and let $\Sigma_{\mathbf{RV}}$ denote the new sorts coming from the $\mathbf{RV}$-enrichment.

Remark II.1.5:
Any quantifier free $\mathcal{L}$-formula $\varphi(\overline{x},\overline{y})$ where $\overline{x}$ are $\mathbf{K}$-variables and $\overline{y}$ are $\mathbf{RV}$-variables, is equivalent modulo $\mathrm{T_{vf}}$ to a formula of the form $\psi(\mathrm{rv}_{\overline{n}}(\overline{u}(\overline{x})),\overline{y})$ where ψ is a quantifier free $\mathcal{L}|_{\mathbf{RV}\cup\Sigma_{\mathbf{RV}}}$-formula and $\overline{u}$ are $\mathcal{L}|_{\mathbf{K}}$-terms. Indeed the only predicate involving $\mathbf{K}$ is the equality and $t(\overline{x}) = s(\overline{x})$ is equivalent to $\mathrm{rv}_1(t(\overline{x}) - s(\overline{x})) = 0$. The statement follows immediately.

Here is an easy lemma that will be very helpful later on to uniformize certain results.

Corollary II.1.6:
Let T be an $\mathcal{L}$-theory that eliminates $\mathbf{K}$-quantifiers, $M \models T$, $C \leqslant M$ — i.e. C is a substructure of M — and $\overline{x}, \overline{y} \in \mathbf{K}(M)$ be such that for all $\mathcal{L}|_{\mathbf{K}}(C)$-terms $\overline{u}$, and all $n \in \mathbb{N}_{>0}$, $\mathrm{rv}_n(\overline{u}(\overline{x})) = \mathrm{rv}_n(\overline{u}(\overline{y}))$. Then $\overline{x}$ and $\overline{y}$ have the same $\mathcal{L}(C)$-type.

Proof. Let $f : M \to M$ be the identity on $\mathbf{RV} \cup \Sigma_{\mathbf{RV}}(M)$ and send $u(\overline{x})$ to $u(\overline{y})$ for all $\mathcal{L}|_{\mathbf{K}}(C)$-term u. By Remark **(II.1.5)**, f is a partial $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism. But $\mathbf{K}$-quantifiers elimination implies that f is in fact elementary. ■

The other kind of language, the one with angular components, essentially boils down to giving oneself a section of the short sequences defining the $\mathbf{RV}_n$. That statement is made explicit in **(II.1.8)**.

Definition II.1.7 ($\mathcal{L}^{\mathrm{ac}}$)**:**
The language $\mathcal{L}^{\mathrm{ac}}$ has the following sorts: sorts $\mathbf{K}$ and $\mathbf{\Gamma}^\infty$ and a family of sorts $(\mathbf{R}_n)_{n\in\mathbb{N}_{>0}}$. The sorts $\mathbf{K}$ and $\mathbf{R}_n$ come with the ring language and the sort $\mathbf{\Gamma}^\infty$ comes with the language of ordered (additive) groups and a constant ∞. The language also contains a function $\mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}^\infty$, for all n, functions $\mathrm{ac}_n : \mathbf{K} \to \mathbf{R}_n$, $\mathrm{res}_n : \mathbf{K} \to \mathbf{R}_n$, $\mathrm{val}_{\mathbf{R},n} : \mathbf{R}_n \to \mathbf{\Gamma}^\infty$, $\mathrm{s}_{\mathbf{R},n} : \mathbf{\Gamma}^\infty \to \mathbf{R}_n$ and for all $m|n$, functions $\mathrm{res}_{m,n} : \mathbf{R}_n \to \mathbf{R}_m$ and $\mathrm{t}_{\mathbf{R},m,n} : \mathbf{R}_n \to \mathbf{R}_m$.

As one might guess, the $\mathbf{R}_n$ are interpreted as the residue rings $\mathcal{O}/n\mathfrak{M}$. As with $\mathbf{RV}$, we will write $\mathbf{R} := \bigcup_n \mathbf{R}_n$. The res_n and $\mathrm{res}_{m,n}$ denote the canonical surjections $\mathcal{O} \to \mathbf{R}_n$ and $\mathbf{R}_n \to \mathbf{R}_m$. The function ac_n denotes an angular component, i.e a multiplicative homomorphism $\mathbf{K}^\star \to \mathbf{R}_n^\star$ that extend the canonical surjection on $\mathcal{O}^\star$ and send 0 to 0_n. Moreover, the system of the ac_n should be consistent, i.e. $\mathrm{res}_{m,n} \circ \mathrm{ac}_n = \mathrm{ac}_m$. The function $\mathrm{val}_{\mathbf{R},n}$ is interpreted as the function induced by val on $\mathbf{R}_n \smallsetminus \{0\}$ and sending 0_n to ∞. The function $\mathrm{s}_{\mathbf{R},n}$ is defined by $\mathrm{s}_{\mathbf{R},n}(\mathrm{val}(x)) = \mathrm{res}_n(x)\mathrm{ac}_n(x)^{-1}$ and finally, the function $\mathrm{t}_{\mathbf{R},m,n}$ is defined by $\mathrm{t}_{\mathbf{R},m,n}(\mathrm{res}_n(x)) = \mathrm{ac}_m(x)$ when $\mathrm{val}(x) \leqslant \mathrm{val}(n) - \mathrm{val}(m)$ and 0_m otherwise (this is well-defined).
It should be noted that any valued field that is saturated enough can be endowed with angular components (cf. [Pas90, Corollary 1.6]).
Let $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$ be the enrichment of $\mathcal{L}^{\mathbf{RV}^+} \cup (\mathcal{L}^{\mathrm{ac}} \smallsetminus \{\mathrm{val}, \mathrm{res}_n, \mathrm{ac}_n : n \in \mathbb{N}_{>0}\})$ with symbols $\mathrm{val}_{\mathbf{RV},n} : \mathbf{RV}_n \to \mathbf{\Gamma}^\infty$ for the functions induced by the valuation, symbols $\mathrm{i}_n : \mathbf{R}_n \to \mathbf{RV}_n$ for the injection of $\mathbf{R}_n^\star \to \mathbf{RV}_n$ extended by 0 outside $\mathbf{R}_n^\star$, symbols $\mathrm{res}_{\mathbf{RV},n} : \mathbf{RV}_n \to \mathbf{R}_n$ for the canonical projection, $\mathrm{s}_n : \mathbf{\Gamma}^\infty \to \mathbf{RV}_n$ for a coherent system of sections of $\mathrm{val}_{\mathbf{RV},n}$ compatible with the $\mathrm{rv}_{m,n}$ and symbols $\mathrm{t}_n : \mathbf{RV}_n \to \mathbf{R}_n$ interpreted as $\mathrm{t}_n(x) = \mathrm{i}_n{}^{-1}(x\mathrm{s}_n(\mathrm{val}_{\mathbf{RV},n}(x))^{-1})$. Let $\mathrm{T}^{\mathrm{s}}_{\mathrm{vf}}$ be the $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$-theory of characteristic zero valued fields and $\mathrm{T}^{\mathrm{ac}}_{\mathrm{vf}}$ the $\mathcal{L}^{\mathrm{ac}}$-theory of characteristic zero valued fields.

Let $\mathcal{L}^{\mathrm{s},e}$ be an $\mathbf{RV}$-enrichment (with potentially new sorts $\Sigma_{\mathbf{RV}}$) of a $\mathbf{K}$-enrichment (with potentially new sorts $\Sigma_{\mathbf{K}}$) of $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$ and T^e be an $\mathcal{L}^{\mathrm{s},e}$-theory extending $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$. We define $\mathcal{L}^{\mathrm{ac},e}$ to be the language containing:

1. $\mathcal{L}^{\mathrm{ac}} \cup \mathcal{L}^{\mathrm{s},e}|_{\mathbf{K}\cup\Sigma_{\mathbf{K}}}$;
2. The new sorts $\Sigma_{\mathbf{RV}}$;
3. For each new function symbol $f : \prod S_i \to \mathbf{RV}_n$, two functions symbols $f_{\mathbf{R}} : \prod T_i \to \mathbf{R}_n$ and $f_{\mathbf{\Gamma}} : \prod T_i \to \mathbf{\Gamma}^\infty$ where $T_i = \mathbf{R}_m \times \mathbf{\Gamma}^\infty$ whenever $S_i = \mathbf{RV}_m$ and $T_i = S_i$ otherwise;
4. For each new function symbol $f : \prod S_i \to S$, where $S \neq \mathbf{RV}_n$, the same symbol f but with domain $\prod T_i$ as above;
5. For each new predicate $R \subseteq \prod S_i$, the same symbol R but as a predicate in $\prod T_i$ for T_i as above.

We also define $\mathrm{T}^{\mathrm{ac},e}$ to be the theory containing:

1. $\mathrm{T}^{\mathrm{ac}}_{\mathrm{vf}}$;
2. For all new function symbol f, whenever f or $f_{\mathbf{R}}$ and $f_{\mathbf{\Gamma}}$ (depending on the case) is applied to an argument — corresponding to an $\mathbf{RV}_n$-variable of f — outside of $\mathbf{R}^\star_n \times \mathbf{\Gamma} \cup \{0, \infty\}$, then f has the same value as if f were applied to $(0, \infty)$ instead;
3. For all new symbol f with image $\mathbf{RV}_n$, $\mathrm{Im}(f_{\mathbf{R}}, f_{\mathbf{\Gamma}}) \subseteq \mathbf{R}^\star_n \times \mathbf{\Gamma} \cup (0, \infty)$;
4. For all new predicate R, R applied to an argument outside of $\mathbf{R}^\star_n \times \mathbf{\Gamma} \cup \{0, \infty\}$ is equivalent to R applied to $(0, \infty)$ instead;
5. The theory T^e translated in $\mathcal{L}^{\mathrm{ac},e}$ as explained in the following proposition.

In the following proposition, $\mathrm{Str}(T)$ denote the category of substructures of models of T, i.e. models of $T_\forall$. See Appendix II.B, for precise definitions.

Proposition II.1.8:
There exist functors $F : \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e}) \to \mathrm{Str}(T^e)$ and $G : \mathrm{Str}(T^e) \to \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ that respect models, cardinality up to 1 and elementary submodels and induce an equivalence of categories between $\mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ and $\mathrm{Str}(T^e)$. Moreover G sends $\mathbf{R} \cup \mathbf{\Gamma}^\infty$ to $\mathbf{RV} \cup \mathbf{R} \cup \mathbf{\Gamma}^\infty$.

Proof. Let C be an $\mathcal{L}^{\mathrm{ac},e}$-structure (inside some $M \models \mathrm{T}^{\mathrm{ac},e}$), we define $F(C)$ to have the same underlying sets for all sorts common to $\mathcal{L}^{\mathrm{ac},e}$ and $\mathcal{L}^{\mathrm{s},e}$ and $\mathbf{RV}_n(F(C)) = (\mathbf{R}^\star_n < C > \times(\mathbf{\Gamma}^\infty(C) \smallsetminus \{\infty\})) \cup \{(0_n, \infty)\}$. All the structure on the sorts common to $\mathcal{L}^{\mathrm{s},e}$ and $\mathcal{L}^{\mathrm{ac},e}$ is inherited from C. We define $\mathrm{rv}_m(x) = (\mathrm{ac}_n(x), \mathrm{val}(x))$ and $\mathrm{rv}_{m,n}(x, \gamma) = (\mathrm{res}_{m,n}(x), \gamma)$. The (semi-)group structure on $\mathbf{RV}_n$ is the product (semi-)group structure, 0_n is interpreted as $(0_n, \infty)$. We set $(x, \gamma)|_n(y, \delta)$ to hold if and only if $\gamma \leqslant \delta$ and we define $(x, \gamma) +_{m,n} (y, \delta)$ as $(\mathrm{res}_{m,n}(x), \gamma)$ if $\gamma < \delta$, $(\mathrm{res}_{m,n}(y), \delta)$ if $\delta < \gamma$ and $(\mathrm{t}_{\mathbf{R},m,n}(x + y), \gamma + \mathrm{val}_{\mathbf{R},n}(x + y))$ if $\delta = \gamma$. The functions $\mathrm{val}_{\mathbf{RV},n}$ are interpreted as the right projections and the functions t_n

as the left projections. Finally, define $\mathrm{i}_n(x) = (x, 0)$ on $\mathbf{R}_n^\star$ and $\mathrm{i}_n(x) = (0, \infty)$ otherwise, $\mathrm{res}_{\mathbf{RV},n}(x,\gamma) = x\mathrm{s}_{\mathbf{R},n}(\gamma)$, $\mathrm{s}_n(\gamma) = (1,\gamma)$ if $\gamma \neq \infty$ and $\mathrm{s}_n(\infty) = (0,\infty)$. For each function $f : \prod S_i \to \mathbf{RV}_n$ for some n, define $\overline{u} : \prod S_i \to \prod T_i$ to be such that $u_i(\overline{x}) = x_i$ if $S_i \neq \mathbf{RV}_m$ and $u_i(\overline{x}) = (\mathrm{t}_m(x_i), \mathrm{val}_{\mathbf{RV},m}(x_i))$ if $S_i = \mathbf{RV}_m$. Then $f^{F(C)}(\overline{x}) = (f_{\mathbf{R}}^C(\overline{u}(\overline{x})), f_{\mathbf{\Gamma}}^C(\overline{u}(\overline{x})))$. If $f : \prod S_i \to S$ where $S \neq \mathbf{RV}_n$ for any n, then define $f^{F(C)}(\overline{x}) = f^C(\overline{u}(\overline{x}))$ and finally $F(C) \vDash R(\overline{x})$ if and only if $C \vDash R(\overline{u}(\overline{x}))$.
If $f : C_1 \to C_2$ is an $\mathcal{L}^{\mathrm{ac},e}$-isomorphism, we define $F(f)$ to be f on all sorts common to $\mathcal{L}^{\mathrm{ac},e}$ and $\mathcal{L}^{\mathrm{s},e}$ and $F(f)(x,\gamma) = (f(x), f(\gamma))$. It is easy to check that $F(f)$ is an $\mathcal{L}^{\mathrm{s},e}$-isomorphism.
Let D be an $\mathcal{L}^{\mathrm{s},e}$-structure (inside some $N \vDash T^e$), define $G(D)$ to be the restriction of D to all $\mathcal{L}^{\mathrm{ac},e}$-sorts enriched with $\mathrm{val} = \mathrm{val}_{\mathbf{RV},n} \circ \mathrm{rv}_1$, $\mathrm{res}_n = \mathrm{res}_{\mathbf{RV},n} \circ \mathrm{rv}_n$, $\mathrm{ac}_n = \mathrm{t}_n \circ \mathrm{rv}_n$. Moreover, for any function $f : \prod S_i \to \mathbf{RV}_n$ for some n, let $\overline{v} : \prod T_i \to \prod S_i$ to be such that $v_i(\overline{x}) = x_i$ if $S_i \neq \mathbf{RV}_m$ for any m and $v_i(\overline{x}) = \mathrm{i}_m(y_i)\mathrm{s}_m(\gamma_i)$ where $x_i = (y_i, \gamma_i)$, if $S_i = \mathbf{RV}_m$. Then define $f_{\mathbf{R}}^{G(D)}(\overline{x}) = \mathrm{t}_n(f^D(\overline{v}(\overline{x})))$ and $f_{\mathbf{\Gamma}}^{G(D)}(\overline{x}) = \mathrm{val}_{\mathbf{RV},n}(f^D(\overline{v}(\overline{x})))$. If $f : \prod S_i \to S$ where $S \neq \mathbf{RV}_n$ for any n, then $f^{G(D)}(\overline{x}) = f^D(\overline{v}(\overline{x}))$ and finally $G(D) \vDash R(\overline{x})$ if and only if $D \vDash R(\overline{v}(\overline{x}))$. If $f : D_1 \to D_2$ is an $\mathcal{L}^{\mathrm{s},e}$-isomorphism, it is easy to show that the restriction of f to the $\mathcal{L}^{\mathrm{ac},e}$-sorts is an $\mathcal{L}^{\mathrm{ac},e}$-isomorphism.
Now, one can check that for any $\mathcal{L}^{\mathrm{s},e}$-formula $\varphi(\overline{x})$ there exists an $\mathcal{L}^{\mathrm{ac},e}$-formula $\varphi^{\mathrm{ac},e}(\overline{y})$ such that for any $C \in \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ and $\overline{c} \in C$, $C \vDash \varphi(\overline{c})$ if and only if $F(C) \vDash \varphi^{\mathrm{ac},e}(\overline{u}(\overline{c}))$ where u is as above (for the sorts corresponding to $\overline{x}$). Similarly, to any $\mathcal{L}^{\mathrm{ac},e}$-formula $\psi(\overline{x})$ we can associate an $\mathcal{L}^{\mathrm{s},e}$-formula $\psi^{\mathrm{s},e}(\overline{x})$ such that for any $D \in \mathrm{Str}(T)$ and $d \in D$, $D \vDash \psi(\overline{d})$ if and only if $G(D) \vDash \psi^{\mathrm{s},e}(\overline{d})$. One can also check that for all $\mathcal{L}^{\mathrm{s},e}$-formula φ, $T \vDash (\varphi^{\mathrm{ac},e})^{\mathrm{s},e}(\overline{u}(\overline{x})) \iff \varphi(\overline{x})$ and for all $\mathcal{L}^{\mathrm{ac},e}$-formula ψ, $\mathrm{T}^{\mathrm{ac},e} \vDash (\psi^{\mathrm{s},e})^{\mathrm{ac},e} \iff \psi$. The rest of the proposition follows. ∎

Remark II.1.9:

1. The functions $\mathrm{t}_{\mathbf{R},m,n}$ are actually not needed, if we Morleyize on $\mathbf{R} \cup \mathbf{\Gamma}^\infty$, as they are definable using only quantification in the $\mathbf{R}_n$.

2. As with leading terms structure, in equicharacteristic zero, the angular component structure is a lot simpler. We only need val and ac_1 (and none of the $\mathrm{val}_{\mathbf{RV},n}$, $\mathrm{s}_{\mathbf{R},n}$ or $\mathrm{t}_{\mathbf{R},m,n}$).

3. In mixed characteristic with finite ramification — i.e. $\mathbf{\Gamma}$ has a smallest positive element 1 and $\mathrm{val}(p) = k \cdot 1$ for some $k \in \mathbb{N}_{>0}$ — the structure is also simpler. The functions $\mathrm{val}_{\mathbf{R},n}$, $\mathrm{s}_{\mathbf{R},n}$ and $\mathrm{t}_{\mathbf{R},m,n}$ can be redefined (without $\mathbf{K}$-quantifiers) knowing only $\mathrm{s}_{\mathbf{R},n}(1)$. Let $\mathcal{L}^{\mathrm{ac},fr}$ be the language $(\mathcal{L}^{\mathrm{ac}} \smallsetminus \{\mathrm{val}_{\mathbf{R},n}, \mathrm{s}_{\mathbf{R},n}, \mathrm{t}_{\mathbf{R},m,n} : m, n \in \mathbb{N}_{>0}\}) \cup \{c_n\}$ where c_n will be interpreted as $\mathrm{s}_{\mathbf{R},n}(1)$ — i.e. as $\mathrm{res}_n(x)\mathrm{ac}_n(x)^{-1}$ for x with minimal positive valuation. This is the language in which finitely ramified mixed characteristic fields with angular components are usually considered — and eliminate field quantifiers.

To finish this section let us define balls and Swiss cheeses.

Definition II.1.10 (Balls and Swiss cheeses)**:**
Let (K,v) be a valued field, $\gamma \in \mathrm{val}(K)$ and $a \in K$. Write $\mathring{\mathcal{B}}_\gamma(a) := \{x \in \mathbf{K}(M) : \mathrm{val}(x-a) > \gamma\}$ for the open ball of center a and radius γ, and $\overline{\mathcal{B}}_\gamma(a) := \{x \in \mathbf{K}(M) : \mathrm{val}(x-a) \geqslant \gamma\}$ for the closed ball of center a and radius γ.
A Swiss cheese is a set of the form $b \smallsetminus (\bigcup_{i=1,\dots,n} b_i)$ where b and the b_i are open or closed balls.

We allow closed balls to have radius ∞ — i.e. singletons are balls — and we allow open balls to have radius $-\infty$ — i.e. $\mathbf{K}$ itself is an open ball.

Definition II.1.11 ($\mathcal{L}_{\mathrm{div}}$)**:**
The language $\mathcal{L}_{\mathrm{div}}$ has a unique sort $\mathbf{K}$ equipped with the ring language and a binary predicate $|$.

In a valued field (K, val), the predicate $x|y$ will denote $\mathrm{val}(x) \leqslant \mathrm{val}(y)$. If $C \subseteq K$, we will denote by $\mathcal{SC}(C)$, the set of all quantifier free $\mathcal{L}_{\mathrm{div}}(C)$-definable sets in one variable. Note that all those sets are finite unions of swiss cheeses.
Note that later on, our valued fields may be endowed with more than one valuation. In that case, we will write $\mathring{\mathcal{B}}^{\mathcal{O}}_\gamma(a)$ or $\mathcal{SC}^{\mathcal{O}}(C)$ to specify that we are considering the valuation associated to $\mathcal{O}$. We will also extend the notation for balls by writing $\mathring{\mathcal{B}}_\gamma(\overline{a}) := \{\overline{b} : \mathrm{val}(\overline{b} - \overline{a}) > \gamma\}$ and $\overline{\mathcal{B}}_\gamma(\overline{a}) := \{\overline{b} : \mathrm{val}(\overline{b} - \overline{a}) \geqslant \gamma\}$ where $\mathrm{val}(\overline{a}) := \min_i\{\mathrm{val}(a_i)\}$.

II.2. Coarsening

The goal of this section is to provide the necessary tools for the reduction to the equicharacteristic zero case. This is a classical method , which underlies most existing proofs of K-quantifier elimination for enriched mixed characteristic Henselian fields. We present it here on its own, as a general transfer principle which we will then be able to invoke directly, in order, I hope, to make the proofs clearer.

Definition II.2.1 (Coarsening valuations)**:**
Let (K, val) be a valued field, $\Delta \subseteq \mathbf{\Gamma}(K)$ a convex subgroup and $\pi : \mathbf{\Gamma}(K) \to \mathbf{\Gamma}(K)/\Delta$ the canonical projection. Let $\mathrm{val}^\Delta := \pi \circ \mathrm{val}$, extended to 0 by $\mathrm{val}^\Delta(0) = \infty$.

Remark II.2.2:
The valuation val^Δ is a valuation coarser than val. Its valuation ring is $\mathcal{O}^\Delta := \{x \in K : \exists \delta \in \Delta,\ \delta < \mathrm{val}(x)\} \supseteq \mathcal{O}(K)$ and its maximal ideal is $\mathfrak{M}^\Delta := \{x \in K : \mathrm{val}(x) > \Delta\} \subseteq \mathfrak{M}(K)$. Its residue field $\mathbf{R}^\Delta_1$ is in fact a valued field for the valuation $\widetilde{\mathrm{val}}^\Delta$ defined by $\widetilde{\mathrm{val}}^\Delta(x + \mathfrak{M}^\Delta) := \mathrm{val}(x)$ for all $x \in \mathcal{O}^\Delta \smallsetminus \mathfrak{M}^\Delta$ and $\widetilde{\mathrm{val}}^\Delta(\mathfrak{M}^\Delta) = \infty$. Then $\widetilde{\mathrm{val}}^\Delta(\mathbf{R}^\Delta_1) = \Delta^\infty = \Delta \cup \{\infty\}$. The valuation ring of $\mathbf{R}^\Delta_1$ is $\widetilde{\mathcal{O}}^\Delta := \mathcal{O}/\mathfrak{M}^\Delta$, its maximal ideal is $\mathfrak{M}/\mathfrak{M}^\Delta$ and its residue field is $\mathbf{R}_1$. Moreover, if $\mathrm{rv}^\Delta_n : K \to K^\star/(1 + n\mathfrak{M}^\Delta) \cup \{0\} =: \mathbf{RV}^\Delta_n$ is the canonical projection, rv_n factorizes through rv^Δ_n; i.e. there is a function $\pi_n : \mathbf{RV}^\Delta_n \to \mathbf{RV}_n$ such that $\mathrm{rv}_n = \pi_n \circ \mathrm{rv}^\Delta_n$.

$$
\begin{tikzcd}
\mathcal{O}^{\star} \arrow[r, hook] \arrow[d] \arrow[dd, bend right, "\mathrm{res}_1"'] & (\mathcal{O}^{\Delta})^{\star} \arrow[r, hook] \arrow[d, "\mathrm{res}_1^{\Delta}"] & \mathbf{K}^{\star} \arrow[d, "\mathrm{rv}_1^{\Delta}"] \\
(\widetilde{\mathcal{O}}^{\Delta})^{\star} \arrow[r, hook] \arrow[d, "\widetilde{\mathrm{res}}_1^{\Delta}"] & (\mathbf{R}_1^{\Delta})^{\star} \arrow[r, hook] \arrow[d, "\widetilde{\mathrm{val}}^{\Delta}"] & (\mathbf{RV}_1^{\Delta})^{\star} \\
\mathbf{R}_1^{\star} & \Delta &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{K} \arrow[d, "\mathrm{rv}_n"] \arrow[drr, "\mathrm{val}"] \arrow[ddl, "\mathrm{rv}_n^{\Delta}"'] \arrow[ddr, "\mathrm{val}^{\Delta}"] & & \\
& \mathbf{RV}_n \arrow[rr, "\mathrm{val}_{\mathbf{RV},n}"] & & \mathbf{\Gamma}^{\infty} \arrow[dl, "\pi"] \\
\mathbf{RV}_n^{\Delta} \arrow[ur, "\pi_n"] \arrow[rr, "{\mathrm{val}_{\mathbf{RV},n}}^{\Delta}"'] & & (\mathbf{\Gamma}/\Delta)^{\infty} &
\end{tikzcd}
$$

Before we go on let us explain the link between open balls for the coarsened valuations and open balls for the original valuation.

Proposition II.2.3:
Let (K, val) be a valued field and Δ a convex subgroup of its valuation group. Let S be an $\mathcal{O}$-Swiss cheese, b an $\mathcal{O}^{\Delta}$-ball, c, $d \in K$ such that $b = \mathring{\mathcal{B}}^{\mathcal{O}^{\Delta}}_{\mathrm{val}^{\Delta}(d)}(c)$. If $b \subseteq S$, there exists $d' \in K$ such that $\mathrm{val}^{\Delta}(d') = \mathrm{val}^{\Delta}(d)$ *and* $b \subseteq \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d')}(c) \subseteq S$.

Proof. Let (g_{α}) be a cofinal (ordinal indexed) sequence in Δ. We have $b = \bigcap_{\alpha} \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha})}(c)$. Indeed, $\mathrm{val}^{\Delta}(dg_{\alpha}) = \mathrm{val}^{\Delta}(d)$ and hence $b = \mathring{\mathcal{B}}^{\mathcal{O}^{\Delta}}_{\mathrm{val}^{\Delta}(dg_{\alpha})}(c) \subseteq \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha})}(c)$. Conversely, if $x \in \bigcap_{\alpha} \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha})}(c)$, then $\mathrm{val}((x-c)/d) > \mathrm{val}(g_{\alpha})$ for all α, hence $(x-c)/d \in \mathfrak{M}^{\Delta}$.
Let b' be any $\mathcal{O}$-ball, then $b = \bigcap_{\alpha} \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha})}(c) \subseteq b'$ if and only if there exists α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \subseteq b'$ and $b \cap b' = \bigcap_{\alpha} \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha})}(c) \cap b' = \varnothing$ if and only if there exists α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \cap b' = \varnothing$. These statements still hold for Boolean combinations of balls hence there is some α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \subseteq S$. ∎

When (K, val) is a mixed characteristic valued field, the coarsened valuation we are interested in is the one associated to Δ_p the convex group generated by $\mathrm{val}(p)$ as $(K, \mathrm{val}^{\Delta_p})$ has equicharacteristic zero. We will write $\mathrm{val}_{\infty} := \mathrm{val}^{\Delta_p}$, $\mathbf{R}_{\infty} := \mathbf{R}_1^{\Delta_p}$, $\mathcal{O}_{\infty} := \mathcal{O}^{\Delta_p} = \mathcal{O}_{p^{-1}}$ and $\mathfrak{M}_{\infty} := \mathfrak{M}^{\Delta_p} = \bigcap_{n \in \mathbb{N}} p^n \mathfrak{M}$. As the coarsened field has equicharacteristic zero, all $\mathbf{RV}_n^{\Delta_p}$ are the same and we will write $\mathbf{RV}_{\infty} := K^{\star}/(1 + \mathfrak{M}_{\infty}) \cup \{0\} = \mathbf{RV}_1^{\Delta_p}$.

Remark II.2.4:
We can — and we will — identify $\mathbf{RV}_{\infty}$ (canonically) with a subgroup of $\varprojlim \mathbf{RV}_n$ and the canonical projection $K \to \mathbf{RV}_{\infty}$ then coincides with $\varprojlim \mathrm{rv}_n : K \to \varprojlim \mathbf{RV}_n$, in particular, $\mathbf{RV}_{\infty} = (\varprojlim \mathrm{rv}_n)(K)$. Similarly, $\widetilde{\mathcal{O}}^{\Delta_p}$ can be identified with a subring of $\varprojlim \mathbf{R}_n$ and $\mathbf{R}_{\infty} = \mathrm{Frac}(\widetilde{\mathcal{O}}^{\Delta_p}) \subseteq \mathrm{Frac}(\varprojlim \mathbf{R}_n) = (\varprojlim \mathbf{R}_n)[\mathrm{rv}_{\infty}(p)^{-1}]$. The inclusions are equalities if K is $\aleph_1$-saturated. In particular, $\varprojlim \mathrm{rv}_n$ is surjective.

$$\begin{tikzcd}
& \mathbf{K} \arrow[r, "\mathrm{rv}_\infty"] \arrow[dl, "\mathrm{rv}_m"'] \arrow[d, "\mathrm{rv}_n"] \arrow[dr, "\varprojlim \mathrm{rv}_n"] & \mathbf{RV}_\infty \arrow[d, hook] \\
\mathbf{RV}_m \arrow[r, "\mathrm{rv}_{m,n}"'] & \mathbf{RV}_n & \varprojlim \mathbf{RV}_n \arrow[l, "\pi_n"] \arrow[ll, bend left, "\pi_m"]
\end{tikzcd}$$

Hence (K, val_∞) is prodefinable — i.e. a prolimit of definable sets — in (K, val) with its $\mathcal{L}^{\mathbf{RV}}$-structure.
Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of a $\mathbf{K}$-enrichment of $\mathcal{L}^{\mathbf{RV}}$ with new sorts $\Sigma_{\mathbf{K}}$ and $\Sigma_{\mathbf{RV}}$ resp.. We will write still $\mathbf{K}$ for $\mathbf{K} \cup \Sigma_{\mathbf{K}}$ and $\mathbf{RV}$ for $\bigcup_n \mathbf{RV}_n \cup \Sigma_{\mathbf{RV}}$ (and rely on the context for it to make sense). Let $T \supseteq \mathrm{T}_{\mathrm{vf},0,p}$ an $\mathcal{L}$-theory. Let $\mathcal{L}^{\mathbf{RV}_\infty}$ be a copy of $\mathcal{L}^{\mathbf{RV}}$ (as $\mathcal{L}^{\mathbf{RV}_\infty}$ will only be used in equicharacteristic zero, we will only need its $\mathbf{RV}_1$, which we will denote $\mathbf{RV}_\infty$ to avoid confusion with the original $\mathbf{RV}_1$). Let $\mathcal{L}^\infty$ be $\mathcal{L}^{\mathbf{RV}_\infty} \cup \mathcal{L}|_{\mathbf{K}} \cup \mathcal{L}|_{\mathbf{RV}} \cup \{\pi_n : n \in \mathbb{N}_{>0}\}$ where π_n is a function symbol $\mathbf{RV}_n \to \mathbf{RV}_\infty$. Let T^∞ be the theory containing:

- $\mathrm{T}^\infty_{\mathrm{vf},0,0}$, i.e. the theory of equicharacteristic zero valued fields in $\mathcal{L}^{\mathbf{RV}_\infty}$;
- The translation of T into $\mathcal{L}^\infty$ by replacing rv_n by $\pi_n \circ \mathrm{rv}_\infty$.

Recall that $\mathrm{Str}(T)$ is the category of substructures of models of T. See Appendix II.B for precise definitions. The main goal of the following proposition is to show that quantifier elimination results in equicharacteristic zero can be transferred to mixed characteristic using result from Appendix II.B.

Proposition II.2.5 (Reduction to equicharacteristic zero)**:**
We can define functors $\mathfrak{C}^\infty : \mathrm{Str}(T) \to \mathrm{Str}(\mathrm{T}^\infty)$ *and* $\mathfrak{U}\mathfrak{C}^\infty : \mathrm{Str}(\mathrm{T}^\infty) \to \mathrm{Str}(T)$ *which respect cardinality up to* $\aleph_0$ *and induce an equivalence of categories between* $\mathrm{Str}(T)$ *and* $\mathrm{Str}_{\mathfrak{C}^\infty,\aleph_1}(\mathrm{T}^\infty)$*. Moreover,* $\mathfrak{C}^\infty$ *respects* $\aleph_1$*-saturated models and* $\mathfrak{U}\mathfrak{C}^\infty$ *respects models and elementary submodels and sends* $\mathbf{RV}$ *to* $\mathbf{RV}_\infty \cup \mathbf{RV}$ *(which are closed).*

Proof. Let $C \leqslant M \vDash T$ be $\mathcal{L}$-structures. Then $\mathfrak{C}^\infty(C)$ has underlying sets $\mathbf{K}(\mathfrak{C}^\infty(C)) = K(C)$, $\mathbf{RV}_\infty(\mathfrak{C}^\infty(C)) = \varprojlim \mathbf{RV}_n(C)$ and $\mathbf{RV}(\mathfrak{C}^\infty(C)) = \mathbf{RV}(C)$, keeping the same structure on $\mathbf{K}$ and $\mathbf{RV}$, defining rv_∞ to be $\varprojlim \mathrm{rv}_n$ and π_n to be the canonical projection $\mathbf{RV}_\infty \to \mathbf{RV}_n$. Now, if $f : C_1 \to C_2$ is an $\mathcal{L}$-embedding, let us write $f_\infty := \varprojlim f|_{\mathbf{RV}_n}$. By definition, we have $\pi_n \circ f_\infty = f|_{\mathbf{RV}_n} \circ \pi_n$ and by immediate diagrammatic considerations, $\mathrm{rv}_\infty \circ f|_{\mathbf{K}} = f_\infty \circ \mathrm{rv}_\infty$ and f_∞ is injective. Then, let $\mathfrak{C}^\infty(f)$ be $f|_{\mathbf{K}} \cup f_\infty \cup f|_{\mathbf{RV}}$. As f is an $\mathcal{L}$-embedding, $f|_{\mathbf{K}}$ respects the structure on $\mathbf{K}$, $f|_{\mathbf{RV}}$ respects the structure on $\mathbf{RV}$ and, as we have already seen, $\mathfrak{C}^\infty(f)$ respects rv_∞ and π_n. Hence $\mathfrak{C}^\infty(f)$ is an $\mathcal{L}^\infty$-embedding.
If $M \vDash T$ is $\aleph_1$-saturated, it follows from Remark **(II.2.4)** that $\mathfrak{C}^\infty(M) \vDash \mathrm{T}^\infty$. Beware though that $\mathfrak{C}^\infty(M)$ is never $\aleph_0$-saturated because if it were, we would find $x \neq y \in \mathbf{RV}_\infty(M_1)$ such that for all $n \in \mathbb{N}_{>0}$, $\pi_n(x) = \pi_n(y)$, contradicting the fact that $\mathbf{RV}_\infty(M_1) = \varprojlim \mathbf{RV}_n(M_1)$.
Let C be a substructure of M. We will denote i the injection. Then $\mathfrak{C}^\infty(i)$ is an embedding of $\mathfrak{C}^\infty(C)$ into $\mathfrak{C}^\infty(M)$ and $\mathfrak{C}^\infty$ is indeed a functor into $\mathrm{Str}(T)$.
The functor $\mathfrak{U}\mathfrak{C}^\infty$ is defined as the restriction to $\mathbf{K} \cup \mathbf{RV}$. It is clear that if C is an $\mathcal{L}$-structure in some model of T, then $\mathfrak{U}\mathfrak{C}^\infty \circ \mathfrak{C}^\infty(C)$ is trivially isomorphic to C. Now if D

is in $\mathrm{Str}(\mathrm{T}^\infty)$ there will be three leading term structures (and hence valuations): the one associated with the $\mathcal{L}^{\mathbf{RV}_\infty}$-structure of C (which is definable), whose valuation ring is $\mathcal{O}$, the one given by $\mathrm{rv}_n = \pi_n \circ \mathrm{rv}_\infty$ (which is definable), whose valuation ring is $\mathcal{O}_\infty$, and the one given by $\varprojlim \mathrm{rv}_n$ (which is only prodefinable), whose valuation ring is $\mathcal{O}_{p^{-1}}$. In general, we have $\mathcal{O} \subsetneq \mathcal{O}_{p^{-1}} \subsetneq \mathcal{O}_\infty$, but if $D = \mathfrak{C}^\infty(C)$ — or D embeds in some $\mathfrak{C}^\infty(C)$ — $\mathcal{O}_{p^{-1}} = \mathcal{O}_\infty$ and $\varprojlim \mathrm{rv}_n(D) = \mathrm{rv}_\infty(D)$. Hence, if C embeds in some $\mathfrak{C}^\infty(M)$ then $\mathfrak{C}^\infty \circ \mathfrak{U}\mathfrak{C}^\infty(C)$ is (naturally) isomorphic to C.
Functoriality of all the previous constructions is a (tedious but) easy verification ∎

II.3. Analytic structure

In [CL11], Cluckers and Lipshitz study valued fields with analytic structure. Let us recall some of their results. From now on, A will be a Noetherian ring separated and complete in its I-adic topology for some ideal I. Let $A\langle \overline{X}\rangle$ be the ring of power series with coefficients in A whose coefficients I-adically converge to 0. Let us also define $\mathcal{A}_{m,n} := A\langle \overline{X}\rangle[[\overline{Y}]]$ where $|\overline{X}| = m$ and $|\overline{Y}| = n$ and $\mathcal{A} := \bigcup_{m,n} \mathcal{A}_{m,n}$. Note that $\mathcal{A}$ is a separated Weierstrass system over (A, I) as in [CL11, Example 4.4.(1)]. The main example to keep in mind here will be $\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}]\langle \overline{X}\rangle[[\overline{Y}]]$ which is a separated Weierstrass system over $(\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}], p\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}])$.

Definition II.3.1 ($\mathcal{Q}$)**:**
We will extensively use a quotient symbol $\mathcal{Q} : \mathbf{K}^2 \to \mathbf{K}$ that is interpreted as $\mathcal{Q}(x, y) = x/y$, when $y \neq 0$ and $\mathcal{Q}(x, 0) = 0$.

Definition II.3.2 ($\mathcal{R}$)**:**
Let $\mathcal{R}$ be a valuation ring of K included in $\mathcal{O}$, let $\mathfrak{N}$ be its maximal ideal and $\mathrm{val}^{\mathcal{R}}$ its valuation. We have $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathcal{R} \subseteq \mathcal{O}$. Also, note that $1 + n\mathfrak{M} \subseteq 1 + n\mathfrak{N} \subseteq \mathcal{R}^\star$ and hence the valuation $\mathrm{val}^{\mathcal{R}}$ corresponding to $\mathcal{R}$ factors through rv_n, i.e. there is some function f_n such that $\mathrm{val}^{\mathcal{R}} = f_n \circ \mathrm{rv}_n$. We will also be using a new predicate $x|^{\mathcal{R}}_1 y$ on $\mathbf{RV}_1$ interpreted by $f_1(x) \leqslant f_1(y)$.

Note that $\mathcal{O}$ is the coarsening of $\mathcal{R}$ associated to the convex subgroup $\mathcal{O}^\star/\mathcal{R}^\star$ of $\mathbf{K}^\star/\mathcal{R}^\star$. Note also that $\mathcal{R}$ is then definable by the (quantifier free) formula, $\mathrm{rv}_1(1)|^{\mathcal{R}}_1 \mathrm{rv}_1(x)$. In fact the whole leading term structure associated to $\mathcal{R}$ is quantifier free interpretable in $\mathcal{L}^{\mathbf{RV}} \cup \{|^{\mathcal{R}}_1\}$.

Definition II.3.3 (Fields with separated analytic $\mathcal{A}$-structure)**:**
Let $\mathcal{L}_{\mathcal{A}}$ be the language $\mathcal{L}^{\mathbf{RV}^+}$ enriched with a symbol for each element in $\mathcal{A}$ (we will identify the elements in $\mathcal{A}$ and the corresponding symbols). For each $E \in \mathcal{A}^\star_{m,n}$ let also $E_k : \mathbf{RV}_k^{m+n} \to \mathbf{RV}_k$ be a new symbol and $\mathcal{L}_{\mathcal{A},\mathcal{Q}} := \mathcal{L}_{\mathcal{A}} \cup \{|^{\mathcal{R}}_1, \mathcal{Q}\} \cup \{E_k : E \in \mathcal{A}^\star_{m,n}, m, n, k \in \mathbb{N}\}$. The theory $\mathrm{T}_{\mathcal{A}}$ of fields with separated analytic $\mathcal{A}$-structure consists of the following:

(i) T_{vf}*;*

*(ii) $\mathcal{Q}$ is interpreted as in Definition **(II.3.1)**;*

(iii) $|^{\mathcal{R}}_1$ comes from a valuation subring $\mathcal{R} \subseteq \mathcal{O}$ with fraction field $\mathbf{K}$;

(iv) *Each symbol $f \in \mathcal{A}_{m,n}$ is interpreted as a function $\mathcal{R}^m \times \mathfrak{N}^n \to \mathcal{R}$ (the symbols will be interpreted as 0 outside $\mathcal{R}^m \times \mathfrak{N}^n$);*

(v) *The interpretations $i_{m,n} : \mathcal{A}_{m,n} \to \mathcal{R}^{\mathcal{R}^m \times \mathfrak{N}^n}$ are morphisms of the inductive system of rings $\bigcup_{m,n} \mathcal{A}_{m,n}$ to $\bigcup_{m,n} \mathcal{R}^{\mathcal{R}^m \times \mathfrak{N}^n}$, where the inclusions are the obvious ones.*

(vi) *$i_{0,0}(I) \subseteq \mathfrak{N}$;*

(vii) *$i_{m,n}(X_i)$ is the i-th coordinate function and $i_{m,n}(Y_j)$ is the $(m+j)$-th coordinate function;*

(viii) *For every $E \in \mathcal{A}^\star_{m,n}$, E_k is interpreted as the function induced by E on $\mathbf{RV}_k$ (we will see in Corollary **(II.3.19)** that E does induce a well defined function on $\mathbf{RV}_k$).*

To specify the characteristic we will write $\mathrm{T}_{\mathcal{A},0,0}$ or $\mathrm{T}_{\mathcal{A},0,p}$.

Remark II.3.4:

- These axioms imply a certain number of properties that it seems reasonable to require. First (iv) implies that every constant in $A = \mathcal{A}_{0,0}$ is interpreted in $\mathcal{R}$. By (v) and (vii) polynomials in $\mathcal{A}$ are interpreted as polynomials. And (v) implies that any ring equality between functions in $\mathcal{A}_{m,n}$ for some m and n are also true in models of $\mathrm{T}_{\mathcal{A}}$. Using Weierstrass division (see Proposition **(II.3.9)**) one can also show that compositional identities in $\mathcal{A}$ are also true in models of $\mathrm{T}_{\mathcal{A}}$.

- We have the analytic structure over a smaller valuation ring in order to be able to coarsen the valuation while staying in our setting of analytic structures.

From now on, we will write $\langle C\rangle := \langle C\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q}}}$ and $C\langle\overline{c}\rangle := C\langle\overline{c}\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q}}}$ for the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structures generated by C and $C\overline{c}$ (cf. Definition **(II.A.12)**).
We could be working in a larger context here. What we really need in the proof is not that $\mathcal{A}$ is a separated Weierstrass system, as in [CL11], but the consequences of this fact, namely: Henselianity, (uniform) Weierstrass preparation, differentiability of the new function symbols and extension of the analytic structure to algebraic extensions. One could give an axiomatic treatment along those lines, but to simplify the exposition, we restrict to a more concrete case.
Also note that if $\mathcal{A}$ is not countable we may now work with an uncountable language
Let us now describe all the nice properties of models of $\mathrm{T}_{\mathcal{A}}$.

Proposition II.3.5:
Let $M \models \mathrm{T}_{\mathcal{A}}$, then M is Henselian.

Proof. If $\mathcal{O} = \mathcal{R}$, this is proved exactly as in [LR99, Lemma 3.3]. The case $\mathcal{R} \neq \mathcal{O}$ follows as coarsening preserves Henselianity. ∎

Remark II.3.6:
As $\mathrm{T}_{\mathcal{A}}$ implies $\mathrm{T}_{\mathrm{Hen}}$, by resplendent elimination of quantifiers in $\mathrm{T}_{\mathrm{Hen}}$ (cf. Theorem **(II.1.4)**), any $\mathcal{L}_{\mathcal{A},\mathcal{Q}} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$-formula is equivalent modulo $\mathrm{T}_{\mathcal{A}}$ to a $\mathbf{K}$-quantifier free formula.

Let us now (re)prove a well-known result from papers by Cluckers, Lipshitz and Robinson. There are two main reasons for which I reprove this result. The first reason is that although the proof I give here is very close to the classical Denef-van den Dries proof as explained in [LR05, Theorem 4.2], the proof there only shows quantifier elimination for algebraically closed fields with analytic structures over $(\mathbb{Z}, 0)$. The second reason is to make sure that $\mathcal{O} \neq \mathcal{R}$ does not interfere.

Theorem II.3.7:
$\mathrm{T}_{\mathcal{A}}$ *eliminates* **K**-*quantifiers resplendently.*

The proof of this theorem will need many definitions and properties that will only be used here and I will introduce them now.
For all $m, n \in \mathbb{N}$, we define $J_{m,n}$ to be the ideal $\{\sum_{\mu,\nu} a_{\mu,\nu}\overline{X}^{\mu}\overline{Y}^{\nu} \in \mathcal{A}_{m,n} : a_{\mu,\nu} \in I\}$ of $\mathcal{A}_{m,n}$. Most of the time we will only write J and rely on context for the indices. We will also write $\overline{X}_{\neq n}$ for the tuple $\overline{X}$ without its n-th component.

Definition II.3.8 (Regularity)**:**
Let $f \in \mathcal{A}_{m_0,n_0}$, $m < m_0$, $n < n_0$. *We say that:*

(i) $f = \sum_i a_i(\overline{X}_{\neq m}, \overline{Y})X_m^i$ *is regular in* X_m *of degree* d *if* f *is congruent to a monic polynomial in* X_m *of degree* d *modulo* $J + (\overline{Y})$;

(ii) $f = \sum_i a_i(\overline{X}, \overline{Y}_{\neq n})Y_n^i$ *is regular in* Y_n *of degree* d *if* f *is congruent to* Y_n^d *modulo* $J + (\overline{Y}_{\neq n}) + (Y_n^{d+1})$.

If we do not want to specify the degree, we will just say that f is regular in X_m (resp. Y_n).

Proposition II.3.9 (Weierstrass division and preparation)**:**
Let $f, g \in \mathcal{A}_{m_0,n_0}$ *and suppose* f *is regular in* X_m *(resp. in* Y_n*) of degree* d*, then there exists unique* $q \in \mathcal{A}_{m,n}$ *and* $r \in A\langle\overline{X}_{\neq m}\rangle[[\overline{Y}]][X_m]$ *(resp.* $r \in A\langle\overline{X}\rangle[[\overline{Y}_{\neq n}]][Y_n]$*) of degree strictly lower than* d *such that* $g = qf + r$.
Moreover, there exists unique $P \in A\langle\overline{X}_{\neq m}\rangle[[\overline{Y}]][X_m]$ *(resp.* $P \in A\langle\overline{X}\rangle[[\overline{Y}_{\neq n}]][Y_n]$*) regular in* X_m *(resp. in* Y_n*) of degree at most* d *and* $u \in \mathcal{A}_{m,n}^{\star}$ *such that* $f = uP$.

Proof. See [LR05, Corollary 3.3]. ■

We will be ordering multi-indices μ of the same length by lexicographic order and we write $|\mu| = \sum_i \mu_i$.

Definition II.3.10 (Preregularity)**:**
Let $f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2, \overline{Y}_2)\overline{X}_1^{\mu}\overline{Y}_1^{\nu} \in \mathcal{A}_{m_1+m_2,n_1+n_2}$. *We say that* f *is preregular in* $(\overline{X}_1, \overline{Y}_1)$ *of degree* (μ_0, ν_0, d) *when:*

(i) $f_{\mu_0,\nu_0} = 1$;

(ii) For all μ*,and* ν *such that* $|\mu| + |\nu| \geqslant d$, $f_{\mu,\nu} \in J + (\overline{Y}_2)$;

(iii) For all $\nu < \nu_0$ *and for all* μ, $f_{\mu,\nu} \in J + (\overline{Y}_2)$;

(iv) For all $\mu > \mu_0$, $f_{\mu,\nu_0} \in J + (\overline{Y}_2)$.

Remark II.3.11:
Note that if $f = \sum_\nu f_\nu(\overline{X})\overline{Y}^\nu$ is preregular in $(\overline{X},\overline{Y})$ of degree (μ_0,ν_0,d) then f_{ν_0} is preregular in $\overline{X}$ of degree $(\mu_0,0,d)$.

Let $T_d(\overline{X}) := (X_0 + X_{m-1}^{d^{m-1}},\dots,X_i + X_{m-1}^{d^{m-1-i}},\dots,X_{m-2}+X_{m-1}^d, X_{m-1})$ where $m = |\overline{X}|$. We call T_d a Weierstrass change of variables. Note that Weierstrass changes of variables are bijective.

Proposition II.3.12:
Let $f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2,\overline{Y}_2)\overline{X}_1^\mu\overline{Y}_1^\nu \in \mathcal{A}_{m_1+m_2,n_1+n_2}$. Then:

(i) If f is preregular in $(\overline{X}_1,\overline{Y}_1)$ of degree $(\mu_0,0,d)$ then $f(T_d(\overline{X}_1),\overline{X}_2,\overline{Y})$ is regular in X_{1,m_1-1}.

(ii) If f is preregular in $(\overline{X}_1,\overline{Y}_1)$ of degree $(0,\nu_0,d)$ then $f(\overline{X},T_d(\overline{Y}_1),\overline{Y}_2)$ is regular in Y_{1,n_1-1}.

Proof. Let $m = m_1 - 1$ and $n = n_1 - 1$. First assume f is preregular in $(\overline{X}_1,\overline{Y}_1)$ of degree $(\mu_0,0,d)$, then

$$f \equiv \sum_{\mu<\mu_0,|\mu|<d} f_{\mu,0}\overline{X}_1^\mu \mod J + (\overline{Y_2}) + (\overline{Y_1}).$$

Furthermore, $T_d(\overline{X}_1)^\mu = (\prod_{i=0}^{m-1}(X_{1,i}+X_{1,m}^{d^{m-i}})^{\mu_i})X_{1,m}^{\mu_m}$ is a sum of monomials whose highest degree monomial only contains the variable $X_{1,m}$ and has degree $\sum_{i=0}^m d^{m-i}\mu_i$. It now suffices to show that this degree is maximal when $\mu = \mu_0$, but that is exactly what is shown in the following claim.

Claim II.3.13: *Let μ and ν be two multi-indices such that $\mu < \nu$ and $|\mu| < d$ then*

$$\sum_{i=0}^m d^{m-i}\mu_i < \sum_{i=0}^m d^{m-i}\nu_i.$$

Proof. Let i_0 be minimal such that $\mu_i < \nu_i$. Then for all $j < i_0$, $\mu_j = \nu_j$. Moreover,

$$\begin{array}{rcl} \sum_{i=i_0+1}^m d^{m-i}\mu_i & \leqslant & \sum_{i=i_0+1}^m d^{m-i}(d-1) \\ & = & d^{m-i_0}-1 \\ & < & d^{m-i_0}, \end{array}$$

hence

$$\begin{array}{rcl} \sum_{i=0}^m d^{m-i}\mu_i & < & \sum_{i=0}^{i_0-1} d^{m-i}\mu_i + d^{m-i_0}\mu_{i_0} + d^{m-i_0} \\ & \leqslant & \sum_{i=0}^{i_0-1} d^{m-i}\mu_i + d^{m-i_0}\nu_{i_0} \\ & \leqslant & \sum_{i=0}^m d^{m-i}\nu_i \end{array}$$

and we have proved our claim. ♦

Let us now suppose that f is preregular in $(\overline{X}_1,\overline{Y}_1)$ of degree $(0,\nu_0,d)$. Then

$$f \equiv \overline{Y}_1^{\nu_0} + \sum_{\nu>\nu_0,\mu} f_{\mu,\nu}\overline{X}_1^\mu\overline{Y}_1^\nu \mod J + (\overline{Y_2}).$$

Now,

$$T_d(\overline{Y}_1)^\nu = (\prod_{i=0}^{n-1}(Y_{1,i} + Y_{1,n}^{d^{n-i}})^{\nu_i})Y_{1,n}^{\nu_n} \equiv Y_{1,n}^{\sum_{i=0}^n d^{n-i}\nu_i} \mod J + (\overline{Y_2}) + (\overline{Y_{1\neq n}})$$

and we conclude again by Claim **(II.3.13)**. ∎

Proposition II.3.14 (Bound on the degree of preregularity)**:**
Let

$$f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2, \overline{Y}_2)\overline{X}_1^\mu \overline{Y}_1^\nu \in \mathcal{A}_{m_1+m_2,n_1+n_2}.$$

There exists d *such that for any* (μ,ν) *with* $|\mu| + |\nu| < d$*, there exists* $g_{\mu,\nu} \in \mathcal{A}_{m_1+m_3,n_1+n_3}$ *preregular in* $(\overline{X}_1, \overline{Y}_1)$ *of degree* (μ,ν,d) *and* $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$*-terms* $\overline{u}_{\mu,\nu}$ *and* $\overline{s}_{\mu,\nu}$ *such that for all* $M \vDash \mathrm{T}_{\mathcal{A}}$ *and every* $\overline{a} \in \mathcal{R}(M)$ *and* $\overline{b} \in \mathfrak{N}(M)$*, if* $f(\overline{X}_1, \overline{a}, \overline{Y}_1, \overline{b})$ *is not the zero function, then there exists* (μ_0,ν_0) *with* $|\mu_0| + |\nu_0| \leqslant d$ *and*

$$f(\overline{X}_1, \overline{a}, \overline{Y}_1, \overline{b}) = f_{\mu_0,\nu_0}(\overline{a}, \overline{b}) g_{\mu_0,\nu_0}(\overline{X}_1, \overline{u}_{\mu_0,\nu_0}(\overline{a}, \overline{b}), \overline{Y}_1, \overline{s}_{\mu_0,\nu_0}(\overline{a}, \overline{b})).$$

Proof. This follows from the strong Noetherian property [CL11, Theorem 4.2.15 and Remark 4.2.16] as in [LR05, Corollary 3.8]. ∎

The natural setting to prove this quantifier elimination is to consider a language with three sorts $\mathcal{R}$, $\mathfrak{N}$ and **RV** and then transport this elimination to the language $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ we have been considering all along. But to avoid introducing yet another language we will prove the result directly in $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ at the cost of a certain heaviness of the proof.
A **K**-quantifier free $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ will be said to be well-formed if $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are **K**-variables and $\overline{R}$ are **RV**-variables, symbols of functions from $\mathcal{A}$ are never applied to anything but variables and $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ implies that $\bigwedge_i \mathrm{val}^{\mathcal{R}}(X_i) \geqslant 0$, $\bigwedge_i \mathrm{val}^{\mathcal{R}}(Z_i) \geqslant 0$ and $\bigwedge_i \mathrm{val}^{\mathcal{R}}(Y_i) > 0$. The $(\overline{X}, \overline{Y})$-rank of φ is the tuple $(|\overline{X}|, |\overline{Y}|)$. We order ranks lexicographically.

Lemma II.3.15:
Let $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ *be a well-formed* **K***-quantifier free* $\mathcal{L}_{\mathcal{A}}$*-formula. Then there exists a finite set of well-formed* **K***-quantifier free* $\mathcal{L}_{\mathcal{A}}$*-formulae* $\varphi_i(\overline{X}_i, \overline{Y}_i, \overline{Z}_i, \overline{R})$ *of* $(\overline{X}_i, \overline{Y}_i)$*-rank strictly smaller than the* $(\overline{X}, \overline{Y})$*-rank of* φ *and* $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$*-terms* $\overline{u}_i(\overline{Z})$ *such that*

$$\mathrm{T}_{\mathcal{A}} \vDash \exists \overline{X} \exists \overline{Y} \varphi \iff \bigvee_i \exists \overline{X}_i \exists \overline{Y}_i \varphi_i(\overline{X}_i, \overline{Y}_i, \overline{u}_i(\overline{Z}), \overline{R}).$$

Proof. Let $m := |X|$ and $n := |Y|$. As polynomials with variables in $\mathcal{R}$ are in fact elements of $\mathcal{A}$ and $\mathcal{A}$ is closed under composition (for the $\mathcal{R}$-variables), we may assumes that any $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}$-term appearing in φ is an element of $\mathcal{A}$. Let $f_i(\overline{X}, \overline{Y}, \overline{Z})$ be the $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}$-terms appearing in φ. Splitting φ into different cases, we may assume that whenever a variable S appears as an $\mathfrak{N}$-variable of an f_i then φ implies that $\mathrm{val}^{\mathcal{R}}(S) > 0$ (in the part of the disjunction where $\mathrm{val}^{\mathcal{R}}(S) \leqslant 0$ we replace this f_i by zero).
If an X_i appears as an $\mathfrak{N}$ variable in an f_i, then φ implies that $\mathrm{val}^{\mathcal{R}}(X_i) > 0$ and hence we can safely rename this X_i into Y_n and we obtain an equivalent formula of lower rank. If Y_i

appears as an $\mathcal{R}$-variable in an f_i, we can change this f_i so that Y_i appears as an $\mathfrak{N}$-variable. Thus we may assume that the X_i only appear as $\mathcal{R}$-variables and the Y_i as $\mathfrak{N}$-variables. Similarly adding new Z_j variables, we may assume that each Z_j appears only once (and in the end we can put the old variables back in) and that φ implies that $\mathrm{val}^{\mathcal{R}}(Z_j) > 0$ if it is an $\mathfrak{N}$-variable.

Applying Proposition **(II.3.14)** to each of the $f_i(\overline{X},\overline{Y},\overline{Z}) = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{Z})\overline{X}^{\nu}\overline{Y}^{\mu}$, we find d, $g_{i,\mu,\nu}$ and $u_{i,\mu,\nu}(\overline{Z})$ such that $g_{i,\mu,\nu}$ is preregular in $(\overline{X},\overline{Y})$ of degree (μ,ν,d) and for every $M \models \mathrm{T}_{\mathcal{A}}$ and $\overline{a} \in M$, if $f_i(\overline{X},\overline{Y},\overline{a})$ is not the zero function, then there exists (μ,ν) such that $|\mu| + |\nu| < d$ and $f_i(\overline{X},\overline{Y},\overline{a}) = f_{i,\mu,\nu}(\overline{a})g_{i,\mu,\nu}(\overline{X},\overline{Y},\overline{u}_{i,\mu,\nu}(\overline{a}))$. Splitting the formula into the different cases, we may assume that for each i, there are μ_i and ν_i such that $f_i(\overline{X},\overline{Y},\overline{a}) = f_{i,\mu_i,\nu_i}(\overline{a})g_{i,\mu_i,\nu_i}(\overline{X},\overline{Y},\overline{u}_i(\overline{a}))$ (in the case where no such μ_i and ν_i exist, then we can replace f_i by 0). Let us consider that every argument of a $g_{i,\mu,\nu}$ that is not in $\overline{X}$ or $\overline{Y}$ is named by a new variable T_j (and for each of these new T_j we add to the formula $\mathrm{val}^{\mathcal{R}}(T_j) \geqslant 0$ if T_j is an $\mathcal{R}$-argument of $g_{i,\nu,\mu}$ or $\mathrm{val}^{\mathcal{R}}(T_j) > 0$ if it is an $\mathfrak{N}$-argument). Let us write $g_{i,\mu_i,\nu_i} = \sum_\nu g_{i,\nu}\overline{Y}^{\nu}$. Note that g_{i,ν_i} is preregular in $\overline{X}$ of degree $(\mu_i,0,d)$. We can split the formula some more (and still call it φ) so that for each i, one of the the two conditions $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) > 0$ or $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) = 0$ holds.

If a condition $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) > 0$ occurs, let us add $\mathrm{val}^{\mathcal{R}}(Y_n) > 0 \wedge g_{i,\nu_i} - Y_n = 0$ to the formula. By Proposition **(II.3.12)**, after a Weierstrass change of variable on the $\overline{X}$, we may assume that $g_{i,\nu_i} - Y_n$ is regular in X_{m-1}. By Weierstrass division, we can replace every f_j by a term polynomial in X_{m-1} and by Weierstrass preparation we can replace the equality $g_{i,\nu_i} - Y_n = 0$ by the equality of a term polynomial in X_{m-1} to 0. In the resulting formula, no $f \in \mathcal{A}$ is ever applied to a term containing X_{m-1} and we can apply Remark **(II.3.6)** to the formula where every $f \in \mathcal{A}$ is replaced by a new variable S_f to obtain a **K**-quantifier free formula $\psi(\overline{X}_{\neq m-1},\overline{Y},\overline{Z},\overline{T},\overline{S},\overline{R})$ such that

$$\mathrm{T}_{\mathcal{A}} \models \exists X_{m-1}\varphi \iff \psi(\overline{X}_{\neq m-1},\overline{Y},\overline{Z},\overline{u}(\overline{Z}),\overline{f}(\overline{X}_{\neq m-1},\overline{Y},\overline{Z}),\overline{R})$$

and $\psi(\overline{X}_{\neq m-1},\overline{Y},\overline{Z},\overline{T},\overline{f}(\overline{X}_{\neq m-1},\overline{Y},\overline{Z}),\overline{R})$ is well-formed of $(\overline{X},\overline{Y})$-rank $(m-1,n+1)$.

If for all i we have $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) = 0$, we add $\mathrm{val}^{\mathcal{R}}(X_m) \geqslant 0 \wedge X_m \prod_i g_{i,\nu_i} - 1 = 0$ to the formula. As every g_{i,ν_i} is preregular in $\overline{X}$ of degree $(\mu_i,0,d)$, $g = X_m \prod_i g_{i,\nu_i} - 1$ is preregular in $\overline{X}$ of degree $(\mu,0,d')$ for some μ and d'. After a Weierstrass change of variables in $\overline{X}$, we may assume that g and each g_{i,ν_i} are in fact regular in X_m. Hence by Weierstrass preparation we may replace g in $g = 0$ by a term polynomial in X_m. Furthermore, by Remark **(II.1.5)** the f_i appear as $\mathrm{rv}_{n_i}(f_i)$ for some n_i in the formula. Replacing f_i by $f_{\mu_i,\nu_i}g_{i,\mu_i,\nu_i}$, we only have to show that $\mathrm{rv}_{n_i}(g_{i,\mu_i,\nu_i})$ can be replaced by a term polynomial in Y_{n-1} (and X_n). Let $h_i = X_n(\prod_{j\neq i} g_{j,\nu_j})g_{i,\nu_i,\mu_i} = \sum_\nu h_{i,\nu}Y^\nu$. Then $h_{i,\nu_i} = X_n \prod_i g_{i,\nu_i} = 1$ and if $\nu < \nu_i$, $h_{i,\nu} = X_n(\prod_{j\neq i} g_{j,\nu_j})g_{i,\nu} \equiv 0 \mod J + (Z_j : Z_j$ is an $\mathfrak{N}$-argument). Hence h_i is preregular in $(\overline{X},\overline{Y})$ of degree $(0,\nu_i,d)$. After a Weierstrass change of variables of the $\overline{Y}$, we may assume that h_i is in fact regular in Y_{n-1}.

Note that $\mathrm{rv}_{n_i}(g_{i,\nu_i,\mu_i}) = \mathrm{rv}_{n_i}(X_n)^{-1}\prod_{j\neq i}\mathrm{rv}_{n_i}(g_{i,\nu_i})^{-1}\mathrm{rv}_{n_i}(h_i)$. By Weierstrass preparation we can replace h_i by the product of a unit and p_i a polynomial in Y_{n-1}. As we have included the trace of units on the $\mathbf{RV}_n$ in our language, the unit is taken care of and by Weierstrass division by g, we can replace each coefficients in the p_i and each of the g_{i,ν_i} by a term polynomial in X_n. Note that because we allow quantification on **RV**, although the language does not contain the inverse on $\mathbf{RV}$, the inverses can be taken care of by quantifying over $\mathbf{RV}$. Hence we obtain a formula where X_n and Y_{n-1} only occur polynomially and we can proceed as in the previous case to eliminate them. ∎

Corollary II.3.16:
Let $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ be a well-formed $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathcal{A}}$-formula. Then there exists an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula $\psi(\overline{Z}, \overline{R})$ such that $\mathrm{T}_{\mathcal{A}} \vDash \exists \overline{X} \exists \overline{Y} \varphi \iff \psi$.

Proof. This follows from Lemma **(II.3.15)** and an immediate induction. ∎

Proof (Theorem **(II.3.7)**). Resplendence comes for free (see Proposition **(II.A.9)**). Hence, it suffices to show that if $\varphi(X, \overline{Z})$ is a quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula, then there exists a quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula $\psi(\overline{Z})$ such that $\mathrm{T}_{\mathcal{A}} \vDash \exists X \varphi \iff \psi$. First, splitting the formula φ, we can assume that for any of its variables S, φ implies either $\mathrm{val}^{\mathcal{R}}(S) \geqslant 0$ or $\mathrm{val}^{\mathcal{R}}(S) < 0$, in the second case replacing S by S^{-1} we also have $\mathrm{val}^{\mathcal{R}}(S) > 0$. We also add one variable X_i (resp. Y_i) per $\mathcal{R}$-argument (resp. $\mathfrak{N}$-argument) of any $f \in \mathcal{A}$ applied to some non variable term u and we add the corresponding equality $X_i = u$ (resp. $Y_i = u$) and the corresponding inequalities $\mathrm{val}^{\mathcal{R}}(X_i) \geqslant 0$ (resp. $\mathrm{val}^{\mathcal{R}}(Y_i) > 0$) and quantify existentially over this variable. Splitting the formula further — whether denominators in occurrences of $\mathcal{Q}$ are zero or not — we can transform φ such that it contains no $\mathcal{Q}$. Now $\exists X \varphi$ is equivalent to a disjunction of formulas $\exists \overline{X} \exists \overline{Y} \psi$ where ψ is well-formed and we conclude by applying Corollary **(II.3.16)**.
This concludes the proof of Theorem **(II.3.7)**. ∎

Let us now show that functions from $\mathcal{A}$ have nice differential properties.

Definition II.3.17:
Let K be a valued field and $f : K^n \to K$. We say that f is differentiable at $\overline{a} \in K^n$ if there exists $\overline{d} \in K^n$ and ξ and $\gamma \in \mathrm{val}(\mathbf{K}^\star)$ such that for all $\overline{\varepsilon} \in \mathring{\mathcal{B}}_\xi(\overline{a})$,

$$\mathrm{val}(f(\overline{a} + \overline{\varepsilon}) - f(\overline{a}) - \overline{d} \cdot \overline{\varepsilon}) \geqslant 2\mathrm{val}(\overline{\varepsilon}) + \gamma.$$

There is a unique such $\overline{d}$ and we will denote it $\mathrm{d}f_{\overline{a}}$. The d_i are usually called the derivatives of f at $\overline{a}$. We will denote them $\partial f / \partial x_i(\overline{a})$.

Proposition II.3.18:
Let $M \vDash \mathrm{T}_{\mathcal{A}}$ and $f \in \mathcal{A}_{m,n}$ for some m and n. Then for all $i < m+n$ there is $g_i \in \mathcal{A}_{m,n}$ such that for all $\overline{a} \in K^{m+n}$, f is differentiable at $\overline{a}$ and $\partial f / \partial x_i(\overline{a}) = g_i(\overline{a})$.

Proof. If $\overline{a} \notin \mathcal{R}^m \times \mathfrak{N}^n$ then f is equal to 0 on $\mathring{\mathcal{B}}_0(\overline{a})$ and the statement is trivial. If not, as $f \in A\langle X\rangle[[\overline{Y}]]$, it has a (formal) Taylor development which implies differentiability of f in $\mathbf{K}(M)$ at $\overline{a}$. ∎

Corollary II.3.19:
Let $M \vDash \mathrm{T}_{\mathcal{A}}$, $E(\overline{x}) \in \mathcal{A}_{m,n}$ and $S \subseteq \mathbf{K}(M)^{m+n}$. If, for all $\overline{x} \in S$, $\mathrm{val}(E(\overline{x})) = 0$ then, for all $\overline{x} \in S$, $\mathrm{rv}_n(E(\overline{x}))$ only depends on $\mathrm{res}_n(\overline{x})$.

In particular if $E \in \mathcal{A}^{\star}_{m,n}$, then for all $\overline{x} \in \mathcal{R}^m \times \mathfrak{N}^n$, $\mathrm{val}^{\mathcal{R}}(E(\overline{x})) = 0$ and hence $\mathrm{val}(E(\overline{x})) = 0$ and thus $\mathrm{rv}_n(E(\overline{x}))$ is a function of $\mathrm{res}_n(\overline{x})$ which is a function of $\mathrm{rv}_n(\overline{x})$. Outside of $\mathcal{R}^m \times \mathfrak{N}^n$, $\mathrm{rv}_n(E(\overline{x}))$ is constant equal to 0 and hence it is also a function of $\mathrm{rv}_n(\overline{x})$. Hence, as announced earlier, E does induce a function on $\mathbf{RV}_k$ for any k.

Proof (Corollary **(II.3.19)**). Any element with the same res_n residue as $\overline{x}$ is of the form $\overline{x}+n\overline{m}$ for some $\overline{m} \in \mathfrak{M}$. By Proposition **(II.3.18)**, $E(\overline{x}+n\overline{m}) = E(\overline{x}) + \overline{G}(\overline{x}) \cdot (n\overline{m}) + H(\overline{x}, n\overline{m})$ where $\overline{G}(\overline{x}) \in \mathcal{R} \subseteq \mathcal{O}$ and $\mathrm{val}(H(\overline{x}, n\overline{m})) \geqslant 2\mathrm{val}(n\overline{m}) > \mathrm{val}(n)$, hence $\mathrm{res}_n(E(\overline{x}+n\overline{m})) = \mathrm{res}_n(E(\overline{x}))$. As for all $\overline{z} \in S$, $\mathrm{val}(E(\overline{z})) = 0$, $\mathrm{rv}_n(E(\overline{z})) = \mathrm{res}_n(E(\overline{z}))$ and we have the expected result. ■

Recall that we denote by $\mathcal{SC}^{\mathcal{R}}(C)$ the set of all quantifier free $\mathcal{L}_{\mathrm{div}}(C)$-definable sets (where $x|y$ is interpreted by $\mathrm{val}^{\mathcal{R}}(x) \leqslant \mathrm{val}^{\mathcal{R}}(y)$).

Definition II.3.20 (Strong unit):
Let $M \vDash \mathrm{T}_{\mathcal{A}}$, $C = \mathbf{K}(\langle C\rangle)$ and $S \in \mathcal{SC}^{\mathcal{R}}(C)$. We say that an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term $E : \mathbf{K} \to \mathbf{K}$ is a strong unit on S if for any open $\mathcal{O}$-ball $b := \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c) \subseteq S$, there exists $\overline{a}$, $e \in C\langle cd\rangle$ and $F(t,\overline{z}) \in \mathcal{A}$ such that $e \neq 0$ and for all $x \in b$,

$$\mathrm{val}(F((x-c)/d, \overline{a})) = 0$$

and

$$E(x) = eF((x-c)/d, \overline{a}).$$

It is not quite clear that being a strong unit is a first order property but if M is taken saturated enough — i.e. at least $(|\mathcal{A}|+|C|)^{+}$-saturated — if E is a strong unit on S then, by compactness, there exist a tuple $\overline{a}(y,z)$ of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms, a finite number of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms $e_i(y,z)$ and $F_i[t,\overline{u}] \in \mathcal{A}$ such that for all balls $b = \mathring{\mathcal{B}}_{\mathrm{val}(d)}(c) \subseteq S$, there is an i such that for all $x \in b$,

$$E(x) = e_i(c,d)F_i((x-c)/d, \overline{a}(c,d))$$

and

$$F_i((x-c)/d, \overline{a}(c,d)) \in \mathcal{O}^{\star}.$$

Hence if E is a strong unit on S there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C)$-formula that says so. If E and S are defined using some parameters $\overline{y}$ and for all $\overline{y}$ in some definable set Y, $E = E_{\overline{y}}$ is a strong unit on $S = S_{\overline{y}}$ then we can choose this formula uniformly in $\overline{y}$.
We will say that E is an $\mathcal{R}$-strong unit on S if it verifies all the requirements of a strong unit, where all references to $\mathcal{O}$ are replaced by references to $\mathcal{R}$ (and references to $\mathcal{R}$ remain the same).

Proposition II.3.21:
If E is an $\mathcal{R}$-strong unit on S then it is also a strong unit on S.

Proof. If $b \subseteq S$ is an $\mathcal{O}$-ball, then by Proposition **(II.2.3)** there exists d and c such that $b = \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c) \subseteq \mathring{\mathcal{B}}^{\mathcal{R}}_{\mathrm{val}^{\mathcal{R}}(d)}(c) \subseteq S$. But E being a strong unit on S for $\mathcal{R}$, it has the expected form on $\mathring{\mathcal{B}}^{\mathcal{R}}_{\mathrm{val}^{\mathcal{R}}(d)}(c)$ and hence also on $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c)$. ■

Definition II.3.22 (Weierstrass preparation for terms)**:**
Let M be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure, $C = \mathbf{K}(\langle C\rangle) \subseteq M$, $t : \mathbf{K} \to \mathbf{K}$ an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term and $S \in \mathcal{SC}^{\mathcal{R}}(C)$. We say that t has a Weierstrass preparation on S if there exists an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term E that is a strong unit on S and a rational function $R \in C(X)$ with no poles in $S(\overline{\mathbf{K}(M)}^{\mathrm{alg}})$ such that for all $x \in S$, $t(x) = E(x)R(x)$.
The structure M has a Weierstrass preparation if for any $C = \mathbf{K}(\langle C\rangle)$ and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms t and $u : \mathcal{R} \to K$ we have:

(i) There exists a finite number of $S_i \in \mathcal{SC}^{\mathcal{R}}(C)$ that cover $\mathcal{R}$ such that t has a Weierstrass preparation on each of the S_i.

(ii) If t and u have a Weierstrass preparation on some open ball b, and for all $x \in b$, $\mathrm{val}(t(x)) \geqslant \mathrm{val}(u(x))$, then $t + u$ also has a Weierstrass preparation on b.

Remark II.3.23:

1. An immediate consequence of Weierstrass preparation is that all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-terms in one variable have only finitely many isolated zeroes. Indeed a zero of t is the zero of one of the R_i appearing in its Weierstrass preparation. That zero is isolated if R_i is non-zero or the corresponding S_i is discrete, i.e. is a finite set. In particular, let $\overline{m}$ be the parameters of t, then any isolated zero of t is in the algebraic closure (in ACVF) of $\mathbf{K}(\langle\overline{m}\rangle)$. As the algebraic closure in ACVF coincides with the field theoretic algebraic closure, any isolated zero of t is in fact also the zero of a polynomial (with coefficients in $\mathbf{K}(\langle\overline{m}\rangle)$).

2. As for strong units, for each choice of term $t_{\overline{y}}$ (with parameters $\overline{y}$), there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(\overline{y})$-formula that states that (i) holds for $t_{\overline{y}}$ in M and we can choose this formula to be uniform in $\overline{y}$. For each choice of terms t, u and formula defining S, there also is a (uniform) formula saying that (ii) holds for t, u and b in M.

Proposition II.3.24:
Any $M \vDash \mathrm{T}_{\mathcal{A}}$ has Weierstrass preparation.

Proof. If $\mathcal{R} = \mathcal{O}$, then the proposition is shown in [CL11, Theorem 5.5.3] and (ii) — called from now on invariance under addition — is clear from the proof given there. The one difference in the Weierstrass preparation is that in [CL11], there is a finite set of points algebraic over the parameters where the behavior of the term is unknown. But this finite set can be replaced by discrete S_i and as these exceptional points are common zeroes of terms u and v such that $Q(u, v)$ is a subterm of t, it suffices to replace $Q(u, v)$ by 0 and apply the theorem to the new term to obtain the Weierstrass preparation also on the discrete S_i. The fact that the strong units in [CL11] have the proper form on open balls follows, for example, from the proof of [CL11, Lemma 6.3.12].
If $\mathcal{R} \neq \mathcal{O}$, the proposition follows from the $\mathcal{O} = \mathcal{R}$ case and Proposition **(II.3.21)**. ∎

Remark II.3.25:

1. Let $t_{\overline{y}}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-term with parameters $\overline{y}$. As shown in Remark **II.3.23.2**, there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula θ that states that Weierstrass preparation holds for $t_{\overline{y}}$ in models of

T. More explicitly, there are finitely many choices of S_i^k, E_i^k and R_i^k (with parameters $\overline{u}(\overline{y})$ where $\overline{u}$ are $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-terms) such that for each $\overline{y}$ there is a k such that the S_i^k, E_i^k and R_i^k work for $t_{\overline{y}}$. As $\mathrm{T}_{\mathcal{A}}$ eliminates $\mathbf{K}$-quantifiers, for each k there is a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ formula $\theta_k(\overline{y})$ that is true when the k-th choice works for t (and not the ones before). Hence taking $S_{i,k}$ to be $S_i^k \wedge \theta_k$, we could suppose that Weierstrass preparation for terms is uniform, but we will not be using that fact.

2. The converse is also true, i.e. the proof of Proposition **(II.5.3)** can be adapted to show that uniform Weierstrass preparation for terms implies $\mathbf{K}$-quantifier elimination. This is exactly the proof of quantifier elimination given in [CL11], although its authors did not see at the time that they were relying on a more uniform version of Weierstrass preparation for terms than what they had actually showed. Hence it would be interesting to know if one could prove uniform Weierstrass preparation for terms without using $\mathbf{K}$-quantifier elimination to recover their proof (see [CL] for more on this subject).

Proposition II.3.26:
Let $M \vDash \mathrm{T}_{\mathcal{A}}$, then the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure of M can be extended (uniquely) to any algebraic extension of $\mathbf{K}(M)$, so that it remains a model of $\mathrm{T}_{\mathcal{A}}$. Moreover, if $C \leqslant M$ and $a \in \mathbf{K}(M)$ is algebraic over $\mathbf{K}(C)$, then $\mathbf{K}(C\langle a\rangle) = \mathbf{K}(C)[a]$.

Proof. The case $\mathcal{R} = \mathcal{O}$ is proved in [CLR06, Theorem 2.18]. The same proof applies when $\mathcal{R} \neq \mathcal{O}$. ∎

To finish this section, let us show that under certain circumstances analytic terms have a linear behavior.

Proposition II.3.27:
Let $M \vDash \mathrm{T}_{\mathcal{A}}$ and suppose that $\mathbf{K}(M)$ is algebraically closed. Let $t : \mathbf{K} \to \mathbf{K}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(M)$-term and b be an open ball in M with radius $\xi \neq \infty$. Suppose that t has a Weierstrass preparation on b — hence t is differentiable at any $a \in b$ — and $\mathrm{rv}(\mathrm{d}t_x)$ is constant on b. Also assume that $\mathrm{val}(t(x))$ is constant or $t(x)$ is polynomial. Then for all a, $e \in b$, $\mathrm{rv}(t(a) - t(e)) = \mathrm{rv}(\mathrm{d}t_a) \cdot \mathrm{rv}(a - e)$.
Moreover, if $v(t(x))$ is constant on b then $\mathrm{val}(t(a)) \leqslant \mathrm{val}(\mathrm{d}t_a) + \xi$.

Proof. If $\mathrm{val}(t(x))$ is constant on b, then $\mathrm{val}(t(x)) - \mathrm{val}(t(a)) \geqslant 0$ and by invariance under addition, $t(x) - t(a)$ has a Weierstrass preparation on b. If $t(x)$ is polynomial this is also clear. Hence there is $F_a \in \mathcal{A}$ (with other parameters in $\mathbf{K}(M)$), P_a, $Q_a \in \mathbf{K}(M)[X]$ such that for all $x \in b$,

$$t(x) - t(a) = F_a\left(\frac{x-a}{g}\right)\frac{P_a(x)}{Q_a(x)}$$

where $\mathrm{val}(F_a(y)) = 0$ for all $y \in \mathfrak{M}$ and $\mathrm{val}(g) = \xi$. If t is constant on b, i.e. $P_a = 0$, then the proposition follows easily. If not, P_a has only finitely many zeroes. Let a_i be the zeros of P_a in $\mathbf{K}(M)$ — recall that M is assumed algebraically closed — and m_i be the multiplicity of a_i. Let c_j be the zeroes of Q_a and n_j be their multiplicities. Note that every zero of $Q_a(x)$ is outside b, hence for all j, $\mathrm{val}(c_j - a) \leqslant \xi$. For all $e \in b$, note that $t(x) - t(a)$ is also differentiable at e with differential $\mathrm{d}t_e$ and hence, if e is distinct from all a_i, then:

$$\begin{aligned}
\operatorname{rv}\left(\frac{\mathrm{d}t_a}{t(e)-t(a)}\right) &= \operatorname{rv}\left(\frac{\mathrm{d}t_e}{t(e)-t(a)}\right)\\
&= \operatorname{rv}\left(\frac{\partial\left(F_a\left(\frac{x-a}{g}\right)\right)/\partial x_x(e)}{F_a\left(\frac{e-a}{g}\right)} + \frac{\mathrm{d}(P_a)_e}{P_a(e)} + \frac{\mathrm{d}(Q_a)_e}{Q_a(e)}\right)\\
&= \operatorname{rv}\left(\frac{\mathrm{d}(F_a)_{\frac{e-a}{g}}}{gF_a\left(\frac{e-a}{g}\right)} + \sum_i \frac{m_i}{e-a_i} + \sum_j \frac{n_j}{e-c_j}\right).
\end{aligned}$$

For any $y \in \mathfrak{M}$, $\operatorname{val}(\mathrm{d}(F_a)_y) \geqslant 0 = \operatorname{val}(F_a(y))$, hence $\operatorname{val}(\mathrm{d}(F_a)_y/(gF_a(y))) \geqslant -\operatorname{val}(g) > -\operatorname{val}(e-a)$. We also have that for all j, $\operatorname{val}(1/(e-c_j)) = -\operatorname{val}(e-c_j) > -\operatorname{val}(e-a)$. Finally, suppose that there is a unique a_{i_0} such that $\operatorname{val}(e-a_{i_0})$ is maximal, then, for all $i \neq i_0$, $\operatorname{val}(1/(e-a_i)) > \operatorname{val}(1/(e-a_{i_0}))$ and hence $\operatorname{rv}(m_{i_0})\operatorname{rv}(e-a_{i_0})^{-1} = \operatorname{rv}(\mathrm{d}t_a)\operatorname{rv}(t(e)-t(a))^{-1}$, i.e. $\operatorname{rv}(t(e)-t(a)) = \operatorname{rv}(\mathrm{d}t_a m_{i_0}^{-1}(e-a_{i_1}))$.
As $t(e) \neq t(a)$, this immediately implies that $\mathrm{d}t_a \neq 0$. Let us now show that if $a_i \in b$ it cannot be a multiple zero.

$$\mathrm{d}t_{a_i} = \mathrm{d}(F_a((x-a)/c)/Q_a(x))_{a_i} P_a(a_i) + P_a'(a_i)F_a((a_i-a)/c)/Q_a(a_i) = 0$$

which is absurd. Hence for all $a_i \in b$, $m_i = 1$ and if we could show that there is a unique $a_i \in b$ — namely a itself — we would be done.
Suppose there are more that one a_i in b and let $\gamma := \min\{\operatorname{val}(a_i-a_j) : a_i, a_j \in b \wedge i \neq j\}$. We may assume $\operatorname{val}(a_0-a_1) = \gamma$. Let us also assume the a_i have been numbered so that there is i_0 such that for all $i \leqslant i_0$, $\operatorname{val}(a_i-a_0) = \gamma$ and for all $i > i_0$, $\operatorname{val}(a_i-a_0) < \gamma$. In particular, for all $i \neq j \leqslant i_0$, $\operatorname{val}(a_i-a_j) = \gamma$. For each $i \leqslant i_0$, let e_i be such that $\operatorname{val}(e_i-a_i) > \gamma$. Then we can apply the previous computation to e_i and we get that $\operatorname{rv}(t(e_i)-t(a)) = \operatorname{rv}(\mathrm{d}t_a)\operatorname{rv}(e_i-a_i)$. But

$$\operatorname{rv}(t(e_i)-t(a)) = \operatorname{rv}(F_a\left(\frac{e_i-x_{\alpha_0+1}}{g}\right))\operatorname{rv}(p)\prod_k(\operatorname{rv}(e_i-a_k))^{m_k}\operatorname{rv}(q)^{-1}\prod_j(\operatorname{rv}(e_i-c_j))^{-n_j}$$

where p and q are the dominant coefficients of respectively P_a and Q_a and hence

$$\operatorname{rv}(\mathrm{d}t_a) = \operatorname{rv}(F_a\left(\frac{e_i-x_{\alpha_0+1}}{g}\right))\operatorname{rv}(p)\prod_{k\neq i}(\operatorname{rv}(e_i-a_k))^{m_k}\operatorname{rv}(q)^{-1}\prod_j(\operatorname{rv}(e_i-c_j))^{-n_j}.$$

As $\operatorname{rv}(F_a((e_i-x_{\alpha_0+1})/g))$, $\operatorname{rv}(e_i-a_k)$ for all $k > i_0$ and $\operatorname{rv}(e_i-c_j)$ do not depend on i, and for all $k \leqslant i_0$, $k \neq i$, $\operatorname{rv}(e_i-a_k) = \operatorname{rv}(a_i-a_k)$, we obtain that for all $i, j \leqslant i_0$:

$$\prod_{i\neq k\leqslant i_0}\operatorname{rv}(a_i-a_k) = \prod_{j\neq k\leqslant i_0}\operatorname{rv}(a_j-a_k).$$

Replacing a_i by $(a_i-a_0)/g$ where $\operatorname{val}(g) = \gamma$, we obtain the same equalities but we may assume that for all $i \leqslant i_0$, $a_i \in \mathcal{O}$ and for all $i \neq j$, $a_i - a_j \in \mathcal{O}^\star$. The equations can now

be rewritten as $\prod_{i\neq k}\mathrm{res}(a_i - a_k) = \prod_{i\neq k}(\mathrm{res}(a_i) - \mathrm{res}(a_k)) = c$ for some $c \in \mathbf{R}(M)$. Let $P = \prod_k(X - \mathrm{res}(a_k))$ then our equations state that $P'(\mathrm{res}(a_i)) - c = 0$ for all $i \leqslant i_0$. But $P' - c$ is a degree i_0 polynomial, it cannot have $i_0 + 1$ roots.
Finally, if $\mathrm{val}(t(x))$ is constant on b, then, for all a and $e \in b$, $\mathrm{val}(t(a)) \leqslant \mathrm{val}(t(a) - t(e)) = \mathrm{val}(\mathrm{d}t_a) + \mathrm{val}(a - e)$. As this holds for any e, we must have $\mathrm{val}(t(a)) \leqslant \mathrm{val}(\mathrm{d}t_a) + \zeta$. ∎

Remark II.3.28:
The conclusion of Proposition **(II.3.27)** seems very close to the Jacobian property (e.g. [CL11, Definition 6.3.5]). In fact, this lemmas is very similar (both in its hypothesis and its conclusion) to [CL11, Lemma 6.3.9].

II.4. σ-Henselian fields

Definition II.4.1 (Analytic field with an automorphism)**:**
Let us suppose that each $\mathcal{A}_{m,n}$ is given with an automorphism of the inductive system $t \mapsto t^\sigma : \mathcal{A}_{m,n} \to \mathcal{A}_{m,n}$. An analytic field M with an automorphism is a model of $\mathrm{T}_\mathcal{A}$ with a distinguished $\mathcal{L}^{\mathbf{RV}} \cup \{|^{\mathcal{R}}_1\}$-automorphism σ such that for symbols $t \in \mathcal{A}_{m,n}$ and $\overline{x} \in \mathbf{K}(M)^{m+n}$, $\sigma(t(\overline{x})) = t^\sigma(\sigma(\overline{x}))$.

Let $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma} := \mathcal{L}_{\mathcal{A},\mathcal{Q}} \cup \{\sigma\} \cup \{\sigma_n : n \in \mathbb{N}\}$. An analytic field M with an automorphism τ can be made into an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-structure by interpreting σ as $\tau|_\mathbf{K}$ and σ_n as $\tau|_{\mathbf{RV}_n}$. Note that σ also induces a ring automorphism on every $\mathbf{R}_n$ and an ordered group morphism on $\mathbf{\Gamma}$. We will write $\mathrm{T}_{\mathcal{A},\sigma}$ for the $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of analytic fields with an automorphism. We will most often write σ instead of σ_n and $\sigma_\mathbf{\Gamma}$ as there should not be any confusion.
If K is a field with an automorphism σ, we will write $\mathrm{Fix}(K) := \{x \in K : \sigma(x) = x\}$ for its fixed field. For all $x \in K$, we will write $\overline{\sigma}(x)$ for the tuple $x, \sigma(x), \dots, \sigma^n(x)$ where the n should be explicit from the context.

Remark II.4.2:
In fact σ induces an action on all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_\mathbf{K}$-terms and we have $\mathrm{T}_{\mathcal{A},\sigma} \vDash \sigma(t(\overline{x})) = t^\sigma(\sigma(\overline{x}))$. It follows immediately that for any $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}|_\mathbf{K}$-term t there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_\mathbf{K}$-term u such that $\mathrm{T}_{\mathcal{A},\sigma} \vDash t(\overline{x}) = u(\overline{\sigma}(\overline{x}))$.

Definition II.4.3 (Linearly closed difference field)**:**
A difference field (K, σ) is called linearly closed if every equation of the form $\sum_{i=0}^n a_i\sigma^i(x) = b$, where $a_n \neq 0$, has a solution.

Definition II.4.4 (Linear approximation)**:**
Let K be a valued field with an automorphism σ, $f : K^n \to K^n$ a (partial) function and $\overline{d} \in K^n$.

(i) Let $\overline{b}$ be a tuple of open balls in M. We say that $\overline{d}$ linearly approximates f on $\overline{b}$ if for all $\overline{a}$ and $\overline{c} \in \overline{b}$ we have:

$$\mathrm{val}(f(\overline{c}) - f(\overline{a}) - \overline{d}\cdot(\overline{c} - \overline{a})) > \min_i\{\mathrm{val}(d_i) + \mathrm{val}(c_i - a_i)\}.$$

(ii) Let b an open ball of M. We say that $\overline{d}$ linearly approximates f at prolongations on b if

for all $a, c \in b$ *we have:*

$$\operatorname{val}(f(\overline{\sigma}(c)) - f(\overline{\sigma}(a)) - \overline{d} \cdot \overline{\sigma}(c-a)) > \min_i \{\operatorname{val}(d_i) + \operatorname{val}(\sigma^i(c-a))\}.$$

Remark II.4.5:

1. Let $M \vDash \mathrm{T}_{\mathcal{A},\sigma}$. Suppose that σ is an isometry, i.e. $\sigma_{\mathbf{\Gamma}} = \mathrm{id}$. Let t be an $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}(\mathcal{O}(M))$-term and $a \in \mathcal{O}(M)$, we can show that $\mathrm{d}t_{\overline{a}}$ linearly approximates t on $\mathring{\mathcal{B}}_\gamma(a)$ where $\gamma = \min_i\{\operatorname{val}(\partial f / \partial x_i(\overline{a}))\}$.

2. We allow a slight abuse of notation by saying that terms constant on a ball is linearly approximated (at prolongations) by the zero tuple, even though the required inequality does not hold as $\infty \not> \infty$.

Let us first show that it suffices to show linear approximation variable by variable to obtain linear approximation for the whole function. We will write $(\overline{a}_{\neq i}, x_i)$ for the tuple $\overline{a}$ where the i-th component is replaced by x_i (with a slight abuse of notations as the x_i does not appear in the right place) and $\overline{a}^{\leqslant i}$ for the tuple $\overline{a}$ where the j-th components for $j > i$ are replaced by zeroes.

Proposition II.4.6:
Let (K, val) *be a valued field,* $f : K^n \to K$, $\overline{d} \in K^n$ *and* $\overline{b}$ *a tuple of balls. If for all* $\overline{a} \in \overline{b}$ *and* $j < n$, d_j *linearly approximates* $f(\overline{a}_{\neq j}, x_j)$ *on* b_j*, then* $\overline{d}$ *linearly approximates* f *on* $\overline{b}$*.*

Proof. Let $\overline{a}$ and $\overline{e} \in \overline{b}$ and $\overline{\varepsilon} = \overline{e} - \overline{a}$. Then, we have

$$\begin{array}{rcl} \operatorname{val}(f(\overline{a}+\overline{\varepsilon}) - f(\overline{a}) - \overline{d}\cdot\overline{\varepsilon}) & = & \operatorname{val}(\sum_j f(\overline{a}+\overline{\varepsilon}^{\leqslant j}) - f(\overline{a}+\overline{\varepsilon}^{\leqslant j-1}) - d_j\varepsilon_j) \\ & \geqslant & \min_j\{f(\overline{a}+\overline{\varepsilon}^{\leqslant j}) - f(\overline{a}+\overline{\varepsilon}^{\leqslant j-1}) - d_j\varepsilon_j\} \\ & > & \min_j\{\operatorname{val}(d_j) + \operatorname{val}(\varepsilon_j)\}. \end{array}$$

And that concludes the proof. ■

Although linear approximation (at prolongations) looks like differentiability, one must be aware that linear approximations are not uniquely determined, because, among other reasons, we are only looking at tuples that are prolongations but also because the error term is only linear. But when σ is an isometry, we can recover some uniqueness, and give an alternative definition (perhaps of a more geometric flavor) of linear approximation at prolongations.

Definition II.4.7 ($\mathbf{R}_{1,\gamma}$):
Let (K, val) *be a valued field and* $\gamma \in \operatorname{val}(K)$*. We define* $\mathbf{R}_{1,\gamma} := \overline{\mathcal{B}}_\gamma(0)/\mathring{\mathcal{B}}_\gamma(0)$ *and let* $\operatorname{res}_{1,\gamma}$ *denote the canonical projection* $\overline{\mathcal{B}}_\gamma(0) \to \mathbf{R}_{1,\gamma}$*. Note that* $\mathbf{R}_{1,\gamma}$ *can be identified (canonically) with* ${\operatorname{val}_{\mathbf{RV},1}}^{-1}(\gamma) \cup \{0\} \subseteq \mathbf{RV}_1$*.*

Proposition II.4.8:
Let (K, val) *be a valued field with an isometry* σ *and a linearly closed residue field. Let* $f : K^n \to K$, $\overline{d}$ *be a linear approximation of* f *at prolongations on some open ball* b *with radius* ξ, $\overline{e} \in K^n$, $\delta := \operatorname{val}(\overline{d})$ *and* $\eta := \operatorname{val}(\overline{e})$*. The following are equivalent:*

(i) $\overline{e}$ *is a linear approximation of* f *at prolongations on* b*;*

(ii) $\mathrm{val}(\overline{d}-\overline{e}) > \min\{\delta,\eta\}$*;*

(iii) $\eta = \delta$ ***and*** $\mathrm{res}_{1,\delta}(\overline{d}) = \mathrm{res}_{1,\delta}(\overline{e})$*.*

Proof.

(i)⇒(ii) Suppose $\overline{d} \neq \overline{e}$. Let ε be such that $\mathrm{val}(\varepsilon) > \xi$ and let $g \in K$ be such that $\mathrm{val}(g) = \mathrm{val}(\overline{d}-\overline{e})$. Then $P(\overline{\sigma}(x)) := \sum_i (d_i - e_i)\sigma^i(\varepsilon)g^{-1}\varepsilon^{-1}\sigma^i(x)$ is a linear difference polynomial with a non zero residue. As K is residually linearly closed, the residue of P cannot always be zero and hence there exists $c \in \mathcal{O}^\star$ such that $\mathrm{val}(P(\overline{\sigma}(c))) = 0$, i.e. $\mathrm{val}((\overline{d}-\overline{e})\cdot\overline{\sigma}(\varepsilon c)) = \mathrm{val}(g) + \mathrm{val}(\varepsilon)$. But then, for all $a \in b$:

$$\begin{aligned}
&\mathrm{val}(g) + \mathrm{val}(\varepsilon)\\
&\quad = \mathrm{val}((\overline{d}-\overline{e})\cdot\overline{\sigma}(\varepsilon c))\\
&\quad = \mathrm{val}(f(\overline{\sigma}(a+\varepsilon c)) - f(\overline{\sigma}(a)) - \overline{e}\cdot\overline{\sigma}(\varepsilon b))\\
&\quad -f(\overline{\sigma}(a+\varepsilon c)) + f(\overline{\sigma}(a)) + \overline{d}\cdot\overline{\sigma}(\varepsilon b)\\
&\quad > \mathrm{val}(\varepsilon) + \min\{\delta,\eta\}
\end{aligned}$$

i.e. $\mathrm{val}(\overline{d}-\overline{e}) > \min\{\delta,\eta\}$.

(ii)⇒(iii) First, suppose that $\delta < \eta$, then if $\mathrm{val}(d_i)$ is minimal, $\mathrm{val}(d_i) = \delta < \eta \leqslant \mathrm{val}(e_i)$ and hence $\mathrm{val}(d_i - e_i) = \mathrm{val}(d_i) = \delta = \min\{\delta,\eta\}$ contradicting our previous inequality. Hence we must have, by symmetry, $\delta = \eta$. Now inequality (ii) can be rewritten $\mathrm{val}(\overline{d}-\overline{e}) > \delta$ which exactly means that $\mathrm{res}_{1,\delta}(\overline{d}) = \mathrm{res}_{1,\delta}(\overline{e})$.

(iii)⇒(i) For all ε such that $\mathrm{val}(\varepsilon) > \xi$, as $\mathrm{val}(\overline{d}-\overline{e}) > \delta$, we have:

$$\begin{aligned}
&\mathrm{val}(f(\overline{\sigma}(a+\varepsilon)) - f(\overline{\sigma}(a)) - \overline{e}\cdot\overline{\sigma}(\varepsilon))\\
&\quad = \mathrm{val}(f(\overline{\sigma}(a+\varepsilon)) - f(\overline{\sigma}(a)) - \overline{d}\cdot\overline{\sigma}(\varepsilon) + (\overline{d}-\overline{e})\cdot\overline{\sigma}(\varepsilon))\\
&\quad > \delta + \mathrm{val}(\varepsilon)\\
&\quad = \eta + \mathrm{val}(\varepsilon).
\end{aligned}$$

This concludes the proof. ■

■

Remark II.4.9:

1. In the isometry case, linear approximations describe the trace of a given function on $\mathbf{RV}_1$. More precisely, a function f is linearly approximated at prolongations on some open ball b with radius ξ if and only if there exists $\delta \in \mathrm{val}(K)$ and $\overline{d} \in \mathbf{R}_{1,\delta}(K)$ such that for all $\gamma > \xi$ t and $a \in b$ he function $\mathrm{res}_{1,\gamma}(\varepsilon) \mapsto \mathrm{res}_{1,\gamma+\delta}(f(\overline{\sigma}(a+\varepsilon)) - f(\overline{\sigma}(a))) : \mathbf{R}_{1,\gamma} \to \mathbf{R}_{1,\gamma+\delta}$ is well defined and coincides with the function $x \mapsto \overline{d}\cdot\overline{\sigma}(x)$ (where the sum is given by $+_{1,1}$).

2. If we are working in a valued field with a linearly closed residue field, it follows from Proposition **(II.4.8)**, that δ and $\overline{d}$ from (i) are actually uniquely defined.

Definition II.4.10 (σ-Henselianity)**:**
Let $M \vDash \mathrm{T}_{\mathcal{A},\sigma}$, t be a $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$, $\overline{d} \in \mathbf{K}(M)$, $a \in \mathbf{K}(M)$ and $\xi \in \mathbf{\Gamma}(M)$. Say that $(t, a, \overline{d}, \xi)$ is in σ-Hensel configuration if $\overline{d}$ linearly approximates f at prolongations on $\mathring{\mathcal{B}}_\xi(a)$ and:

$$\mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \sigma^i(\xi)\}.$$

We say that M is σ-Henselian if for all $(t, a, \overline{d}, \xi)$ in σ-Hensel configuration, there exists $c \in \mathbf{K}(M)$ such that $t(\overline{\sigma}(c)) = 0$ and $\mathrm{val}(c - a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$.

Remark II.4.11:
By Remark **II.4.5.1**, when σ is an isometry, this form of the σ-Hensel lemma is equivalent to classical forms for difference polynomials — i.e without any analytic structure — as stated in [Sca00; Sca03; AD10] for example. In particular, it implies Hensel's lemma (for polynomials).

Definition II.4.12 (Pseudo-convergence)**:**
Let $M \vDash \mathrm{T}_{\mathcal{A},\sigma}$.

(i) A sequence $(x_\alpha)_{\alpha\in\beta}$ of (distinct) points in $\mathbf{K}(M)$ indexed by an ordinal is said to be pseudo-convergent if for all α, γ, $\delta \in \beta$ such that $\alpha < \gamma < \delta$ we have $\mathrm{val}(x_\alpha - x_\delta) < \mathrm{val}(x_\gamma - x_\delta)$;

(ii) We say that $a \in \mathbf{K}(M)$ is a pseudo-limit of the pseudo-convergent sequence (x_α) — and we write $x_\alpha \rightsquigarrow a$ — if for all $\alpha < \gamma < \beta$, $\mathrm{val}(x_\alpha - a) < \mathrm{val}(x_\gamma - a)$;

(iii) A pseudo-convergent sequence of elements of $C \subseteq \mathbf{K}(M)$ is said to be maximal if it has no pseudo-limit in C;

(iv) We say that a sequence $(\overline{x}_\alpha)$ of tuples pseudo-solves an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term t if $t = 0$ or for $\alpha \gg 0$ — i.e. for α in a final segment — $t(\overline{x}_\alpha) \rightsquigarrow 0$.

(v) We say that a sequence (x_α) σ-pseudo-solves an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term t if $(\overline{\sigma}(x_\alpha))$ pseudo-solves t.

(vi) We say that M is maximally complete if any pseudo-convergent sequence in M (indexed by a limit ordinal) has a pseudo-limit in M;

(vii) We say M is σ-algebraically maximally complete if any pseudo-sequence (x_α) from M (indexed by a limit ordinal) σ-pseudo-solving an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term $t \neq 0$ has a pseudo-limit in M.

Remark II.4.13:

1. Let (x_α) be a pseudo-convergent sequence, then for all $\alpha < \beta$, $\mathrm{val}(x_\alpha - x_\beta) = \mathrm{val}(x_\alpha - x_{\alpha+1}) =: \gamma_\alpha$. The γ_α form a strictly increasing sequence. If $x_\alpha \rightsquigarrow a$ then $\mathrm{val}(a - x_\alpha) = \gamma_\alpha$ and if b is such that for all i, $\mathrm{val}(b - a) > \gamma_\alpha$ then we also have $x_\alpha \rightsquigarrow b$.

2. As, in any valued field, balls with a non infinite radius always have more than one point, if (x_α) is maximal pseudo-convergent sequence then either γ_α is cofinal in $\mathrm{val}(\mathbf{K}^\star)$ and (x_α) is indexed by the successor of a limit ordinal or (x_α) is indexed by

a limit ordinal.

Proposition II.4.14:
If M is σ-algebraically maximally complete and $\mathbf{R}_1(M)$ is linearly closed then M is σ-Henselian.

Proof. First an easy claim:

Claim II.4.15: *Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration, then*

$$\max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \xi.$$

Proof. As $(t, a, \overline{d}, \xi)$ is in σ-Hensel configuration, there exists an i_0 such that $\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}(d_{i_0}) + \sigma^{i_0}(\xi)$ and hence $\mathrm{val}(\sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1})) = \sigma^{-i_0}(\mathrm{val}(t(\overline{\sigma}(a))) - \mathrm{val}(d_{i_0})) > \xi$. ♦

And now, two lemmas about finding better approximations to zeros of terms.

Lemma II.4.16:
Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration such that $t(\overline{\sigma}(a)) \neq 0$. Then there exists c such that $\mathrm{val}(c - a) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$, $\mathrm{val}(t(\overline{\sigma}(c))) > \mathrm{val}(t(\overline{\sigma}(a)))$ and $(t, c, \overline{d}, \xi)$ is also in σ-Hensel configuration.

Proof. Pick any $\varepsilon \in \mathbf{K}(M)$ with $\mathrm{val}(\varepsilon) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$. By Claim **(II.4.15)**, $\mathrm{val}(\varepsilon) > \xi$. For all $x \in \mathcal{O}$, let $R(a, \varepsilon, x) := t(\overline{\sigma}(a) + \overline{\sigma}(\varepsilon x)) - t(\overline{\sigma}(a)) - \overline{d} \cdot \overline{\sigma}(\varepsilon x)$ and

$$u(x) := \frac{t(\overline{\sigma}(a) + \overline{\sigma}(\varepsilon x))}{t(\overline{\sigma}(a))} = 1 + \sum_i \frac{d_i(\overline{\sigma}(a))\sigma^i(\varepsilon)}{t(\overline{\sigma}(a))}\sigma^i(x) + \frac{R(a, \varepsilon, x)}{t(\overline{\sigma}(a))}.$$

For all i,

$$\mathrm{val}\left(\frac{d_i\sigma^i(\varepsilon)}{t(\overline{\sigma}(a))}\right) \geqslant \mathrm{val}(d_i) + \mathrm{val}(t(a)) - \mathrm{val}(d_i) - \mathrm{val}(t(a)) = 0$$

and for any i_0 such that $\mathrm{val}(\varepsilon) = \mathrm{val}(\sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1}))$ it is an equality. As $\overline{d}$ linearly approximates t at prolongations on $\mathring{\mathcal{B}}_\xi(a)$, we also have

$$\mathrm{val}(R(a, \varepsilon, x)) > \min_i\{\mathrm{val}(\sigma^i(\varepsilon)) + \mathrm{val}(d_i)\} \geqslant \mathrm{val}(t(\overline{\sigma}(a)))$$

and $\mathrm{res}_1(u(x)) = 0$ is a non trivial linear equation in the residue field. As $\mathbf{R}_1(M)$ is linearly closed, this equation has a solution $\mathrm{res}_1(e)$. Note that we must have $\mathrm{res}_1(e) \neq 0$.
Let $c = a + \varepsilon e$. It is clear that $\mathrm{val}(c - a) = \mathrm{val}(\varepsilon) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \xi$ and that $\mathrm{val}(t(\overline{\sigma}(c))) = \mathrm{val}(t(\overline{\sigma}(a))u(e)) > \mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \sigma_i(\xi)\}$. ♦

Lemma II.4.17:
Let (x_α) be a pseudo-convergent sequence (indexed by a limit ordinal). Assume that for all α, $(t, x_\alpha, \overline{d}, \xi)$ is in σ-Hensel configuration, $\mathrm{val}(x_{\alpha+1} - x_\alpha) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(x_\alpha))d_i^{-1}))\}$ and for all $\beta > \alpha$, $\mathrm{val}(t(\overline{\sigma}(x_\beta))) > \mathrm{val}(t(\overline{\sigma}(x_\alpha)))$. If c is such that $x_\alpha \rightsquigarrow c$, then $(t, c, \overline{d}, \xi)$ is in σ-Hensel configuration and for all α, $\mathrm{val}(t(\overline{\sigma}(c))) > \mathrm{val}(t(\overline{\sigma}(x_\alpha)))$.

Proof. First of all, as $(t, x_0, \overline{d}, \xi)$ is in σ-Hensel configuration, $\overline{d}$ continuously linearly approximates t at prolongations on $\mathring{\mathcal{B}}_\xi(x_0)$. By Claim **(II.4.15)**, $\mathrm{val}(c - x_0) = \mathrm{val}(x_1 - x_0) > \xi$. Moreover, let $R(x, c) := t(\overline{\sigma}(c)) - t(\overline{\sigma}(x)) - \overline{d} \cdot \overline{\sigma}(c - x)$. Then for all α,

$$\begin{array}{rcl} \mathrm{val}(t(\overline{\sigma}(c))) & = & \mathrm{val}(t(\overline{\sigma}(x_\alpha)) + \overline{d}(\overline{\sigma}(x_\alpha)) \cdot \overline{\sigma}(c - x_\alpha) + R(x_\alpha, c)) \\ & \geqslant & \min_i\{\mathrm{val}(t(\overline{\sigma}(x_\alpha))), \mathrm{val}(d_i) + \mathrm{val}(\sigma^i(c - x_\alpha))\} \\ & \geqslant & \mathrm{val}(t(\overline{\sigma}(x_\alpha))). \end{array}$$

Finally, as $\mathrm{val}(t(\overline{\sigma}(c))) \geqslant \mathrm{val}(t(\overline{\sigma}(x_0))) > \min_i\{\mathrm{val}(d_i) + \sigma^i(\xi)\}$, $(t, c, \overline{d}, \xi)$ is also in σ-Hensel configuration. ♦

Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration. If $t = 0$, we are done, if not let $(x_\alpha)_{\alpha\in\beta}$ be a maximal sequence (with respect to the length) such that $x_0 = a$ and for all α, $(t, x_\alpha, \overline{d}, \xi)$ is in σ-Hensel configuration, $\mathrm{val}(x_{\alpha+1} - x_\alpha) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(x_\alpha))d_i^{-1}))\}$ and $t(\overline{\sigma}(x_\alpha)) \rightsquigarrow 0$. If α is a limit ordinal, as M is σ-algebraically maximally complete, and $t \neq 0$, (x_α) has a pseudo-limit x_β. By Lemma **(II.4.17)**, the sequence $(x_\alpha)_{\alpha\in\beta+1}$ still meets the same requirements, contradicting the maximality of $(x_\alpha)_{\alpha\in\beta}$. It follows that $\beta = \gamma + 1$. If $t(\overline{\sigma}(x_\gamma)) \neq 0$, then applying Lemma **(II.4.16)**, to (t, x_γ), we obtain an element x_β such that $(x_\alpha)_{\alpha\in\beta+1}$ still meets the same requirements, contradiction the maximality of $(x_\alpha)_{\alpha\in\beta}$ once again. Hence we must have that $t(\overline{\sigma}(x_\gamma)) = 0$ and $c = x_\gamma$ is a solution to the σ-Hensel configuration $(t, a, \overline{d}, \xi)$. ■

Definition II.4.18 ($\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$)**:**
Let $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be the $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of analytic fields with an automorphism that are σ-Henselian and have a non-trivial valuation group. To specify the characteristic we will write $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ or $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$.

Proposition II.4.19:
Let $\mathcal{A} = \bigcup_{\overline{X},\overline{Y}} \mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}]\langle\overline{X}\rangle[[\overline{Y}]]$ and let W_p be the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure with base set $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$, the obvious valued field structure and analytic structure and taking σ to be the lifting to $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ of the Frobenius automorphism on $\overline{\mathbb{F}_p}^{\mathrm{alg}}$. Then $\mathrm{W}_p \models \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$.

Proof. It is clear that $\mathrm{W}_p \models \mathrm{T}_{\mathcal{A}}$. As $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ is complete with a discrete valuation it is maximally complete and σ-Henselianity follows from Proposition **(II.4.14)**. ■

In the definition of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, we have not required the residue field to be linearly closed, since it comes for free:

Proposition II.4.20:
Let $M \models \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, then $\mathbf{K}(M)$ is linearly closed.

Proof. Let $P(x) = \sum_i a_i x^i = c$ be a non zero linear equation. Let $\varepsilon \in \mathbf{K}(M)$ be such that $\mathrm{val}(\varepsilon) < \mathrm{val}(c) - \mathrm{val}(a_0)$. Then $P(\varepsilon x) = \sum a_i \sigma^i(\varepsilon)\sigma^i(x)$ and $\min_i\{\mathrm{val}(a_i\sigma^i(\varepsilon))\} \leqslant \mathrm{val}(a_0) + \mathrm{val}(\varepsilon) < \mathrm{val}(c)$. Finding a solution to $P(\varepsilon x) = c$ being the same as finding one for $P(x) = c$, we may assume that $\min_i\{\mathrm{val}(a_i)\} < c = \mathrm{val}(P(0) - c)$. But, as P is linear, $\overline{a}$ linearly approximates P at prolongations on $\mathfrak{M}$ and $(P - c, 0, \overline{a}, 0)$ is in σ-Hensel configuration. As M is σ-Henselian, there exists $e \in \mathbf{K}(M)$ such that $P(e) = c$. ■

To conclude this section, let us show that $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ behaves well with respect to coarsening. Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$ and T be an $\mathcal{L}$-theory containing $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$ Morleyized on $\mathbf{RV}$. By Section II.2 we can find an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty$ of $\mathcal{L}^{\mathbf{RV}_\infty}$ — the ∞ in $\mathcal{L}^{\mathbf{RV}_\infty}$ is there to recall that the leading term structure is given by $\mathbf{RV}_\infty$ and not the $\mathbf{RV}_n$, although, to add to the general confusion, the $\mathbf{RV}_n$ are indeed present in the enrichment — an $\mathcal{L}^\infty$-theory $\mathrm{T}_1^\infty \supseteq \mathrm{T}_{\mathrm{vf},0,0}^\infty$ and two functors $\mathfrak{C}_1^\infty : \mathrm{Str}(T) \to \mathrm{Str}(\mathrm{T}_1^\infty)$ and $\mathfrak{UC}_1^\infty : \mathrm{Str}(\mathrm{T}_1^\infty) \to \mathrm{Str}(T)$. For any C in $\mathrm{Str}(T)$ we enrich $\mathfrak{C}_1^\infty(C)$ by defining:

- $\cdot_\infty$ and 1_∞ to be the multiplicative group structure of $\mathbf{RV}_\infty$;
- 0_∞ to be $(0_n)_{n\in\mathbb{N}_{>0}}$;
- $x|_\infty y$ to hold if for some n, $\pi_1(x)|_1\mathrm{rv}_1(p^{-n})\pi_1(y)$ holds;
- $x+_{\infty,\infty} y$ to be $(\pi_{mn}(x)+_{mn,m}\pi_{mn}(y))_{m\in\mathbb{N}_{>0}}$ if there exists $n\in\mathbb{N}_{>0}$ such that $\pi_n(x)+_{n,1}\pi_n(y)\neq 0_1$ and 0_∞ otherwise;
- $x|_\infty^{\mathcal{R}} y$ to hold if $\pi_1(x)|_1^{\mathcal{R}}\pi_1(y)$ holds;
- $E_\infty(x)$ to be $(E_k(x))_{k\in\mathbb{N}_{>0}}$ for all E in some $\mathcal{A}_{m,n}^\star$;
- σ_∞ to be $(\sigma_n(x))_{n\in\mathbb{N}_{>0}}$;

and we obtain a new functor $\mathfrak{C}_2^\infty : \mathrm{Str}(T) \to \mathrm{Str}(\mathrm{T}^\infty{}_2)$ where $\mathrm{T}_2^\infty := \mathrm{T}_1^\infty \cup \mathrm{T}_{\mathcal{A},\sigma,0,0}^\infty$. One can check that we still have an equivalence of categories induced by $\mathfrak{C}_2^\infty$ and $\mathfrak{UC}_1^\infty$ and that $\mathfrak{C}_2^\infty$ also respects cardinality up to $\aleph_0$ and $\aleph_1$-saturated models. Finally, by Corollary **(II.B.4)**, as T is Morleyized on $\mathbf{RV}$, we obtain functors $\mathfrak{C}_3^\infty : \mathrm{Str}(T) \to \mathrm{Str}(T_2^{(\mathbf{RV}_\infty\cup\mathbf{RV})-\mathrm{Mor}})$ and $\mathfrak{UC}_3^\infty : \mathrm{Str}(T_2^{(\mathbf{RV}_\infty\cup\mathbf{RV})-\mathrm{Mor}}) \to \mathrm{Str}(T)$. Note that in this case, because we only enrich by predicates, the full subcategory $\mathfrak{F}$ of $\mathrm{Str}(T)$ is not actually needed.
Let us now show that for all $M\vDash T$, $\mathfrak{C}_3^\infty(M)\vDash \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}^\infty$.

Proposition II.4.21:
Let $M\vDash T$ and $t:\mathbf{K}^n\to\mathbf{K}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term, $\overline{d}\in\mathbf{K}(M)$ and b an open $\mathcal{O}_\infty$-ball. Then if, in $\mathfrak{C}_3^\infty(M)$, $\overline{d}$ linearly approximates t at prolongations on b, then for any open $\mathcal{O}$-ball $b'\subseteq b$, $\overline{d}$ also linearly approximates t at prolongations on b' in M.

Proof. For all a and $e\in b'\subseteq b$, we have $\mathrm{val}_\infty(t(\overline{\sigma}(a))-t(\overline{\sigma}(e))-\overline{d}\cdot\overline{\sigma}(a-e)) > \min_i\{\mathrm{val}_\infty(d_i)+\mathrm{val}_\infty(\sigma^i(a-e))\}$. Let i_0 be such that $\mathrm{val}_\infty(d_{i_0})+\mathrm{val}_\infty(\sigma^{i_0}(a-e))$ is minimal, then we have $\mathrm{val}(t(\overline{\sigma}(a))-t(\overline{\sigma}(e))-\overline{d}\cdot\overline{\sigma}(a-e)) > \mathrm{val}(d_{i_0})+\mathrm{val}(\sigma^{i_0}(a-e)) \geqslant \min_i\{\mathrm{val}(d_i)+\mathrm{val}(\sigma^i(a-e))\}$. ∎

Proposition II.4.22:
Let $M\vDash T$, then $\mathfrak{C}_3^\infty(M)$ is σ-Henselian (for the valuation val_∞).

Proof. Let $(t,a,\overline{d},\xi)$ be in σ-Hensel configuration in $\mathfrak{C}_3^\infty(M)$ and i_0 be such that $\mathrm{val}_\infty(d_{i_0})+\sigma^{i_0}(\xi)$ is minimal. As $(t,a,\overline{d},\xi)$ be in σ-Hensel configuration, $\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}_\infty(d_{i_0})+\sigma^{i_0}(\xi)$. Let $r=\sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1}p^{-1})$. Then

$$\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}(d_{i_0})+\mathrm{val}(\sigma^{i_0}(r)) \geqslant \min_i\{\mathrm{val}(d_i)+\mathrm{val}(\sigma^i(r))\}.$$

Moreover,

$$\mathrm{val}_\infty(\sigma^{i_0}(r)) = \mathrm{val}_\infty(t(a)) - \mathrm{val}_\infty(d_{i_0}) > \sigma^{i_0}(\xi),$$

i.e. $\mathrm{val}_\infty(r) > \xi$. It follows that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(r)}(a) \subseteq \mathring{\mathcal{B}}^{\mathcal{O}_\infty}_{\xi}(a)$ and hence, by Proposition **(II.4.21)**, $\overline{d}$ linearly approximates t at prolongations on $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(r)}(a)$ and $(t, a, \overline{d}, \mathrm{val}(r))$ is in σ-Hensel configuration.
It follows that we can find $c \in \mathbf{K}(M)$ such that $\mathrm{val}(c - a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$ and $t(\overline{\sigma}(c)) = 0$. But then we have that for all i, $\mathrm{val}_\infty(c - a) \geqslant \mathrm{val}_\infty(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))$. ■

It follows from those two propositions that we can further enrich $T_2^{(\mathbf{RV}_\infty\cup\mathbf{RV})-\mathrm{Mor}}$ so that it is an $\mathbf{RV}$-enrichment of $\mathrm{T}^\infty_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$. Hence we have proved:

Proposition II.4.23:
Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$ and T be an $\mathcal{L}$-theory containing $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$ Morleyized on $\mathbf{RV}$. There exists an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty$ of $\mathcal{L}^\infty_{\mathcal{A},\mathcal{Q},\sigma}$ — with new sorts $\mathbf{RV} = \bigcup_n \mathbf{RV}_n$ — and an $\mathcal{L}^\infty$-theory $\mathrm{T}^\infty \supseteq \mathrm{T}^\infty_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty : \mathrm{Str}(T) \to \mathrm{Str}(\mathrm{T}^\infty)$ and $\mathfrak{UC}^\infty : \mathrm{Str}(\mathrm{T}^\infty) \to \mathrm{Str}(T)$ that respect cardinality up to $\aleph_0$ and induce an equivalence of categories between $\mathrm{Str}(T)$ and $\mathrm{Str}_{\mathfrak{C}^\infty,(|\mathcal{L}|^{\aleph_1})^+}(\mathrm{T}^\infty)$ and such that $\mathfrak{UC}^\infty$ respects models and elementary submodels and sends $\mathbf{RV}_\infty \cup \mathbf{RV}$ to $\mathbf{RV}$ and $\mathfrak{C}^\infty$ respects $(|\mathcal{A}|^{\aleph_1})^+$-saturated models.

Similarly, we can prove the existence of these functors in the analytic and in the algebraic setting, and these functors are actually induced by those in the analytic difference case.

Proposition II.4.24:
Let $\mathcal{L}_{\mathrm{an}}$ be any $\mathbf{RV}$-extension of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ contained in $\mathcal{L}$ and $\mathcal{L}_{\mathrm{alg}}$ be any $\mathbf{RV}$-extension of $\mathcal{L}^{\mathbf{RV}^+}$ contained in $\mathcal{L}_{\mathrm{an}}$. Define $T_{\mathrm{an}} := T|_{\mathcal{L}_{\mathrm{an}}}$, and $T_{\mathrm{alg}} := T|_{\mathcal{L}_{\mathrm{alg}}}$. Assume that both T_{an} and T_{alg} are Morleyized on $\mathbf{RV}$.

*(i) There exists an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty_{\mathrm{an}}$ of $\mathcal{L}^\infty_{\mathcal{A},\mathcal{Q}}$ and an $\mathcal{L}^\infty_{\mathrm{an}}$-theory $\mathrm{T}^\infty_{\mathrm{an}} \supseteq \mathrm{T}^\infty_{\mathcal{A},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty_{\mathrm{an}} : \mathrm{Str}(T_{\mathrm{an}}) \to \mathrm{Str}(\mathrm{T}^\infty_{\mathrm{an}})$ and $\mathfrak{UC}^\infty_{\mathrm{an}} : \mathrm{Str}(T_{\mathrm{an}}) \to \mathrm{Str}(T_{\mathrm{an}})$ with the same properties as in Proposition **(II.4.23)**.*

Moreover $\mathfrak{C}^\infty_{\mathrm{an}}(\cdot\,|_{\mathcal{L}_{\mathrm{an}}}) = \mathfrak{C}^\infty(\cdot)|_{\mathcal{L}^\infty_{\mathrm{an}}}$ and similarly for $\mathfrak{UC}^\infty_{\mathrm{an}}$.

*(ii) There exists an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty_{\mathrm{alg}}$ of $\mathcal{L}^{\mathbf{RV}_\infty{}^+}$ and an $\mathcal{L}^\infty_{\mathrm{alg}}$-theory $\mathrm{T}^\infty_{\mathrm{alg}} \supseteq \mathrm{T}^\infty_{\mathrm{Hen},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty_{\mathrm{alg}} : \mathrm{Str}(T_{\mathrm{alg}}) \to \mathrm{Str}(\mathrm{T}^\infty_{\mathrm{alg}})$ and $\mathfrak{UC}^\infty_{\mathrm{alg}} : \mathrm{Str}(\mathrm{T}^\infty_{\mathrm{alg}}) \to \mathrm{Str}(T_{\mathrm{alg}})$ with the same properties as in Proposition **(II.4.23)**.*

Moreover $\mathfrak{C}^\infty_{\mathrm{alg}}(\cdot\,|_{\mathcal{L}_{\mathrm{alg}}}) = \mathfrak{C}^\infty_{\mathrm{an}}(\cdot\,|_{\mathcal{L}_{\mathrm{an}}})\big|_{\mathcal{L}^\infty_{\mathrm{alg}}} = \mathfrak{C}^\infty(\cdot)|_{\mathcal{L}^\infty_{\mathrm{alg}}}$ and similarly for $\mathfrak{UC}^\infty_{\mathrm{alg}}$.

II.5. Reduction to the algebraic case

In the following section, let $\mathcal{L}_{\mathrm{an}}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ and let T_{an} be an $\mathcal{L}_{\mathrm{an}}$-theory containing $\mathrm{T}_{\mathcal{A}}$, Morleyized on $\mathbf{RV}$. We define $\mathcal{L}_{\mathrm{alg}} := \mathcal{L}_{\mathrm{an}} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$ — it is an $\mathbf{RV}$-enrichment of $\mathcal{L}^{\mathbf{RV}^+}$ — and $T_{\mathrm{alg}} = T_{\mathrm{an}}|_{\mathcal{L}_{\mathrm{alg}}}$. As previously, if there are new sorts $\Sigma_{\mathbf{RV}}$, we write $\mathbf{RV}$ for $\mathbf{RV} \cup \Sigma_{\mathbf{RV}}$.

Remark II.5.1:
Let M_1 and $M_2 \vDash T_{\mathrm{an}}$, $C_i \subseteq M_i$ and $f : C_1 \to C_2$ an $\mathcal{L}_{\mathrm{an}}$-isomorphism. Then f extends uniquely to $\langle C_1 \rangle$. As $\mathcal{L}_{\mathrm{an}}$ contains $\mathcal{Q}$, $\mathbf{K}(\langle C_1 \rangle)$ is a field. Hence any partial $\mathcal{L}_{\mathrm{an}}$-isomorphism with domain C has a unique extension to $\mathrm{Frac}(\mathbf{K}(C))$.

Although it is well-known, the algebraic case (i.e. in $\mathcal{L}_{\mathrm{alg}}$) is a bit more complicated because we do not have $\mathcal{Q}$ in $\mathcal{L}_{\mathrm{alg}}$.

Proposition II.5.2:
Let M_1 and $M_2 \vDash T_{\mathrm{alg}}$ be two $\mathcal{L}_{\mathrm{alg}}$-structures, $C_i \subseteq M_i$ and $f : C_1 \to C_2$ an $\mathcal{L}^{\mathbf{RV}^+}$-isomorphism. If $\mathrm{rv}(\mathrm{Frac}(\mathbf{K}(C_1))) \subseteq \mathbf{RV}(C_1)$, then f has a unique extension to $\mathrm{Frac}(\mathbf{K}(C_1))$.

Proof. Let $f'|_{\mathbf{K}}$ be the unique extension of $f|_{\mathbf{K}}$ to $\mathrm{Frac}(\mathbf{K}(C_1))$. It is a ring morphism. By Lemma **(II.A.13)**, it suffices to show that $f'|_{\mathbf{K}} \cup f|_{\mathbf{RV}}$ respects the rv_n. As $\mathrm{rv}(\mathrm{Frac}(\mathbf{K}(C_1))) \subseteq \mathbf{RV}(C_1)$, $f|_{\mathbf{RV}}$ commutes with the inverse on any rv_n and hence

$$\mathrm{rv}_n(f'(a/b)) = \mathrm{rv}_n(f(a)f(b)^{-1}) = f(\mathrm{rv}_n(a))f(\mathrm{rv}_n(b)^{-1}) = f(\mathrm{rv}_n(a/b)).$$

This concludes the proof. ∎

In the following proposition we will be working in equicharacteristic zero, hence, to avoid needlessly cluttered notations, we will write $\mathbf{R}$, res, $\mathbf{RV}$ and rv for $\mathbf{R}_1$, res_1, $\mathbf{RV}_1$ and rv_1.

Proposition II.5.3 (Reduction to the algebraic case)**:**
Suppose $T_{\mathrm{an}} \supseteq \mathrm{T}_{\mathcal{A},0,0}$. Let M_1 and $M_2 \vDash T_{\mathrm{an}}$, $f : M_1 \to M_2$ a partial $\mathcal{L}_{\mathrm{an}}$-isomorphism with domain $C_1 \leqslant M_1$ and $a_1 \in M_1$. If f can be extended to an $\mathcal{L}_{\mathrm{alg}}$-isomorphism f' whose domain contains a_1, then f can be extended to an $\mathcal{L}_{\mathrm{an}}$-isomorphism whose domain contains a_1.

Proof. First, because $T_{\mathrm{alg}}|_{\mathbf{RV}} = T_{\mathrm{an}}|_{\mathbf{RV}}$, T_{alg} is also Morleyized on $\mathbf{RV}$. By Lemma **(II.A.11)**, we can extend f' on $\mathbf{RV}$ and we may assume that $\mathbf{RV}(C_1\langle a_1 \rangle) \subseteq \mathbf{RV}(C_1)$. Moreover, as f' respects $|_1^{\mathcal{R}}$, f' respects $\mathcal{R}$ and by Remark **(II.5.1)** and Proposition **(II.5.2)**, replacing, if need be, a_1 by its inverse, we can assume that $a_1 \in \mathcal{R}$.
Let $a_2 = f'(a_1)$ and let us define f'' on $\mathbf{K}(\langle C_1 \rangle a_1)$ by $f''(t(a_1)) = t^f(a_2)$ — clearly coinciding with f' on $\mathbf{K}(C_1)[a_1]$. This is well defined. Indeed, it suffices to check that if $t(a_1) = 0$ then $t^f(a_2) = 0$. But, by Weierstrass preparation, there exists $S \in \mathcal{SC}^{\mathcal{R}}(C_1)$, an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C_1)$ term E (a strong unit on S) and $P, Q \in \mathbf{K}(C_1)[X]$ such that Q does not have any zero in $S(\overline{\mathbf{K}(C_1)}^{\mathrm{alg}})$, $a_1 \in S$ and for all $x \in S$, $t(x) = E(x)P(x)/Q(x)$. As $t(a_1) = 0$ and $E(x) \neq 0$, we must have $P(a_1) = 0$. As f' is a partial $\mathcal{L}_{\mathrm{alg}}$-isomorphism, we have $a_2 \in S^f$ and $P^f(a_2) = 0$. As f is an $\mathcal{L}_{\mathrm{an}}$-isomorphism, by Theorem **(II.3.7)** it is in fact an elementary partial $\mathcal{L}_{\mathrm{an}}$-isomorphism and we also have that for all $x \in S^f$, $t^f(x) = E^f(x)P^f(x)/Q^f(x)$ and E^f is a strong unit on S^f. Hence, $t^f(a_2) = E^f(a_2)P^f(a_2)/Q^f(a_2) = 0$.
Let us show that $f'' \cup f|_{\mathbf{RV}}$ is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-isomorphism. By Lemma **(II.A.13)**, it suffices to show that for all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C_1)$-terms t, $\mathrm{rv}(t^f(a_2)) = f(\mathrm{rv}(t(a_1)))$. By Remark **(II.1.5)**, S is defined by a formula of the form $\theta(\mathrm{rv}(\overline{R}(x)))$ where θ is an $\mathcal{L}_{\mathrm{alg}}|_{\mathbf{RV}}$-formula and the R_i are polynomials in $\mathbf{K}(C_1)[X]$. By [CL07, proof of Theorem 7.5], there exists an $\mathcal{L}_{\mathrm{alg}}(C_1)$-definable function $g : K \to \prod_i \mathbf{RV}_{n_i}$ such that every fiber is an open $\mathcal{O}$-ball and for any polynomial T equal to P, Q or one of the R_i, $\mathrm{rv}(T(x))$ is constant on any fiber of g. It

follows immediately that every fiber of g is either in S or in its complement. Let $\overline{\alpha} = g(a_1)$ and $\beta = \mathrm{rv}(t(a_1))$. As E is a strong unit, on $g^{-1}(\overline{\alpha}) = \mathring{\mathcal{B}}_{\mathrm{val}(d)}(c)$ it is of the form $eF((x-c)/d)$ with $\mathrm{val}(F((x-c)/d)) = 0$. As $\mathrm{res}((x-c)/d) = 0$ on all of $g^{-1}(\overline{\alpha})$, by Corollary **(II.3.19)**, $\mathrm{rv}(E(x))$ is constant on $g^{-1}(\overline{\alpha})$, and hence $\mathrm{rv}(t(x))$ is constant on $g^{-1}(\overline{\alpha})$. As f is a partial elementary $\mathcal{L}_{\mathrm{an}}$-isomorphism and $\overline{\alpha}$ and $\beta \in \mathbf{RV}(C_1)$, the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-formula $\forall x, g(x) = \overline{\alpha} \Rightarrow \mathrm{rv}(t(x)) = \beta$ is preserved by f. And as f' is a partial elementary $\mathcal{L}_{\mathrm{alg}}$-isomorphism (by Theorem **(II.1.4)**) and g is $\mathcal{L}_{\mathrm{alg}}(C_1)$-definable, $g^f(a_2) = f(\overline{\alpha})$ and we have that $\mathrm{rv}(t^f(a_2)) = f(\beta) = f(\mathrm{rv}(t(a_1)))$. ∎

Corollary II.5.4:
The previous proposition holds without any assumption on residue characteristic.

Proof. Recall Proposition **(II.4.24)** and assume M_1 and M_2 have mixed characteristic and f and f' are as in Proposition **(II.5.3)**.
Then $\mathfrak{C}^{\infty}_{\mathrm{alg}}(f')$ is an extension of $\mathfrak{C}^{\infty}_{\mathrm{an}}(f)$ whose domain contains a_1. Applying Proposition **(II.5.3)**, we obtain f'' an $\mathcal{L}^{\infty}_{\mathrm{an}}$-isomorphism extending $\mathfrak{C}^{\infty}_{\mathrm{an}}(f)$ whose domain contains a_1 and we conclude by applying $\mathfrak{U}\mathfrak{C}^{\infty}_{\mathrm{an}}$. ∎

Corollary II.5.5:
Let $\varphi(x,\overline{y},\overline{r})$ be any $\mathcal{L}_{\mathrm{an}}$-formula where x and $\overline{y}$ are $\mathbf{K}$-variables and $\overline{r}$ are $\mathbf{RV}\cup\Sigma_{\mathbf{RV}}$-variables, then there exists a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathrm{alg}}$-formula $\psi(x,\overline{z},\overline{r})$ and $\mathcal{L}_{\mathrm{an}}|_{\mathbf{K}}$-terms $\overline{u}(\overline{y})$ such that $T_{\mathrm{an}} \vDash \varphi(x,\overline{y},\overline{r}) \iff \psi(x,\overline{u}(\overline{y}),\overline{r})$.

Proof. This follows from the previous corollary by a (classic) compactness argument. For the sake of completeness (and also because the uniformization part of that argument may be less usual), let us state it. Consider the set of formulae

$$T_{\mathrm{an}} \cup \{\varphi(x_1,\overline{y},\overline{r}), \neg\varphi(x_2,\overline{y},\overline{r})\}\cup$$
$$\{\psi(x_1,\overline{u}(\overline{y}),\overline{r}) \iff \psi(x_2,\overline{u}(\overline{y}),\overline{r}) : \psi \text{ is a } \mathcal{L}_{\mathrm{alg}}\text{-formula and the } \overline{u} \text{ are } \mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}\text{-terms}\}.$$

By Corollary **(II.5.4)**, this set of formulas cannot be consistent. Hence there is a finite set of $\mathcal{L}_{\mathrm{alg}}$-formulae $(\psi_i)_{0\leqslant i<n}$ — that we can take $\mathbf{K}$-quantifier free by Theorem **(II.1.4)** — and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-terms $\overline{u}_i$ such that:

$$T_{\mathrm{an}} \vDash \forall\overline{y}x_1x_2(\bigwedge_i \psi_i(x_1,\overline{u}_i(\overline{y}),\overline{r}) \iff \psi_i(x_2,\overline{u}_i(\overline{y}),\overline{r})) \Rightarrow (\varphi(x_1,\overline{y},\overline{r}) \iff \varphi(x_2,\overline{y},\overline{r})).$$

For all $\varepsilon \in 2^n$, let $\theta_\varepsilon := \bigwedge \psi_i(x,\overline{u}_i(\overline{y}),\overline{r})^{\varepsilon(i)}$ where $\psi^1 = \psi$ and $\psi^0 = \neg\psi$. For fixed $\overline{y}$ and $\overline{r}$, the $\theta_\varepsilon(x,\overline{y},\overline{y})$ form a partition of $\mathbf{K}$ compatible with $\varphi(x,\overline{y},\overline{r})$. For all $\eta \in 2^{2^n}$, let $\chi_\eta(\overline{y},\overline{r})$ be a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathrm{an}}$-formula equivalent to $\bigwedge_\varepsilon(\exists x\,\theta_\varepsilon(x,\overline{y},\overline{r}) \wedge \varphi(x,\overline{y},\overline{r}))^{\eta(\varepsilon)}$. Note that for any choice of $\overline{y}$ and $\overline{r}$ there is exactly one η such that $\chi_\eta(\overline{y},\overline{r})$ holds. It is now quite easy to show that $\varphi(x,\overline{y},\overline{r}) \iff \bigvee_\eta(\chi_\eta(\overline{y},\overline{r}) \wedge \bigvee_{\varepsilon\in\eta}\theta_\varepsilon(x,\overline{y},\overline{r}))$. ∎

Remark II.5.6:

1. This corollary is a stronger version of [DHM99, Theorem B]. Not only is it resplendent but it also has better control of the parameters (essentially due to a better control of the parameters in Weierstrass preparation in [CL11]). In particular, it is uniform.

2. Let $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ be $\mathcal{L}^{\mathrm{ac}}$ enriched with symbols for all the functions from $\mathcal{A}$, a symbol $\mathcal{Q} : \mathbf{K}^2 \to K$, for all units $E \in \mathcal{A}$ a symbol $E_k : \mathbf{R}_k \to \mathbf{R}_k$, a symbol $|^{\mathcal{R}} \subseteq (\mathbf{\Gamma}^{\infty})^2$. Then, any $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$-formula (or even formulae in an $\mathbf{R} \cup \mathbf{\Gamma}$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$) can be translated into an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ (see Proposition **(II.1.8)**), and hence Corollary **(II.5.5)** also holds (resplendently) for the $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$-theory $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathrm{Hen}}$ of Henselian valued fields with separated $\mathcal{A}$-structure and angular components. Note that some of the symbols we should have added have disappeared, like the trace of E_k on $\mathbf{\Gamma}^{\infty}$ which is constant equal to 0. Similarly the E_k and $|^{\mathcal{R}}_1$ are missing one of their arguments — the $\mathbf{\Gamma}^{\infty}$-argument in the case of E_k and the $\mathbf{R}_n$-argument for $|^{\mathcal{R}}$ — but they depend trivially on it.

II.6. K-quantifier elimination in $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$

Until Section II.6.3, we will be working in equicharacteristic zero, hence, we will once again write $\mathbf{RV}$ and rv for $\mathbf{RV}_1$ and rv_1. We will also be considering that variables are indexed by $\mathbb{N}$ and we will sometime identify a variable and its index. But hopefully no confusion should arise.
Let $M \models \mathrm{T}_{\mathcal{A}}$ and $C \leqslant M$.

Definition II.6.1 (Order-degree)**:**
We say that an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term $t = \sum_{i=0}^{d} t_i(\overline{x}_{\neq m})x_m^i$ is polynomial of order (at most) d in x_m. If t is not of this form, we take the convention that t has infinite degree in x_m. Let $\mathcal{T}(C)$ be the set of tuples (t, I, m, d) where I is a finite set of variables, $m \in I$, $d \in \mathbb{N} \cup \{\infty\}$ and $t \neq 0$ is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term whose variables are contained in I and which is polynomial in x_m of degree at most d. Let $\mathcal{T}_0(C) = \mathcal{T}(C) \cup \{0\}$.
We (partially) order $\mathcal{T}(C)$ by saying that (u, J, n, e) has lower order-degree than (t, I, m, d) if one of the following holds:

(i) $\max(J) < \max(I)$*;*

(ii) $\max(J) = \max(I)$ *and* $J \subsetneq I$*;*

(iii) $J = I$ *and* $n > m$*;*

(iv) $J = I$ *and* $n = m$ *and* $e < d$*.*

We extend this order to $\mathcal{T}_0(C)$ by making the zero term greater than any element of $\mathcal{T}(C)$.

Remark II.6.2:
1. This is a well-founded (partial) order.

2. In condition (iii), the order is inverse of what one would expect but that is because we want minimal terms to be polynomial in the last variable.

3. We will also write $J < I$ to mean that conditions (i) or (ii) hold.

When $\overline{a}$ is indexed by some set $I \subseteq \mathbb{N}$ and $n \in I$, we will denote by $\overline{a}_{\neq n}$ the tuple $\overline{a}$ missing its n-th component and $(\overline{a}_{\neq n}, x_n)$ for the tuple $\overline{a}$ where the n-th component is replaced by x_n. We define $\overline{\sigma}_{\neq n}(a)$ and $(\overline{\sigma}_{\neq n}(a), x_n)$ similarly and let $\overline{\sigma}_{\leqslant n}(a) := (a, \sigma(a), \ldots, \sigma^n(a))$. Finally, we will write $\langle C\rangle_\sigma := \langle C\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}}$ and $C\langle\overline{c}\rangle_\sigma := C\langle\overline{c}\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}}$ (cf. Definition **(II.A.12)**).

II.6.1. Residual and ramified extensions

Definition II.6.3 (Regularity)**:**
Let $t(\overline{x}) = \sum_i t_i(\overline{x}_{\neq m})x_m^i$ *be an* $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(M)$ *term and* $\overline{a} \in \mathbf{K}(M)$*. We say that* t *is regular at* $\overline{a}$ *in* x_m *if*

$$\mathrm{val}(t(\overline{a})) = \min_i\{\mathrm{val}(t_i(\overline{a}_{\neq m})) + i\mathrm{val}(a_m)\}.$$

By convention the zero term is never regular.

First, we state a proposition which has nothing to do with automorphisms:

Proposition II.6.4:
Let $\overline{\alpha} \in \mathrm{rv}(\mathcal{R}(M))$*,* $\overline{a} \in \mathrm{rv}^{-1}(\overline{\alpha})$ *and* $(t, I, m, d) \in \mathcal{T}_0(C)$ *be of minimal order-degree such that* $t(\overline{x})$ *is polynomial in* x_m *and* t *is not regular at* $\overline{a}$*. Then for all* $(u, J, n, e) < (t, I, m, d)$*,* $\mathrm{rv}(u(\overline{x}))$ *is constant on* $\mathrm{rv}^{-1}(\overline{\alpha})$*. Moreover for all* $\overline{a}_{\neq n} \in \mathrm{rv}^{-1}(\overline{\alpha}_{\neq n})$*,* $u(\overline{a}_{\neq n}, x_n)$ *has Weierstrass division on* $\mathrm{rv}^{-1}(\alpha_n)$*.*

Proof. First, we may assume that $\overline{\mathbf{K}(M)}^{\mathrm{alg}} = \mathbf{K}(M)$ (see Proposition **(II.3.26)**). We work by induction on J. The proposition is trivial for constant terms. Now, assume the proposition is true for any (v, K, p, f) with $K < J$. Let us first assume that u is polynomial in x_n. Then, $u = \sum_i u_i(\overline{x}_{\neq n})x_n^i$ must be regular at $\overline{a}$ and hence $\mathrm{val}(u(\overline{a})) = \min_i\{\mathrm{val}(u_i(\overline{a}_{\neq n})) + i\mathrm{val}(a_n)\}$ and hence $\mathrm{rv}(u(\overline{a})) = \sum_i \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i \neq 0$. For any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\alpha})$ and any i, $\mathrm{rv}(u_i(\overline{e}_{\neq n}))\mathrm{rv}(e_n)^i = \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i$. Moreover, if $\sum_i \mathrm{rv}(c_i) \neq 0$ then $\mathrm{rv}(\sum_i c_i) = \sum_i \mathrm{rv}(c_i)$ hence we must also have $\mathrm{rv}(u(\overline{e})) = \sum_i \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i \neq 0$. As u is polynomial in x_n, it has a Weierstrass preparation. Hence for polynomial u, the proposition is proved.
Suppose now that u is of infinite degree in x_n and hence that all terms (v, J, n, e) with $e \neq \infty$ are smaller than (t, I, m, d) (and thus have been taken care of in the previous paragraph). By Weierstrass preparation, there exists $S \in \mathcal{SC}^{\mathcal{R}}(C\langle\overline{a}_{\neq n}\rangle)$ such that $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on S and $a_n \in S$. But then either $\mathrm{rv}^{-1}(\alpha_n) \subseteq S$ or $\mathrm{rv}^{-1}(\alpha_n)$ contains a $\overline{\mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$-ball and hence a point $c \in \overline{\mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$. Let $P = \sum p_i(\overline{a}_{\neq n})X^i \in \mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)[X]$ be its minimal polynomial, then for all $e \in \mathrm{rv}^{-1}(\alpha_n)$, $\mathrm{rv}(P(e)) = 0$ — i.e. $P(e) = 0$ — but that is absurd. Hence t has a Weierstrass preparation on $\mathrm{rv}^{-1}(\alpha_n)$ and there exists $F(x, \overline{z}) \in \mathcal{A}$, $\overline{c} \in \mathbf{K}(C\langle\overline{a}\rangle)$, P and $Q \in \mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)[X]$ such that for all $x_n \in \mathrm{rv}^{-1}(\alpha_n)$:

$$u(\overline{a}_{\neq n}, x_n) = F\left(\frac{x_n - a_n}{a_n}, \overline{c}\right)\frac{P(x_n)}{Q(x_n)}$$

and $\mathrm{val}(F((x_n - a_n)/a_n, \overline{c})) = 0$. But $\mathrm{rv}(P(x_n))$ and $\mathrm{rv}(Q(x_n))$ do not depend on x_n and $\mathrm{rv}(F((x_n - a_n)/a_n, \overline{c}))$ only depends on $\mathrm{res}((x_n - a_n)/a_n) = 0$ (see Corollary **(II.3.19)**). Hence $\beta := \mathrm{rv}(u(\overline{a}_{\neq n}, x_n))$ does not depend on $x_n \in \mathrm{rv}^{-1}(\alpha_n)$. The $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula

$$\forall x_n\, \mathrm{rv}(x_n) = \alpha_n \Rightarrow \mathrm{rv}(u(\overline{a}_{\neq n}, x_n)) = \beta$$

is in the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{a}_{\neq n}$ over $C\alpha_n\beta$. By induction (and Corollary **(II.1.6)**), all tuples $\overline{a}_{\neq n} \in \mathrm{rv}^{-1}(\overline{\alpha}_{\neq n})$ have the same $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C\alpha\beta)$-type and $\mathrm{rv}(u(\overline{a})) = \beta$ for all $\overline{a} \in \mathrm{rv}^{-1}(\overline{\alpha})$. ∎

Let us now prove the first embedding theorem we will need for elimination of quantifiers. Let M_1 and M_2 be models of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, $C_i \leqslant M_i$ and $f : C_1 \to C_2$ an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism.

Proposition II.6.5:
Let $\alpha \in \mathrm{rv}(\mathcal{R}(M_1)) \cap \mathbf{RV}(C_1)$, $a \in \mathrm{rv}^{-1}(\alpha)$ and $(t, I, m, d) \in \mathcal{T}_0(C)$ be polynomial in x_m for some $m \in \mathbb{N}$. Assume that (t, I, m, d) is of minimal order-degree such that t is not regular at $\overline{\sigma}(a)$. Then:

(i) There exists $a_1 \in \mathcal{R}(M_1)$ and $a_2 \in \mathcal{R}(M_2)$ such that $t(\overline{\sigma}(a_1)) = 0 = t^f(\overline{\sigma}(a_2))$, $\mathrm{rv}(a_1) = \alpha$ and $\mathrm{rv}(a_2) = f(\alpha)$.

(ii) For any such a_i, f can be extended to an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism sending a_1 to a_2.

Proof. Let $t = \sum_{i=0}^{d} t_i(\overline{x}_{\neq m})x_m^i$. By minimality of t, we cannot have $t_d(\overline{\sigma}_{\neq m}(a)) = 0$. Dividing by t_d, we may assume that $t_d = 1$.

Claim II.6.6: *There exists $\overline{c} \in \mathbf{K}(M)$ that linearly approximates t on $\mathrm{rv}^{-1}(\alpha)$ at prolongations and such that*

$$\min_j\{\mathrm{val}(c_j) + \mathrm{val}(\sigma^j(a))\} = \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\mathrm{val}(\sigma^m(a))\}.$$

Proof. Let $N_i = \overline{M_i}^{\mathrm{alg}}$ (see Proposition **(II.3.26)**). Let $s_e := \sum_{i<e} t_i x_m^i$ and $s := s_d$. For all i and $j \neq m$, by Proposition **(II.6.4)** applied in N_1, $t_i(\overline{\sigma}_{\neq j}(a), x_j)$ has Weiestrass preparation on the ball $b_j := \mathrm{rv}^{-1}(\alpha_j)$ and constant valuation. By Proposition **(II.6.4)**, for all $e \leqslant d$, s_e also has constant valuation on $\mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. By invariance under addition — and an induction on e — we can show that $s(\overline{\sigma}_{\neq j}(a), x_j)$ also has Weiestrass preparation on b_j. Moreover $\partial s/\partial x_j(\overline{x})$ is also given by an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-term of degree $d-1$ in x_m hence $\mathrm{rv}(\partial s/\partial x_j(\overline{\sigma}_{\neq j}(a), x_j))$ is constant on b_j (equal to some $\mathrm{rv}(c_j)$, where $c_j \in \mathbf{K}(M_1)$). By Proposition **(II.3.27)**, for all y_j and $z_j \in b_j$:

$$\mathrm{rv}(t(\overline{\sigma}_{\neq j}(a), y_j) - t(\overline{\sigma}_{\neq j}(a), z_j)) = \mathrm{rv}(s(\overline{\sigma}_{\neq j}(a), y_j) - s(\overline{\sigma}_{\neq j}(a), z_j)) = \mathrm{rv}(c_j)\mathrm{rv}(y-z).$$

This last statement is in the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{\sigma}_{\neq j}(a)$ over $C_1\mathrm{rv}(c_j)$. By Proposition **(II.6.4)** and Corollary **(II.1.6)**, any $\overline{e}_{\neq j} \in \mathrm{rv}^{-1}(\overline{\sigma}_{\neq j}(\alpha))$ has the same $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1\mathrm{rv}(c_j))$-type and hence the same c_j works for any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$.
As $\mathrm{val}(s(\overline{\sigma}_{\neq j}(a), x_j)) = \min_{i<d}\{\mathrm{val}(t_i(\overline{\sigma}_{\neq j}(a))) + i\mathrm{val}(x_j)\}$ is constant on b_j, we also have:

$$\mathrm{val}(c_j) + \mathrm{val}(\sigma^j(a)) \geqslant \mathrm{val}(s(\overline{\sigma}(a))) \geqslant \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\mathrm{val}(\sigma^m(a))\}.$$

When $j = m$, as t is polynomial in x_m and $\partial t/\partial x_m(\overline{x})$ is of degree $d-1$ in x_m, we can also find $c_m \in \mathbf{K}(M_1)$ that linearly approximates $t(\overline{e}_{\neq m}, x_m)$ on $\mathrm{rv}^{-1}(\sigma^m(\alpha))$ for any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. And

$$\mathrm{val}(c_m) + \mathrm{val}(\sigma^m(a)) = \mathrm{val}(\partial t/\partial x_m(\overline{x})) + \mathrm{val}(\sigma^m(a)) = \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\mathrm{val}(\sigma^m(a))\}.$$

It now follows from Proposition **(II.4.6)** that $\overline{c}$ linearly approximates t on $\mathrm{rv}^{-1}(\alpha)$ at prolongations. ♦

If $t \neq 0$, we have proved that $(t, a, \overline{c}, \mathrm{val}_{\mathbf{RV}}(\alpha))$ is in σ-Hensel configuration. Hence there exists $a_1 \in M_1$ such that $t(a_1) = 0$ and $\mathrm{val}(a_1 - a) \geqslant \max_i\{\sigma^{-i}(t(\overline{\sigma}(a)c_i^{-1}))\}$. In particular, $\mathrm{val}(\sigma^m(a_1 - a)) \geqslant \mathrm{val}(t(\overline{\sigma}(a))) - \mathrm{val}(c_m) > \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\mathrm{val}(\sigma^m(a))\} - \mathrm{val}(c_m) = \mathrm{val}(\sigma^m(a))$, i.e. $\mathrm{rv}(a_1) = \mathrm{rv}(a)$.
If x_m is not the highest variable appearing in t — that we call x_n — then, applying Proposition **(II.6.4)** to $(t, I, n, \infty) < (t, I, m, d)$, we get that $\mathrm{rv}(t(\overline{x}))$ is constant equal to 0 on all of $\mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. As $t(\overline{\sigma}_{\neq m}(a), x_m)$ is polynomial and has infinitely many zeros, we must have $t_i(\overline{\sigma}(a)) = 0$ for all i, but that contradicts the non-regularity of t in x_m at $\overline{\sigma}(a)$. Thus x_m must be the highest variable appearing in t. For the same reasons, we cannot have $\mathrm{rv}(c_m) = 0$.
Note that we have also proved that for all $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$, t is minimal such that it is not regular in $\overline{e}$, hence the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{\sigma}(\alpha)$ says so and hence, as $\mathrm{T}_{\mathcal{A}}$ eliminates field quantifiers, t^f has the same minimality property (relative to $f(\alpha)$) and we find a_2 in the exact same way. If $t = 0$ then any a_1 and $a_2 \in \mathrm{rv}^{-1}(\alpha)$ will work.
Let us now show that f can be extended to send a_1 to a_2. First, extending f on $\mathbf{RV}$, we can assume that $\mathbf{RV}(C_1\langle a_1\rangle_\sigma) \subseteq \mathbf{RV}(C_1)$. Let $C_{i,n} := C_i\langle\overline{\sigma}_{\leqslant n}(a_1)\rangle$ and $f_{-1} := f : C_1 \to C_2$. Let us show that, for all n, we can extend f_{n-1} to $f_n : C_{1,n} \to C_{2,n}$ sending $\sigma^n(a_1)$ to $\sigma^n(a_2)$. If $n < m$, for any term u polynomial in x_n of order-degree strictly smaller than (t, I, m, d), let us define $f_n(\sum_i u_i(\overline{\sigma}_{\neq n}(a_1))\sigma^n(a_1)^i) = \sum_i f_{n-1}(u_i(\overline{\sigma}_{\neq n}(a_1)))\sigma^n(a_2)^i$. It follows from regularity of u in x_n at $\overline{\sigma}(a_1)$ that if for some i $u_i(\overline{\sigma}_{\neq n}(a_1)) \neq 0$ then $u(\overline{\sigma}(a_1)) \neq 0$. Thus $\sigma^n(a_i)$ is transcendental over $C_{i,n-1}$ and f_n is a field isomorphism. To show that this is an $\mathcal{L}^{\mathbf{RV}^+}$-isomorphism, it suffices, by Lemma **(II.A.13)**, to show that it respects rv. But this is true, as for all terms of order-degree strictly smaller than (t, I, m, d), $\mathrm{rv}(u(\overline{\sigma}(a_1))) =: \beta$ does not depend on the choice of a_1 and the formula "$\forall\overline{x}\,\mathrm{rv}(\overline{x}) = \overline{\sigma}(\alpha) \Rightarrow \mathrm{rv}(u(\overline{x})) = \beta$" is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-formula respected by f.

Claim II.6.7: *Let* $P := t(\overline{\sigma}_{\neq m}(a_1), x_m) \in \mathbf{K}(C_{1,m-1})[X]$. *For all* $n \geqslant m$, $\sigma^n(a_1)$ *is the only zero of* $P^{\sigma^{n-m}}$ *whose leading term is* $\sigma^n(\alpha)$.

Proof. Because σ is an automorphism of valued fields, it suffices to prove the case $n = m$. Let $e \in \mathrm{rv}^{-1}(\sigma^m(\alpha))$, then $\mathrm{rv}(P(e)) = \mathrm{rv}(P(e) - P(a)) = \mathrm{rv}(c_m)\mathrm{rv}(e - \sigma^m(a_1)) \neq 0$. ♦

The same claim is true of $\sigma^m(a_2)$ with respect to $f(P^{\sigma^{n-m}})$ and $f(\sigma^n(\alpha))$. Thus, it suffices to extend f_{m-1} to the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-definable closure of $C_{1,m-1}$, which we can certainly do as f_{m-1} is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-elementary isomorphism (by resplendent field quantifier elimination in $\mathrm{T}_{\mathcal{A}}$).
Then $f' = \bigcup_n f_n$ is an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism between $C_1\langle a_1\rangle_\sigma$ and $C_2\langle a_2\rangle_\sigma$. It is also an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism by Lemma **(II.A.13)**, ■

Corollary II.6.8:
Let $\alpha \in \mathbf{RV}(C_1)$, *then there exists* $a_1 \in M_1$ *such that* $\mathrm{rv}(a_1) = \alpha$ *and* f *extends to an isomorphism on* $C_1\langle a_1\rangle_\sigma$.

Proof. If $\alpha \in \mathrm{res}(\mathcal{R}(M_1))$, then Proposition **(II.6.5)** applies. If not apply Proposition **(II.6.5)** to α^{-1} and conclude by extending the isomorphism to the analytic field generated by its domain by Remark **(II.5.1)**. ■

II.6.2. Immediate extensions

Let $M \vDash \mathrm{T}_{\mathcal{A}}$ be saturated enough and $C \leqslant M$.

Definition II.6.9 (pseudo-convergent $\star$-sequences)**:**
Let $(\overline{x}_\alpha)$ be a sequence of tuples of the same length. We say that it is a pseudo-convergent sequence if for all i, $(\overline{x}_{i,\alpha})$ is pseudo-convergent. Moreover, we will say that $\overline{a}$ is a pseudo-limit of $(\overline{x}_\alpha)$ if for all i, $\overline{x}_{i,\alpha} \rightsquigarrow a_i$.

Definition II.6.10 (Equivalent pseudo-convergent sequences)**:**
We will say that two pseudo-convergent sequences are equivalent if they have the same pseudo-limits.

Lemma II.6.11:
Let $\overline{x}_\alpha$ be a pseudo-convergent sequence, $\overline{a}$ a pseudo-limit of this sequence and $\overline{y}_\alpha$ such that for all i, $\mathrm{val}(a_i - y_{i,\alpha}) = \mathrm{val}(a_i - x_{i,\alpha})$, then (y_α) is also a pseudo-convergent sequence that is equivalent to (x_α).

Proof. We may assume that $|x_\alpha| = 1$. Note that for all $\beta > \alpha$, $\mathrm{val}(y_\beta - y_\alpha) = \mathrm{val}(y_\beta - a + a - y_\alpha) = \mathrm{val}(a - x_\alpha) = \mathrm{val}(x_\beta - x_\alpha)$, as $\mathrm{val}(a - x_\beta) > \mathrm{val}(a - x_\alpha)$. Hence (y_α) is also pseudo-convergent. Moreover, if b is any pseudo-limit of (x_α), then $\mathrm{val}(b - y_\alpha) = \mathrm{val}(b - x_{\alpha+1} + x_{\alpha+1} - a + a - y_\alpha) = \mathrm{val}(a - y_\alpha) = \mathrm{val}(a - x_\alpha) = \mathrm{val}(b - x_\alpha)$ and $y_\alpha \rightsquigarrow b$. The symmetric argument shows that if $y_\alpha \rightsquigarrow b$ then $x_\alpha \rightsquigarrow b$. ■

Definition II.6.12 (Rich enough families)**:**
We say that a family $\mathcal{F}$ of equivalent pseudo-convergent sequences of C is rich enough if for any linear polynomial $P(\overline{X}) = \sum_i \pi_i X_i \in \mathrm{rv}(\mathbf{K}(C))[\overline{X}]$, there exists $(\overline{x}_\alpha) \in \mathcal{F}$ such that for all pseudo-limit $\overline{a}$ and all α, $P(\mathrm{rv}(\overline{a} - \overline{x}_\alpha)) \neq 0$, i.e. if $\mathrm{rv}(p_i) = \pi_i$ then $\mathrm{val}(\sum_i p_i(a_i - x_{i,\alpha})) = \min_i\{\mathrm{val}(p_i) + \mathrm{val}(a_i - x_{i,\alpha})\}$.

We will say that a term $u = \sum_{i=0}^{d} u_i(\overline{x}_{\neq m})\sigma^m(x)^i$ is monic if $u_d = 1$. As in section II.6.1, let us begin by a proposition that does not seem to have anything to do with automorphisms.

Proposition II.6.13:
Let $\mathcal{F}$ be a rich enough family of equivalent pseudo-convergent sequences of C that are eventually in $\mathcal{R}$ and $(t, I, m, d) \in \mathcal{T}_0(C)$. Suppose that (t, I, m, d) has minimal order-degree such that t is a monic polynomial in x_m and there exists a pseudo-convergent sequence $(\overline{x}_\alpha) \in \mathcal{F}$ that pseudo-solves t. Then for all $(u, J, n, e) < (t, I, m, d)$, there exists α_0 such $\mathrm{rv}(u(\overline{x}))$ is constant on $\overline{b}_0 := \mathring{\mathcal{B}}_{\overline{\gamma}_0}(\overline{x}_{\alpha_0+1})$, where $\overline{\gamma}_0 := \mathrm{val}(\overline{x}_{\alpha_0+1} - \overline{x}_{\alpha_0})$ — it follows immediately that $\mathrm{rv}(u(\overline{x})) \in \mathrm{rv}(\mathbf{K}(C))$ — and for any $\overline{a} \in \overline{b}_0$, $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on $b_{n,0}$.

Proof. We may assume that $\overline{\mathbf{K}(M)}^{\mathrm{alg}} = \mathbf{K}(M)$. The proof proceeds by induction on J. Suppose that Proposition **(II.6.13)** holds for any term (v, K, p, f) such that $K < J$. Let us prove a few claims to take care of some induction steps.

Claim II.6.14: *Fix e and $n \in \mathbb{N}$. Suppose the lemma holds for all (u, J, n, e), then it holds for any $(u, J, n, e+1)$ where u is a monic polynomial in x_n.*

Note that the case $e = 0$ does not require any hypothesis (other than the induction hypothesis on J).

Proof. Let $u = x_n^{e+1} + \sum_{i \leqslant e} u_i(\overline{x}_{\neq n})x_n^i$ and for all $f \leqslant e$, $s_f := \sum_{i \leqslant f} u_i(\overline{x}_{\neq n})x_n^i$. Let $\overline{a}$ be a pseudo limit of $(\overline{x}_\alpha)$. Then we can find α_0 such that, for all $j \neq n$, $\mathrm{val}(s_f(\overline{a}_{\neq j}, x_j))$ and $\mathrm{val}(u_{f+1}(\overline{a}_{\neq j}, x_j))$ are constant on $b_{j,0}$ and $u_{f+1}(\overline{a}_{\neq j}, x_j)$ has a Weierstrass preparation. By induction on f and invariance under addition, $s_e(\overline{a}_{\neq j}, x_j)$ has a Weierstrass preparation on $b_{j,0}$. Let $s := s_e$. Making α_0 bigger we can also assume that $\mathrm{val}(\partial s/\partial x_j(\overline{a}_{\neq j}, x_j))$ is constant on $b_{j,0}$. By Proposition **(II.3.27)**, we find $c_j \in \mathbf{K}(C)$ such that for all y_j and $z_j \in b_{j,0}$, $\mathrm{rv}(u(\overline{a}_{\neq j}, y_j) - u(\overline{a}_{\neq j}, z_j)) = \mathrm{rv}(s(\overline{a}_{\neq j}, y_j) - s(\overline{a}_{\neq j}, z_j)) = \mathrm{rv}(c_j)\mathrm{rv}(y_j - z_j)$. By field quantifier elimination, this statement only depends on the value of $\mathrm{rv}(v(\overline{a}_{\neq j}))$ for a finite number of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms v and hence, by induction, making α_0 bigger, we may assume that this statement is true of all $\overline{a} \in \overline{b}_0$. When $j = n$, the same arguments yields some $c_n \in \mathbf{K}(C)$ as u is already polynomial in x_n and $\partial u/\partial x_n(\overline{x})$ is polynomial in x_n of degree at most e. It now follows from Proposition **(II.4.6)** that $\overline{c}$ linearly approximates u on $\overline{b}_0$.
Let $(\overline{y}_\alpha) \in \mathcal{F}$ be such that for any pseudo-limit $\overline{a}$, $\mathrm{val}(\overline{c} \cdot (\overline{a} - \overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$. Then $\mathrm{val}(u(\overline{a}) - u(\overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$. If for all α, $\mathrm{val}(u(\overline{a})) > \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$, then $\mathrm{val}(u(\overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$ and $u(\overline{y}_\alpha) \rightsquigarrow 0$ contradicting the minimality of t. Hence $\mathrm{val}(u(\overline{a})) < \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$ for $\alpha \gg 0$ and $\mathrm{rv}(u(\overline{a})) = \mathrm{rv}(u(\overline{y_\alpha})) \in \mathrm{rv}(\mathbf{K}(C))$. By compactness, making α_0 bigger, this is true for any $\overline{a} \in \overline{b}_0$. ♦

Claim II.6.15: *Fix e and $n \in \mathbb{N}$. Suppose the lemma holds for (u, J, n, e) monic polynomial in x_n, then it holds for any (u, J, n, e).*

Proof. Dividing by the dominant coefficient u_e (which has constant rv on $\overline{b}_0$ by induction), we obtain a term v monic polynomial of degree at most e in x_n and which must also have constant rv on $\overline{b}_0$ if we take α_0 big enough. ♦

Claim II.6.16: *Fix $n \in \mathbb{N}$. Suppose that for all $e \in \mathbb{N}$, the lemma holds for all (u, J, n, e). Then it also holds for all (u, J, n, ∞).*

Proof. Let $\overline{a}$ be a pseudo-limit of (x_α). Any $S \in \mathcal{SC}^{\mathcal{R}}(C\langle\overline{a}_{\neq n}\rangle)$ that contains a_n must contain $b_{n,0}$ for α_0 big enough. If not, there exists $c \in \overline{\mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$ such that $x_{n,\alpha} \rightsquigarrow c$. Let $P(\overline{a}_{\neq n}, x_n) = \sum_i p_i(\overline{a}_{\neq n})x_n^i$ be its minimal polynomial. Then, by hypothesis, for all $\overline{e} \in \overline{b}_0$ (for α_0 big enough), $\mathrm{rv}(P(\overline{e})) = 0$ and we must have $p_i(\overline{a}_{\neq n}) = 0$ for all i, but that is absurd. It follows that we can find α_0 such that, $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on $b_{n,0}$, i.e. there exists $F \in \mathcal{A}$, $\overline{c} \in \mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)$ and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}(C)}$-terms P and Q polynomial in x_n such that for all $x_n \in b_{j,0}$,

$$u(\overline{a}_{\neq n}, x_n) = F\left(\frac{x_n - x_{n,\alpha_0+1}}{x_{n,\alpha_0+1} - x_{n,\alpha_0}}, \overline{c}\right)\frac{P(\overline{a}_{\neq n}, x_n)}{Q(\overline{a}_{\neq n}, x_n)}$$

and $\mathrm{val}(F((x_n - x_{n,\alpha_0+1})(x_{n,\alpha_0+1} - x_{n,\alpha_0}), \overline{c})) = 0$. In turn, this implies that $\mathrm{rv}(u(\overline{a}_{\neq n}, x_n))$ does not depend on $x_n \in b_{n,0}$ and by the usual uniformization argument (making α_0 bigger), we can ensure that $\mathrm{rv}(u(\overline{e}))$ does not depend on $\overline{e} \in \overline{b}_0$. ♦

Proposition **(II.6.13)** follows by induction. ■

Remark II.6.17:
Note that the proof of Claim **(II.6.14)** also shows that there exists $\overline{d} \in \mathbf{K}(C)$ that linearly approximates t on $\overline{b}_0$ for α_0 big enough.

Let $M \vDash \mathrm{T}_{\mathcal{A},\sigma}$ be saturated enough and $C \leqslant M$ such that $\mathrm{res}(\mathbf{K}(C))$ is linearly closed.

Proposition II.6.18:
Let x_α be a pseudo-convergent sequence of C. The family $\{\overline{\sigma}(y_\alpha) : y_\alpha$ is a pseudo-convergent sequence of C equivalent to $x_\alpha\}$ is a rich enough family of equivalent pseudo-convergent sequences of C.

Proof. Let $P(\overline{X}) = \sum_i p_i X_i \in \mathbf{K}(C)[\overline{X}]$. If for all i $p_i = 0$, we are done. Otherwise, let $\varepsilon_\alpha = x_{\alpha+1} - x_\alpha$, let i_0 such that $\mathrm{val}(p_{i_0}) + \mathrm{val}(\sigma^{i_0}(\varepsilon_\alpha))$ is minimal, and let

$$Q_\alpha(\overline{\sigma}(X)) = p_{i_0}^{-1}\sigma^{i_0}(\varepsilon_\alpha)^{-1}P(\overline{\sigma}(\varepsilon_\alpha X)) = \sum_i p_i p_{i_0}^{-1}\sigma^i(\varepsilon_\alpha)\sigma^{i_0}(\varepsilon_\alpha^{-1})\sigma^i(X).$$

As $\mathrm{res}(Q_\alpha)$ is linear with coefficients in $\mathrm{res}(\mathbf{K}(C))$, which is linearly closed, we can find $d_\alpha \in \mathbf{K}(C)$ such that $\mathrm{res}(Q_\alpha(\overline{\sigma}(d_\alpha))) \neq \mathrm{res}(Q_\alpha(\overline{\sigma}(1)))$. In particular, $\mathrm{res}(d_\alpha) \neq \mathrm{res}(1)$ and $\mathrm{val}(d_\alpha - 1) = 0$. Let $y_\alpha = x_\alpha + \varepsilon_\alpha d_\alpha$.
Let $\overline{a}$ be such that $\overline{\sigma}(x_\alpha) \rightsquigarrow \overline{a}$, then

$$\begin{aligned}\mathrm{rv}(a_i - \sigma^i(y_\alpha)) &= \mathrm{rv}(a_i - \sigma^i(x_{\alpha+1}) + \sigma^i(x_{\alpha+1}) - \sigma^i(x_\alpha) + \sigma^i(x_\alpha) - \sigma^i(y_\alpha))\\ &= \mathrm{rv}(\sigma^i(\varepsilon_\alpha))\mathrm{rv}(1 - \sigma^i(d_\alpha)).\end{aligned}$$

It follows that $\mathrm{val}(a_0 - y_\alpha) = \mathrm{val}(\varepsilon_\alpha) = \mathrm{val}(a_0 - x_\alpha)$. By Lemma **(II.6.11)**, (y_α) is equivalent to (x_α). Let $c_i = (a_i - \sigma^i(y_\alpha))/\sigma^i(\varepsilon_\alpha)$. Then

$$\begin{aligned}\mathrm{res}(P(\overline{a} - \overline{\sigma}(y_\alpha))p_{i_0}^{-1}\varepsilon_\alpha^{-1}) &= \mathrm{res}(Q)(\mathrm{res}(\overline{c}))\\ &= \mathrm{res}(Q)(\mathrm{res}(\overline{\sigma}(1) - \overline{\sigma}(d_\alpha)))\\ &= \mathrm{res}(Q(\overline{\sigma}(1))) - \mathrm{res}(Q(\overline{\sigma}(d_\alpha)))\\ &\neq 0.\end{aligned}$$

Hence, we have $\mathrm{val}(P(\overline{a} - \overline{\sigma}(y_\alpha))) = \mathrm{val}(p_{i_0}) + \mathrm{val}(\sigma^{i_0}(\varepsilon_\alpha)) = \min_i\{\mathrm{val}(p_i) + \mathrm{val}(a_i - \sigma^i(y_\alpha))\}$. ■

And now let us prove another embedding theorem for immediate extensions. Let M_1 and $M_2 \vDash \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be saturated enough, $N_i \leqslant M_i$ have no immediate extension in M_i and be σ-Henselian — as we will see in Remark **(II.6.21)** this second hypothesis follows from the first one —, $C_i \leqslant N_i$ be such that $\mathrm{res}(\mathbf{K}(C_1))$ is linearly closed and $f : C_1 \to C_2$ an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism.

Definition II.6.19 (Minimal term of a pseudo-convergent sequence):
Let (x_α) be a pseudo-convergent sequence of C_1. We say that $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is its minimal term if it is minimal such that it is monic polynomial in x_n and it is σ-pseudo-solved by a pseudo-convergent sequence equivalent to (x_α).

Note that any pseudo-convergent sequence has a minimal term, as any pseudo-convergent sequence σ-pseudo-solves 0.

Proposition II.6.20:
Let (x_α) be a pseudo-convergent sequence of $\mathbf{K}(C_1)$ (indexed by a limit ordinal) which is eventually in $\mathcal{R}$. Let (t, I, m, d) be its minimal term. Then:

(i) There exists $a_1 \in N_1$ and $a_2 \in N_2$ such that $x_\alpha \rightsquigarrow a_1$, $f(x_\alpha) \rightsquigarrow a_2$ and $t(\overline{\sigma}(a_1)) = 0 = t^f(\overline{\sigma}(a_2))$.

(ii) For any such a_1, $C_1\langle a_1\rangle_\sigma$ is an immediate extension of C_1;

(iii) For any such a_i, f can be extended into an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism sending a_1 to a_2.

Proof. If t is zero, it suffices to choose any a_1 and a_2 such that $x_\alpha \rightsquigarrow a_1$ and $f(x_\alpha) \rightsquigarrow a_2$. These exist in M_i and we will see in the end why they exist in N_i. Let us now assume that t is not zero. By Remark **(II.6.17)** — and Propositions **(II.6.13)** and **(II.6.18)** — we find α_0 and $\overline{d} \in \mathbf{K}(C_1)$ that linearly approximate t at prolongations on $b_0 := \mathring{\mathcal{B}}_{\mathrm{val}(x_{\alpha_0+1}-x_{\alpha_0})}(x_{\alpha_0+1})$. By Proposition **(II.6.18)**, we can find a pseudo-convergent sequence (z_α) of C_1 equivalent to (x_α) such that for all pseudo-limit a of (x_α), $\mathrm{val}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(z_\alpha))) = \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\}$. If for all such a, $\mathrm{val}(t(\overline{\sigma}(a))) < \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\}$ for α big enough, then $\mathrm{val}(t(\overline{\sigma}(a))) = \mathrm{val}(t(\overline{\sigma}(y_\alpha)))$. By compactness, $\mathrm{val}(t(\overline{\sigma}(x)))$ is constant on some b_0. But this contradicts the fact that we can find y_α equivalent to x_α that σ-pseudo-solves t.
Hence there exists a pseudo-limit a such that $\mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\} \geqslant \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(\xi_0))\}$ where ξ_0 is the radius of b_0 and $(t, a, \overline{d}, \xi_0)$ is in σ-Hensel configuration. As N_1 is σ-Henselian, we can find $a_1 \in \mathbf{K}(N_1)$ such that $t(a_1) = 0$ and $\mathrm{val}(a_1 - a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \mathrm{val}(x_{\alpha+1} - x_\alpha)$, i.e. $x_\alpha \rightsquigarrow a_1$. As f is an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism, (t^f, I, m, d) is the minimal term of $(f(x_\alpha))$ and the same argument shows that there is $a_2 \in \mathbf{K}(N_2)$ such that $t^f(a_2) = 0$ and $f(x_\alpha) \rightsquigarrow a_2$.
If $t \neq 0$, let us now show that x_m must be the last variable appearing in t. If it is not, let x_n be that last variable. By Proposition **(II.6.13)**, we can find $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(A)$-terms E, P and Q such that E is a strong unit in x_n, P and Q are polynomial in x_n and for all $\overline{a} \in \overline{b}_0$, $t(\overline{a}) = E(\overline{a})P(\overline{a})/Q(\overline{a})$. As $t(\overline{\sigma}(a_1)) = 0$ we also have $P(\overline{\sigma}(a_1)) = 0$ and because $(P, I, n, \infty) < (t, I, m, d)$, we have $\mathrm{rv}(P(\overline{a})) = 0$ — and hence $\mathrm{rv}(t(\overline{a})) = 0$ — for all $\overline{a} \in \overline{b}_0$, contradicting the fact that we can find y_α equivalent to x_α that σ-pseudo-solves t.
We can now conclude as in Proposition **(II.6.5)** by extending f to $C_{1,n} := C_1\langle\overline{\sigma}_{\leqslant n}(a_1)\rangle$ progressively, by sending $\sigma^n(a_1)$ to $\sigma^n(a_2)$. For $n < m$, it is exactly the same and for $n \geqslant m$, use the fact that $\sigma^n(a_1)$ is the only zero of $P^{\sigma^{n-m}}(X)$ in $(b_{n,0})$ for $\alpha_0 \gg 0$, where $P(X_m) = t(\overline{\sigma}_{\neq m}(a_1), X_m)$.
If $n < m$, we have proved in Proposition **(II.6.13)** that the extension is immediate. If $n \geqslant m$,

we have just seen that $\sigma^n(a_1)$ is ACVF-definable over $\mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)$. It follows that $\sigma^n(a_1) \in \mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)^h$ which is an immediate extension of $\mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)$ and we conclude by Proposition **(II.3.26)**.
In the case where t is zero, we have yet to show that we can take $a_i \in N_i$. Let (u, J, n, e) be minimal over N_1 that is σ-pseudo solved by a pseudo-converging sequence equivalent to x_α. We can find a_1 in M_1 such that $u(\overline{\sigma}(a_1)) = 0$ and $x_\alpha \rightsquigarrow a_1$. But then $\mathbf{K}(N_1\langle a_1\rangle_\sigma)$ is an immediate extension of N_1 and we must have $a_1 \in N_1$. ∎

Remark II.6.21:
Note that we have just shown that if we only assume that N_1 has no immediate extension in M_1 (and not that N_1 is σ-Henselian), then N_1 is maximally complete and hence, by Proposition **(II.4.14)** it is σ-Henselian.

Definition II.6.22 (Minimal term of a point)**:**
Let $a \in M_1$. We say that $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is the minimal term of a over C_1 if it is minimal such that it is monic polynomial in x_m and $t(\overline{\sigma}(a)) = 0$.

Note that because of Weierstrass preparation, minimal terms will always be polynomial in their last variable.

Definition II.6.23 ((t, I, m, d)-fullness)**:**
Let $(t, I, m, d) \in \mathcal{T}_0(C_1)$. We will say that C_1 is (t, I, m, d)-full if for all pseudo-convergent sequences (x_α) (indexed by a limit ordinal) of elements in C_1 that are eventually in $\mathcal{R}$ with minimal term $(u, J, n, e) < (t, I, m, d)$, (x_α) has a pseudo-limit in C_1.

Corollary II.6.24:
Let (x_α) be a maximal pseudo-convergent sequence in C_1 (indexed by a limit ordinal) pseudo-converging to some $a_1 \in \mathcal{R}(M_1)$. If $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is its minimal term over C_1 and C_1 is (t, I, m, d)-full, then $\mathbf{K}(C_1\langle a_1\rangle_\sigma)$ is an immediate extension of $\mathbf{K}(C_1)$ and f extends from $C_1\langle a_1\rangle_\sigma$ into N_2.

Proof. Since C_1 is (t, I, m, d)-full, (x_α) (or any equivalent pseudo-convergent sequence) cannot pseudo-solve a term of order-degree strictly less than (t, I, m, d) (this would contradict either (t, I, m, d)-fullness of C_1 or maximality of (x_α)). By Propositions **(II.6.13)** and **(II.6.18)**, there is a tuple $\overline{d}$ and a sequence (y_α) equivalent to (x_α) such that

$$\mathrm{val}(t(\overline{\sigma}(y_\alpha))) = \mathrm{val}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(y_\alpha))) = \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - y_\alpha))\},$$

i.e. $t(\overline{\sigma}(y_\alpha)) \rightsquigarrow 0$. We have just showed that t is the minimal term of the pseudo-convergent sequence (x_α) and thus we can now apply Proposition **(II.6.20)**. ∎

From now on, suppose that $\mathbf{K}(N_i)$ is an immediate extension of $\mathbf{K}(C_i)$, hence it is a maximal immediate extension of $\mathbf{K}(C_i)$ in M_i.

Corollary II.6.25:
Suppose that all $a \in \mathcal{R}(N_1)$ with a minimal term of order-degree strictly smaller than (t, I, m, d) are already in C_1, then C_1 is (t, I, m, d)-full.

Proof. Let (x_α) be a pseudo-convergent sequence of C_1 (indexed by a limit ordinal) that is eventually in $\mathcal{R}$ and $(u, J, n, e) < (t, I, m, d)$ that is σ-pseudo-solved by (x_α). We may assume that (u, J, n, e) is its minimal term. By Proposition **(II.6.20)**, there is $a_1 \in N_1$ such that $x_\alpha \rightsquigarrow a_1$ and $u(a_1) = 0$. Hence a_1 has a minimal polynomial of order-degree strictly lower than (t, I, m, d), so $a_1 \in C_1$ and C_1 is indeed (t, I, m, d)-full. ■

Corollary II.6.26:
The isomorphism f extends to an isomorphism $N_1 \to N_2$, i.e. maximum immediate extensions (in some saturated model) — and hence maximally complete extensions — are unique up to isomorphism.

We could prove this corollary without using the notion of fullness and without doing the extensions in the right order — just pick any maximal pseudo-convergent sequence indexed by a limit ordinal, find its minimal term and apply Proposition **(II.6.20)** to extend f some more and iterate. But I find the following proof more informative in terms of the information you need to describe the type of a given point in an immediate extension.

Proof. Let us consider the extensions $C_1 \leqslant L_\alpha \leqslant N_1$ defined by taking $L_{\alpha+1} = L_\alpha\langle c_\alpha\rangle_\sigma$ where $c_\alpha \in \mathcal{R}(N_1) \smallsetminus L_\alpha$ has a minimal term of minimal order-degree over $L_{\alpha+1}$ and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for λ limit. Then we can show by induction that we can extend f to L_α in a coherent way. Let us suppose we have extended f to f_α on L_α. Let $a = c_\alpha$. Let $x_\beta \rightsquigarrow a$ be a maximal pseudo-converging sequence of L_α. Then if (t, I, m, d) is a minimal term of a, then by Corollary **(II.6.25)**, L_α is (t, I, m, d)-full. Applying Corollary **(II.6.24)**, we obtain that f_α can be extended to $L_\alpha\langle a\rangle_\sigma = L_{\alpha+1}$. The limit case is trivial.
As N_1 is the field generated by $\bigcup_\alpha L_\alpha$, by Remark **(II.5.1)** we can extend f from N_1 into N_2. Now if f is not onto, pick $a \in \mathbf{K}(N_2) \smallsetminus \mathbf{K}(f(N_1))$, (x_α) maximal pseudo-converging to a and (t, I, m, d) its minimal term. Then applying Proposition **(II.6.20)** the other way round, we would find an immediate extension of N_1 in M_1, but that is absurd. ■

II.6.3. Relative quantifier elimination

Theorem D:
The theory $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ eliminates quantifiers resplendently relative to $\mathbf{RV}$.

Proof. By Proposition **(II.A.9)**, it suffices to show that $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ eliminates quantifiers relative to $\mathbf{RV}$. Note that if two models of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ contain isomorphic substructures they have the same characteristic and residual characteristic, hence it also suffices to prove the result for $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ and $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$. Let us first consider the equicharacteristic zero case. It suffices to show that if M_1 and M_2 are sufficiently saturated models of $\mathrm{T}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$, f a partial $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q}}$-isomorphism with (small) domain C_1, and $a_1 \in K(M_1)$, f can be extended to $C_1\langle a_1\rangle_\sigma$. Let $N_1 \leqslant M_1$ with no immediate extension in M_1 and containing both C_1 and a_1. By Morleyization on $\mathbf{RV}$ and Lemma **(II.A.11)** we can extend f to $D_1 \leqslant N_1$ such that $\mathbf{RV}(D_1) = \mathbf{RV}(N_1)$. Then applying Corollary **(II.6.8)** repetitively we can extend f to $E_1 \leqslant N_1$ such that $\mathrm{rv}(\mathbf{K}(E_1)) = \mathbf{RV}(E_1)$. Now $\mathbf{K}(N_1)$ is a maximal immediate extension of $\mathbf{K}(E_1)$ and we can extend f to N_1 by Proposition **(II.6.20)**.

Now that we know the equicharacteristic zero case, the mixed characteristic case follows from Propositions **(II.B.5)** and **(II.4.23)**. ∎

We also obtain the corresponding results when there are angular components. Let $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ be $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ enriched with a symbol $\sigma : \mathbf{K} \to \mathbf{K}$, symbols $\sigma^n : \mathbf{R}^n \to \mathbf{R}^n$ and a symbol $\sigma_{\mathbf{\Gamma}} : \mathbf{\Gamma}^\infty \to \mathbf{\Gamma}^\infty$. Let $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be the $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of σ-Henselian analytic difference fields with a linearly closed residue field and angular components that are compatible with σ, i.e. $\mathrm{ac}_n \circ \sigma = \sigma_n \circ \mathrm{ac}_n$. Let $\mathcal{L}^{\mathrm{ac},fr}_{\mathcal{A},\mathcal{Q},\sigma}$ be the enrichment of $\mathcal{L}^{\mathrm{ac},fr}$ with the same symbols and $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ be the theory of finitely ramified characteristic $(0,p)$ valued fields as above with ramification index at most e, i.e. $e \cdot 1 \geqslant \mathrm{val}(p)$.

Corollary II.6.27:
$\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ and $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ for all p and e, eliminate $\mathbf{K}$-quantifiers resplendently.

Proof. By Proposition **(II.A.9)**, resplendence comes for free once we have $\mathbf{K}$-quantifier elimination. Moreover, by Propositions **(II.1.8)** and **(II.B.5)**, we can transfer quantifier elimination in an $\mathbf{RV}$-enrichment of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ (cf. Theorem **D**) to quantifier elimination in a definable $\mathbf{R} \cup \mathbf{\Gamma}$-enrichment of $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ and hence $\mathbf{K}$-quantifier elimination in $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$. The proof for $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ now follows by Remark **II.1.9.3**. ∎

Remark II.6.28:

1. In a valued field with an isometry and $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$, angular components that are compatible with σ are determined by their restriction to the fixed field. Indeed if $\mathrm{val}(x) = \mathrm{val}(\varepsilon)$ where $\varepsilon \in \mathrm{Fix}(\mathbf{K})$, then $\mathrm{ac}_n(x) = \mathbf{R}_n(x\varepsilon^{-1})\mathrm{ac}_n(\varepsilon)$. In fact, any angular components on the fixed field can be extended using this formula to angular components on the whole field that are compatible with σ and hence any valued field with an isometry and $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$ can be elementarily embedded into a valued field with an isometry and compatible angular components.

2. As a matter of fact, the existence of angular components in a σ-Henselian valued field with an isometry implies that $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$.

Until the end of this section, we will add constants to $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ and $\mathcal{L}^{\mathrm{ac},fr}_{\mathcal{A},\mathcal{Q},\sigma}$ for $\mathrm{ac}_n(t)$ and $\mathrm{val}(t)$ where t is any $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}|_{\mathbf{K}}$-term without any free variables. The reason for which we need to add theses constants is that although these are $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$-terms, we may have no trace of them in $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}|_{\mathbf{R}}$ and $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}|_{\mathbf{\Gamma}}$. Ax-Kochen-Eršov type results now follow by the same arguments as usual.

Corollary II.6.29 (Ax-Kochen-Eršov principle for analytic difference fields)**:**

(i) Let $\mathcal{L}$ be an $\mathbf{R}$-extension of a $\mathbf{\Gamma}$-extension of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$, T an $\mathcal{L}$-theory containing $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ and M and $N \vDash T$ then:

  (a) $M \equiv N$ if and only if $\mathbf{R}_0(M) \equiv \mathbf{R}_0(N)$ as $\mathcal{L}|_{\mathbf{R}_0}$-structures and $\mathbf{\Gamma}^\infty(M) \equiv \mathbf{\Gamma}^\infty(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^\infty}$-structures;

  (b) Suppose $M \leqslant N$ then $M \preccurlyeq N$ if and only if $\mathbf{R}_0(M) \preccurlyeq \mathbf{R}_0(M)$ as $\mathcal{L}|_{\mathbf{R}_0}$-structures and $\mathbf{\Gamma}^\infty(M) \preccurlyeq \mathbf{\Gamma}^\infty(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^\infty}$-structures.

(ii) Let $\mathcal{L}$ be an $\mathbf{R}$-extension of a $\mathbf{\Gamma}$-extension of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$, T an $\mathcal{L}$-theory containing $\mathrm{T}^{\mathrm{ac},e-fr}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ and M and $N \vDash T$ then:

(a) $M \equiv N$ if and only if $\mathbf{R}(M) \equiv \mathbf{R}(N)$ as $\mathcal{L}|_{\mathbf{R}}$-structures and $\mathbf{\Gamma}^{\infty}(M) \equiv \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures;

(b) Suppose $M \leqslant N$ then $M \preccurlyeq N$ if and only if $\mathbf{R}(M) \preccurlyeq \mathbf{R}(N)$ as $\mathcal{L}|_{\mathbf{R}}$-structures and $\mathbf{\Gamma}^{\infty}(M) \preccurlyeq \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures.

Remark II.6.30:

1. In mixed characteristic with finite ramification and an isometry, if $\mathcal{R} = \mathcal{O}$, we have better results. Indeed, the trace of any unit E on any $\mathbf{RV}_k$ is given by the trace of a polynomial (which depends only on E and not on its interpretation) and the E_k are in fact useless. Hence the $\mathbf{R}_n$ are pure rings with an automorphism. If there is no ramification (i.e. $e = 1$), the $\mathbf{R}_n$ are ring schemes over $\mathbf{R}_1$ (the Witt vectors of length n) — the ring scheme structure does not depend on the actual model we are looking contrary to the general finite ramification case — and the automorphism on $\mathbf{R}_n$ can be defined using the automorphism on $\mathbf{R}_1$, hence $\mathbf{R}$ is definable in $\mathbf{R}_1$. Finally if σ is a lifting of the Frobenius, σ_1 is definable in the ring structure of $\mathbf{R}_1$. It follows that we obtain Ax-Kochen-Eršov results looking only at $\mathbf{R}_1$ as a ring and $\mathbf{\Gamma}^{\infty}$ as an ordered abelian group (after adding some constants).

2. The fact that the E_1 are useless is also true in equicharacteristic zero when $\mathcal{R} = \mathcal{O}$.

3. It also follows that in equicharacteristic zero or mixed characteristic with finite ramification (with angular component), $\mathbf{R}$ and $\mathbf{\Gamma}^{\infty}$ are stably embedded and have pure $\mathcal{L}|_{\mathbf{R}}$-structure (resp. $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structure) where $\mathcal{L}$ is either $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ or $\mathcal{L}^{\mathrm{ac},fr}_{\mathcal{A},\mathcal{Q},\sigma}$. In particular it will make sense to speak of the theory induced on $\mathbf{R}$ or $\mathbf{\Gamma}^{\infty}$.

Proposition II.6.31:
Let $\mathcal{L}$ be the language $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$ enriched with predicates P_n on $\mathbf{RV}_1$ interpreted as $n|\mathrm{val}_{\mathbf{RV},1}(x)$. The $\mathcal{L}$-theory of W_p is axiomatized by $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, σ_1 is the Frobenius and, the induced theory on $\mathbf{R}_1$ is ACF_p, p has minimal positive valuation, $\mathbf{\Gamma}$ is a $\mathbb{Z}$-group and $\sigma_{\mathbf{\Gamma}}$ is the identity. Moreover $\mathbf{R}_0$ is a pure algebraically closed valued field and $\mathbf{\Gamma}$ is a pure $\mathbb{Z}$-group and they are stably embedded.

Proof. Any model of that theory has definable angular components compatible with σ. And these angular components extend the usual ones on the field of constants $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$. Hence the only constants we add are for elements of $\overline{\mathbb{F}_p}^{\mathrm{alg}} \subseteq \mathbf{R}_0$ and $\mathbb{Z} \subseteq \mathbf{\Gamma}$. The proposition now follows from the discussion above (and the fact that ACF and $\mathbb{Z}$-groups are model complete). ∎

II.7. The NIP property in analytic difference fields

Let us first recall what is shown by Bélair and Delon in the algebraic case [Del81; Bél99]. Let $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ be the $\mathcal{L}^{\mathrm{ac}}$-theory of Henselian valued fields with angular component maps.

Theorem II.7.1:
Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^\infty$-enrichment of $\mathcal{L}^{\mathrm{ac}}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ be an $\mathcal{L}$-theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^\infty$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^\infty}$-structure) are* NIP.

Proof. See [Bél99, Théorème 7.4]. The resplendence of the theorem is not stated there but the proof is exactly the same after enriching on $\mathbf{R}$ and $\mathbf{\Gamma}^\infty$. ■

This result can be extended first to analytic fields then to analytic fields with an automorphism.

Corollary II.7.2:
Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^\infty$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathrm{Hen}}$ be an $\mathcal{L}$ theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^\infty$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^\infty}$-structure) are* NIP.

Proof. Suppose T is not NIP. Then there is a formula $\varphi(x,\overline{y})$ which has the independence property. Note that, since for any sort there is an $\varnothing$-definable function from $\mathbf{K}$ onto that sort, we may assume that x and $\overline{y}$ are $\mathbf{K}$-variables. By Remark **II.5.6.2**, there is an $\mathcal{L}\smallsetminus(\mathcal{A}\cup\{\mathcal{Q}\})$-formula $\psi(x,\overline{z})$ and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$terms $\overline{u}(\overline{y})$ such that $\varphi(x,\overline{y})$ is equivalent to a $\psi(x,\overline{u}(\overline{y}))$. But then ψ would have the independence property too, contradicting Theorem **(II.7.1)**. ■

Corollary II.7.3:
Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^\infty$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be an $\mathcal{L}$ theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^\infty$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^\infty}$-structure) are* NIP.

Proof. Suppose T is not NIP, then there is a formula $\varphi(x,\overline{y})$ which has the independence property (where x and the $\overline{y}$ are $\mathbf{K}$-variables). By Corollary **(II.6.27)**, we may assume that φ is without $\mathbf{K}$-quantifiers, i.e. there is a $\mathbf{K}$-quantifier free $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}\smallsetminus\{\sigma\}$-formula $\psi(\overline{x},\overline{z})$ such that $\varphi(x,\overline{y})$ is equivalent to $\psi(\overline{\sigma}(x),\overline{\sigma}(\overline{y}))$. But then ψ would have the independence property too, contradicting Corollary **(II.7.2)**. ■

Remark II.7.4:
In the isometry case with $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$, this last result also holds without angular components because any such valued field can be elementarily embedded into a valued field with angular components compatible with σ.

Corollary II.7.5:
The $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of W_p is NIP.

Proof. This is an immediate corollary of Remark **(II.7.4)**, Corollary **(II.7.3)** and the fact that $\mathbf{R}$ is definable in $\mathbf{R}_0$ which is a pure algebraically closed field (where the Frobenius automorphism is definable) and that $\mathbf{\Gamma}$ is a pure $\mathbb{Z}$-group (see Proposition **(II.6.31)**). ■

II.A. Resplendent relative quantifier elimination

The following section, although it may appear fastidious and nitpicking, is actually an attempt at clarifying some notions and properties that are often assumed to be clear when studying model theory of valued fields, but may actually need precise and careful presentation. In all this section, $\mathcal{L}$ will denote a language and Σ, Π a partition of its sorts.

Definition II.A.1 (Restriction)**:**
If $\mathcal{L}' \subseteq \mathcal{L}$ be another language and T an $\mathcal{L}$-theory we will denote by $T|_{\mathcal{L}'}$ the $\mathcal{L}'$-theory $\{\varphi$ an $\mathcal{L}'$-formula : $T \vDash \varphi\}$ and if C is an $\mathcal{L}$-structure, $C|_{\mathcal{L}'}$ will have underlying set $\bigcup_{S\in\mathcal{L}'} S(C)$ with the obvious $\mathcal{L}'$-structure. In particular, when Σ is a set of $\mathcal{L}$ sorts, let $\mathcal{L}|_\Sigma$ be the restriction of $\mathcal{L}$ to the predicate and function symbols that only concern the sorts in Σ. Then we will write $T|_\Sigma := T|_{\mathcal{L}|_\Sigma}$ and $C|_\Sigma := C|_{\mathcal{L}|_\Sigma}$.

Note that the restriction is a functor from $\mathrm{Str}(T)$ to $\mathrm{Str}(T|_{\mathcal{L}'})$ respecting models, cardinality and elementary submodels (see Section II.B for the definitions).

Definition II.A.2 (Enrichment)**:**
Let $\mathcal{L}_e \supseteq \mathcal{L}$ be a another language and Σ_e the set of new $\mathcal{L}_e$-sorts, i.e. the $\mathcal{L}_e$-sorts that are not $\mathcal{L}$-sorts. The language $\mathcal{L}_e$ is said to be a Σ-enrichment of $\mathcal{L}$ if $\mathcal{L}_e \smallsetminus \mathcal{L}_e|_{\Sigma\cup\Sigma_e} \subseteq \mathcal{L}$, i.e. the enrichment is limited to the new sorts and the sorts in Σ. If, moreover, $\Sigma_e = \varnothing$ and $\mathcal{L}_e \smallsetminus \mathcal{L}$ consists only of function symbols, we will say that $\mathcal{L}_e$ is a Σ-term enrichment of $\mathcal{L}$.
Let T be an $\mathcal{L}$-theory. An $\mathcal{L}_e$-theory $T_e \supseteq T$ is said to be a definable enrichment of T if there are no new sorts and for every predicate $P(\overline{x})$ (resp. function $f(\overline{x})$) symbol in $\mathcal{L}_e \smallsetminus \mathcal{L}$, there is an $\mathcal{L}$-formula $\varphi_P(\overline{x})$ (resp. $\varphi_f(\overline{x}, y)$ such that $T \vDash \forall\overline{x}\exists^{=1}y,\ \varphi_f(\overline{x}, y)$) and that $T_e = T \cup \{P(\overline{x}) \leftrightarrow \varphi_P(\overline{x})\} \cup \{\varphi_f(\overline{x}, f(\overline{x}))\}$.

Definition II.A.3 (Morleyization)**:**
The Morleyization of $\mathcal{L}$ on Σ is the language $\mathcal{L}^{\Sigma-\mathrm{Mor}} := \mathcal{L} \cup \{P_\varphi(\overline{x}) : \varphi(\overline{x})$ an $\mathcal{L}|_\Sigma$-formula$\}$. If T is an $\mathcal{L}$-theory, the Morleyization of T on Σ is the following $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-theory $T^{\Sigma-\mathrm{Mor}} := T \cup \{P_\varphi(\overline{x}) \leftrightarrow \varphi(\overline{x})\}$ and if M is an $\mathcal{L}$-structure, $M^{\Sigma-\mathrm{Mor}}$ is the $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-structure with the same $\mathcal{L}$-structure as M and where P_φ is interpreted by $\varphi(M)$.
On the other hand, we will say that an $\mathcal{L}$-theory T is Morleyized on Σ if every $\mathcal{L}|_\Sigma$-formula is equivalent, modulo T, to a quantifier free $\mathcal{L}|_\Sigma$-formula.

Note that $T^{\Sigma-\mathrm{Mor}}$ is a definable Σ-enrichment of T and if $M \vDash T$ then $M^{\Sigma-\mathrm{Mor}} \vDash T^{\Sigma-\mathrm{Mor}}$.

Definition II.A.4 (Elementary on Σ)**:**
Let M_1 and M_2 be two $\mathcal{L}$-structures. A partial isomorphism $M_1 \to M_2$ is said to be Σ-elementary if it is a partial $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-isomorphism.

Definition II.A.5 (Resplendent relative elimination of quantifiers)**:**
Let T be an $\mathcal{L}$-theory. We say that T eliminates quantifiers relative to Σ if $T^{\Sigma-\mathrm{Mor}}$ eliminates quantifiers.
We say that T eliminates quantifiers resplendently relative to Σ if for any Σ-enrichment $\mathcal{L}_e$ of $\mathcal{L}$ (with possibly new sorts Σ_e) and any $\mathcal{L}_e$-theory $T_e \supseteq T$, T_e eliminates quantifiers relative to $\Sigma \cup \Sigma_e$.

Definition II.A.6 (Resplendent elimination of quantifiers from a sort)**:**
We will say that an $\mathcal{L}$-theory T eliminates Π-quantifiers if every $\mathcal{L}$-formula is equivalent modulo T to a formula where quantification only occurs on variables from the sorts in Σ.
We will say that T eliminates Π-quantifiers resplendently if for any Σ-enrichment $\mathcal{L}_e$ of $\mathcal{L}$ and any $\mathcal{L}_e$-theory $T_e \supseteq T$, T_e eliminates Π-quantifiers.

Definition II.A.7 (Closed sorts)**:**
We will say that Σ is closed if $\mathcal{L} \smallsetminus (\mathcal{L}|_\Pi \cup \mathcal{L}|_\Sigma)$ only consists of function symbols $f : \prod_i P_i \to S$ where $P_i \in \Pi$ and $S \in \Sigma$. Equivalently, any predicate involving a sort in Σ and any function with a domain involving a sort in Σ only involves sorts in Σ.

Remark II.A.8:

1. Note that if the sorts Σ are closed then in any Σ-enrichment — with possibly new sorts Σ^e — of a Π-enrichment of $\mathcal{L}$ (or vice-versa), the sorts $\Sigma \cup \Sigma^e$ are still closed.
2. Elimination of quantifiers relative to Σ implies elimination of Π-quantifiers. But the converse is in general not true. Indeed, if $\mathcal{L}$ is a language with two sorts S_1 and S_2 and a predicate on $S_1 \times S_2$, then the formula $\exists x R(x, y)$ is an S_2-quantifier free formula but there is no reason for it to be equivalent to any quantifier free $\mathcal{L}^{S_1-\mathrm{Mor}}$-formula.
3. However, if the sorts Σ are closed, then it follows from Remark **II.A.10.1** that T eliminates Π-quantifiers if and only if T eliminates quantifiers relative to Σ. If $\mathcal{L}_e$ is a Σ-enrichment of $\mathcal{L}$ with new sorts Σ_e, then $\Sigma \cup \Sigma_e$ is still closed, thus the equivalence is also true resplendently.

We will now suppose that Σ is *closed* and we will denote by $\mathcal{F}$ the set of functions $f : \prod_i P_i \to S$ where $P_i \in \Pi$ and $S \in \Sigma$.

Proposition II.A.9:
Let T be an $\mathcal{L}$-theory. If T eliminates quantifiers relative to Σ then T eliminates quantifiers resplendently relative to Σ.

Let us begin with some remarks and lemmas that will have a more general interest.

Remark II.A.10:

1. Any atomic $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ where $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables, is either of the form $\psi(\overline{x})$ where ψ is an atomic $\mathcal{L}|_\Pi$-formula or of the form $\psi(\overline{f}(\overline{u}(\overline{x})), \overline{y})$ where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms and $\overline{f}$ are functions from $\mathcal{F}$.
2. If T eliminates quantifiers relative to Σ, it follows from Remark **II.A.10.1** above that for any $M \vDash T$, any $\mathcal{L}(M)$-definable set in a product of sorts from Σ is defined by a formula of the form $\varphi(\overline{x}, \overline{f}(\overline{a}), \overline{b})$ where φ is a $\mathcal{L}|_\Sigma$-formula. Hence Σ is stably embedded in T, i.e. any $\mathcal{L}(M)$-definable subset of Σ is in fact $\mathcal{L}(\Sigma(M))$-definable. Moreover, these sets are in fact $\mathcal{L}|_\Sigma(\Sigma(M))$-definable. In that case, we say that Σ is a pure $\mathcal{L}|_\Sigma$-structure.

Lemma II.A.11:
Suppose T is an $\mathcal{L}$-theory Morleyized on Σ, then for any sufficiently saturated M_1, $M_2 \models T$, any partial $\mathcal{L}$-isomorphism $f : M_1 \to M_2$ with small domain C_1 and any $c_1 \in \Sigma(M_1)$, f can be extended to a partial $\mathcal{L}$-isomorphism whose domain contains c_1.

Proof. First we may assume that $C_1 \leqslant M_1$ and in particular for all $g \in \mathcal{F}$, $g(C_1) \subseteq \Sigma(C_1)$. Because f is a partial $\mathcal{L}$-isomorphism and T is Morleyized on Σ, $f|_\Sigma$ is a partial elementary $\mathcal{L}|_\Sigma$-isomorphism. By saturation of M_2 we can extend $f|_\Sigma$ to $f'|_\Sigma : M_1|_\Sigma \to M_2|_\Sigma$ a partial elementary $\mathcal{L}|_\Sigma$-isomorphism whose domain contains c_1. Let $f' = f|_\Pi \cup f'|_\Sigma$.
As $f|_\Pi$ is a partial $\mathcal{L}|_\Pi$-isomorphism, f' respects formulae $\varphi(\overline{x})$ where φ is an atomic $\mathcal{L}|_\Pi$-formula ($f|_\Pi$ also respects $\mathcal{L}|_\Pi$-terms). Moreover, as for all $g \in \mathcal{F}$, $f'|_{g(C_1)} = f|_{g(C_1)}$, f' still respects g. As $f'|_\Sigma$ is a partial $\mathcal{L}|_\Sigma$-isomorphism, it respects all atomic $\mathcal{L}|_\Sigma$-formulae. It follows that f' also respects formulae of the form $\psi(\overline{g}(\overline{u}(\overline{x})), \overline{y})$ where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms and $\overline{g} \in \mathcal{F}$. By Remark **II.A.10.1**, f' respects all atomic $\mathcal{L}$-formulae and hence is a partial $\mathcal{L}$-isomorphism. ■

Definition II.A.12 (Generated structure)**:**
Let $\mathcal{L}$ be a language, M an $\mathcal{L}$-structure and $C \subseteq M$. The $\mathcal{L}$-structure generated by C will be denoted $\langle C \rangle_\mathcal{L}$. If C is an $\mathcal{L}$-structure and $\overline{c} \in M$, the $\mathcal{L}$-structure generated by C and $\overline{c}$ will be denoted $C\langle \overline{c} \rangle_\mathcal{L}$.

Lemma II.A.13:
Let M_1, $M_2 \models T$, $f : M_1 \to M_2$ a partial $\mathcal{L}$-isomorphism with domain $C_1 \leqslant M_1$ and $c_1 \in \Pi(M_1)$ such that $\Sigma(C_1\langle c_1 \rangle_\mathcal{L}) \subseteq \Sigma(C_1)$. Suppose that f' is a partial $\mathcal{L}|_\Pi \cup \mathcal{F}$-isomorphism extending f whose domain is $C_1\langle c_1 \rangle_\mathcal{L}$, then f' is also a partial $\mathcal{L}$-isomorphism.

Proof. First, by hypothesis, f' respects atomic $\mathcal{L}|_\Pi$-formulae. Moreover as $\Sigma(C_1\langle c_1 \rangle_\mathcal{L}) \subseteq \Sigma(C_1)$, $f'|_\Sigma = f|_\Sigma$ and it is a partial $\mathcal{L}|_\Sigma$-isomorphism. As, by hypothesis, f' respects $g \in \mathcal{F}$, it respects all formulae of the form $\psi(\overline{g}(\overline{u}(\overline{x})), \overline{y})$ where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms and $g \in \mathcal{F}$. Hence by Remark **II.A.10.1**, f' is a partial $\mathcal{L}$-isomorphism. ■

Proof (Proposition **(II.A.9)**). We want to show that if $\mathcal{L}_e$ is a Σ-enrichment of $\mathcal{L}$ (with new sorts Σ_e) and $T_e \supseteq T$ an $\mathcal{L}_e$-theory, then $T_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$ eliminates quantifiers. It suffices to show that for all M_1 and $M_2 \models T_e$ that are $|\mathcal{L}_e|^+$-saturated, for all partial $\mathcal{L}_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$-isomorphism $f : M_1 \to M_2$ of domain C_1 with $|C_1| \leqslant |\mathcal{L}_e|$, and for all $c_1 \in M_1$, f can be extended to a partial $\mathcal{L}_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$-isomorphism whose domain contains c_1.
Note first that $\Sigma \cup \Sigma_e$ is closed. If $c_1 \in \Sigma\cup\Sigma_e(M_1)$, then we can conclude by Lemma **(II.A.11)** (where $\mathcal{L}$ is now $\mathcal{L}_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$). If $c_1 \in \Pi(M_1)$, by repetitively applying Lemma **(II.A.11)**, we can extend f to f' whose domain contains all of $\Sigma \cup \Sigma_e(C_1\langle c_1 \rangle_{\mathcal{L}_e})$. Then f' is in particular an $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-isomorphism and, as T eliminates quantifiers relative to Σ, f' is in fact a partial elementary $\mathcal{L}$-isomorphism that can be extended to a partial $\mathcal{L}$-isomorphism f'' whose domain contain c_1. But, by Lemma **(II.A.13)**, $f''|_{C_1\langle c_1 \rangle_{\mathcal{L}_e}}$ is also a partial $\mathcal{L}_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$-isomorphism. ■

II.B. Categories of structures

May I recall that structures are always non empty.

Definition II.B.1 ($\mathrm{Str}(T)$)**:**
Let $\mathcal{L}$ be a language, T an $\mathcal{L}$-theory. We will denote by $\mathrm{Str}(T)$ the category whose objects are the $\mathcal{L}$-structures that can be embedded in a model of T — i.e. models of $T_\forall$ — and whose morphisms are the $\mathcal{L}$-embeddings between those structures.
Moreover, let T_i be an $\mathcal{L}_i$-theory for $i = 1, 2$, $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ be a functor and κ be a cardinal. We will denote by $\mathrm{Str}_{F,\kappa}(T_2)$ the full sub category of $\mathrm{Str}(T_2)$ of structures that embed into some $F(M)$ for $M \vDash T_1$ κ-saturated.

A functor $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ is said to respect:

- models if for all $M \vDash T_1$, $F(M) \vDash T_2$;
- κ-saturated models if for all κ-saturated $M \vDash T_1$, $F(M) \vDash T_2$;
- cardinality up to κ if for all $C \vDash T_\forall$, $|F(C)| \leqslant |C|^\kappa$;
- elementary submodels if for all $M_1 \preccurlyeq M_2 \vDash T_1$, $F(M_1) \preccurlyeq F(M_2)$.

Let Σ_i be a closed set of $\mathcal{L}_i$-sorts for $i = 1, 2$. We say that $f : C_1 \to C_2$ in $\mathrm{Str}(T_1)$ is a Σ_1-extension if $C_2 \smallsetminus f(C_1) \subseteq \Sigma_1(C_2)$. We say that the functor F sends Σ_1 to Σ_2 if for all Σ_1-extensions $C_1 \to C_2$, $F(C_1) \to F(C_2)$ is a Σ_2-extension.
Let me recall some basic notions of category theory. A natural transformation α between functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ associates a morphism $\alpha_c \in \mathrm{Hom}_{\mathcal{C}_2}(F(c), G(c))$ to every object $c \in \mathcal{C}_1$ such that for all morphism $f \in \mathrm{Hom}_{\mathcal{C}_1}(c, d)$, we have $G(f) \circ \alpha_c = \alpha_d \circ F(f)$. A natural transformation is said to be a natural isomorphism if for all $c \in \mathcal{C}_1$, α_c is an isomorphism in $\mathcal{C}_2$. It is easy to check that when α is a natural isomorphism, its inverse — namely the transformation that associates α_c^{-1} to any $c \in \mathcal{C}_1$ — is also natural.
A pair of functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ are said to be an equivalence of categories between $\mathcal{C}_1$ and $\mathcal{C}_2$ if GF and FG are naturally isomorphic to the identity functor of resp. $\mathcal{C}_1$ and $\mathcal{C}_2$. We can always choose the natural isomorphisms $\alpha : FG \to \mathrm{Id}$ and $\beta : GF \to \mathrm{Id}$ such that $\alpha_F = F(\beta)$ and $\beta_G = G(\alpha)$ where $\alpha_F : c \mapsto \alpha_{F(c)}$ and $F(\alpha) : c \mapsto F(\alpha_c)$.
Until the end of this section, let κ be a cardinal, T_i be an $\mathcal{L}_i$-theory and Σ_i be a set of closed $\mathcal{L}_i$-sorts for $i = 1, 2$ and $\mathfrak{F}$ be a full subcategory of $\mathrm{Str}(T_1)$ containing κ^+-saturated models such that for any $C \to M_1 \vDash T_1$ where M_1 is κ^+-saturated and $|C| \leqslant \kappa$, there is some D in $\mathfrak{F}$ such that $C \to D \to M_1$ and $C \to D$ is a Σ_1-extension. Let $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ and $G : \mathrm{Str}(T_2) \to \mathrm{Str}(T_1)$ be functors that respect cardinality up to κ and induce an equivalence of categories between $\mathfrak{F}$ and $\mathrm{Str}_{F,\kappa^+}(T_2)$. We will also suppose that G respects models and elementary submodels and sends Σ_2 to Σ_1 and F respects κ^+-saturated models. The goal of this section is to show that these (somewhat technical) requirements are a way to transfer elimination of quantifiers results from one theory to another and to give a meaning to — and in fact extend — the impression that if theories are quantifier free bi-definable (whatever that means) then elimination of quantifiers in one theory should imply elimination in the other. Proposition **(II.B.5)** will be used, for example, to deduce valued field quantifiers elimination with angular components from valued field quantifiers elimination with sectioned leading terms. It will also be used to reduce the mixed characteristic case to the equicharacteristic zero case.

Proposition **(II.B.2)** is only used to prove Corollary **(II.B.4)** which in turn will be very useful to show that the functors between mixed characteristic and equicharacteristic zero can be modified to take in account Morleyization on **RV** while remaining in the right setting to transfer elimination of quantifiers.

Proposition II.B.2:
Suppose T_1 is Morleyized on Σ_1 and let M_1 and $M_2 \vDash T_1$ be $(|\mathcal{L}_2|^{\kappa})^{+}$-saturated. Then any partial $\mathcal{L}_2$-isomorphism $f : F(M_1) \to F(M_2)$ is Σ_2-elementary.

Proof. To show that f is Σ_2-elementary, it suffices to show that the restriction of f to any finitely generated structure is Σ_2-elementary. To do so it suffices to show that the restriction of f can be extended (on both its domain and its image) to any finitely generated Σ_2-extension. By symmetry, it suffices to prove the following property: if $D_1, D_2 \leqslant F(M_1)$ are such that $D_1 \to D_2$ is a Σ_2-extension, $|D_2| \leqslant |\mathcal{L}_2|$ and $f : D_1 \to F(M_2)$ is an $\mathcal{L}_2$-embedding, then f can be extended to some $g : D_2 \to F(M_2)$.
Applying G to the initial data, we obtain the following diagram:

$$\begin{tikzcd}
GF(M_1) & M_2 & GF(M_2) \arrow[l, "\beta_{M_2}"'] \\
G(D_2) \arrow[u] \arrow[ur, dotted, "g"] & & \\
G(D_1) \arrow[u] \arrow[uurr, "G(f)"'] & &
\end{tikzcd}$$

where g comes from the fact that, as T is Morleyized on Σ_1, $\beta_{M_2} \circ G(f)|_{\Sigma_1}$ is in fact elementary and, as $|G(D_2)| \leqslant |\mathcal{L}_2|^{\kappa}$, M_2 is $(|\mathcal{L}_2|^{\kappa})^{+}$-saturated and $G(D_1) \to G(D_2)$ is a Σ_1-extension, by Lemma **(II.A.11)**, $\beta_{M_2} \circ G(f)$ can be extended to $g : G(D_2) \to M_2$. Applying F, we now obtain:

$$\begin{tikzcd}
D_2 \arrow[r, "\alpha_{D_2}^{-1}"] & FG(D_2) \arrow[r, "F(g)"] & F(M_2) \\
D_1 \arrow[u] \arrow[r] \arrow[urr, bend right, "f"'] & FG(D_1) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and $F(g) \circ \alpha_{D_2}^{-1}$ is the extension we were looking for. ∎

Remark II.B.3:

1. One could hope the proposition to be true without the saturation hypothesis. But without some saturation, it is not even true that $M_1 \preccurlyeq M_2$ implies $F(M_1) \preccurlyeq F(M_2)$. Take for example the coarsening functor $\mathfrak{C}^{\infty}$ of Section II.2 and $\mathbb{Q}_p \preccurlyeq M$ where M is $\aleph_0$-saturated, then $\mathfrak{C}^{\infty}(\mathbb{Q}_p)$ is trivially valued but $\mathfrak{C}^{\infty}(M)$ is not.

2. One should beware that as $F(M_1)$ and $F(M_2)$ are not saturated, we have not proved that T_2 eliminates quantifiers.

3. We have proved nonetheless that, if Σ_i is the set of all $\mathcal{L}_i$ sorts (in that case we ask that T_2 eliminates all quantifiers) then for all M_1 and $M_2 \vDash T_1$ sufficiently saturated, $M_1 \equiv M_2$ implies $F(M_1) \equiv F(M_2)$.

Corollary II.B.4:
Let T_2^e be a definable Σ_2-enrichment of T_2 (in the language $\mathcal{L}_2^e$). Then F induces a functor $F^e : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2^e)$ and G induces a functor $G^e : \mathrm{Str}(T_2^e) \to \mathrm{Str}(T_1)$. We can also find a full subcategory $\mathfrak{F}^e$ of $\mathfrak{F}$ such that F^e and G^e induce an equivalence of categories between $\mathfrak{F}^e$ and $\mathrm{Str}_{F^e,(|\mathcal{L}_2|^\kappa)^+}(T_2^e)$. The functor G^e still respects cardinality up to κ, models and elementary submodels and sends Σ_2 to Σ_1 and F^e respects cardinality up to $\kappa + |\mathcal{L}_2|$ and $(|\mathcal{L}_2|^\kappa)^+$-saturated models. Finally, $\mathfrak{F}^e$ contains all $(|\mathcal{L}_2|^\kappa)^+$-saturated models and any C in $\mathrm{Str}(T_1)$ has a Σ_1-extension D in $\mathfrak{F}^e$. Moreover, if $C \leqslant M_1 \vDash T_1$ and M_1 is $(|\mathcal{L}_2|^\kappa)^+$-saturated, then we can find such a $D \leqslant M_1$.

Proof. Let $C \leqslant M \vDash T_1$. We can suppose that M is $(|\mathcal{L}_2|^\kappa)^+$-saturated. As $F(M) \vDash T_2$, we can enrich $F(M)$ to make it into an $\mathcal{L}_2^e$-structure $F(M)^e \vDash T_2^e$ and we take $F^e(C) = \langle C \rangle_{\mathcal{L}_2^e}$. Note that if M_1 and M_2 are two $(|\mathcal{L}_2|^\kappa)^+$-saturated models containing C, then Proposition **(II.B.2)** implies that $\mathrm{id}_{F(C)}$ is a partial isomorphism $F(M_1) \to F(M_2)$ Σ_2-elementary and hence the generated $\mathcal{L}_2^e$-structures are $\mathcal{L}_2^e$-isomorphic. As $F^e(C)$ does not depend (up to $\mathcal{L}_2^e$-isomorphism) on the choice of $(|\mathcal{L}_2|^\kappa)^+$-saturated model containing C, F^e is well-defined on objects. If $f : C_1 \to C_2$ is a morphism in $\mathrm{Str}(T_1)$, by the same Proposition **(II.B.2)**, $F(f)$ is Σ_2-elementary and can be extended to a $\mathcal{L}_2^e$-isomorphism on the $\mathcal{L}_2^e$-structure generated by its domain. Note that if we denote by i_C the embedding $F(C) \to F^e(C)$, we have also defined a natural transformation from F to F^e (a meticulous reader might want to add the forgetful functor $\mathrm{Str}(T_2^e) \to \mathrm{Str}(T_2)$ for it all to make sense).
We define G^e to be G (precomposed by the same forgetful functor). All the statements about G^e follow immediately from those about G. As $\langle F(C) \rangle_{\mathcal{L}_2^e}$ has cardinality at most $|C|^\kappa |\mathcal{L}_2| \leqslant |C|^{\kappa + |\mathcal{L}_2|}$, F respect cardinality up to $\kappa + \mathcal{L}_2$ and if $M \vDash T_1$ is $(|\mathcal{L}_2|^\kappa)^+$-saturated then seeing it as a substructure of itself we obtain that $F^e(M) \vDash T_2^e$.
We define $\mathfrak{F}^e$ to be the full-subcategory of $\mathfrak{F}$ containing the C such that i_C is an isomorphism. In particular, it contains $(|\mathcal{L}_2|^\kappa)^+$-saturated models. Let D be an $\mathcal{L}_2^e$-substructure of $F^e(M)$ for some $(|\mathcal{L}_2|^\kappa)^+$-saturated $M \vDash T_1$. Then $F^e G^e(D) = \langle FG(D) \rangle_{\mathcal{L}_2^e}$, where the generated structure is taken in $F(M)$. By Proposition **(II.B.2)**, the (natural) isomorphism $D \to FG(D)$ is Σ_2-elementary and can be extended (uniquely) into an $\mathcal{L}_2^e$-isomorphism between $D = \langle D \rangle_{\mathcal{L}_2^e}$ and $F^e G^e(D)$. This new isomorphism is also natural. It follows that $FG(D) = F^e G^e(D)$ and that $i_{G(D)}$ is in fact an isomorphism, hence $G(D) \in \mathfrak{F}^e$.
If $C \in \mathfrak{F}^e$, $\beta_C \circ G(i_C^{-1}) : G^e F^e(C) \to C$ is a natural isomorphism. Finally, there remains to show that any $C \to M \vDash T_1$, where M is $(|\mathcal{L}_2|^\kappa)^+$-saturated, can be embedded in some $E \in \mathfrak{F}^e$ such that $C \to E$ is a Σ_1-extension and $E \to M$. We already know that there exists $D \in \mathfrak{F}$ such that $C \to D \to M$ and $C \to D$ is a Σ_1-extension. Now $F(D) \to F^e(D)$ is a Σ_2-extension hence $D \cong GF(D) \to GF^e(D)$ is a Σ_1-extension. Moreover $GF^e(D) \to GF^e(M) \cong M$ and, as $F^e(D)$ is an $\mathcal{L}_2^e$-structure of $F^e(M)$, $GF^e(D) \in \mathfrak{F}^e$. Thus we can take $E = GF^e(D)$. ∎

Let us now prove a second result in the spirit of Proposition **(II.B.2)**, but the other way round.

Proposition II.B.5:
If T_1 is Morleyized on Σ_1 and T_2 eliminates quantifiers, then T_1 eliminates quantifiers.

Proof. To show that T_1 eliminates quantifiers it suffices to show that for all κ^+-saturated $M_i \models T_1$, $i = 1,2$, and $C_1 \preccurlyeq C_2 \subseteq M_1$ and $f : C_1 \to M_2$ an $\mathcal{L}_1$-embedding, then f can be extended to an embedding from C_2 into some elementary extension of M_2. Let $D_1 \in \mathfrak{F}$ be such that $C_1 \to D_1 \to M_1$ and $C_1 \to D_1$ is a Σ_1-extension. As T_1 is Morleyized on Σ_1, by Lemma **(II.A.11)**, we can extend f to an embedding from D_1 into an elementary extension of M_2. Replacing C_1 by D_1, C_2 by $\langle D_1 C_2\rangle_{\mathcal{L}_1}$ and M_2 by its elementary extension, we can consider that $C_1 \in \mathfrak{F}$. Applying F, we obtain the following diagram:

$$\begin{tikzcd}
F(M_1) & M_2^\star \\
F(C_2) \arrow[u] \arrow[ur, dotted, "g"] & F(M_2) \arrow[u, "\preccurlyeq"'] \\
F(C_1) \arrow[u] \arrow[ur, "F(f)"'] &
\end{tikzcd}$$

where $M_2^\star$ is a $(|C_1|^{\aleph_0})^+$-saturated extension of $F(M_2)$ and g comes from quantifier elimination in T_2 and saturation of $M_2^\star$. Applying G we obtain:

$$\begin{tikzcd}
C_2 \arrow[r] & GF(C_2) \arrow[r, "G(g)"'] & G(M_2^\star) & \\
C_1 \arrow[u] \arrow[r] \arrow[rrr, bend right, "f"'] & GF(C_1) \arrow[u] \arrow[r, "GF(f)"'] & GF(M_2) \arrow[u, "\preccurlyeq"'] \arrow[r, leftrightarrow] & M_2 \arrow[ul, "\preccurlyeq"']
\end{tikzcd}$$

and we have the required extension. ∎

CHAPTER III

Imaginaries in enrichments of ACVF

Le Logicien, au Vieux Monsieur.
Tous les chats sont mortels. Socrate est mortel. Donc Socrate est un chat.
Le Vieux Monsieur
Et il a quatre pattes.
E. Ionesco, *Rhinocéros*, Acte I

The goal of this chapter is to show that certain enrichments of ACVF have no more imaginaries than ACVF itself — the so called geometric imaginaries of [HHM06] — and have the invariant extension property, that is types over algebraically closed sets have invariant global extensions (cf. Definition **(0.4.13)**). The main example to keep in mind of such an enrichment (and the only one so far where we can prove all the hypotheses that appear along the way) is $\mathrm{VDF}_{\mathcal{EC}}$, the model completion of differential valued fields where the derivation preserves the valuation: for all x, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$ (cf. [Sca00] and Section IV.1). In fact, the main motivation behind these results was to prove the elimination of imaginaries and the invariant extension property in $\mathrm{VDF}_{\mathcal{EC}}$.
To be precise, we will be working in the more general context of a theory $\widetilde{T}$ enriching a theory T which is itself a C-minimal enrichment of ACVF. But for the sake of clarity, in this introduction, we will focus on the example where T is ACVF and $\widetilde{T}$ is $\mathrm{VDF}_{\mathcal{EC}}$. The fact that the abstract result proved in this chapter applies to $\mathrm{VDF}_{\mathcal{EC}}$ is shown in Section IV.1.
Following the general idea of [Hru14; Joh], elimination of imaginaries relative to the geometric sorts is obtained as a consequence of the density of types definable over geometric parameters (see the conclusion of Theorem **E** for a precise statement). Furthermore, the invariant extension property is also a consequence of the density of definable types. Hence the actual goal of this chapter is, given a set X definable in some model of $\mathrm{VDF}_{\mathcal{EC}}$, to find a definable type of elements in X which is the most "generic" possible — in particular which has only boundedly many conjugates under automorphisms that stabilize X globally — and has a canonical basis in the geometric sorts.
Let us fix some notations. Let $\mathcal{L}_{\mathrm{div}}$ be the one sorted language for ACVF and $\mathcal{L}_{\partial,\mathrm{div}} := \mathcal{L}_{\mathrm{div}} \cup \{\partial\}$ be the one sorted language for $\mathrm{VDF}_{\mathcal{EC}}$, where ∂ is a symbol for the derivation. It follows from quantifier elimination in $\mathrm{VDF}_{\mathcal{EC}}$ that to describe the $\mathcal{L}_{\partial,\mathrm{div}}$-type of x (denoted p) it suffices to give the $\mathcal{L}_{\mathrm{div}}$-type of $\partial_\omega(x) := (\partial^n(x))_{n<\omega}$ (denoted $\nabla_\omega(p)$) and that p is consistent with X if and only if $\nabla_\omega(p)$ is consistent with $\partial_\omega(X)$, the image of X under the map ∂_ω. Note that $\nabla_\omega(p)$ is the pushforward of p by ∂_ω restricted to $\mathcal{L}_{\mathrm{div}}$. If $\nabla_\omega(p)$ is definable (in ACVF), because ACVF eliminates imaginaries relative to the geometric sorts, $\nabla_\omega(p)$ has a canonical basis in the geometric sorts and p is definable (in $\mathrm{VDF}_{\mathcal{EC}}$) with the same canonical basis. Hence it will be enough to find a "generic" definable $\mathcal{L}_{\mathrm{div}}$-type q consistent with $\partial_\omega(X)$.
Note that $\partial_\omega(X)$ is not at all definable in ACVF but, in fact, it is $\star$-definable in $\mathrm{VDF}_{\mathcal{EC}}$.

The $\star$-definability is not very hard to deal with, but because $\partial_\omega(X)$ lives in $\mathrm{VDF}_{\mathcal{EC}}$, the definable $\mathcal{L}_{\mathrm{div}}$-types we build will more naturally have a defining scheme of $\mathcal{L}_{\partial,\mathrm{div}}(M)$-formulas, where M is some model of $\mathrm{VDF}_{\mathcal{EC}}$ — what we could call an $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable $\mathcal{L}_{\mathrm{div}}$-type. But, because we do not know yet that $\mathrm{VDF}_{\mathcal{EC}}$ eliminates imaginaries in the geometric sorts, such a type has no reason to have a canonical basis in the geometric sorts. In Section III.1, however, we show a general result about NIP theories which implies that any $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable $\mathcal{L}_{\mathrm{div}}$-type also has a defining scheme of $\mathcal{L}_{\mathrm{div}}$-formulas and hence a geometric canonical basis. Therefore, we need only find an $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable $\mathcal{L}_{\mathrm{div}}$-type "generic" and consistent with $\partial_\omega(X)$ (cf. Corollary **(III.6.7)**). But a definable $\mathcal{L}_{\mathrm{div}}$-type is a consistent collection of definable Δ-types where Δ is a finite set of $\mathcal{L}_{\mathrm{div}}$-formulas and so we can ultimately reduce to finding, for any such finite Δ a "generic" $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable Δ-type consistent with some $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable set (cf. Corollary **(III.6.5)**). It follows that most of the preparatory work in this chapter — Sections III.3 to III.5 — will attempt to better understand Δ-types for finite Δ in ACVF.

An example of this somewhat convoluted back and forth between two languages $\mathcal{L}_{\mathrm{div}}$ and $\mathcal{L}_{\partial,\mathrm{div}}$, is essentially underlying the proof of elimination of imaginaries in DCF_0 (the model completion of characteristic zero differential fields) — although, in the classical proofs, it might not appear clearly. One might think that this example is too simple, but it is, in fact, quite revealing of what is going on in theses pages. Take any set X definable in DCF_0 and let $X_n := \partial_n(X)$ where $\partial_n(x) := (\partial^i(x))_{0\leqslant i\leqslant n}$ and let Y_n be the Zariski closure of X_n. Now, choose a consistent sequence $(p_n)_{n<\omega}$ of ACF-types such that p_n has maximal Morley rank in Y_n. Because ACF is stable all the p_n are definable and they all have canonical bases of field points by elimination of imaginaries in ACF. Then the complete type of points x such that $\partial_n(x) \models p_n$ is also definable with a canonical basis of field points and it is obviously consistent with X.

In ACVF, we cannot use the Zariski closure because we also need to take into account valuative inequalities. But the balls in ACVF are combinatorially well-behaved, and we can approximate sets definable in $\mathrm{VDF}_{\mathcal{EC}}$ by finite fibrations of balls over lower dimensional sets — i.e. cells in the C-minimal setting (cf. Section III.6). And, because C-minimality is really the core property of ACVF that we are using, this chapter generalizes naturally to any C-minimal extension of ACVF. Although that might seem like unnecessary generalization, we hope it might lead in the future to a proof that $\mathrm{VDF}_{\mathcal{EC}}$ with analytic structure has the invariant extension property and has no more imaginaries than ACVF with analytic structure (denoted $\mathrm{ACVF}_{\mathcal{A}}$), even though we have no concrete idea of what those analytic imaginaries are (see [HHM13]).

As for the organization of this chapter, in Section III.1, we show that being externally definable is a first order property in NIP theories (see Proposition **(III.1.2)**) leading to the proof (in Corollary **(III.1.5)**) that if p is a type in an NIP theory T with a defining scheme in an enrichment $\widetilde{T}$ of T such that $\widetilde{T}$ has a "nice" model, then p has a defining scheme in T. This is joint work with Pierre Simon whom I would like to thank for allowing me to include these pages here.

Section III.2 studies the question of uniform stable embeddedness in pairs of valued fields (following [Del89; Cub13]) to show that ACVF has "nice" models. To apply the results of Section III.1 to some enrichment $\widetilde{T}$ of ACVF, it will then suffice to find a model of $\widetilde{T}$ whose

underlying valued field is one of those "nice" models. Note that this is the only section that we do not know how to generalize to $\mathrm{ACVF}_{\mathcal{A}}$.
In Section III.3, we consider definable families of functions into the value group, in ACVF and $\mathrm{ACVF}_{\mathcal{A}}$, and show that their germs are internal to the value group.
In Section III.4 we study certain "generic" Δ-types, for Δ finite, in a C-minimal expansion T of ACVF (see Definition **(III.4.12)**). These generic types are the n-ary generalization of the unary notion of being generic in a ball (cf. [HHM06, Definition 2.3.4] and Section I.3). We show that any Δ-type in these theories is implied by a "generic" type, at the cost of making Δ a little bigger.
In Section III.5, we introduce and study the notion of implicatively definable types. Recall that a (complete) type is definable if for any formula $\varphi(x;s)$ there is a formula $\theta(s)$ such that $\theta(s)$ holds if and only if $\varphi(x;s)$ is in p — equivalently if p implies $\varphi(x;s)$. If p is not complete, it still makes sense to require that $\theta(s)$ holds if and only if $\varphi(x;s)$ is a consequence of p (even if it is possible that the type implies neither a given formula nor its complement). This is implicative definability (see Definition **(III.5.1)**). We show in particular that the "generic" Δ-types introduced in Section III.4 — under some more assumptions — have this property with respect to sets definable in certain reasonable enrichments of T.
In Section III.6, we put everything together to prove, in Proposition **(III.6.1)**, that sets definable in reasonable enrichments of T can be approximated by finite fibrations of balls and then go on to prove, as explained at the beginning of the introduction, in Theorem **E**, the density of types definable over real parameters. We will check in Section IV.1 that Theorem **E** applies to $\mathrm{VDF}_{\mathcal{EC}}$.
Finally, in Section III.7, we show how this density result can be used to give a criterion for elimination of imaginaries and the invariant extension property.

III.1. Definability of externally definable sets in NIP theories

This section is joint work with Pierre Simon.
In previous work (mainly [CS13; CS]) on the subject, it was shown that external definability was rather tractable in NIP theories particularly because of the existence of honest definitions. In stable theories, any externally definable set is definable — this, in fact, is equivalent to the stability of the theory. More precisely, in a stable theory, a set is externally φ-definable (see Definition **(III.1.1)**) if and only if it is definable by an instance of a fixed formula ψ. In NIP theories, the picture is a bit more complicated but we show in Proposition **(III.1.2)** that external φ-definability is — almost — a first order property. We go on to prove (in Theorem **(III.1.4)**) that if a set is externally definable in an NIP theory and definable in some enrichment of the theory — under some hypothesis on the enrichment to get rid of obvious counter-examples — then the set is already definable in the NIP theory.

Definition III.1.1 (Externally φ-definable)**:**
Let M be an $\mathcal{L}$-structure and $\varphi(x;t)$ be an $\mathcal{L}$-formula. We say that X is externally φ-definable if there exists $N \succcurlyeq M$ and a tuple $a \in N$ such that $\varphi(M;a) = X$.

Proposition III.1.2:
Let T be an NIP *$\mathcal{L}$-theory. Let U be a new predicate symbol. Let $\mathcal{L}_U := \mathcal{L} \cup \{U\}$. Then for all $\mathcal{L}$-formulas $\varphi(x;t)$, there is an $\mathcal{L}_U$-sentence θ_U and an $\mathcal{L}$-formula $\psi(x;s)$ such that for all $M \vDash T$ and any enrichment M_U of M to $\mathcal{L}_U$, we have:*

$$\text{if } U(M_U) \text{ is externally } \varphi\text{-definable, then } M_U \vDash \theta_U$$

and

$$\text{if } M_U \vDash \theta_U, \text{ then } U(M_U) \text{ is externally } \psi\text{-definable.}$$

Proof. Let $\chi(x;u)$ be a uniform honest definition of φ (see [Sim, Theorem 6.16]) — i.e. for any $M \preccurlyeq N \vDash T$, any tuple $b \in N$ and any $A \subseteq \varphi(M;b)$ finite, there exists a tuple $d \in M$ such that $A \subseteq \chi(M;d) \subseteq \varphi(M;b)$. Let k be the VC-dimension of $\chi(x;u)$. By the dual version of the (p,q)-theorem (see [Sim, Corollary 6.13]) there exists q and n such that for any set X, any finite $A \subseteq X$ and any $\mathcal{S} \subseteq \mathcal{P}(X)$ of VC-dimension at most k, if for all $A_0 \subseteq A$ of size at most q there exists $S \in \mathcal{S}$ containing A_0, then there exists $S_1 \dots S_n \in \mathcal{S}$ such that $A \subseteq \bigcup_i S_i$.
Let

$$\theta_U := \forall x_1 \dots x_q \bigwedge_{i \leqslant q} U(x_i) \Rightarrow \exists u\, (\forall x\, \chi(x;u) \Rightarrow U(x)) \wedge \bigwedge_{i \leqslant q} \chi(x_i;u).$$

Now, let $M \preccurlyeq N \vDash T$ and $b \in N$ be a tuple. Let $U(M_U) := \varphi(M;b)$. For any $A \subseteq \varphi(M;b) = U(M_U)$ of size at most q, as χ is an honest definition of φ, we find a tuple $d \in M$ such that $A \subseteq \chi(M;d) \subseteq \varphi(M;b) = U(M_U)$ i.e $M_U \vDash \theta_U$.
Suppose now that $M_U \vDash \theta_U$. The following set of formulas, where $\mathcal{D}^{\mathrm{el}}(M)$ denotes the elementary diagram of M,

$$\mathcal{D}^{\mathrm{el}}_{\mathcal{L}_U}(M_U) \cup \{\bigvee_{i=1}^{n} \chi(a;u_i) : a \in U(M_U) \text{ a tuple}\} \cup \{\forall x\, \chi(x;u_i) \Rightarrow U(x) : 1 \leqslant i \leqslant n\}$$

is finitely consistent. Indeed, let $A \subseteq U(M_U)$ be finite. The family $\{\chi(M;d) : d \in M \text{ a tuple and } \chi(M;d) \subseteq U(M_U)\}$ has VC-dimension at most k and — as $M_U \vDash \theta_U$ — for any $A_0 \subseteq A$ of size at most q, there exists a tuple $d \in M$ such that $A_0 \subseteq \chi(M;d) \subseteq U(M_U)$. It follows that there are tuples $d_1 \dots d_n \in M$ such that $A \subseteq \bigvee_i \chi(M;d_i) \subseteq U(M_U)$.
We can therefore find $N_U \succcurlyeq M_U$ and $d_1, \dots, d_n$ such that $U(M_U) \subseteq \bigvee_i \chi(M;d_i) \subseteq U(N_U) \cap M = U(M_U)$, i.e. $U(M_U)$ is externally $\bigvee_{i=1}^{n} \chi(x;u_i)$-definable. ∎

Definition III.1.3 (Uniform stable embeddedness)**:**
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that A is uniformly stably embedded if for all formulas $\varphi(x;t)$ there exists a formula $\chi(x;s)$ such that for all tuples $b \in M$ there exists a tuple $a \in A$ such that $\varphi(A,b) = \chi(A,a)$.

Theorem III.1.4:
Let T be an NIP *$\mathcal{L}$-theory that eliminates imaginaries, $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\widetilde{T} \supseteq T$ be a complete theory. Suppose that there exists $\widetilde{M} \vDash \widetilde{T}$ such that $\widetilde{M}|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Then for all $\widetilde{N} \vDash \widetilde{T}$ and $\widetilde{A} = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}) \subseteq \widetilde{N}^{\mathrm{eq}}$, any externally $\mathcal{L}$-definable set X that is also $\widetilde{\mathcal{L}}^{\mathrm{eq}}(\widetilde{A})$-definable is in fact $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable where $\mathcal{R}$ denotes the set of all $\mathcal{L}$-sorts.*

Recall that when X is $\widetilde{\mathcal{L}}(\widetilde{N})$-definable for some $\widetilde{N} \models \widetilde{T}$, we denote its code in $\widetilde{N}^{\mathrm{eq}}$ by $\ulcorner X \urcorner$.

Proof. Let $\varphi(x;t)$ be an $\mathcal{L}$-formula and $\chi(x;s)$ be an $\widetilde{\mathcal{L}}$-formula. Let $\theta(s)$ be the $\widetilde{\mathcal{L}}$-formula obtained from the formula θ_U of Proposition **(III.1.2)** by replacing $U(x)$ by $\chi(x;s)$. Equivalently, for any $\widetilde{\mathcal{L}}$-structure $\widetilde{N}$ and any tuple $d \in \widetilde{N}$, we can make $N := \widetilde{N}|_{\mathcal{L}}$ into an $\mathcal{L}_U$-structure N_d by interpreting U as $\chi(\widetilde{N};d)$. Then there exists an $\widetilde{\mathcal{L}}$-formula $\theta(s)$ such that $N_d \models \theta_U \iff \widetilde{N} \models \theta(d)$. By Proposition **(III.1.2)** there also exists an $\mathcal{L}$-formula $\psi(x;u)$ such that for all $\widetilde{N} \models \widetilde{T}$ and tuple $d \in \widetilde{N}$,

$$\chi(\widetilde{N};d) \text{ externally } \varphi\text{-definable implies } \widetilde{N} \models \theta(d)$$

and

$$\widetilde{N} \models \theta(d) \text{ implies } \chi(\widetilde{N};d) \text{ externally } \psi\text{-definable.}$$

Let $\widetilde{M} \models \widetilde{T}$ be as in the hypothesis, $M := \widetilde{M}|_{\mathcal{L}}$, $\mathfrak{U} \succcurlyeq M$ be saturated enough and $\xi(x;v)$ be an $\mathcal{L}$-formula such for any tuple $c \in \mathfrak{U}$, there is a tuple $a \in M$ such that $\psi(M,c) = \xi(M,a)$. Then for all tuples $d \in \widetilde{M}$ such that $\widetilde{M} \models \theta(d)$, there is some tuple $c \in \mathfrak{U}$ such that $\chi(\widetilde{M};d) = \psi(M;c)$ and hence some tuple $a \in M$ such that $\chi(\widetilde{M};d) = \xi(M;a) = \xi(\widetilde{M};a)$, i.e.

$$\widetilde{M} \models \forall s\, \theta(s) \Rightarrow \exists u\, (\forall x\, (\chi(x;s) \iff \xi(x,u))).$$

But as $\widetilde{T}$ is complete, this holds in any $\widetilde{N} \models \widetilde{T}$ and for any tuple $d \in \widetilde{N}$ such that $X := \chi(\widetilde{N},d)$ is externally φ-definable — and hence $\widetilde{N} \models \theta(d)$ — there exists a tuple $a \in N := \widetilde{N}|_{\mathcal{L}}$ such that $X = \xi(N;a)$, i.e. X is $\mathcal{L}(N)$-definable.
As for X being $\mathcal{L}(A)$-definable, we have just shown that we can find $\ulcorner X \urcorner^{\mathcal{L}} \in N$ but because X is also $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable, we also have that $\ulcorner X \urcorner^{\mathcal{L}} \in \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}) = \widetilde{A}$, i.e. $\ulcorner X \urcorner^{\mathcal{L}} \in \widetilde{A} \cap N = \mathcal{R}(\widetilde{A})$. ∎

Corollary III.1.5:
Let T be an NIP *$\mathcal{L}$-theory that eliminates imaginaries and let $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$ be some language and $\widetilde{T} \supseteq T$ be a complete $\widetilde{\mathcal{L}}$-theory. Suppose that there exists $\widetilde{M} \models \widetilde{T}$ such that $\widetilde{M}|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension. Let $\Delta(x;t)$ be a set of $\mathcal{L}$-formulas, $\widetilde{N} \models \widetilde{T}$, $\widetilde{A} = \mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}) \subseteq \widetilde{N}^{\mathrm{eq}}$, $N := \widetilde{N}|_{\mathcal{L}}$ and $p \in \mathcal{S}^{\Delta}_x(N)$. If p is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable, then it is in fact $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable where $\mathcal{R}$ denotes the set of all $\mathcal{L}$-sorts.*

Proof. Let $a \models p$ and $\varphi(x;t) \in \Delta$. Then $\{m \in N : \varphi(x;m) \in p\} = \{m \in N :\models \varphi(a;m)\}$ is $\mathcal{L}$-externally definable and $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. It follows from Theorem **(III.1.4)** that it is in fact $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable. ∎

Remark III.1.6:

1. The assumption that T is NIP is not enough for the conclusion of Theorem **(III.1.4)** to hold. Indeed, let T be the theory of dense linear orders — in the language $\mathcal{L} := \{<\}$ — $\widetilde{\mathcal{L}} = \mathcal{L}_U$, $\widetilde{M} := (\mathbb{Q}, <, U)$ where $U(\widetilde{M})$ is the initial segment of a non definable cut — i.e. U has no maximal element and its complement does not have a minimal element — and $\widetilde{T} := \mathrm{Th}(\widetilde{M})$. Then the conclusion of Theorem **(III.1.4)** cannot hold as $U(\widetilde{M})$ is not $\mathcal{L}$-definable, but like all cuts it is externally $\mathcal{L}$-definable.

But, it is also clear that in all $\widetilde{N} \equiv_{\widetilde{\mathcal{L}}} \widetilde{M}$, the cut defined by U is never $\mathcal{L}$-definable and hence $\widetilde{N}|_{\mathcal{L}}$ cannot be stably embedded in all its elementary extensions.

2. The second assumption is not sufficient either. Take $\mathcal{L} := \{E, \leqslant\}$ and T_E be the theory that states that E is an equivalence relation and $x \leqslant y$ implies xEy. For all $\{\leqslant\}$-formula $\varphi(x)$, let $\varphi^E(x,y)$be the relativization of φ to the E class of y:
   a. $\varphi^E(x,y) := \varphi(x) \wedge \bigwedge_i x_i E y$ for any quantifier free φ;
   b. $(\exists x\, \varphi)^E(z,y) := \exists x\, xEy \wedge \varphi^E(x,z,y)$;
   c. $(\forall x\, \varphi)^E(z,y) := \forall x\, xEy \Rightarrow \varphi^E(x,z,y)$.

   Let $R_\varphi(x,y)$ be a new relation symbol $\mathcal{L}^\star := \mathcal{L} \cup \{R_\varphi : \varphi(x)$ is an $\{\leqslant\}$-formula$\}$ and $T_E^\star := T_E \cup \{\forall x(R_\varphi(x,y) \iff \varphi^E(x,y))\}$. Then for all $\{\leqslant\}$-formula φ, $M \vDash T_E^\star$ tuples $a \in M$ and elements $b \in M$, $M \vDash R_\varphi(a,b)$ if and only if all the a_i are in the E-class of b — denoted $\widehat{b}$ — and $\widehat{b}(M) \vDash \varphi(a)$.

Claim III.1.7: *Any complete $\mathcal{L}^\star$-theory $T \supseteq T_E^\star$ eliminates quantifiers.*

Proof. Let M and $N \vDash T$ be saturated enough, $A \subseteq M$, $B \subseteq N$, $f : A \to B$ an $\mathcal{L}^\star$-isomorphism and $c \in M$. Let $A_0 := \{a \in A : aEc\}$. Let us first assume that $A_0 \neq \varnothing$ and let $\widehat{A_0}(M)$ the E-class of A_0 in M, $B_0 := f(A_0)$ and $\widehat{B_0}(N)$ the E-class of B_0 in N. Then $f|_{A_0}$ is a partial elementary $\{\leqslant\}$-isomorphism between $\widehat{A_0}(M)$ and $\widehat{B_0}(N)$. It follows that there exists a partial elementary $\{\leqslant\}$-isomorphism $g : \widehat{A_0}(M) \to \widehat{B_0}(N)$ with domain $A_0 \cap a$. It is then easy to check, that $g \cup f$ is an $\mathcal{L}^\star$-isomorphism.

If $A_0 = \varnothing$, let $\varphi(x)$ be any $\{\leqslant\}$-formula such that $\widehat{c}(M) \vDash \varphi(c)$, i.e. $M \vDash \varphi^E(c,c)$. Let n be the number of E-classes containing some $a \in A$ such that $M \vDash \varphi^E(a,a)$. In M, there are at least $n+1$ E-classes where $\varphi^E(x,x)$ is realized. As T is complete, this is also the case in N and there exists a class with no points in B where this formula is realized. By compactness, we can find d in a class with no intersection with B such that for all $\{\leqslant\}$-formulas φ, $\widehat{c}(M) \vDash \varphi(c)$ if and only if $\widehat{d}(N) \vDash \varphi(d)$, i.e. there exists a partial elementary $\{\leqslant\}$-isomorphism $g : \widehat{c}(M) \to \widehat{c}(N)$ with domain c. As before, $g \cup f$ is an $\mathcal{L}^\star$-isomorphism. ∎

Claim III.1.8: *Any quantifier free $\mathcal{L}^\star$-formula is equivalent modulo $T_E^\star$ to a formula of the form $\bigwedge_i(\varphi_i(x) \wedge \bigwedge_j R_{\varphi_{i,j}}(x, x_{i,j}))$ where φ_i are quantifier free $\{E\}$-formulas and $\varphi_{i,j}$ are $\{\leqslant\}$-formulas.*

Proof. As formulas have a conjunctive normal form, it suffices to prove that for all $\{\leqslant\}$-formulas $\varphi(x)$, $\neg R_\varphi(x,y)$ is as in the claim. But for all $M \vDash T^\star$, tuples $a \in M$ and elements $b \in M$, $M \vDash \neg R_\varphi(a,b)$ if and only if there exists i such that $\neg a_i E b$ or for all i, $a_i \in \widehat{b}$ and $\widehat{b}(M) \vDash \neg\varphi(a)$, i.e. $M \vDash \bigvee_i \neg b E a_i \vee R_{\neg\varphi}(a,b)$. ∎

Let M_n be $\mathcal{P}(\{0,\ldots,n\})$ where $\leqslant$ is interpreted as the inclusion and $M := \coprod_{n\in\mathbb{N}} M_n$ be the $\mathcal{L}$-structure where the classes of E are exactly the M_n. Then $\mathrm{Th}(M)$ has IP. But because of Claims **(III.1.7)** and **(III.1.8)** and because the classes in M do not have new points in elementary extensions of M, M is uniformly stably embedded in every elementary extension.

Now, let $S(x)$ be the set of singletons, i.e. points x such that $\exists^{=1} y\, y \leqslant x$. In M, every (finite) subset of $S(M)$ in a given E-class is definable by a formula of the form $x \leqslant a$. By compactness, it follows that in every infinite E-class $C \subseteq N \succcurlyeq M$, every subset of $S(C)$ is externally definable by a formula of the form $x \leqslant a$ for some external parameter a. Let $\varphi_i(x; y_i)$ be an enumeration of all $\mathcal{L}$-formulas such that $|x| = 1$ and $|y_i| < 1 - \log_2(i)/i$. Then by a simple counting argument, for all $n \in \mathbb{N}$, we can find $U_n \subseteq S(M_n)$ such that $U_n \notin \{\varphi_i(M_n; m) : i < n \text{ and } m \in M_n \text{ a tuple}\}$. Let $\widetilde{\mathcal{L}} := \mathcal{L}_U$ and $\widetilde{M}$ be M where $U(\widetilde{M}) := \bigcup_n U_n$. Let $\widetilde{N} := \widetilde{M}^{\mathbb{N}}/\mathcal{U}$ be some non principal ultrapower. For all $n \in \mathbb{N}$, pick $c_n \in M_n$ and let $c = (c_n)_{n \in \mathbb{N}}/\mathcal{U}$ and $\widetilde{c}(\widetilde{N})$ be its E-class in $\widetilde{N}$. As we have seen above, $U(\widetilde{N}) \cap \widetilde{c}(\widetilde{N}) \subseteq S(\widetilde{c}(\widetilde{N}))$ is externally $\mathcal{L}$-definable, it is $\widetilde{\mathcal{L}}(\widetilde{N})$-definable, but it is not $\mathcal{L}(\widetilde{N}|_{\mathcal{L}})$-definable.

III.2. Uniform stable embeddedness of Henselian valued fields

The goal of this section is to study stable embeddedness in pairs of valued fields and in particular show that there exist models of ACVF uniformly stably embedded in every elementary extension. These models will then be used to prove that there are models of $\mathrm{VDF}_{\mathcal{EC}}$ whose under lying valued field is stably embedded in every elementary extension (see the proof of Theorem **F**). Some of the results given here are not necessary to attain this goal though, but they are of a similar nature and will be used in Section IV.2.
Following Baur, let us first introduce the notion of a separated pair of valued fields.

Definition III.2.1 (Separated pair)**:**
Let $L|K$ be an extension of valued fields. Call a tuple $a \in L$ K-separated if for any tuple $\lambda \in K$, $\mathrm{val}(\sum_i \lambda_i a_i) = \min_i\{\mathrm{val}(\lambda_i a_i)\}$. The pair $L|K$ is said to be separated if any finite dimensional sub-K-vector space of L has a K-separated basis.

Recall that a maximally complete field is a field where every chain of balls has a point, se Definition **(0.4.15)**. Let us now reprove a well known result of [Bau82].

Proposition III.2.2:
Let K be a maximally complete field. Then any extension $L|K$ is separated.

Proof. Let us first prove the following result:

Claim III.2.3: *Let x_0 and $x_1 \in L$. Then the set $\{\mathrm{val}(x_0 - \lambda x_1) : \lambda \in K\}$ has a greatest element.*

Proof. Let $t = x_0/x_1$. For all $y \in K$, let $b_y = \overline{\mathcal{B}}_{\mathrm{val}(y-t)}(x)$. As $y \in b_y(K)$, the balls $b_y(K)$ form a chain of non empty balls from K, hence $z \in \bigcup b_y(K)$. In particular, for all $y \in K$, $\mathrm{val}(t - z) \geqslant \mathrm{val}(t - y)$ and hence $\mathrm{val}(x_0 - zx_1)$ is the maximal value we were looking for. ♦

Let us now prove by induction on $|\overline{x}|$ that for any choice of $\overline{x} \in L$, we can find $\overline{y} \in L$ separated and generating the same K-vector space as the x_i. The case $|\overline{x}| = 1$ is trivial. Let us now assume that $n > 0$ and by induction we find $\overline{y}$ generating the same K-vector space as $(x_i)_{i>0}$ and such that the y_i are K-separated.

Claim III.2.4: *The set* $\{\mathrm{val}(x_0 - \overline{\lambda} \cdot \overline{y}) : \overline{\lambda} \in K\}$ *has a greatest element.*

Proof. For any choice of $i \leqslant n$ and any choice of $\overline{\lambda}_{\neq i}$, by Claim **(III.2.3)**, there exists $\gamma(i, \overline{\lambda}_{\neq i}) \in \mathrm{val}(L)$ and $z(i, \overline{\lambda}_{\neq i}) \in K$ such that $\gamma(i, \overline{\lambda}_{\neq i}) = \mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z(i, \overline{\lambda}_{\neq i}) y_i) = \max\{\mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - \lambda y_i) : \lambda \in K\}$. Then any $z \in K$ is such that the maximum is reached if and only if $\mathrm{val}(z - z(i, \overline{\lambda}_{\neq i})) \geqslant \gamma(i, \overline{\lambda}_{\neq i})$. Indeed, if z is such that $\mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z y_i) = \gamma(i, \overline{\lambda}_{\neq i})$, then;

$$\mathrm{val}(z - z(i, \overline{\lambda}_{\neq i})) + \mathrm{val}(y_i) = \mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z(i, \overline{\lambda}_{\neq i}) y_i - (x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z y_i)) \geqslant \gamma(i, \overline{\lambda}_{\neq i})$$

and if $\mathrm{val}(z - z(i, \overline{\lambda}_{\neq i})) \geqslant \gamma(i, \overline{\lambda}_{\neq i})$, then:

$$\mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z y_i) = \mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z(i, \overline{\lambda}_{\neq i}) y_i + (z(i, \overline{\lambda}_{\neq i}) - z) y_i) \geqslant \gamma(i, \overline{\lambda}_{\neq i}).$$

Hence the z such that the valuation is maximal form a closed ball with radius $\gamma(i, \overline{\lambda}_{\neq i})$. Let $b(i, \overline{\lambda}_{\neq i})$ be this ball. Note that $b(i, \overline{\lambda}_{\neq i})(K) \neq \varnothing$. Let us show that the $b(i, \overline{\lambda}_{\neq i})$ form a chain. Suppose that $\overline{\mu}_{\neq i}$ is such that $\gamma(i, \overline{\lambda}_{\neq i}) \geqslant \gamma(i, \overline{\mu}_{\neq i})$, and let $\lambda \in b(i, \overline{\lambda}_{\neq i})$ and $\mu \in b(i, \overline{\mu}_{\neq i})$. Then $\mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - \lambda y_i) \geqslant \mathrm{val}(x_0 - \overline{\mu}_{\neq i} \cdot \overline{y}_{\neq i} - \mu y_i)$ and hence

$$\gamma_{\overline{\mu}_{\neq i}} \leqslant \mathrm{val}((\overline{\lambda}_{\neq i} - \overline{\mu}_{\neq i}) \cdot \overline{y}_{\neq i} + (\lambda - \mu) y_i) = \min\{\lambda_j - \mu_j, \lambda - \mu\} \leqslant (\lambda - \mu).$$

It follows that $\lambda \in b(i, \overline{\mu}_{\neq i})$ and $b(i, \overline{\lambda}_{\neq i}) \subseteq b(i, \overline{\mu}_{\neq i})$.
As K is maximally complete, there exists $z_i \in \bigcup_{\overline{\lambda}_{\neq i}} b(i, \overline{\lambda}_{\neq i})(K)$, i.e. $\mathrm{val}(x_0 - \overline{\lambda}_{\neq i} \cdot \overline{y}_{\neq i} - z_i y_i)$ is maximal for any choice of $\overline{\lambda}_{\neq i}$. It follows that $\mathrm{val}(x_0 - \overline{z} \cdot \overline{y})$ is maximal. ♦

Let $\overline{z}$ be such that $\mathrm{val}(x_0 - \overline{z} \cdot \overline{y})$ is maximal and $t = x_0 - \overline{z} \cdot \overline{y}$, then $(t, \overline{y})$ generates the same K-vector space as the x_i. Let us now show that $(t, \overline{y})$ is separated. For all μ and $\overline{\lambda} \in K$, if $\mu = 0$, we have $\mathrm{val}(\mu t + \overline{\lambda} \cdot \overline{y}) = \mathrm{val}(\overline{\lambda} \cdot \overline{y}) = \min\{\mathrm{val}(\lambda_i y_i)\} = \min\{\mathrm{val}(\mu t), \mathrm{val}(\lambda_i y_i)\}$ as $\overline{y}$ is K-separated. Otherwise, suppose $\mathrm{val}(\mu t + \overline{\lambda} \cdot \overline{y}) > \min\{\mathrm{val}(\mu t), \mathrm{val}(\lambda_i y_i)\}$. By maximality of $\mathrm{val}(x_0 - \overline{z} \cdot \overline{y}) = \mathrm{val}(t)$ we have:

$$\mathrm{val}(t) + \mathrm{val}(\mu) \geqslant \mathrm{val}(\mu t + \overline{\lambda} \cdot \overline{y}) > \min\{\mathrm{val}(\mu) + \mathrm{val}(t), \mathrm{val}(\lambda_i y_i)\}$$

and thus $\mathrm{val}(\mu t) > \min\{\mathrm{val}(\lambda_i y_i)\} = \mathrm{val}(\overline{\lambda} \cdot \overline{y})$. But then $\mathrm{val}(\mu t + \overline{\lambda} \cdot \overline{y}) = \min\{\mathrm{val}(\lambda_i y_i)\} = \min\{\mathrm{val}(\mu t), \mathrm{val}(\lambda_i y_i)\}$, a contradiction. ∎

Following [Del89; Cub13], let us give the links between separation of the pair $L|K$ and uniform stable embeddedness of K in L. In fact the proof of Proposition **(III.2.5)** is taken almost word for word from the one in [Cub13], although we have slightly different assumptions and we put more emphasis on uniformity here. Let $\mathcal{L}$ be an $\mathbf{RV}$-extension of $\mathcal{L}^{\mathbf{RV}}$ and let $\mathrm{T}_{\mathrm{Hen}}$ be the $\mathcal{L}^{\mathbf{RV}}$-theory of Henselian valued fields of characteristic zero. Recall that $\mathbf{RV}_n^\star = \mathbf{K}^\star/(1 + n\mathfrak{M})$ and that $\mathrm{T}_{\mathrm{Hen}}$ resplendently eliminates field quantifiers (cf. Theorem **(II.1.4)**).

Proposition III.2.5:
Let $M \vDash \mathrm{T_{Hen}}$ be an $\mathcal{L}$-structure and $\varphi(x;s)$ an $\mathcal{L}$-formula. There exists an $\mathcal{L}|_{\mathbf{RV}}$-formula $\psi(y;u)$ and polynomials $Q_i \in \mathbb{Z}[\overline{X},\overline{T}]$ such that for any $N \leqslant M$, where the pair $\mathbf{K}(M)|\mathbf{K}(N)$ is separated, and any $a \in M$ there exists $b \in \mathbf{K}(N)$ and $c \in \mathbf{RV}(M)$ such that $\varphi(N;a) = \psi(\mathrm{rv}_n(\overline{Q}(N,b));c)$.

Proof. By (resplendent) elimination of field quantifiers (and the fact that $\mathbf{K}$ is dominant), we may assume that $\varphi(x;a)$ is of the form $\theta(\mathbf{RV}_n(\overline{P}(x)))$ where $\overline{P} \in \mathbf{K}(M)[\overline{X}]$, $n \in \mathbb{N}$ and ψ is an $\mathcal{L}|_{\mathbf{RV}}$-formula. Let us write each P_i as $\sum_\mu a_{i,\mu}\overline{X}^\mu$. As the pair $\mathbf{K}(M)|\mathbf{K}(N)$ is separated, the $\mathbf{K}(N)$-vector space generated by the $a_{i,\mu}$ is generated by a $\mathbf{K}(N)$-separated tuple $\overline{d} \in \mathbf{K}(M)$. Note that $|\overline{d}| \leqslant |\overline{a}|$ and adding zeros to $\overline{d}$ we may assume $|\overline{d}| = |\overline{a}|$. For each i and μ, find $\lambda_{i,\mu,j} \in \mathbf{K}(N)$ such that $a_{i,\mu} = \sum_j \lambda_{i,\mu,j}d_j$. We can rewrite each P_i as $\sum_j d_jQ_{i,j}(\overline{X},\overline{\lambda})$, where $Q_{i,j} \in \mathbb{Z}[\overline{X},\overline{T}]$ does not depend on $\overline{a}$, and for all $x \in K(N)$ we have $\mathrm{val}(P_i(x)) = \min_j\{\mathrm{val}(d_jQ_{i,j}(x,\overline{\lambda}))\}$. As $\mathrm{rv}_n(x+y) = \mathrm{rv}_n(x) +_{n,n} \mathrm{rv}_n(y)$ whenever $\mathrm{val}(x+y) = \min\{\mathrm{val}(x),\mathrm{val}(y)\}$, it follows immediately that

$$\mathrm{rv}_n(P_i(x)) = \sum_{j\in J_i(x)} \mathrm{rv}_n(d_j)\mathrm{rv}_n(Q_{i,j}(x,\overline{\lambda}))$$

where $J_i(x) = \{j : \mathrm{val}(d_j)\mathrm{val}(Q_{i,j}(x,\overline{\lambda})) \text{ is minimal}\}$.
The proposition now follow easily with $b = \overline{\lambda}$ and $c = \mathrm{rv}_n(\overline{d})$. ∎

Let $\mathcal{L}_{\mathrm{og}}$ be the language of ordered groups. If M is algebraically closed, we can obtain a stronger statement.

Proposition III.2.6:
Let $M \vDash \mathrm{ACVF}$ and $\varphi(x,y)$ an $\mathcal{L}_{\mathrm{div}}$-formula. There is an $\mathcal{L}_{\mathrm{og}}$-formula $\psi(y;u)$ and polynomials $Q_i \in \mathbb{Z}[\overline{X},\overline{T}]$ such that for any $N \leqslant M$, where the pair $\mathbf{K}(M)|\mathbf{K}(N)$ is separated, and any $a \in M$ there exists $b \in \mathbf{K}(N)$ and $c \in \mathbf{\Gamma}(M)$ such that $\varphi(N;a) = \psi(\mathrm{val}(\overline{Q}(N,b));c)$.

Proof. The proof is essentially the same as for Proposition **(III.2.6)** except that we use the quantifier elimination in the two sorted language. ∎

Let us now give the two consequences of these computations that we will be using later on.

Theorem III.2.7 (AKE for stable embeddedness; algebraically closed case)**:**
Let $L|K$ be a separated pair valued fields such that L is algebraically closed. Then K is stably embedded in L if and only if $\mathbf{\Gamma}(K)$ is stably embedded in $\mathbf{\Gamma}(L)$ — as an ordered abelian group.
Moreover, if $\mathbf{\Gamma}(K)$ is uniformly stably embedded in $\mathbf{\Gamma}(L)$, then K is uniformly stably embedded in L.

Proof. This follows immediately from Proposition **(III.2.6)**. ∎

We will now be considering angular components in mixed characteristic $(0,p)$. Recall that $\mathbf{R}_n = \mathcal{O}/(p^n\mathfrak{M})$, $\mathbf{R} = \bigcup_n \mathbf{R}_n$ and that angular component maps are compatible systems of group morphisms $\mathrm{ac}_n : \mathbf{K}^\star \to \mathbf{R}_n^\star$ such that $\mathrm{ac}_n|_{\mathcal{O}^\star} = \mathrm{res}_n|_{\mathcal{O}^\star}$. Note that we changed conventions slightly compared to Chapter II because we fixed the residual characteristic and hence we can restrict ourselves to the relevant residual rings.

Theorem III.2.8 (AKE for stable embeddedness; unramified mixed characteristic with ac)**:**

Let $L|K$ be a separated pair of unramified mixed characteristic valued fields with angular component maps such that L is Henselian and $\mathbf{R}_0(L)$ is perfect. Then K is stably embedded in L if and only if $\mathbf{\Gamma}(K)$ is stably embedded in $\mathbf{\Gamma}(L)$ — as an ordered abelian group — and $\mathbf{R}_0(K)$ is stably embedded in $\mathbf{R}_0(L)$ — as a ring.
Moreover if $\mathbf{\Gamma}(K)$ is uniformly stably embedded in $\mathbf{\Gamma}(L)$ and $\mathbf{R}_0(K)$ is uniformly stably embedded in $\mathbf{R}_0(L)$, then K is uniformly stably embedded in L.

Proof. As explained in Section II.1, an angular component is nothing more than a section of the short exact sequence $\mathbf{R}_n^\star \to \mathbf{RV}_n^\star \to \mathbf{\Gamma}$. Therefore, it follows from Proposition **(III.2.5)** that we only need to prove that $\mathbf{R} \cup \mathbf{\Gamma}(K)$ is (uniformly) stably embedded in $\mathbf{R} \cup \mathbf{\Gamma}(L)$. Because L is unramified, it follows from the quantifier elimination result mentioned in Remark **II.1.9.3** that $\mathbf{\Gamma}$ and $\mathbf{R}$ are orthogonal — i.e. any definable subset of $\mathbf{R}^n \times \mathbf{\Gamma}^m$ is a union of products $X \times Y$ where $X \subseteq \mathbf{R}^n$ is definable in $\mathbf{R}$ and $Y \subseteq \mathbf{\Gamma}^m$ is definable in $\mathbf{\Gamma}$. Hence it suffices to prove that $\mathbf{\Gamma}(K)$ is (uniformly) stably embedded in $\mathbf{\Gamma}(L)$ and that $\mathbf{R}(K)$ is (uniformly) stably embedded in $\mathbf{R}(L)$.
For all n, the canonical projection $\mathrm{res}_{0,n} : \mathbf{R}_n \to \mathbf{R}_0$ has a definable section defined by $\tau_n(x)$ is the only y such that $\mathbf{R}_{0,n}(y) = x$ and y is a p^n-th power. Using this residual version of the Teichmüller liftings, one can show that $\mathbf{R}_n(L)$ is $\varnothing$-definably isomorphic to $\mathrm{W}_n(\mathbf{R}_0(L))$ and hence that $\mathbf{R}(K)$ is (uniformly) stably embedded in $\mathbf{R}(L)$ if and only if $\mathbf{R}_0(K)$ is (uniformly) stably embedded in $\mathbf{R}_0(L)$. ∎

Corollary III.2.9:
Let k be any algebraically closed field. The Hahn field $K := k((t^{\mathbb{R}}))$ is uniformly stably embedded — as a valued field — in any elementary extension.

Proof. The field K is maximally complete — as are all Hahn fields — and so it is Henselian. Moreover its residue field k is algebraically closed and its value group $\mathbb{R}$ is divisible. It follows that K is algebraically closed. By Proposition **(III.2.2)**, any extension $L|K$ is separated. By Theorem **(III.2.7)**, it suffices to show that $\mathbb{R}$ is uniformly stably embedded — as an ordered group — in any elementary extension. But that follows from the fact that $(\mathbb{R}, <)$ is complete and $(\mathbb{R}, +, <)$ is o-minimal, see [CS, Corollary 64]. ∎

III.3. $\mathbf{\Gamma}$-reparametrization

Let $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$, $T \supseteq \mathrm{ACVF}$ be an $\mathcal{L}$-theory that eliminates imaginaries. Assume that T is C-minimal, i.e. $\mathbf{K}$ is dominant — every $\mathcal{L}$-sort is the image of an $\mathcal{L}$-definable map with domain some $\mathbf{K}^n$ — and for all $M \vDash T$, every $\mathcal{L}(M)$-definable unary set $X \subseteq \mathbf{K}$ is a finite union of swiss cheeses (cf. Definition **(0.2.8)**). The two main examples of such theories are $\mathrm{ACVF}^{\mathcal{G}}$ and $\mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$ for some separated Weierstrass system $\mathcal{A}$ (cf. Section II.3).
Let $M \vDash T$, $f = (f_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$, $\Delta(x;t)$ be a finite set of $\mathcal{L}$-formulas and $p \in \mathcal{S}^{\Delta}_x(M)$. We wish to study the family f and in particular

its germs over p (see Definition **(III.3.3)**), to show that they are internal to $\mathbf{\Gamma}$. This is later used as a partial elimination of imaginaries result in enrichments $\widetilde{T}$ of T where $\mathbf{\Gamma}$ is stably embedded: any subset of these germs definable in $\widetilde{T}$ is coded in $\mathbf{\Gamma}^{\mathrm{eq}}$ — where eq is taken relative to the theory induced by $\widetilde{T}$. We only achieve this goal in $\mathrm{ACVF}^{\mathcal{G}}$ and $\mathrm{ACVF}_{\mathcal{A}}{}^{\mathrm{eq}} := (\mathrm{ACVF} \cup \mathrm{T}_{\mathcal{A}})^{\mathrm{eq}}$. The idea of the proof is to reparametrize the family of functions (see Definition **(III.3.2)**).

Remark III.3.1:

1. Recall that $\mathbf{\Gamma}$ is stably embedded and o-minimal in T (cf. Proposition **(0.3.23)**).

2. As $\mathbf{\Gamma}$ is an o-minimal group, the induced structure on $\mathbf{\Gamma}$ eliminates imaginaries.

Let $g = (g_\gamma)_{\gamma \in G}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$, where $G \subset \mathbf{\Gamma}^k$ for some k.

Definition III.3.2 ($\mathbf{\Gamma}$-reparametrization)**:**
We say that g $\mathbf{\Gamma}$-reparametrizes f over p if for all $\lambda \in \Lambda(M)$, there is $\gamma \in G(M)$ such that

$$p(x) \vdash f_\lambda(x) = g_\gamma(x).$$

We say that T admits $\mathbf{\Gamma}$-reparametrizations if for all $\mathcal{L}(M)$-definable families $f = (f_\lambda)_{\lambda \in \Lambda}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ there exists a finite set of $\mathcal{L}$-formulas $\Delta(x;s)$ such that for all $M \vDash T$ and $p \in \mathcal{S}_x^{\Delta}(M)$, there exists an $\mathcal{L}(M)$-definable family $g = (g_\gamma)_{\gamma \in G}$ of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ that $\mathbf{\Gamma}$-reparametrizes f over p.

We will say that Δ is adapted to f (resp. to g) when any Δ-type decides when $f_{\lambda_1}(x) = f_{\lambda_2}(x)$ (resp. $g_{\gamma_1}(x) = g_{\gamma_2}(x)$).

Definition III.3.3 (p-germ)**:**
Assume that Δ is adapted to f and that p is $\mathcal{L}(M)$-definable. We say that f_{λ_1} and f_{λ_2} have the same p-germ if $p(x) \vdash f_{\lambda_1}(x) = f_{\lambda_2}(x)$. Let us denote $\partial_p f_\lambda \in M^{\mathrm{eq}}$ the code of the equivalence class of λ under the equivalence relation "having the same p-germ".

Proposition III.3.4:
Let us assume that g is a $\mathbf{\Gamma}$-reparametrization of f over p, that Δ is adapted to both f and g and that p is $\mathcal{L}(M)$-definable. The set $\{\partial_p f_\lambda : \lambda \in \Lambda\}$ is internal to $\mathbf{\Gamma}$, i.e. there is an $\mathcal{L}(M)$-definable one to one map from this set into some cartesian power of $\mathbf{\Gamma}$.

Proof. As γ is a tuple from $\mathbf{\Gamma}$ and $\mathbf{\Gamma}$ is stably embedded in T and eliminates imaginaries (see Remark **(III.3.1)**), we may assume that $\partial_p g_\gamma \in \mathbf{\Gamma}$. Now pick any λ. Let γ be such that $p(x) \vdash f_\lambda(x) = g_\gamma(x)$. Then $\partial_p g_\gamma$ only depends on $\partial_p f_\lambda$ and not on λ or γ. It follows that the set $\{\partial_p f_\lambda : \lambda \in \Lambda\}$ is in $\mathcal{L}(M)$-definable one to one correspondence with — a subset of — the set $\{\partial_p g_\gamma : \gamma \in G\}$ which is itself a subset of some cartesian power of $\mathbf{\Gamma}$. ∎

Let T now be either $\mathrm{ACVF}^{\mathcal{G}}$ or $\mathrm{ACVF}_{\mathcal{A}}^{\mathrm{eq}}$ and $\mathcal{L}$ be respectively $\mathcal{L}^{\mathcal{G}}$ or $\mathcal{L}_{\mathcal{A},\mathcal{Q}}^{\mathrm{eq}}$. Let us now show that we can reparametrize the valuation of certain terms.
If Z_1 and $Z_2 \subseteq \mathbf{K}$ are finite sets, we will denote $D(Z_1, Z_2) := \{\mathrm{val}(z_1 - z_2) : z_1 \in Z_1$ and $z_2 \in Z_2\}$. Let us order the elements in $D(Z_1, Z_2)$ as $d_1 > d_2 > \cdots > d_k$ and let $d_i(Z_1, Z_2) := d_i$. If $Z_1 = \{z\}$ is singleton we will write $d_i(z, Z_2)$.

Proposition III.3.5:
Let $t(x,y,\lambda):\mathbf{K}^{n+1+l}\to\mathbf{K}$ be an $\mathcal{L}|_{\mathbf{K}}(M)$-term polynomial in y where $|x|=n$, $|y|=1$ and $|\lambda|=l$. Let $Z_\lambda(x):=\{y:t(x,y,\lambda)=0\}$. Then there exists an $\mathcal{L}(M)$-definable family $q=(q_\eta)_{\eta\in\mathrm{H}}$ of functions $\mathbf{K}^n\to\mathbf{\Gamma}$ such that for all $N\succcurlyeq M$, $x\in\mathbf{K}^n(N)$ and $y\in\mathbf{K}(N)$, there exists $\mu_0\in\Lambda(M)$ such that for all $\lambda\in\Lambda(M)$ there exists $\eta\in\mathrm{H}(M)$ and n smaller than the degree of t in y such that:

$$\mathrm{val}(t(x,y,\lambda))=q_\eta(x)+n\cdot d_1(y,Z_{\mu_0}(x)).$$

Proof. For all $\alpha\in Z_\lambda(x)$, let m_α be its multiplicity and let us define:

$$u(x,\lambda):=\frac{t(x,y,\lambda)}{\prod\limits_{\alpha\in Z_\lambda(x)}(y-\alpha)^{m_\alpha}}$$

which is $\mathcal{L}$-definable and does not depend on y, and

$$q_{\lambda,k,\overline{j},\eta}(x):=\mathrm{val}(u(x,\lambda))+\sum_{i=0}^{k}d_{j_i}(Z_\lambda(x),Z_\eta(x)).$$

where k is at most the degree of t in y and $j_i\leqslant l^2$. Note that because we can code disjunctions on a finite number of integers, q can be considered as an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n\to\mathbf{\Gamma}$.
Let $N\succcurlyeq M$, $x\in\mathbf{K}^n(N)$ and $y\in\mathbf{K}(N)$ and let us first assume that there exists $\mu_0\in\Lambda(M)$ such that $d_1(y,Z_{\mu_0}(x))=\max_\mu\{d_1(y,Z_\mu(x))\}$ and let $\alpha_0\in Z_{\mu_0}(x)$ be such that $\mathrm{val}(y-\alpha_0)=d_1(y,Z_{\mu_0}(x))$. Now pick any $\lambda\in\Lambda(M)$ and $\alpha\in Z_\lambda(x)$. Let $\eta=\mu_0$.

Claim III.3.6: *Either* $\mathrm{val}(y-\alpha)=d_1(y,Z_{\mu_0}(x))$ *or* $\mathrm{val}(y-\alpha)=d_j(Z_\lambda(x),Z_\eta(x))$ *for some* j.

Proof. If $\mathrm{val}(y-\alpha)\neq d_1(y,Z_{\mu_0}(x))$, then $\mathrm{val}(y-\alpha)<d_1(y,Z_{\mu_0}(x))$ and $\mathrm{val}(y-\alpha)=\mathrm{val}(\alpha-\alpha_0)=d_j(Z_\lambda(x),Z_{\mu_0}(x))$ for some j. ♦

In the other case, if there does not exist a maximum in $\{d_1(y,Z_\mu(x))\}$, pick any $\lambda\in\Lambda(M)$. Then there exists $\eta\in\Lambda(M)$ such that $d_1(y,Z_\eta(x))>d_1(y,Z_\lambda(x))$. Let α_0 be such that $\mathrm{val}(y-\alpha_0)=d_1(y,Z_\eta(x))$, then for all $\alpha\in Z_\lambda(x)$, $\mathrm{val}(y-\alpha)=\mathrm{val}(\alpha-\alpha_0)=d_j(Z_\lambda(x),Z_\eta(x))$ for some j and that concludes the proof.
In both cases, as $\mathrm{val}(t(x,y,\lambda))=\mathrm{val}(u(x,\lambda))+\sum_{\alpha\in Z_\lambda(x)}m_\alpha\mathrm{val}(y-\alpha)$, it follows that

$$\begin{aligned}\mathrm{val}(t(x,y,\lambda)) &= \mathrm{val}(u(x,\lambda))+\textstyle\sum_{i=0}^{k}d_{j_i}(Z_\lambda(x),Z_\eta(x))+n\cdot d_1(y,Z_{\mu_0}(x))\\ &= q_{\lambda,k,\overline{j},\mu_0}(x)+n\cdot d_1(y,Z_{\mu_0}(x))\end{aligned}$$

for some n, k and $\overline{j}$. ■

Proposition III.3.7 (Existence of $\mathbf{\Gamma}$-reparametrization)**:**
Let $f=(f_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n\to\mathbf{\Gamma}$. Then there exists an $\mathcal{L}(M)$-definable family $g=(g_{\omega,\gamma})_{\omega\in\Omega,\gamma\in G}$ of functions $\mathbf{K}^n\to\mathbf{\Gamma}$ where $G\subseteq\mathbf{\Gamma}^k$ for some k and a finite set of $\mathcal{L}$-formulas $\Delta(x;t)$ such that for any $p\in\mathcal{S}^{\Delta}_x(M)$ there exists $\omega_0\in\Omega(M)$ such that $(g_{\omega_0,\gamma})_{\gamma\in G}$ is a $\mathbf{\Gamma}$-reparametrization of f over p.

Proof. We work by induction on n. The case $n = 0$ is trivial as f is nothing more than a family of points in $\mathbf{\Gamma}$ that can be reparametrized by themselves. Let us now assume that $n = m + 1$ and $x = (y, z)$ where $|z| = 1$. Because $\mathbf{K}$ is dominant, we may assume up to reparametrization that λ is a tuple from $\mathbf{K}$. If $T = \mathrm{ACVF}^{\mathcal{G}}$, the graph of f_λ is given by an $\mathcal{L}^{\mathcal{G}}(M)$-formula. If $T = \mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$, by Corollary **(II.5.5)** there exists an $\mathcal{L}^{\mathcal{G}}(M)$-formula $\psi(z, \overline{w}, \gamma)$ and $\mathcal{L}|_{\mathbf{K}}$-terms $\overline{r}(x, \lambda)$ such that $M \vDash f_\lambda(y, z) = \gamma$ if and only if $M \vDash \psi(z, \overline{r}(y, \lambda), \gamma)$. Taking $\overline{r}$ to be the identity, the graph of f_λ also has this form when $T = \mathrm{ACVF}^{\mathcal{G}}$. By elimination of quantifiers in $\mathrm{ACVF}^{\mathcal{G}}$ (or in the two sorted language), we know that $\psi(z, \overline{w}, \gamma)$ is of the form $\chi((\mathrm{val}(P_i(z, \overline{w})))_{0 \leqslant i < k}, \gamma)$ where χ is an $\mathcal{L}^{\mathcal{G}}|_{\mathbf{\Gamma}}$-formula and $P_i \in \mathbf{K}(M)[Y, \overline{W}]$. We may also assume that χ defines a function $h : \mathbf{\Gamma}^k \to \mathbf{\Gamma}$.
Let $t_i(y, z, \lambda) = P_i(z, \overline{r}(y, \lambda))$ and $q_i = (q_{i,\eta})_{\eta \in \mathrm{H}_i}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^m \to \mathbf{\Gamma}$ as in Proposition **(III.3.5)** with respect to t_i. By the usual coding tricks we may assume that there is only one family $q = (q_\eta)_{\eta \in \mathrm{H}}$ such that for all i and $\eta \in \mathrm{H}_i$ there exists $\varepsilon \in \mathrm{H}$ such that $q_{i,\eta} = q_\varepsilon$. By induction, Proposition **(III.3.7)** holds for q and there exist a finite set of $\mathcal{L}$-formulas $\Xi(y; s)$ and an $\mathcal{L}(M)$-definable family $(u_{\varepsilon,\delta})_{\varepsilon \in \mathrm{E}, \delta \in D}$ of functions $\mathbf{K}^m \to \mathbf{\Gamma}$, where $D \subseteq \mathbf{\Gamma}^l$ for some l, such that for any $p \in \mathcal{S}^{\Xi}_y(M)$, for some $\varepsilon_0 \in \mathrm{E}(M)$, $(u_{\varepsilon_0,\delta})_{\delta \in D}$ is a $\mathbf{\Gamma}$-reparametrization of q. Let $Z_{i,\lambda}(y) := \{z : P_i(y, z, \lambda) = 0\}$ and

$$g_{\varepsilon,\overline{\mu},\overline{\delta},\overline{n}}(y, z) := h((u_{\varepsilon,\delta_i}(x) + n_i \cdot d_1(z, Z_{i,\mu_i}(y)))_{0 \leqslant i < k}).$$

Let also $\varphi_{\overline{n}}(y, z; \lambda, \varepsilon, \mu, \overline{\delta})$:="$f_\lambda(y, z) = g_{\varepsilon,\mu,\overline{\delta},\overline{n}}(y, z)$" and $\Delta(y, z; s, \lambda, \varepsilon, \mu, \overline{\delta}, \overline{n}) := \Xi(y; s) \cup \{\varphi_{\overline{n}}(y, z; \lambda, \varepsilon, \mu, \overline{\delta}) : \overline{n} \in \mathbb{N}\}$. For all $p \in \mathcal{S}^{\Delta}_{y,z}(M)$, there exists $\varepsilon_0 \in \mathrm{E}(M)$ such that $(u_{\varepsilon_0,\delta})_{\delta \in D}$ $\mathbf{\Gamma}$-reparametrizes q over $p|_{\Xi}$. Let $(y, z) \vDash p$. By Proposition **(III.3.5)** there exists a tuple $\overline{\mu}_0 \in \Lambda(M)$ such that for all $\lambda \in \Lambda(M)$, there exists tuples $\overline{\eta} \in \mathrm{H}(M)$ and $\overline{n}$ such that $\mathrm{val}(t_i(y, z, \lambda)) = q_{\eta_i}(y) + n_i \cdot d_1(y, Z_{i,\mu_{0,i}}(x))$. As $y \vDash p|_{\Xi}$, there exists $\delta_i \in D(M)$ such that $q_{\eta_i}(y) = u_{\varepsilon_0,\delta_i}(y)$ and hence

$$\begin{aligned} f_\lambda(y, z) &= h((\mathrm{val}(t_i(y, z, \lambda)))_{0 \leqslant i < k}) \\ &= h((u_{\varepsilon_0,\delta_i}(y) + n_i \cdot d_1(y, Z_{i,\mu_{0,i}}(x)))_{0 \leqslant i < k}) \\ &= g_{\varepsilon_0,\overline{\mu_0},\overline{\delta},\overline{n}}(y, z). \end{aligned}$$

Because p decides such equalities, this holds in fact for all realizations of p. We have just shown that $(g_{\varepsilon_0,\overline{\mu_0},\overline{\delta},\overline{n},})_{\overline{\delta} \in D, \overline{n} \in \mathbb{N}}$ reparametrizes f over p. But because $\overline{\delta}$ is a tuple from $\mathbf{\Gamma}$ and disjunctions on a finite number of bounded integers can be coded in $\mathbf{\Gamma}$, it is in fact a $\mathbf{\Gamma}$-reparametrization. ∎

Corollary III.3.8:
The theories $\mathrm{ACVF}^{\mathcal{G}}$ *and* $\mathrm{ACVF}^{\mathrm{eq}}_{\mathcal{A}}$ *admit* $\mathbf{\Gamma}$*-reparametrizations.*

Proof. This is an immediate consequence of Proposition **(III.3.7)**. ∎

In fact, in Proposition **(III.3.7)**, we have proved a slightly stronger result. These theories admit what we could call uniform $\mathbf{\Gamma}$-reparametrizations because the $\mathbf{\Gamma}$-reparametrizations of a given family f come in a definable family that does not depend on the type p. But we will not be needing this stronger result afterwards.

Question III.3.9: Do all C-minimal extensions of ACVF admit $\mathbf{\Gamma}$-reparametrizations?

III.4. Types and uniform families of balls

Let $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ and $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory that eliminates imaginaries. In this section, we wish to make precise the idea that in C-minimal theories, $n+1$-types can be viewed as generic types of balls parametrized over realizations of an n-type. This is an obvious higher dimensional generalization of the unary notion of genericity in a ball (cf. [HHM06, Definition 2.3.4] and Section I.3). To do so, we define a class of Δ-types (see Definition (**III.4.12**)) for Δ a finite set of $\mathcal{L}$-formulas that will play a central role in the rest of this chapter. We also show that at the cost of enlarging Δ, we may assume that all types are of this specific form.
We take the convention that points in $\mathbf{K}$ are closed balls of radius $+\infty$ and $\mathbf{K}$ itself is an open ball of radius $-\infty$.

Definition III.4.1 ($\mathbf{B}^{[l]}$ and $\mathbf{B}^{[l]}_{\mathrm{st}}$)**:**
Let $\overline{\mathbf{B}}$ be the set of all closed balls (potentially with radius $+\infty$), $\dot{\mathbf{B}}$ be the set of all open balls (potentially with radius $-\infty$), $\mathbf{B} := \overline{\mathbf{B}} \cup \dot{\mathbf{B}}$ and $l \in \mathbb{N}_{>0}$. We define $\mathbf{B}^{[l]} := \{B \subseteq \mathbf{B} : |B| \leqslant l\}$. We also define $\mathbf{B}^{[l]}_{\mathrm{st}} := \{B \in \mathbf{B}^{[l]}$: all the balls in B have the same radius and they are either all open or all closed$\}$.

Notation III.4.2:
For all $B \in \mathbf{B}^{[l]}$, we will be denoting $\bigcup_{b \in B} b$ — i.e. the valued field points in the balls in B — by $\mathbb{S}(B)$. Because the balls can be nested, $\mathbb{S}$ is not an injective function. But in each fiber of $\mathbb{S}$ there is a unique element with minimal cardinal — the one where there is no intersection between the balls. We will denote by $\mathbb{B}$ this canonical section of $\mathbb{S}$.

Remark III.4.3:

1. As $\mathbf{B}$ is the disjoint union of the sets of codes for open balls and the set of codes for closed ones, one can decide wether a given code is the code of an open or a closed ball and hence $\mathbf{B}^{[l]}_{\mathrm{st}}$ is indeed an interpretable set. In fact, one can also recognize if a ball b is open or closed by looking if the set $\{\mathrm{val}(x-y) : x, y \in b\}$ has a smallest element or not.

2. Note that $\varnothing \in \mathbf{B}^{[l]}_{\mathrm{st}}$

3. Points in $\mathbf{B}^{[l]}_{\mathrm{st}}$ behave more or less like balls. For example if B_1 and $B_2 \in \mathbf{B}^{[l]}_{\mathrm{st}}$ are such that $\mathbb{S}B_1 \subset \mathbb{S}B_2$, where $\subset$ denotes the strict inclusion, then either all the balls in B_1 have smaller radius than the balls in B_2 or if they have equal radiuses, then the balls in B_1 must be open and those in B_2 must be closed, or else the inclusion would not be strict.

Definition III.4.4 (Generalized radius)**:**
Let $B \in \mathbf{B}^{[l]}_{\mathrm{st}} \smallsetminus \{\varnothing\}$. We define the generalized radius of B — denoted $\mathrm{grad}(B)$ — to be the pair $(\gamma, 0)$ when the balls in B are closed of radius γ and $(\gamma, 1)$ when they are open of radius γ. The set of generalized radii — a subset of $(\mathbf{\Gamma} \cup \{-\infty, +\infty\}) \times \{0,1\}$ — is ordered lexicographically. We also define the generalized radius of $\varnothing$ to be $(+\infty, 1)$, i.e. greater than any generalized radius

of non empty $B \in \mathbf{B}_{\mathrm{st}}^{[l]}$.

Proposition III.4.5:
Let $(B_i)_{i \in I} \subseteq \mathbf{B}_{\mathrm{st}}^{[l]}$. *Assume that there exists* i_0 *such that the balls in* B_{i_0} *have greater — or equal — generalized radius than in all the other* B_i *— in particular this holds if* I *is finite. Then* $\mathbb{B}(\bigcap_i \mathbb{S}(B_i)) \subseteq B_{i_0}$. *Moreover, there exists* $(i_j)_{0<j\leqslant l} \in I$ *such that* $\bigcap_i \mathbb{S}(B_i) = \bigcap_{j=0}^{l} \mathbb{S}(B_{i_j})$.

Proof. For any $b \in B_{i_0}$, if $\bigcap_i \mathbb{S}(B_i) \cap b \neq \varnothing$ then $b \subseteq \bigcap_i \mathbb{S}(B_i)$. Hence $\bigcap_i \mathbb{S}(B_i) = \mathbb{S}(\{b \in B_{i_0} : b \cap \bigcap_i B_i \neq \varnothing\})$ and $\mathbb{B}(\bigcap_i \mathbb{S}(B_i)) \subseteq B_{i_0}$. Moreover, if $\bigcap_i \mathbb{S}(B_i) \cap b = \varnothing$, then there exists i_b such that $b \cap \mathbb{S}(B_{i_b}) = \varnothing$ and $\bigcap_i \mathbb{S}(B_i)$ can be obtained by intersecting B_{i_0} with the B_{i_b} of which there are at most l. ∎

Definition III.4.6 ($d_i(B_1, B_2)$)**:**
Let b_1 *and* $b_2 \in \mathbf{B}$. *When* $b_1 \cap b_2 = \varnothing$, *we define* $d(b_1, b_2)$ *to be* $\mathrm{val}(x_1 - x_2)$, *where* $x_i \in b_i$, *which does not depend on the choice of the* x_i. *When* $b_1 \subseteq b_2$ *(and vice versa), we define* $d(b_1, b_2) = \min\{\mathrm{rad}(b_1), \mathrm{rad}(b_2)\}$.
For all B_1 *and* $B_2 \in \mathbf{B}^{[l]}$, *let us define* $D(B_1, B_2) := \{d(b_1, b_2) : b_1 \in B_1$ *and* $b_2 \in B_2\}$ *and let us list the elements in* $D(B_1, B_2)$ *as* $d_1 > d_2 > \cdots > d_k$. *For all* $i \leqslant k$, *we define* $d_i(B_1, B_2) := d_i$.

This definition coincides with the definition in Section III.3 for finite sets of points. We also define $d_0(B_1, B_2) := \min\{\mathrm{rad}(B_1), \mathrm{rad}(B_2)\}$ — which is equal to $d_1(B_1, B_2)$ when $\mathbb{S}(B_1) \cap \mathbb{S}(B_2) \neq \varnothing$. For later coding purpose we might want $d_i(B_1, B_2)$ to be defined for all $i \leqslant l^2$ in which case, for $i > k$, we set $d_i(B_1, B_2) = d_k$.
Let $M \vDash T$, $F = (F_\lambda)_{\lambda \in \Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ — in particular Λ is an $\mathcal{L}(M)$-definable set — and $\Delta(x, y; t)$ a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$, $y \in \mathbf{K}$ and t is a tuple of variables. To simplify notations, we will be denoting $\mathbb{S}(F_\lambda(x))$ by $F_\lambda^{\mathbb{S}}(x)$. Note that if $n = 0$ all of what we prove in this section and in Section III.5 still hold (and is in fact much simpler because we are considering fixed balls instead of parametrized balls).

Definition III.4.7 (Δ adapted to F)**:**
Say that Δ *is adapted to* F *if there are* $\lambda_\varnothing$ *and* $\lambda_{\mathbf{K}} \in \Lambda$ *such that for all* $x \in \mathbf{K}^n$, $F_{\lambda_\varnothing}(x) = \varnothing$ *and* $F_{\lambda_{\mathbf{K}}}(x) = \{\mathbf{K}\}$ *and for all* $p \in \mathcal{S}_{x,y}^{\Delta}(M)$, $p(x, y)$ *decides:*

(i) *For* $\square \in \{=, \subseteq\}$, λ *and* $(\mu_i)_{0 \leqslant i < l} \in \Lambda(M)$, *if* $F_\lambda^{\mathbb{S}}(x) \square \bigcup_{0 \leqslant i < l} F_{\mu_i}^{\mathbb{S}}(x)$;

(ii) *For* $\square \in \{=, \subseteq\}$, λ *and* $(\mu_i)_{0 \leqslant i < l} \in \Lambda(M)$, *if* $F_\lambda(x) \square \bigcup_{i<l} F_{\mu_i}(x)$;

(iii) *For* λ_1, λ_2 *and* $\mu \in \Lambda(M)$, *if* $F_\mu^{\mathbb{S}}(x) = F_{\lambda_1}^{\mathbb{S}}(x) \cap F_{\lambda_2}^{\mathbb{S}}(x)$;

(iv) *For* $\lambda \in \Lambda(M)$, *if the balls in* $F_\lambda(x)$ *are closed;*

(v) *For* $\square \in \{=, \leqslant\}$, λ, μ_1 *and* $\mu_2 \in \Lambda(M)$ *and* $i \leqslant l^2$, *if* $\mathrm{rad}(F_{\lambda_1}(x)) \square d_i(F_{\mu_1}(x), F_{\mu_2}(x))$;

Note that none of the above formulas (and many later on) actually depends on y so what is really relevant is not p but the closed set induced by p in $\mathcal{S}_x^{\mathcal{L}}(M)$.
Until Proposition **(III.4.15)**, let us assume that Δ is adapted to F and let $p \in \mathcal{S}_{x,y}^{\Delta}(M)$.

Definition III.4.8 (Generic intersection)**:**
We say that F is closed under generic intersection over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$, there exists $\mu \in \Lambda(M)$ such that
$$p(x,y) \vdash F^{\$}_{\mu}(x) = F^{\$}_{\lambda_1}(x) \cap F^{\$}_{\lambda_2}(x).$$

Let us assume, until Proposition **(III.4.15)**, that F is closed under generic intersection over p.

Definition III.4.9 (Generic irreducibility)**:**
For all $\lambda \in \Lambda(M)$, we say that F_λ is generically irreducible over p if for all $\mu \in \Lambda(M)$, if $p(x,y) \vdash F_\mu(x) \subseteq F_\lambda(x)$ and $p(x,y) \vdash F_\mu(x) \neq \varnothing$ then $p(x,y) \vdash F_\mu(x) = F_\lambda(x)$.
We say that F is generically irreducible over p if for every $\lambda \in \Lambda(M)$, F_λ is generically irreducible over p.

Let us now show that generically irreducible families of balls behave nicely under generic intersection.

Proposition III.4.10:
Let λ_1 and $\lambda_2 \in \Lambda(M)$ be such that F_{λ_1} and F_{λ_2} are generically irreducible over p and $p(x,y)$ implies that the balls in $F_{\lambda_1}(x)$ have smaller — or equal — generalized radius than the balls in $F_{\lambda_2}(x)$. Then either $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \cap F^{\$}_{\lambda_2}(x) = \varnothing$ or $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \cap F^{\$}_{\lambda_2}(x) = F^{\$}_{\lambda_1}(x)$.

Proof. Let $(a,c) \vDash p$. By Proposition **(III.4.5)**, we have that $\mathbb{B}(F^{\$}_{\lambda_1}(a) \cap F^{\$}_{\lambda_2}(a)) \subseteq F^{\$}_{\lambda_1}(a)$. By generic intersection, there exists μ such that $p(x,y) \vdash F^{\$}_{\mu}(x) = F^{\$}_{\lambda_1}(x) \cap F^{\$}_{\lambda_2}(x)$. Then $F_\mu(a) \subseteq F_{\lambda_1}(a)$ and hence, if $F_\mu(a) \neq \varnothing$, $F_\mu(a) = F_{\lambda_1}(a)$. ∎

Corollary III.4.11:
Assume p is $\mathcal{L}(M)$-definable. Then $\Lambda_p := \{\lambda \in \Lambda : F_\lambda$ is generically irreducible over $p\}$ is $\mathcal{L}(M)$-definable and the $\mathcal{L}(M)$-definable family $(F_\lambda)_{\lambda \in \Lambda_p}$ is closed under generic intersection over p.

Proof. The definability of Λ_p is a consequence of the definability of p and the closure of $(F_\lambda)_{\lambda \in \Lambda_p}$ under generic intersection follows from Proposition **(III.4.10)**. ∎

Until Proposition **(III.4.15)**, let us also assume that F is generically irreducible over p.

Definition III.4.12 ((Δ, F)-Generic type of E over p)**:**
Let $E \subset \Lambda(M)$. Let $\Psi_{\Delta,F}(x,y;t,\lambda)$ denote the set $\Delta(x,y;t) \cup \{y \in F_\lambda(x) \wedge \lambda \in \Lambda\}$. We define $\alpha_{E/p}(x,y)$ to be the (Δ, F)-generic type of E over p to be the following $\Psi_{\Delta,F}$-type:

$$\begin{aligned} p(x,y) \quad &\cup \quad \{y \in F^{\$}_{\lambda}(x) : \lambda \in E\} \\ &\cup \quad \{y \notin F^{\$}_{\mu}(x) : \mu \in \Lambda(M) \text{ and for all } \lambda \in E,\ p(x,y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)\}. \end{aligned}$$

Note that most of the time, Δ and F will be obvious from the context and it will not be an issue that the notation $\alpha_{E/p}$ mentions neither Δ nor F.

Proposition III.4.13:
Let $E \subset \Lambda(M)$ be such that $\alpha_{E/p}$ is consistent, then $\alpha_{E/p}$ generates a complete $\Psi_{\Delta,F}$-type.

Therefore, when it is consistent, we will identify $\alpha_{E/p}$ with the type it generates.

Proof. Pick any $\mu \in \Lambda(M)$. Either there exists $\lambda \in E$ such that $p(x,y) \vdash F^{\$}_{\mu}(x) \cap F^{\$}_{\lambda}(x) = \emptyset$, in which case $\alpha_{E/p}(x,y) \vdash y \notin F^{\$}_{\mu}(x)$, or there exists $\lambda \in E$ such that $p(x,y) \vdash F^{\$}_{\lambda}(x) \subseteq F^{\$}_{\mu}(x)$, and then $\alpha_{E/p}(x,y) \vdash y \in F^{\$}_{\mu}(x)$, or for all $\lambda \in E$, $p(x,y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)$ and hence $\alpha_{E/p}(x,y) \vdash y \notin F^{\$}_{\mu}(x)$. ∎

Remark III.4.14:
Any $q \in \mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(M)$ is of the form $\alpha_{E/p}$. Indeed let $p := q|_{\Delta}$ and $E = \{\lambda \in \Lambda(M) : q(x,y) \vdash y \in F_{\lambda}(x)\}$, then, quite clearly, $q = \alpha_{E/p}$.

Although this will not be used afterwards, when y does not appear in Δ, we can also prove consistency under some obvious hypothesis:

Proposition III.4.15:
Assume that y does not appear in any of the formulas in $\Delta(x,y;t)$ and let $E \subseteq \Lambda(M)$ be such that for all λ_1 and $\lambda_2 \in E$, $p(x) \vdash F_{\lambda_1}(x) \cap F_{\lambda_2}(x) \neq \emptyset$. Then $\alpha_{E/p}\big|_M$ is consistent and generates a complete $\Psi_{\Delta,F}$-type over M.

Proof. Let us show consistency — completeness then follows from Proposition **(III.4.13)**. If this type is not consistent, there exists finitely many $\lambda_i \in E$ and finitely many $\mu_j \in \Lambda(M)$ such that for all $\lambda \in E$, $p(x) \vdash F^{\$}_{\mu_j}(x) \subset F^{\$}_{\lambda}(x)$ and $p(x) \vdash \bigcap_i F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$. Replacing $\bigcap_i F^{\$}_{\lambda_i}(x)$ by their generic intersection — which is also in E by Proposition **(III.4.10)** — we find $\lambda \in E$ such that $p(x) \vdash F^{\$}_{\lambda}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$. Let $x \vDash p$ and b be one of the balls in $F_{\lambda}(x)$. This ball is covered by finitely subballs from $\bigcup_j F_{\mu_j}(x)$ and, as the residue field is infinite, it must be included in one of those balls. Let us assume that $b \subseteq F^{\$}_{\mu_1}(x)$. Then the balls of $F_{\lambda}(x)$ must have smaller generalized radius than those of $F_{\mu_1}(x)$. Hence by Proposition **(III.4.10)**, $F^{\$}_{\mu_1}(x) \cap F^{\$}_{\lambda}(x) = F^{\$}_{\lambda}(x)$, i.e. $F^{\$}_{\lambda}(x) \subseteq F^{\$}_{\mu_1}(x)$, a contradiction. ∎

Now that we have found finite sets Θ of $\mathcal{L}$-formulas, namely those of the form $\Psi_{\Delta,F}$, for which we understand the Θ-types, let us show that any finite set of formulas with variables in $\mathbf{K}^{n+1}$ can be decided by some $\Psi_{\Delta,F}$ for well chosen Δ and F.

Proposition III.4.16 (Reduction to $\Psi_{\Delta,F}$-types)**:**
Let $\Theta(x,y;t)$ be a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $y \in \mathbf{K}$. Then there exists an $\mathcal{L}$-definable family $(F_{\lambda})_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$ and a finite set of $\mathcal{L}$-formulas $\Delta(x;s)$ such that any $\Psi_{\Delta,F}$-type decides all the formulas in Θ.

Proof. Let $\varphi(x,y;t)$ be a formula in Θ. For all tuples $a \in \mathbf{K}$ and $c \in M$, the set $\varphi(a,M;c)$ has a canonical representation as a swiss cheese, i.e. is of the form $\bigcup_i (b_i \smallsetminus b_{i,j})$ where the b_i and $b_{i,j}$ are algebraic over ac. In particular, there exists $l \in \mathbb{N}_{>0}$ and $\mathcal{L}(c)$-definable functions $H_{\varphi,c} : \mathbf{K}^n \to \mathbf{B}^{[l]}$ and $G_{\varphi,c} : \mathbf{K}^n \to \mathbf{B}^{[l]}$ such that $M \vDash \forall y\,(y \in H^{\$}_{\varphi,c}(a) \smallsetminus G^{\$}_{\varphi,c}(a) \iff \varphi(a,y;c))$. By compactness, we can find finitely many $\mathcal{L}$-definable families $(H_{i,\varphi,c})_c$ and $(G_{i,\varphi,c})_c$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l_{i,\varphi}]}$ such that for any choice of c and a there is an i such that $\varphi(a,y;c) \iff y \in H^{\$}_{i,\varphi,c}(a) \smallsetminus G^{\$}_{i,\varphi,c}(a)$. Choosing l to be the maximum of the $l_{i,\varphi}$ and using any coding trick, one can find an $\mathcal{L}$-definable family $(F_{\lambda})_{\lambda\in\Lambda}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$ such that for any $\varphi \in \Theta$, i and c we find μ and $\nu \in \Lambda$ such that $H_{i,\varphi,c} = F_{\mu}$ and $G_{i,\varphi,c} = F_{\nu}$.

Now let $\Delta(x;t,\mu,\nu) = \{\forall y\,(\varphi(x,y;t) \iff y \in F^{\$}_{\mu}(x) \smallsetminus F^{\$}_{\nu}(x)) : \varphi \in \Theta\}$. Then for any $p \in \mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(M)$, $\varphi \in \Theta$ and tuple $c \in M$, there exists μ and $\nu \in \Lambda(M)$ such that $p(x,y) \vdash \varphi(x,y;c) \iff y \in F^{\$}_{\mu}(x) \smallsetminus F^{\$}_{\nu}(x)$ and either $p(x,y) \vdash y \in F^{\$}_{\mu}(x) \wedge y \notin F^{\$}_{\nu}(x)$ in which case $p(x,y) \vdash \varphi(x,y;c)$ or not, in which case $p(x,y) \vdash \neg\varphi(x,y;c)$. ∎

And now let us show that we can refine any Δ and F into a family verifying all previous hypotheses.

Proposition III.4.17 (Reduction to $\mathbf{B}^{[l]}_{\mathrm{st}}$)**:**
Let $A \subseteq M$ and $(F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}$. Then there exists an $\mathcal{L}(A)$-definable family $(G_\omega)_{\omega\in\Omega}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ such that for all λ there exist $(\omega_i)_{0\leqslant i<l}$ such that $F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$ and for all ω there exists λ such that $G_\omega(x) \subseteq F_\lambda(x)$.

Proof. We define $G_{\lambda,i,j}(x) := \{b \in F_\lambda(x) : b$ is open if $j = 0$ closed otherwise and b has the i-th smallest radius among the balls in $F_\lambda(x)\}$. As we can code disjunctions on a finite number of bounded integers, $G = (G_\omega)_{\omega\in\Omega}$ can indeed be viewed as an $\mathcal{L}(A)$-definable family. Then for all x, $G_\omega(x) \in \mathbf{B}^{[l]}_{\mathrm{st}}$ and for all x and λ, $G_{\lambda,i,j}(x) \subseteq F_\lambda(x)$ and $F_\lambda(x) = \bigcup_{i,j} G_{\lambda,i,j}(x)$ and at most l of them are non empty. ∎

Definition III.4.18 (Generic complement)**:**
We say that F is closed under generic complement over p if for all λ and $\mu \in \Lambda(M)$ such that $p(x) \vdash F_\mu(x) \subseteq F_\lambda(x)$, there exists $\kappa \in \Lambda(M)$ such that

$$p(x) \vdash F_\lambda(x) = F_\mu(x) \uplus F_\kappa(x).$$

Note that p can indeed decide any such statement because it is equivalent to $F_\lambda(x) = F_\mu(x) \cup F_\kappa(x)$ and $F^{\$}_{\mu}(x) \cap F^{\$}_{\kappa}(x) = \varnothing$.

Lemma III.4.19:
Let $F = (F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$, $\Delta(x;t)$ a finite set of $\mathcal{L}$-formulas adapted to F and $p \in \mathcal{S}^{\Delta}_{x}(M)$. Assume that F is closed under generic complement over p. Let $\Lambda_p := \{\lambda \in \Lambda : F_\lambda$ is generically irreducible over $p\}$, then for all $\lambda \in \Lambda(M)$ there exists $(\lambda_i)_{0\leqslant i<l} \in \Lambda_p(M)$ such that $p(x) \vdash F_\lambda(x) = \bigcup_i F_{\lambda_i}(x)$.

Proof. Let $x \models p$. We work by induction on $|F_\lambda(x)|$. If there exists $\mu \in \Lambda(M)$ such that $F_\mu(x) \subset F_\lambda(x)$ and $F_\mu(x) \neq \varnothing$, then there exists $\kappa \in \Lambda(M)$ such that $F_\lambda(x) = F_\mu(x) \uplus F_\kappa(x)$. We now apply the induction hypothesis to $F_\mu(x)$ and $F_\kappa(x)$. Finally, because $|F_\lambda(x)| \leqslant l$, we cannot cut it in more than l distinct pieces. ∎

Proposition III.4.20 (Reduction to irreducible families)**:**
Let $A \subseteq M$, $(F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and $\Delta(x;t)$ a finite set of $\mathcal{L}$-formulas. Then, there exists an $\mathcal{L}(A)$-definable family $(G_\omega)_{\omega\in\Omega}$ of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and a finite set of $\mathcal{L}$-formulas $\Theta(x;t,s) \supseteq \Delta(x;t)$ such that Θ is adapted to G and for any $p \in \mathcal{S}^{\Theta}_{x}(M)$:

(i) G is closed under generic intersection and complement over p;

(ii) For all $\omega \in \Omega(M)$ there exists $\lambda \in \Lambda(M)$ such that $p(x) \vdash G_\omega(x) \subseteq F_\lambda(x)$;

(iii) For all $\lambda \in \Lambda(M)$, there exists $\omega \in \Omega(M)$ such that $p(x) \vdash F_\lambda(x) = G_\omega(x)$;

(iv) For all $\omega \in \Omega(M)$, there exists $(\omega_i)_{0\leqslant i<l} \in \Omega_p(M)$ such that $p(x) \vdash G_\omega(x) = \bigcup_i G_{\omega_i}(x)$;

where $\Omega_p := \{\omega \in \Omega : G_\omega$ is generically irreducible over $p\}$.

Proof. Adding them if necessary, we may assume that F contains the constant functions equal to $\varnothing$ and $\{\mathbf{K}\}$ respectively. Let $H_{\overline{\lambda}}(x) := \mathbb{B}(\bigcap_{i\leqslant l} F^{\mathbb{S}}_{\lambda_i}(x))$. It follows from Proposition **(III.4.5)**, that $H = (H_{\overline{\lambda}})_{\overline{\lambda}\in\Lambda^{l+1}}$ is well-defined and that **III.4.20.(ii)** holds for H. Adding finitely many formulas to $\Delta(x;t)$, we obtain $\Xi(x;s)$ that is adapted to H. Let $p \in \mathcal{S}^{\Xi}_x(M)$. Proposition **(III.4.5)** also implies that for a given x, the intersection of any number of $F^{\mathbb{S}}_\lambda(x)$ is given by the intersection of $r+1$ of them and hence is an instance of H. As Ξ is adapted to H, we have proved that H is closed under generic intersection over any Ξ-type p. Condition **III.4.20.(iii)** also clearly holds for H.
Let $B \in \mathbf{B}^{[l]}_{\mathrm{st}}$, we define B^1 to be B and B^0 to be its complement (in $\mathbf{B}$). As previously, to simplify notations, for $\varepsilon \in \{0,1\}$, we will write $H^{\varepsilon}_\mu(x)$ for $(H_\mu(x))^\varepsilon$.

Claim III.4.21: *Let $B \in \mathbf{B}^{[l]}_{\mathrm{st}}$. Any boolean combination of sets $(C_i)_{i\leqslant r} \subseteq B$ (where the negation is given by the complement in B, i.e. $C^0 \cap B$) lives in $\mathbf{B}^{[l]}_{\mathrm{st}}$ and can be written as $\bigcap_{j<l}\bigcup_{k<l}(C^{\varepsilon_{j,k}}_{j,k} \cap B)$ where the $C_{j,k}$ are taken among the C_i and $\varepsilon_{j,k} \in \{0,1\}$.*

Proof. Such a boolean combination lives in $\mathbf{B}^{[l]}_{\mathrm{st}}$ because it is a subset of B. The fact that it can be written as $\bigcap_j\bigcup_k(C^{\varepsilon_{j,k}}_{j,k} \cap B)$ is just the existence of the conjunctive normal form. Moreover, as in Proposition **(III.4.5)**, any intersection $\bigcap_k C^{\varepsilon_{j,k}}_{j,k} \cap B$ for fixed j can be rewritten as the interaction of at most l of then (for each ball from B missing from the intersection, choose a k such that this ball is not in $C^{\varepsilon_{j,k}}_{j,k} \cap B$). Similarly, the union can be rewritten as the union of at most l of them by choosing, for all $b \in B$ that appears in the union a j such that b appears in $\bigcup_k(C^{\varepsilon_{j,k}}_{j,k} \cap B)$. ♦

Let $G_{\nu,\overline{\mu},\overline{\varepsilon}}(x) = \bigcap_{i<l}\bigcup_{j<l}((H^{\varepsilon_{i,j}}_{\mu_{i,j}}(x)) \cap H_\nu(x))$ whenever all the $H_{\mu_{i,j}} \subseteq H_\nu(x)$ and $H_\nu(x)$ otherwise. Adding some more formulas to Ξ, we obtain a finite set of formulas $\Theta(x;t,s,u)$ that is adapted to G. It is clear that **III.4.20.(ii)** and **III.4.20.(iii)** still hold. Furthermore,

$$G^{\mathbb{S}}_{\nu,\overline{\mu},\overline{\varepsilon}}(x) \cap G^{\mathbb{S}}_{\sigma,\overline{\tau},\overline{\eta}}(x) = \bigcap_{i,k}\bigcup_{j,r}(\mathbb{S}(H^{\varepsilon_{i,j}}_{\mu_{i,j}}(x)) \cap \mathbb{S}(H^{\eta_{k,r}}_{\tau_{k,r}}(x)) \cap H^{\mathbb{S}}_\nu(x) \cap H^{\mathbb{S}}_\sigma(x)).$$

As H is closed under generic intersection there exists ρ such that $H^{\mathbb{S}}_\rho(x) = H^{\mathbb{S}}_\nu(x) \cap H^{\mathbb{S}}_\sigma(x)$. By Proposition **(III.4.5)**, we have both $\mathbb{B}(\mathbb{S}(H^{\varepsilon_{i,j}}_{\mu_{i,j}}(x)) \cap H^{\mathbb{S}}_\rho(x)) \subseteq H_\rho(x)$ and $\mathbb{B}(\mathbb{S}(H^{\eta_{k,r}}_{\tau_{k,r}}(x)) \cap H^{\mathbb{S}}_\rho(x)) \subseteq H_\rho(x)$ and we can conclude by Claim **(III.4.21)** that G is also closed under generic intersection over p. Similarly we show that whenever $G_{\nu,\overline{\mu},\varepsilon}(x) \subseteq G_{\sigma,\overline{\tau},\overline{\eta}}(x)$ then $G^0_{\nu,\overline{\mu},\varepsilon}(x) \cap G_{\sigma,\overline{\tau},\overline{\eta}}(x)$ is also an instance of G, i.e. G is closed under generic complement over p and hence **III.4.20.(iv)** is proved in Lemma **(III.4.19)**. ■

III.5. Implicative definability

Let us begin with the example that motivated the definition of implicatively definable types. Let b be an open ball in some model of ACVF and α_b be its generic type — i.e. the type of points that are in b but avoid all its strict subballs. Let X be any set definable in an enrichment of ACVF. Then $\alpha_b \vdash x \in X$ if and only if there exists $b' \in \mathbf{B}$ such that $b' \subset b$ and $b \smallsetminus b' \subseteq X$. Thus, although for most definable sets X, both X and its complement are consistent with α_b but if it happens that $\alpha_b(x) \vdash x \in X$, then there is a formula that says so. We have just shown that α_b is $\widetilde{\mathcal{L}}$-implicatively definable — see Definition **(III.5.1)** — for any enrichment $\widetilde{\mathcal{L}}$ of ACVF. If $(b_i)_{i\in I}$ is a strict chain of balls, i.e. $P := \bigcap_i b_i$ is not a ball, the exact same proof shows that the generic type of P is also $\widetilde{\mathcal{L}}$-implicatively definable.
If b is a closed ball, the situation is somewhat more complicated because $\alpha_b(x) \vdash x \in X$ if and only there exists finitely many maximal open subballs $(b_i)_{0\leqslant i<k}$ of b such that for all $x \in \mathbf{K}$, $x \in b \smallsetminus \bigcup_i b_i \Rightarrow x \in X$. Because the set of maximal open subballs of a given ball is internal to the residue field, to obtain that α_b is $\widetilde{\mathcal{L}}$-implicatively definable, we need to know that the $\widetilde{\mathcal{L}}$-induced structure on $\mathbf{k}$ eliminates $\exists^\infty$ to bound the number of maximal open subballs we have to remove. Recall that an $\mathcal{L}$-theory T eliminates $\exists^\infty$ if for every $\mathcal{L}$-formulas $\varphi(x;s)$ there is an $n \in \mathbb{N}$ such that for all $M \vDash T$ and $m \in M$, if $|\varphi(M;m)| < \infty$ then $|\varphi(M;m)| \leqslant n$.
The notion of implicative definability will play a fundamental role in Section III.6. The main result of this section is Corollary **(III.5.12)** which says that, under some more hypothesis on the families of parametrized balls we consider, the types of the form $\alpha_{E/p}$ (cf. Definition **(III.4.12)**) are implicatively definable if E is definable and p is implicatively definable. The proof is essentially a parametrized version of the argument above. We then prove that we can refine families of parametrized balls so that they have the necessary properties.
Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure.

Definition III.5.1 ($\mathcal{L}$-implicative definability)**:**
Let p be a partial $\mathcal{L}(M)$-type. We say that p is $\mathcal{L}$-implicatively definable if for all $\mathcal{L}$-formulas $\varphi(x;s)$ there exists an $\mathcal{L}(M)$-formula $\theta(s)$ such that for all tuples $m \in M$,

$$M \vDash \theta(m) \textit{ if and only if } p(x) \vdash \varphi(x;m).$$

Let $A \subseteq M$. If we want to specify that θ is an $\mathcal{L}(A)$-formula, we will say that p is $\mathcal{L}$-implicatively $\mathcal{L}(A)$-definable.

Remark III.5.2:

1. The notion of implicative definability is a generalizes definability of types to partial types. If p is a complete $\mathcal{L}$-type then it is $\mathcal{L}$-definable if and only if it is $\mathcal{L}$-implicatively definable.

2. The partial types we will consider here are Δ-types for some finite set $\Delta(x;t)$ of $\mathcal{L}$-formulas. Note that if $p \in \mathcal{S}^\Delta_x(M)$ is $\mathcal{L}$-implicatively $\mathcal{L}(A)$-definable, it is in particular $\mathcal{L}(A)$-definable as a Δ-type, i.e. for any formula $\varphi(x;t) \in \Delta$, there is an $\mathcal{L}(A)$-formula $\theta(t)$ such that for all tuples $m \in M$, $\varphi(x;m) \in p$ if and only if $M \vDash \theta(m)$.

Let us now prove some results on implicative definability that will not be needed afterwards but that shed some light on this notion.

Proposition III.5.3:
Let $\Delta(x;t)$ be a set of $\mathcal{L}$-formulas and $p \in \mathcal{S}^{\Delta}_{x}(M)$ be $\mathcal{L}$-implicatively definable. Assume that M is $(\aleph_0 + |\Delta|)^{+}$-saturated, then for all $N \succcurlyeq M$, $p|_N$ is $\mathcal{L}$-implicatively definable — using the same formulas.

Proof. Let $\varphi(x;s)$ be any $\mathcal{L}$-formula. By $\mathcal{L}$-implicative definability of p, there exists $\theta(s)$ such that for all tuples $m \in M$, $p|_M(x) \vdash \varphi(x;m)$ if and only if $M \vDash \theta(m)$, which in turn is equivalent to the existence of a finite number of $\psi_i(x;m_i) \in p|_M$ — i.e. $M \vDash d_p x\, \psi_i(x;m_i)$ where $d_p x\, \psi_i(x;t_i)$ is the $\mathcal{L}(M)$-formula in the defining scheme of p relative to ψ_i — such that $M \vDash \forall x \bigwedge_i \psi_i(x;m_i) \Rightarrow \varphi(x;m)$. Hence

$$\theta(s) \Rightarrow \bigvee_{\overline{\psi} \in \Delta} \exists \overline{t}\, (\bigwedge_{i \leqslant d} d_p x\, \psi_i(x;t_i) \wedge (\forall x \bigwedge_{i \leqslant d} \psi_i(x;t_i) \Rightarrow \varphi(x;s))).$$

Because there are at most $\aleph_0 + |\Delta|$ parameters involved in the formulas above and M is $(\aleph_0 + |\Delta|)^{+}$-saturated, there exists finitely many tuples $(\overline{\psi}_j)_{0 \leqslant j < k}$ such that

$$\theta(s) \Rightarrow \bigvee_{0 \leqslant j < k} \exists \overline{t}\, (\bigwedge_{i \leqslant d} d_p x\, \psi_{j,i}(x;t_{j,i}) \wedge (\forall x \bigwedge_{i \leqslant d} \psi_{j,i}(x;t_{j,i}) \Rightarrow \varphi(x;s))).$$

It follows that in any $N \vDash M$, the same implication holds and hence for all $m \in N$, $N \vDash \theta(m)$ implies that $p|_N \vdash \varphi(x;m)$.
Now assume, assume that there exists $m \in N$ such that $N \vDash \neg\theta(m)$ but $p|_N \vdash \varphi(x;m)$. Then there exists $(\psi_i(s;m_i))_{0 \leqslant i < k} \in p|_N$ such that $N \vDash \forall x \bigwedge \psi_i(x;m_i) \Rightarrow \varphi(x;m)$. Therefore

$$N \vDash \exists s\, \neg\theta(s) \wedge \exists \overline{t}\, (\bigwedge_{i} d_p x\, \varphi_i(x;t_i) \wedge (\forall x \bigwedge \psi_i(x;m_i) \Rightarrow \varphi(x;m))).$$

Because $N \succcurlyeq M$, this also holds in M, contradicting the $\mathcal{L}$-implicative definability of p. ∎

Remark III.5.4:
The saturation hypothesis is not superfluous. Indeed, let $\mathcal{L} := E$, M be the $\mathcal{L}$-structure where E is an equivalence relation with exactly one class of every finite cardinality. Let $\Delta(x;t) := \{x = t\}$ and $p := \{x \neq m : m \in M\}$. Then by quantifier elimination and the fact that $\{m \in M : p(x) \vdash xEm\} = \varnothing$ is definable, p is $\mathcal{L}$-implicatively definable. But for all $N \succcurlyeq M$, $\{n \in N : p|_N(x) \vdash \neg xEn\} = M$ is not definable if $N \neq M$.

Proposition III.5.5:
Let $A \subseteq M$. Assume M is $|A|^{+}$-saturated and strongly $|A|^{+}$-homogeneous. If p is $\mathcal{L}$-implicatively definable and $\mathrm{Aut}(M/A)$-invariant, then it is $\mathcal{L}$-implicatively $\mathcal{L}(A)$-definable.

Proof. Let $\varphi(x;s)$ be any $\mathcal{L}$-formula and $\theta(s)$ be the $\mathcal{L}$-formula such that for all tuples $m \in M$, $M \vDash \theta(m)$ if and only if $p(x) \vdash \varphi(x;m)$. Let $\sigma \in \mathrm{Aut}(M/A)$ and $m \in M$ be such that $M \vDash \theta(m)$. Then $p = \sigma(p) \vdash \theta(x;\sigma(m))$ and hence $M \vDash \theta(\sigma(m))$, i.e. $\theta(M)$ is stabilized globally by $\mathrm{Aut}(M/A)$. ∎

As previously, let now $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$, $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory that eliminates imaginaries, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $\widetilde{\mathcal{L}}$ be an enrichment of $\mathcal{L}$, $\widetilde{T}$ an $\widetilde{\mathcal{L}}$-theory containing T, $\widetilde{M} \vDash \widetilde{T}$ and $M := \widetilde{M}|_{\mathcal{L}}$. We will also be assuming that $\mathbf{k}$ is stably embedded in $\widetilde{T}$ and that the induced theory on $\mathbf{k}$ eliminates $\exists^\infty$.
Let $\widetilde{A} \subseteq \widetilde{M}$, $A := \mathcal{R}(\widetilde{A})$, $F = (F_\lambda)_{\lambda\in\Lambda}$ be an $\mathcal{L}(A)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}_{\mathrm{st}}^{[l]}$ and $\Delta(x,y;t)$ a finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $y \in \mathbf{K}$, $p \in \mathcal{S}^{\Delta}_{x,y}(M)$ be definable. Assume that Δ is adapted to F and that F is closed under generic intersection over p and is generically irreducible over p.

Definition III.5.6 (Generic covering property)**:**
We say that F has the generic covering property over p if for any $E \subseteq \Lambda(M)$ and any finite set $(\lambda_i)_{0\leqslant i<k} \in \Lambda(M)$ such that for all $\mu \in E$, $p(x,y) \vdash F^{\$}_{\lambda_i}(x) \subset F^{\$}_{\mu}(x)$, there exists $(\kappa_j)_{0\leqslant j<l} \in \Lambda(M)$ such that:

(i) For all j, $p(x,y) \vdash$ "the balls in $F_{\kappa_j}(x)$ are closed";

(ii) For all $\mu \in E$ and j, $p(x,y) \vdash F^{\$}_{\kappa_j}(x) \subseteq F^{\$}_{\mu}(x)$;

(iii) For all i, $p(x,y) \vdash F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\kappa_j}(x)$;

Note that if $E = \{\lambda_0\}$ and $p(x,y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are closed", then the generic covering property holds trivially as it suffices to take all $\kappa_j = \lambda_0$. It will only be interesting if $p(x,y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are open" or E does not have a smallest element.
Let $\mathcal{E} \subseteq \Lambda$ be $\widetilde{\mathcal{L}}(\widetilde{A})$-definable and assume $\mathcal{E} \subseteq \Lambda$ be $\widetilde{\mathcal{L}}(\widetilde{A})$-definable.

Proposition III.5.7:
Assume that one of the following holds:

(i) $\mathcal{E}(\widetilde{M})$ does not have a smallest element over p, i.e. for all $\lambda \in \mathcal{E}(\widetilde{M})$ there exists $\mu \in \mathcal{E}(\widetilde{M})$ such that $p(x,y) \vdash F^{\$}_{\mu}(x) \subset F^{\$}_{\lambda}(x)$;

(ii) there is a $\lambda_0 \in \mathcal{E}(\widetilde{M})$ such that for all $\lambda \in \mathcal{E}(\widetilde{M})$, $p(x,y) \vdash F^{\$}_{\lambda_0}(x) \subseteq F^{\$}_{\lambda}(x)$ and $p(x,y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are open".

If p is $\widetilde{\mathcal{L}}$-implicatively $\widetilde{\mathcal{L}}(\widetilde{A})$-definable and F has generic covering property over p, then $\alpha_{\mathcal{E}(\widetilde{M})/p}$ is $\widetilde{\mathcal{L}}$-implicatively $\widetilde{\mathcal{L}}(\widetilde{A})$-definable.

Proof. Let $\varphi(x,y;t)$ be an $\widetilde{\mathcal{L}}$-formula. Then, for all tuples $m \in \widetilde{M}$ such that $\alpha_{\mathcal{E}(\widetilde{M})/p}(x,y) \vdash \varphi(x,y;m)$, there exists $\lambda_0 \in \mathcal{E}(\widetilde{M})$ and a finite number of $(\lambda_i)_{0<i<k} \in \Lambda(M)$ such that for all $\mu \in \mathcal{E}(\widetilde{M})$ and $i > 0$, $p(x,y) \vdash F^{\$}_{\lambda_i}(x) \subset F^{\$}_{\mu}(x)$ and $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \setminus \bigcup_{i>0} F^{\$}_{\lambda_i}(x) \Rightarrow \varphi(x,y;m)$. By the generic covering property, we can find $(\kappa_j)_{0\leqslant j<l} \in \Lambda(M)$ such that, for all j, $p(x,y) \vdash$ "the balls in $F_{\kappa_j}(x)$ are closed", for all $\mu \in \mathcal{E}(\widetilde{M})$ and j, $p(x,y) \vdash F^{\$}_{\kappa_j}(x) \subseteq F^{\$}_{\mu}(x)$ and for all $i > 0$, $p(x,y) \vdash F^{\$}_{\lambda_i}(x) \subseteq \bigcup_j F^{\$}_{\kappa_j}(x)$.
If $\mathcal{E}(\widetilde{M})$ does not have a smallest element over p, for all $\mu \in \mathcal{E}(\widetilde{M})$ and j, we have that $p(x,y) \vdash F^{\$}_{\kappa_j}(x) \subset F^{\$}_{\mu}(x)$. If $\mathcal{E}(\widetilde{M})$ has a smallest element, because the balls in $F_{\lambda_0}(x)$ are open and those in $F_{\kappa_i}(x)$ are closed, we also have $p(x,y) \vdash F^{\$}_{\kappa_j}(x) \subset F^{\$}_{\lambda_0}(x)$. As the

$\bigcup_j F^{\$}_{\kappa_j}(x)$ covers $\bigcup_i F^{\$}_{\lambda_i}(x)$, it follows that:

$$p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{0 \leqslant j < l} F^{\$}_{\kappa_j}(x) \Rightarrow \varphi(x,y;m).$$

By $\widetilde{\mathcal{L}}$-implicative $\widetilde{\mathcal{L}}(\widetilde{A})$-definability of p and $\mathcal{L}(A)$-definability of F there exists an $\widetilde{\mathcal{L}}(\widetilde{A})$-formula $\delta_1(\kappa,\mu)$ equivalent to $p(x,y) \vdash F^{\$}_{\kappa}(x) \subset F^{\$}_{\mu}(x)$ and an $\widetilde{\mathcal{L}}(\widetilde{A})$-formula $\delta_2(\lambda_0,\overline{\kappa},m)$ equivalent to $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \smallsetminus \bigcup_{j<l} F^{\$}_{\kappa_j}(x) \Rightarrow \varphi(x,y;m)$. We have just shown that, for all tuples $m \in \widetilde{M}$, $\alpha_{\mathcal{E}(\widetilde{M})/p}(x,y) \vdash \varphi(x,y;m)$ implies that:

$$\widetilde{M} \vDash \exists \lambda_0 \in \mathcal{E}\, \exists \overline{\kappa} \in \Lambda \bigwedge_{j<l} \forall \mu \in \mathcal{E}\, \delta_1(\kappa_j,\mu) \wedge \delta_2(\lambda_0,\overline{\kappa},m).$$

The converse is trivial. ■

Definition III.5.8 (Maximal open subball property)**:**
Say that F has the maximal open subball property over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$ such that $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subset F^{\$}_{\lambda_2}(x)$, there exists $(\mu_i)_{0 \leqslant i < l} \in \Lambda(M)$ such that:

(i) For all i, $p(x,y) \vdash$ "the balls in $F_{\mu_i}(x)$ are open";

(ii) For all i, $p(x,y) \vdash \mathrm{rad}(F_{\lambda_2}(x)) = \mathrm{rad}(F_{\mu_i}(x))$.

(iii) $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_i F^{\$}_{\mu_i}(x)$;

Note that when the balls in $F_{\lambda_2}(x)$ are open, it suffices to take all $\mu_i = \lambda_2$. Hence this property is only really useful when the balls in $F_{\lambda_2}(x)$ are closed.

Proposition III.5.9:
Assume that there is a $\lambda_0 \in \mathcal{E}(\widetilde{M})$ such that for all $\lambda \in \mathcal{E}(\widetilde{M})$, $p(x,y) \vdash F^{\$}_{\lambda_0}(x) \subseteq F^{\$}_{\lambda}(x)$ and that $p(x,y) \vdash$ "the balls in $F_{\lambda_0}(x)$ are closed". Assume also that p is $\widetilde{\mathcal{L}}$-implicatively $\widetilde{\mathcal{L}}(\widetilde{A})$-definable and that F has the maximal open subball property over p, then the type $\alpha_{\mathcal{E}(\widetilde{M})/p}$ is $\widetilde{\mathcal{L}}$-implicatively $\widetilde{\mathcal{L}}(\widetilde{A})$-definable.

Proof. If the balls in $F_{\lambda_0}(x)$ have radius $+\infty$, they are singletons. By irreducibility, $F_{\lambda_0}(x)$ does not have any strict subset of the form $F_{\lambda}(x)$ and $\alpha_{\mathcal{E}(\widetilde{M})/p} \vdash \varphi(x,y;m)$ if and only if $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \Rightarrow \varphi(x,y;m)$. We can conclude immediately by $\widetilde{\mathcal{L}}$-implicative $\widetilde{\mathcal{L}}(\widetilde{A})$-definability of p. We may now assume that the balls in $F_{\lambda_0}(x)$ have a radius different from $+\infty$. Let us begin with some preliminary results.

Claim III.5.10: *Let $(Y_{\alpha,x})_{\alpha \in \mathrm{A}, x \in \mathbf{K}^n} \subseteq \{b : b$ is a maximal open subball of some $b' \in F_{\lambda_0}(x)\}$ be an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable family of sets. Then there exists k such that for all $\alpha \in A$ and $x \in \mathbf{K}^n$, either $|Y_{\alpha,x}| \geqslant \infty$ or $|Y_{\alpha,x}| \leqslant k$.*

Let $\overline{\mathcal{B}}_{\gamma}(a)$ denote the closed ball of radius γ around a.

Proof. Let $Y_{1,\alpha,x,a,c} := \{b \in \mathbf{B} : b \in Y_{\alpha,x}$, b is a maximal open subball of $\overline{\mathcal{B}}_{\mathrm{val}(c)}(a)\}$. Note that for any maximal open subball b of $\overline{\mathcal{B}}_{\mathrm{val}(c)}(a)$, the set $\{(x-a)/c : x \in b\}$ is a coset of $\mathfrak{M}$

in $\mathcal{O}$, i.e. an element of $\mathbf{k}$ that we denote $\mathrm{res}_{a,c}(b)$. The function $\mathrm{res}_{a,c}$ is one to one. Let $Y_{2,\alpha,x,a,c} := \mathrm{res}_{a,c}(Y_{1,\alpha,x,a,c})$.
Then $Y_2 = (Y_{2,\alpha,x,a,c})_{\alpha,x,a,c}$ is an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable family of subsets of $\mathbf{k}$ and hence by stable embeddedness of $\mathbf{k}$ in T — as well as compactness and some coding — there exists an $\widetilde{\mathcal{L}}(\mathbf{k}(\widetilde{M}))$-definable family $(X_d)_{d\in D}$ where $D \subseteq \mathbf{k}^r$ for some r such that for all (α, x, a, c), there exists $d \in D$ such that $Y_{2,\alpha,x,a,c} = X_d$. Moreover as the theory induced on $\mathbf{k}$ eliminates $\exists^\infty$, there exists k such that for all $d \in D$, either $|X_d| \geqslant \infty$ or $|X_d| \leqslant k$. It follows that for all (α, x, a, c), either $|Y_{1,\alpha,x,a,c}| \geqslant \infty$ or $|Y_{1,\alpha,x,a,c}| \leqslant k$. But, as there are at most l balls in $F_{\lambda_0}(x)$ and that each of these balls contains infinitely or at most k maximal open subballs from $Y_{\alpha,x}$, we have that for all x and α, $|Y_{\alpha,x}| \geqslant \infty$ or $|Y_{\alpha,x}| \leqslant lk$. ♦

Let $X_m := \{\lambda : p(x,y) \nvdash y \in F^{\$}_\lambda(x) \Rightarrow \varphi(x,y;m)$ and $p(x,y) \vdash$ "the balls in $F_\lambda(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$" $\}$. By $\widetilde{\mathcal{L}}$-implicative definability of p, X_m is an $\widetilde{\mathcal{L}}(\widetilde{M})$-definable family. Let $Y_{m,x} := \{b : \exists \lambda \in X_m,\, b \in F_\lambda(x)\}$. Then by Claim **(III.5.10)**, there exists k such that for all m and x, $|Y_{m,x}| < \infty$ implies $|Y_{m,x}| \leqslant k$.
Let us now assume that $\alpha_{\mathcal{E}(\widetilde{M})/p}(x,y) \vdash \varphi(x,y;m)$. Then there exists a finite number of $(\mu_i)_{0\leqslant i<r} \in \Lambda(M)$ such that $p(x,y) \vdash F^{\$}_{\mu_i}(x) \subset F^{\$}_{\lambda_0}(x)$ and $p(x,y) \vdash y \in F^{\$}_{\lambda_0}(x) \setminus \bigcup_i F^{\$}_{\mu_i}(x) \Rightarrow \varphi(x,y;m)$. As F has the maximal open subball property over p and is closed under generic intersection, we may assume that $p(x,y) \vdash$ "the balls in the $F_{\mu_i}(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$".

Claim III.5.11: $X_m(M) \subseteq \{\lambda \in \Lambda(M) :$ *for some* i, $p(x,y) \vdash F_\lambda(x) = F_{\mu_i}(x)\}$. *In particular* $|Y_{m,x}| < \infty$ *and hence* $|Y_{m,x}| \leqslant k$.

Proof. Let $\lambda \in X_m$. There exists $x, y \vDash p$ such that $y \in F^{\$}_\lambda(x)$, the balls in $F_\lambda(x)$ are maximal open subballs of the balls in $F_{\lambda_0}(x)$ and $\vDash \neg\varphi(x,y;m)$. Hence $y \in \bigcup_i F^{\$}_{\mu_i}(x)$. We may assume that $y \in F^{\$}_{\mu_0}(x)$ and hence that $F^{\$}_{\mu_0}(x) \cap F^{\$}_\lambda(x) \neq \varnothing$. By Proposition **(III.4.10)**, we must have $F^{\$}_{\mu_0}(x) \cap F^{\$}_\lambda(x) = F^{\$}_\kappa(x)$ for both $\kappa = \lambda$ and $\kappa = \mu_0$, i.e. $F_\lambda(x) = F_{\mu_0}(x)$ and because such an equality is decided by p this holds for all realizations of p.
It follows that $Y_{m,x} \subseteq \bigcup_i F_{\mu_i}(x)$ and $|Y_{m,x}| \leqslant rl < \infty$. ♦

Thus for all $(x,y) \vDash p$, only k balls among the ones in $\bigcup_i F_{\mu_i}(x)$ cover $\varphi(x, F^{\$}{}_{\lambda_0}(x); m)$. By similar arguments as in Proposition **(III.4.5)**, we may assume that for all i, $F_{\mu_i}(x) \subseteq \bigcup_{j=1}^k F_{\mu_j}(x)$. It follows that:

$$p(x,y) \vdash \bigwedge_{j=1}^{k} F^{\$}_{\mu_j}(x) \subset F^{\$}_{\lambda_0}(x) \wedge \Big(y \in F^{\$}_{\lambda_0}(x) \setminus \bigcup_{i=1}^{k} F^{\$}_{\mu_i}(x) \Rightarrow \varphi(x,y;m)\Big).$$

We can now conclude as in Proposition **(III.5.7)**. ■

Corollary III.5.12:
If p *is* $\widetilde{\mathcal{L}}$*-implicatively* $\widetilde{\mathcal{L}}(\widetilde{A})$*-definable and* F *has the generic covering property and the maximal open subball property over* p*, then* $\alpha_{\mathcal{E}(\widetilde{M})/p}$ *is* $\widetilde{\mathcal{L}}$*-implicatively* $\widetilde{\mathcal{L}}(\widetilde{A})$*-definable.*

Proof. This follows immediately from Propositions **(III.5.7)** and **(III.5.9)** and the fact that either $\mathcal{E}(\widetilde{M})$ is non empty and has no smallest element or it has a smallest element that consists of open balls or it has a smallest element that consists of closed balls or it is empty in which case we could also take $\mathcal{E}$ to consist of all the $\lambda \in \Lambda$ such that F_λ is constant equal to $\mathbf{K}$. ∎

Let us conclude this section by showing that, as previously, we can find families of balls verifying all the necessary hypotheses. But because both the generic covering property and the maximal open subball property are instances of more generally being able to find large balls in the family, let us first consider the following definition. Recall that $d_i(B_1, B_2)$ is the i-th distance between balls of B_1 and balls of B_2 (see Definition **(III.4.6)**)

Definition III.5.13 (Generic large ball property)**:**
We say that F has the generic large ball property over p if for all λ_1 and $\lambda_2 \in \Lambda(M)$ and $i \in \mathbb{N}$, there exists $(\mu_j)_{0\leqslant j<l} \in \Lambda(M)$ such that:

(i) For all j, $p(x,y) \vdash$ "the balls in $F_{\mu_j}(x)$ are closed";

(ii) For all j, $p(x,y) \vdash \mathrm{rad}(F_{\mu_j}(x)) = d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$.

(iii) $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$;

and, if $p(x,y) \vdash$ "the balls in $F_{\lambda_1}(x)$ are open" or $p(x,y) \vdash \mathrm{rad}(F_{\lambda_1}(x)) < d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$, there exists $(\rho_j)_{j<l} \in \Lambda(M)$ such that:

(i) For all j, $p(x,y) \vdash$ "the balls in $F_{\rho_j}(x)$ are open";

(ii) For all j, $p(x,y) \vdash \mathrm{rad}(F_{\rho_j}(x)) = d_i(F_{\lambda_1}(x), F_{\lambda_2}(x))$.

(iii) $p(x,y) \vdash F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\rho_j}(x)$;

Definition III.5.14 (Good representation)**:**
Let $\Delta(x,y;t)$ and $\Theta(x,y;s)$ be two finite sets of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$ and $(F_\lambda)_{\lambda\in\Lambda}$ and $(G_\omega)_{\omega\in\Omega}$ be two $\mathcal{L}$-definable families of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$. We say that (Θ, G, x) is a good representation of (Δ, F, x) if for all $\mathcal{L}(M)$-definable $p \in \mathcal{S}^{\Theta}_x(M)$:

(i) Θ is adapted to G;

(ii) $(G_\omega)_{\omega\in\Omega_p}$ is closed under generic intersection over p;

(iii) $(G_\omega)_{\omega\in\Omega_p}$ has the generic large ball property over p;

(iv) p decides all formulas in Δ;

(v) For all $\lambda \in \Lambda(M)$, there exists a finite number of $(\omega_i)_{0\leqslant i<l} \in \Omega_p(M)$ such that $p(x,y) \vdash F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$.

where $\Omega_p := \{\omega \in \Omega : G_\omega$ is generically irreducible over $p\}$.

If we only want to say that **III.5.14.(i)** to **III.5.14.(iii)** hold we will say that (Θ, G, x) is a good representation.

Proposition III.5.15 (Existence of good representations)**:**
Let $(F_\lambda)_{\lambda\in\Lambda}$ be any $\mathcal{L}$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and $\Delta(x;t)$ any finite set of $\mathcal{L}$-formulas where $x \in \mathbf{K}^n$. Then, there exists a good representation (Ψ, G, x) of (Δ, F, x).

Proof. Let us begin with some lemmas.

Lemma III.5.16:
There exists $(H_\rho)_{\rho\in\mathrm{P}}$ an $\mathcal{L}$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and $\Xi(x;t,s) \supseteq \Delta(x;t)$ a finite set of $\mathcal{L}$-formulas adapted to H such that H has the generic large ball property over any Ξ-type and for all $\lambda \in \Lambda$, there exists $\rho \in \mathrm{P}$ such that $H_\rho = F_\lambda$.

Proof. For all λ, μ and $\eta \in \Lambda$ and $i \leqslant l^2$, define $H_{\lambda,\mu,\eta,i,1}(x)$ to be the closed balls with radius $d_i(F_\mu(x), F_\eta(x))$ around the balls in $F_\lambda(x)$. If the balls in $F_\lambda(x)$ are open or if they are closed of radius strictly smaller than $d_i(F_\mu(x), F_\eta(x))$, define $H_{\lambda,\mu,\eta,i,0}(x)$ to be the open balls with radius $d_i(F_\mu(x), F_\eta(x))$ around the balls in $F_\lambda(x)$ — which does exist. Otherwise, define $H_{\lambda,\mu,\eta,i,0}(x)$ to be the closed balls with radius $d_i(F_\mu(x), F_\eta(x))$ around the balls in $F_\lambda(x)$. By usual coding tricks, we may assume that H is an $\mathcal{L}$-definable family of functions. Adding finitely many formulas to Δ we obtain $\Xi(x;t,s)$ which is adapted to H. Let $p \in \mathcal{S}^{\Xi}_x(M)$ and $x \vDash p$.
Let us first show the closed ball case of the generic large ball property. For all λ_k, μ_k and $\eta_k \in \Lambda(M)$ and i_k and $j_k \in \mathbb{N}$ for $k \in \{1,2\}$ and $r \in \mathbb{N}$, $d_r(H_{\lambda_1,\mu_1,\eta_1,i_1,j_1}(x), H_{\lambda_2,\mu_2,\eta_2,i_2,j_2}(x))$ is either the radius of the balls in $H_{\lambda_k,\mu_k,\eta_k,i_k,j_k}(x)$, i.e. $d_{i_k}(F_{\mu_k}(x), F_{\eta_k}(x))$ or the distance between two disjoint balls from the $H_{\lambda_k,\mu_k,\eta_k,i_k,j_k}(x)$ in which case it is also the distance between some disjoint balls in the $F_{\lambda_k}(x)$. In the first case, it is easy to check that $H_{\lambda_1,\eta_k,\mu_k,i_k,1}$ has all the suitable properties — and that this one instance suffices. In the second case there exists some m such that $H_{\lambda_1,\lambda_1,\lambda_2,m,1}(x)$ is suitable.
The same reasoning applies to open ball case (the extra conditions under which we have to work are just here to ensure that the balls in $F_{\lambda_1}(x)$ are indeed smaller than those we are trying to build around them). ♦

Lemma III.5.17:
Assume that F has the generic large ball property over any Δ-type. Let $(G_\omega)_{\omega\in\Omega}$ be any $\mathcal{L}(M)$-definable family of functions $\mathbf{K}^n \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and $\Theta(x;s)$ be any finite set of $\mathcal{L}$-formulas adapted to G such that for all $p \in \mathcal{S}^{\Theta}_x(M)$, we have:

(i) For all $\omega \in \Omega(M)$, there exists $\lambda \in \Lambda(M)$ such that $p(x) \vdash G_\omega(x) \subseteq F_\lambda(x)$;

(ii) For all $\lambda \in \Lambda(M)$, there exists $(\omega_i)_{0\leqslant i<l} \in \Omega(M)$ such that $p(x) \vdash F_\lambda(x) = \bigcup_i G_{\omega_i}(x)$.

Then G also has the generic large ball property over any Θ-type.

Proof. Let ω_1 and $\omega_2 \in \Omega(M)$, $i \in \mathbb{N}_{>0}$ and $x \vDash p$. Then there exists λ_1 and $\lambda_2 \in \Lambda(M)$ such that $G_{\omega_k}(x) \subseteq F_{\lambda_k}(x)$. Then $d_i(G_{\omega_1}(x), G_{\omega_2}(x))$ is either the radius of one of the balls involved and hence is the radius of one of $F_{\lambda_k}(x)$ or the distance between a ball in $G_{\omega_1}(x)$ and a ball in $G_{\omega_2}(x)$, i.e. the distance between a ball in $F_{\lambda_1}(x)$ and one in $F_{\lambda_2}(x)$. In both cases, the large closed ball property in F allows us to find $(\mu_j)_{0\leqslant j<l} \in \Lambda(M)$ such that $G^{\$}_{\omega_1}(x) \subseteq F^{\$}_{\lambda_1}(x) \subseteq \bigcup_j F^{\$}_{\mu_j}(x)$, for all j, the balls in $F_{\mu_j}(x)$ are closed and their radius

is $d_i(G_{\omega_1}(x), G_{\omega_2}(x))$. But, by hypothesis there are $(\rho_{j,k})_{0\leqslant k<l} \in \Omega(M)$ such that $F_{\mu_j}(x) = \bigcup_k G_{\rho_{j,k}}(x)$. By picking one ρ per ball in $G_{\omega_1}(x)$, we see that l of them are enough to cover $G_{\omega_1}(x)$ and we are done. The open ball case is proved similarly as the extra conditions hold for G_{ω_1} and G_{ω_2} if and only if they hold for F_{λ_1} and F_{λ_2}. ♦

Adding them if we have to, we may assume that there is an instance of F constant equal to $\varnothing$ and another constant one equal to $\{\mathbf{K}\}$. Let $(H_\rho)_{\rho\in\mathrm{P}}$ and Ξ be as in Lemma **(III.5.16)**, $(G_\omega)_{\omega\in\Omega}$ and $\Theta(x;u)$ be as given by Proposition **(III.4.20)** and $p \in \mathcal{S}_x^{\Theta}(M)$. Then Conditions **III.5.14.(i)**, **III.5.14.(iv)** and **III.5.14.(v)** hold. Condition **III.5.14.(ii)** also holds, by Corollary **(III.4.11)**, and by Lemma **(III.5.17)** applied to $(G_\omega)_{\omega\in\Omega_p}$, **III.5.14.(iii)** also holds. ■

Proposition III.5.18:
Let $(\Delta(x;t), (F_\lambda)_{\lambda\in\Lambda}, x)$ be a good representation and $p \in \mathcal{S}_x^{\Delta}(M)$ be $\mathcal{L}(M)$-definable. Then $F_p = (F_\lambda)_{\lambda\in\Lambda_p}$ has the generic covering property and the maximal open subball property over p.

Proof. Let $x \vDash p$, λ_1 and $\lambda_2 \in \Lambda_p(M)$ be such that $F_{\lambda_1}^{\$}(x) \subset F_{\lambda_2}^{\$}$. Then by the generic large ball property — because $F_{\lambda_1}^{\$}(x) \subset F_{\lambda_2}^{\$}$, the necessary conditions hold — there exists $\mu_j \in \Lambda_p(M)$ such that the balls in $F_{\mu_j}(x)$ are open of radius $\mathrm{rad}(F_{\lambda_2})$ and $F_{\lambda_1}^{\$}(x) \subseteq \bigcup_j F_{\mu_j}(x)^{\$}$ and we have proved the maximal open subball property.
Let now $E \subseteq \Lambda_p(M)$ and $(\lambda_i)_{0\leqslant i<k} \in \Lambda_p(M)$ be such that for all $\mu \in E$, $F_{\lambda_i}^{\$}(x) \subset F_\mu^{\$}(x)$. For any two μ_1 and $\mu_2 \in E$, if the balls in $F_{\mu_1}(x)$ are smaller than the balls in $F_{\mu_2}(x)$, by irreducibility, as $F^{\$}{}_{\mu_1}(x) \cap F^{\$}{}_{\mu_2}(x) \supseteq F_{\lambda_0}(x) \neq \varnothing$, we must have $F_{\mu_1}^{\$}(x) \subseteq F_{\mu_2}^{\$}(x)$. Let us define the following equivalence relation on $\bigcap_{\lambda\in E} F_\lambda^{\$}(x)$: $y_1 \equiv y_2$ if for all $\mu \in E$, y_1 and y_2 are in the same ball from $F_\mu(x)$. If we take two non equivalent points y_1 and y_2, there exists $\mu \in E$ such that y_1 and y_2 are not in the same ball from $F_\mu(x)$ and in fact this also holds for any η such that $F_\eta^{\$}(x) \subseteq F_\mu^{\$}(x)$. In particular it follows that there are at most l equivalence classes and that there exists μ_0 such that each equivalence class is contained in a different ball from $F_{\mu_0}(x)$. Moreover each of these equivalence classes is in fact the intersection of balls — from the $F_\mu(x)$ for $\mu \in E$. We will denote these equivalence classes by $(P_j)_{j\in J}$.
For any j, let $B_j = \{b \in \bigcup_i F_{\lambda_i}(x) : b \subseteq P_j\}$. Then the set $R_j := \{d(b_1,b_2) : b_1, b_2 \in B_j\} \cup \{\mathrm{rad}(b) : b \in B_j\}$ is finite and hence has a minimum γ. By the generic large ball property, there exists $\mu_j \in \Lambda_p(M)$ such that the balls in $F_{\mu_j}(x)$ are closed of radius γ and one of its balls — call it b_0 — contains one of the balls in B_j. In fact b_0 contains all of them as γ is the minimum of R_j. For all $\kappa \in E$, all $b \in B_j$ are such that $b \subset F_\kappa^{\$}(x)$. If $\mathrm{rad}(b_0) = d(b_1,b_2)$ for some some b_1 and $b_2 \in B_j$ then , because b_1 and b_2 are in the same ball from $F_\kappa(x)$, $\mathrm{rad}(b_0) = d(b_1,b_2) \leqslant \mathrm{rad}(F_\kappa(x))$. If $\mathrm{rad}(b_0) = \mathrm{rad}(b)$ for some $b \in B_j$, then because b is inside one of the balls from $F_\kappa(x)$, $\mathrm{rad}(b_0) = \mathrm{rad}(b) \leqslant \mathrm{rad}(F_\kappa(x))$. In both cases, $b_0 \subseteq F_\mu^{\$}(x)$. Let η_j be such that $F_{\eta_j}^{\$}(x) = F_{\mu_j}^{\$}(x) \cap \bigcap_{\kappa\in E} F_\kappa^{\$}(x)$. Such an η_j exists by generic intersection and because, by Proposition **(III.4.5)**, this intersection is given by the intersection of a finite numbers of its elements.
Then, as $F_{\eta_j}(x) \subseteq F_{\mu_j}(x)$, the balls in $F_{\eta_j}(x)$ are closed. Obviously, for all $\kappa \in E$, $F_{\eta_j}^{\$}(x) \subseteq F_\kappa^{\$}(x)$. Moreover, for all i, $F_{\lambda_i}^{\$}(x) \subseteq \bigcup_j F_{\mu_j}^{\$}(x)$ and for all $\kappa \in E$, $F_{\lambda_i}^{\$}(x) \subseteq F_\kappa^{\$}(x)$, hence we also have $F_{\lambda_i}^{\$}(x) \subseteq \bigcup_j F_{\eta_j}^{\$}(x)$. As there are at most r of the η_j, we are done. ■

III.6. Approximating sets with balls

In this section we bring together all the work we have done in Sections III.1, III.4 and III.5 to actually construct definable types. The core of the work is done in Proposition **(III.6.1)**, after that, it is only a question of proving the various reductions sketched in the introduction.

As above, let $\widetilde{\mathcal{L}} \supseteq \mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ be languages, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory that eliminates imaginaries and admits $\mathbf{\Gamma}$-reparametrizations, $\widetilde{T}$ a complete $\widetilde{\mathcal{L}}$-theory containing T, $\widetilde{N} \vDash \widetilde{T}$, $N := \widetilde{N}|_{\mathcal{L}}$, $\widetilde{A} = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}) \subseteq \widetilde{N}^{\mathrm{eq}}$ and $A := \mathcal{R}(\widetilde{A})$. Let us assume that $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded in $\widetilde{T}$ and that the induced theories on $\mathbf{k}$ and $\mathbf{\Gamma}^{\mathrm{eq}}$ eliminate $\exists^\infty$ — where eq is taken for the $\widetilde{\mathcal{L}}$-induced structure on $\mathbf{\Gamma}$. Finally we will also assume that there exists $\widetilde{M} \vDash \widetilde{T}$ such that $\widetilde{M}|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.

Proposition III.6.1:
Let $Y \subset \mathbf{K}^{n+1}$ be a non empty $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable set. Let $(\Delta(x,y;t),(F_\lambda)_{\lambda\in\Lambda},x)$ be a good representation where $x \in \mathbf{K}^n$. Let $p(x,y) \in \mathcal{S}^{\Delta}_{x,y}(N)$ be $\mathcal{L}(A)$-definable, $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with Y. Let $g = (g_\gamma)_{\gamma\in G}$ be an $\mathcal{L}(N)$-definable family of functions $\mathbf{K}^n \to \mathbf{\Gamma}$ that $\mathbf{\Gamma}$-reparametrizes the family $(\mathrm{rad}\circ F_\lambda)_{\lambda\in\Lambda}$ over p.
Then there exists an $\mathcal{L}(A)$-definable $q(x,y) \in \mathcal{S}^{\Psi_{\Delta,F}}_{x,y}(N)$ which is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with p and Y.

We are looking for a type $q = \alpha_{E/p}$ (see Definition **(III.4.12)**) so most of the work consists in finding the right E.

Proof. We define the preorder $\trianglelefteq$ on $\Lambda_p := \{\lambda \in \Lambda : F_\lambda \text{ is generically irreducible over } p\}$ by $\lambda \trianglelefteq \mu$ if and only if $p(x,y) \vdash y \in F^{\$}_\lambda(x) \wedge (x,y) \in Y \Rightarrow y \in F^{\$}_\mu(x)$. Note that, by $\widetilde{\mathcal{L}}^{eq}$-implicative $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definability of p, $\trianglelefteq$ is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. Let $\sim$ be the associated equivalence relation, i.e $\lambda \sim \mu$ if and only if $p(x,y) \vdash (y \in F^{\$}_\lambda(x) \wedge (x,y) \in Y) \iff (y \in F^{\$}_\mu(x) \wedge (x,y) \in Y)$. Then $\trianglelefteq$ induces a (partial) order on $\Lambda_p/{\sim}$ that we will also denote $\trianglelefteq$. For any λ, let us denote by $\widehat{\lambda} \subseteq \Lambda_p$ the $\sim$-class of λ. The set $\Lambda_p/{\sim}$ has a greatest element given by the class of any $\lambda \in \Lambda_p(N)$ such that $F_\lambda(x) = \{\mathbf{K}\}$ for all x and a smallest element given by the class of any $\lambda \in \Lambda_p(N)$ such that $F_\lambda(x) = \varnothing$ for all x. Let $\widehat{\mathbf{K}}$ be the greatest element of $\Lambda_p/{\sim}$ and $\widehat{\varnothing}$ be its smallest element. Because p is consistent with Y, $\widehat{\mathbf{K}} \neq \widehat{\varnothing}$.

Claim III.6.2: *Let $\lambda \in \Lambda_p \smallsetminus \widehat{\varnothing}$, then $\trianglelefteq$ totally orders $\{\widehat{\mu} : \mu \in \Lambda_p \wedge \lambda \trianglelefteq \mu\}$.*

Proof. It suffices to prove this statement in $\widetilde{N}$. Let μ_1 and $\mu_2 \in \Lambda_p(N)$ such that $\lambda \trianglelefteq \mu_i$. Because $\lambda \notin \widehat{\varnothing}$ there exists $(x,y) \vDash p$ such that $y \in F^{\$}_\lambda(x)$ and $(x,y) \in Y$. As $\lambda \trianglelefteq \mu_i$, we also have $y \in F^{\$}_{\mu_i}(x)$ and hence $F^{\$}_{\mu_1}(x) \cap F^{\$}_{\mu_2}(x) \neq \varnothing$. By Proposition **(III.4.10)**, we have $F^{\$}_{\mu_1}(x) \subseteq F^{\$}_{\mu_2}(x)$ or $F^{\$}_{\mu_2}(x) \subseteq F^{\$}_{\mu_1}(x)$ and we may assume that the first one holds. Then $\mu_1 \trianglelefteq \mu_2$. ♦

Hence $((\Lambda_p/{\sim})\smallsetminus\{\widehat{\varnothing}\},\trianglelefteq)$ is a tree — with the root on top. Let us now show that the branches of this tree are internal to $\mathbf{\Gamma}$. Let $h(\lambda) := (\partial_p\mathrm{rad}(F_\lambda),0)$ if $p(x,y)$ implies that the balls in $F_\lambda(x)$ are closed and $h(\lambda) := (\partial_p\mathrm{rad}(F_\lambda),1)$ otherwise — where the p-germ ∂_p is defined in **(III.3.3)**. By Proposition **(III.3.4)**, we may assume — after adding some parameters — that the image of h is in some cartesian power of $\mathbf{\Gamma}$. Let us also define $h_\star : \widehat{\lambda} \mapsto \ulcorner h(\widehat{\lambda})\urcorner$. By stable embeddedness of $\mathbf{\Gamma}$, $h_\star$ takes its values in $\mathbf{\Gamma}^{\mathrm{eq}}$.

Claim III.6.3: *Pick any $\lambda \in \Lambda_p \smallsetminus \widehat{\varnothing}$, then the function $h_\star$ is injective on $\{\widehat{\mu} : \lambda \trianglelefteq \mu\}$.*

Proof. It suffices to prove this statement in $\widetilde{N}$. Let μ_1 and μ_2 be such that $\lambda \trianglelefteq \mu_i$. We have seen in Claim **(III.6.2)**, that we may assume for all $(x,y) \vDash p$, $F^{\$}_{\mu_1}(x) \subseteq F^{\$}_{\mu_2}(x)$. Let $(x,y) \vDash p$. If $\widehat{\mu_1} \neq \widehat{\mu_2}$ then we must have $F^{\$}_{\mu_1}(x) \subset F^{\$}_{\mu_2}(x)$. Hence either $\mathrm{rad}(F_{\mu_1}(x)) < \mathrm{rad}(F_{\mu_2}(x))$ or the balls in $F_{\mu_1}(x)$ are open and those in $F_{\mu_2}(x)$ are closed. In any case, $h(\mu_1) \neq h(\mu_2)$. In fact for all $\omega_i \in \widehat{\mu_i}$ we obtain by the same argument that $h(\omega_1) \neq h(\omega_2)$ and hence $h_\star(\widehat{\mu_1}) \neq h_\star(\widehat{\mu_2})$. ♦

Let $\lambda \in \Lambda_p(N)$ be such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$. If $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ the generic type of $\widehat{\lambda}(\widetilde{N})$ over p is consistent with Y, it is consistent and it is consistent with p. By Proposition **(III.4.13)**, it is a complete $\Psi_{\Delta,F}$-type. By Corollary **(III.5.12)**, $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. Then it is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. By Corollary **(III.1.5)**, it is in fact $\mathcal{L}(A)$-definable. It follows that taking $q = \alpha_{\widehat{\lambda}(\widetilde{N})/p}$ would work. Therefore, it suffices to find a $\lambda \in \Lambda_p(N)$ such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is consistent with Y.

Claim III.6.4: *Let $\lambda \in \Lambda_p(N)$. If $\widehat{\lambda} \neq \widehat{\varnothing}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is not consistent with Y, there exists μ such that $\widehat{\mu}$ is an immediate $\trianglelefteq$-predecessor of $\widehat{\lambda}$ and $\ulcorner\widehat{\mu}\urcorner \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}\ulcorner\widehat{\lambda}\urcorner)$.*

Proof. If $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is not consistent with Y, there exists $\mu_0 \in \widehat{\lambda}(\widetilde{N})$ and $(\mu_i)_{0<i<k} \in \Lambda_p(N)$ such that for all $\mu \in \widehat{\lambda}(\widetilde{N})$, $p(x,y) \vdash F^{\$}_{\mu_i}(x) \subset F^{\$}_{\mu}(x)$ and

$$p(x,y) \vdash y \in F^{\$}_{\mu_0}(x) \wedge (x,y) \in Y \Rightarrow y \in \bigcup_{i=1}^{k} F^{\$}{}_{\mu_i}(x).$$

In particular $\mu_i \lhd \lambda$. Removing some μ_i, we may assume that for all i, $\mu_i \notin \widehat{\varnothing}$ and that $p(x,y) \vdash F^{\$}_{\mu_i}(x) \cap F^{\$}_{\mu_j}(x) = \varnothing$ for all $i \neq j$.
Let $\kappa \in \Lambda_p(N)$ be such that $\mu_{i_0} \trianglelefteq \kappa \trianglelefteq \lambda$ for some i_0. As $\kappa \trianglelefteq \lambda$, we have $p(x,y) \vdash (y \in F^{\$}_{\kappa}(x) \wedge (x,y) \in Y) \Rightarrow (y \in F^{\$}_{\lambda}(x) \wedge (x,y) \in Y) \Rightarrow \bigvee_{i=1}^{k}(y \in F^{\$}{}_{\mu_i}(x) \wedge (x,y) \in X)$. Because $\mu_{i_0} \trianglelefteq \kappa$, we have $p(x,y) \vdash F^{\$}_{\kappa}(x) \cap F^{\$}_{\mu_{i_0}}(x) \neq \varnothing$. If $p(x,y) \vdash F^{\$}_{\kappa}(x) \subseteq F^{\$}_{\mu_{i_0}}(x)$ then $\kappa \trianglelefteq \mu_{i_0}$ and hence $\kappa \sim \mu_i$.
Otherwise, for any $i \neq i_0$, if $p(x,y) \vdash F^{\$}_{\kappa}(x) \cap F^{\$}_{\mu_i}(x) \neq \varnothing$ then we must have $p(x,y) \vdash F^{\$}_{\mu_i}(x) \subseteq F^{\$}_{\kappa}(x)$ and hence for $I = \{i : F^{\$}_{\mu_i}(x) \cap F_{\kappa}(x)^{\$} \neq \varnothing\}$ we have $p(x,y) \vdash (y \in F^{\$}_{\kappa}(x) \wedge (x,y) \in Y) \iff \bigvee_{i\in I}(y \in F^{\$}_{\mu_i}(x) \wedge (x,y) \in Y)$.
It follows that the set $\{\widehat{\kappa} : \mu_i \trianglelefteq \kappa \trianglelefteq \lambda \text{ for some } i\}$ is finite. In particular we could choose μ_i such that there is no κ such that $\widehat{\mu}_i \lhd \widehat{\kappa} \lhd \widehat{\lambda}$. The $\widehat{\mu}_i$ are then the — finite number of — direct $\trianglelefteq$-predecessors of $\widehat{\lambda}$ and for all i, $\widehat{\mu}_i \in \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A}\ulcorner\widehat{\lambda}\urcorner)$. ♦

If there does not exist λ such that $\ulcorner\widehat{\lambda}\urcorner \in \widetilde{A}$ and $\alpha_{\widehat{\lambda}(\widetilde{N})/p}$ is consistent with Y, then starting with $\lambda_0 \in \widehat{\mathbf{K}}(\widetilde{N})$ we construct by induction — using Claim **(III.6.4)** — a sequence $(\lambda_i)_{i\in\omega}$ such that $\widehat{\lambda}_{i+1}$ is a direct $\trianglelefteq$-predecessor of $\widehat{\lambda}_i$. For all i, $|\{\widehat{\mu} : \widehat{\lambda}_i \trianglelefteq \widehat{\mu}\}| = i + 1 = |h_\star(\{\widehat{\mu} : \widehat{\lambda}_i \trianglelefteq \widehat{\mu}\})|$ but that contradicts the elimination of $\exists^\infty$ in $\mathbf{\Gamma}^{\mathrm{eq}}$. ∎

Corollary III.6.5:
Let $Y \subseteq \mathbf{K}^{n+m}$ be an $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable set, $\Delta(x, y; t)$ and $\Theta(y; s)$ be finite sets of $\mathcal{L}$-formulas where $|x| = n$ and $|y| = m$. Let $p \in \mathcal{S}^{\Delta}_{x,y}(N)$ be $\mathcal{L}(A)$-definable, $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with Y. Then there exists a finite set of $\mathcal{L}$-formulas $\Xi(x, y; s, t, r) \supseteq \Delta \cup \Theta$ and an $\mathcal{L}(A)$-definable type $q \in \mathcal{S}^{\Xi}_{x,y}(N)$ which is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with p and Y.

Proof. We proceed by induction on $|y|$. The case $|y| = 0$ is trivial. Let us now assume that $y = (z, w)$ where $|w| = 1$. By Proposition **(III.4.16)** there exists $\Phi(z; u)$ a finite set of $\mathcal{L}$-formulas and $F = (F_\lambda)_{\lambda\in\Lambda}$ an $\mathcal{L}$-definable family of functions $\mathbf{K}^{m-1} \to \mathbf{B}^{[l]}$ such that $\Psi_{\Phi,F}$ decides any formula in Θ. By Propositions **(III.4.17)** and **(III.5.15)** we can assume that $F_\lambda : \mathbf{K}^{m-1} \to \mathbf{B}^{[l]}_{\mathrm{st}}$ and (Φ, F, z) is a good representation. We can easily make F into an $\mathcal{L}$-definable family of functions $\mathbf{K}^{n+m-1} \to \mathbf{B}^{[l]}_{\mathrm{st}}$ by setting $G_\lambda(x, z) = F_\lambda(z)$. As T admits $\mathbf{\Gamma}$-reparametrizations, there exists $\Upsilon(x, z; v)$ such that for any $p \in \mathcal{S}^{\Upsilon}_{y}(N)$, there exists a $\mathbf{\Gamma}$-reparametrization $(g_\gamma)_\gamma$ of $(\mathrm{rad} \circ G_\lambda)_{\lambda\in\Lambda}$ over p.
By induction applied to $\Delta_0((x, w), z; t) \coloneqq \Delta(x, z, w; t)$, $\Theta_0(z; u, v) \coloneqq \Phi(z; u) \cup \Upsilon(z; v)$ and p, we obtain a finite set of $\mathcal{L}$-formulas $\Omega(x, w, z; r) \supseteq \Delta \cup \Phi \cup \Upsilon$ and an $\mathcal{L}(A)$-definable $q_1 \in \mathcal{S}^{\Omega}_{x,z,w}(N)$ which is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with p and Y. Let $g = (g_\gamma)_\gamma$ be a $\mathbf{\Gamma}$-reparametrization of $(\mathrm{rad}\circ G_\lambda)_{\lambda\in\Lambda}$ over $q_1|_\Upsilon$. We can now apply Proposition **(III.6.1)** to Y, $(\Omega, G, (x, z))$, q_1 and g to find an $\mathcal{L}(A)$-definable type $q_2 \in \mathcal{S}^{\Psi_{\Omega,F}}_{x,w,z}(N)$ which is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with q_1 and Y. As all the formulas in Θ are decided by $\Psi_{\Phi,F}$ — and hence by $\Psi_{\Omega,F}$ — we may assume that q_2 is in fact a $(\Psi_{\Omega,F} \cup \Theta)$-type. ∎

Definition III.6.6 (Strict $\star$-definable sets)**:**
Let $\mathcal{L}$ be a language, N an $\mathcal{L}$-structure and $x = (x_i)_{i\in I}$ an (potentially infinite) tuple of variables. Let P be a set of $\mathcal{L}$-formulas with variables. The set $P(N) \coloneqq \{m \in N^x : \forall\varphi \in P,\ N \models \varphi(m)\}$ is said to be $(\mathcal{L}, x)$-definable — or simply $(\mathcal{L}, \star)$-definable if we do not want to specify x. We say that an $(\mathcal{L}, \star)$-definable set is strict $(\mathcal{L}, \star)$-definable if the projection on any finite subset of x is $\mathcal{L}$-definable.

When x is finite we often call these sets ∞-definable sets.

Corollary III.6.7:
Let X be non empty strict $(\widetilde{\mathcal{L}}^{eq}(\widetilde{A}), x)$-definable where all the variables in x are $\mathbf{K}$-variables and $|x| \leqslant \aleph_0$. Assume also that $|\mathcal{L}| \leqslant \aleph_0$. Then there exists an $\mathcal{L}(A)$-definable type $p \in \mathcal{S}^{\mathcal{L}}_{x}(N)$ consistent with X.

Note that this is the only proof where we need a cardinality hypothesis on $\mathcal{L}$.

Proof. Let $\{\varphi_j(x_j; t_j) : j < \omega\}$ be an enumeration of all $\mathcal{L}$-formulas such that x_j is a tuple of variables from x. Let $\Delta_{-1} \coloneqq \varnothing$ and $p_{-1} \coloneqq \varnothing$ and we construct by induction on j a finite set $\Delta(x_{\leqslant j}; s_j)$ of $\mathcal{L}$-formulas and an $\mathcal{L}(A)$-definable type $p_j \in \mathcal{S}^{\Delta_j}_{\leqslant x_j}(N)$ such that for all $j < \omega$, $\Delta_j \cup \{\varphi_j\} \subseteq \Delta_{j+1}$, p_{j+1} is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable and consistent with p_j and X. Let us assume that p_j and Δ_j have been constructed. Let Y_{j+1} be the projection of X on the variables $x_{\leqslant j+1}$. Then Y_{j+1} is $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable. We can then apply Corollary **(III.6.5)** to

$\Delta_j(x_{\leqslant j}; s_j)$, $\{\varphi(x_{j+1}; t_{j+1})\}$, p_j and Y_{j+1}. As Y_{j+1} is the projection of X on the variables that appear in p_j and p_{j+1}, and p_1, p_{j+1} and Y are consistent, p_j, p_{j+1} and X are also consistent. We can now take $p := \bigcup_{j<\omega} p_j$. As the p_j are $\mathcal{L}(A)$-definable so is p. ∎

Note that p might not be $\widetilde{\mathcal{L}}^{eq}$-implicatively definable anymore. Although the fact that p_j is $\widetilde{\mathcal{L}}^{eq}$-implicatively $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definable is necessary to carry out the induction, we will not need $\widetilde{\mathcal{L}}^{eq}$-implicative $\widetilde{\mathcal{L}}^{eq}(\widetilde{A})$-definability afterwards, except if we were to continue the induction. This is exactly why we cannot prove Corollary **(III.6.7)**, and hence Theorem **E**, if $\mathcal{L}$ is not countable. Nevertheless, we will see later that Theorem **E** is stronger than what is needed to prove elimination of imaginaries which we will be able to show even when $\mathcal{L}$ is not countable.
We now prove the main result we have been aiming for.

Theorem E:

Let $\widetilde{\mathcal{L}} \supseteq \mathcal{L} \supseteq \mathcal{L}_{\mathrm{div}}$ be languages such that $|\widetilde{\mathcal{L}}| \leqslant \aleph_0$, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, $T \supseteq \mathrm{ACVF}$ be a C-minimal $\mathcal{L}$-theory that eliminates imaginaries and admits $\mathbf{\Gamma}$-reparametrizations and $\widetilde{T}$ a complete $\widetilde{\mathcal{L}}$-theory containing T such that, $\mathbf{K}$ is dominant in $\widetilde{T}$ and:

(i) The sets $\mathbf{k}$ and $\mathbf{\Gamma}$ are stably embedded in $\widetilde{T}$ and the induced theories on $\mathbf{k}$ and $\mathbf{\Gamma}^{\mathrm{eq}}$ eliminate $\exists^\infty$;

(ii) There exists $\widetilde{M} \vDash \widetilde{T}$ such that $\widetilde{M}\big|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension;

(iii) For any $\widetilde{N} \vDash \widetilde{T}$, $A = \mathbf{K}(\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)) \subseteq \widetilde{N}$ and any $\widetilde{\mathcal{L}}(A)$-definable set $X \subseteq \mathbf{K}^n$, there exists an $\widetilde{\mathcal{L}}$-definable bijection $f : \mathbf{K}^n \to Y$ such that $f(X) = Y \cap Z$ where Z is $\mathcal{L}(A)$-definable — note that f has to be defined without parameters.

Then for all $\widetilde{N} \vDash \widetilde{T}$ and all non empty $\widetilde{\mathcal{L}}(\widetilde{N})$-definable sets X, there exists $p \in \mathcal{S}^{\mathcal{L}}(\widetilde{N})$ which is consistent with X and $\widetilde{\mathcal{L}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)))$-definable.

This really is a result on the density of definable types with canonical basis in $\mathcal{R}$. Indeed let $\widetilde{A} = \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})$, the conclusion of Theorem **E** states that the set $\{p \in \mathcal{S}^{\mathcal{L}}(\widetilde{A}) : p$ is $\widetilde{\mathcal{L}}(\mathcal{R}(\widetilde{A}))$-definable$\}$ is dense in $\mathcal{S}^{\mathcal{L}}(\widetilde{A})$. Here, by definable type, we mean that the definable type has a global definable extension given by the defining scheme.
As we will see later this result is important in two respects. On the one hand, it shows the density of definable types (and hence invariant types) over algebraically closed sets, and on the other hand, because these definable types have a canonical basis in $\mathcal{R}$, it also gives information on the imaginaries in $\widetilde{T}$.

Proof. Let $\widetilde{A} := \mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)$. It suffices to prove this for $X \subset \mathbf{K}^n$ for some n. Indeed, as $\mathbf{K}$ is dominant, there is an $\widetilde{\mathcal{L}}$-definable surjection $\pi : \mathbf{K}^n \to \prod S_i$ for some n where the sorts S_i are such that $X \subseteq \prod S_i$. If we find p consistent with $Y := \pi^{-1}(X)$ and $\widetilde{\mathcal{L}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner Y \urcorner)))$-definable, then $\pi_\star p$ is consistent with X and $\widetilde{\mathcal{L}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)))$-definable.
Let $F := \{f$ is an $\widetilde{\mathcal{L}}$-definable bijection whose domain is $\mathbf{K}^n\}$ and $\partial_\omega(X) = \prod_{f \in F} f(X)$.

Then $\partial_\omega(X)$ is strict $(\widetilde{\mathcal{L}}^{eq}(\widetilde{A}), I)$-definable for some I with $|I| \leqslant \aleph_0$. By Corollary **(III.6.7)**, there exists an $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable type $p \in \mathcal{S}_I^{\mathcal{L}}(N)$ consistent with $\partial_\omega(X)$.
Let $\partial_\omega(x) = (f(x))_{f\in F}$. Then $q = \{x : \partial_\omega(x) \vDash p\}$. Then q is consistent with X. There only remains to show that it is a complete type and that it is $\widetilde{\mathcal{L}}(\mathcal{R}(\widetilde{A}))$-definable. Let $\varphi(x;s)$ be an $\widetilde{\mathcal{L}}$-formula where $x \in \mathbf{K}^n$. As $\mathbf{K}$ is dominant we may assume s is a tuple of variables from $\mathbf{K}$ too. By (iii), for all tuples $m \in \mathbf{K}(\widetilde{N})$, there exists $(f : \mathbf{K}^n \to Y) \in F$ and an $\widetilde{\mathcal{L}}$-definable map g (into $\mathbf{K}^l$ for some l) such that $f(\varphi(\widetilde{N};m)) = Y(\widetilde{N}) \cap Z(\widetilde{N})$ where Z is $\mathcal{L}(g(m))$-definable. As $\widetilde{N}$ is arbitrary, we may assume that it is saturated enough and by compactness there exists a finite number of $(f_i : \mathbf{K}^n \to Y_i) \in F$, $\widetilde{\mathcal{L}}$-definable maps g_i and $\mathcal{L}$-formulas $\psi_i(y_i;t)$ such that for any tuple $m \in \mathbf{K}(\widetilde{N})$ there exists i_0 such that $f_{i_0}(\varphi(\widetilde{N};m)) = \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$.
Let c_1 and $c_2 \vDash q$ — i.e. $\partial_\omega(c_j) \vDash p$, for $j \in \{1,2\}$ — and assume that $\vDash \varphi(c_1;m)$. Then $f_{i_0}(c_1) \in \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$. As $f_{i_0}(c_1) \equiv_{\mathcal{L}} f_{i_0}(c_2)$ and $f_{i_0}(c_2) \in Y_{i_0}(\widetilde{N})$ we also have $f_{i_0}(c_2) \in \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N}) = f_{i_0}(\varphi(\widetilde{N};m))$ and, because f_{i_0} is a bijection, $\vDash \varphi(c_2;m)$.
As for definability, we have just shown that $\varphi(x;m) \in q$ if and only if $\psi_{i_0}(y_i;g_{i_0}(m)) \in p$ for some i_0 such that $f_{i_0}(\varphi(\widetilde{N};m)) = \psi_{i_0}(\widetilde{N};g_{i_0}(m)) \cap Y_{i_0}(\widetilde{N})$ but that can be stated with an $\widetilde{\mathcal{L}}(\mathcal{R}(\widetilde{A}))$-formula. ∎

III.7. Imaginaries and invariant extensions

As stated earlier, the conclusion of Theorem **E** is very strong. We will show in this section that it implies both elimination of imaginaries and the invariant extension property (cf. Definition **(0.4.13)**). I am very much indebted to [Hru14; Joh] for making me realize that the density of definable types could play an important role in proving elimination of imaginaries. To be precise, both elimination of imaginaries and the invariant extension property follow from the density of types invariant over real parameters.

Remark III.7.1:
Because types definable over some parameters A are also A-invariant, the density of definable types over some parameters implies the density of invariant types. But the converse is false even in NIP theories. Consider $M \equiv \mathbb{Q}_p$ in the three sorted language with angular components (cf. Section II.1). Assume M is $\aleph_0$-saturated. Let $\gamma \in \mathbf{\Gamma}(M)$ be such that $\gamma > n \cdot \mathrm{val}(p)$ for all $n \in \mathbb{N}$ and $b := \{x \in \mathbf{K}(M) : \mathrm{val}(x) = \gamma \wedge \mathrm{ac}_1(x) = 1\}$. Note that b is a ball. Because the residue field is finite, in $\mathrm{acl}(\ulcorner b \urcorner)$ there are all the balls $b' \subseteq b$ such that $\mathrm{rad}(b') - \mathrm{rad}(b) \in \mathbb{Z} \cdot \mathrm{val}(p)$. Let $(b_i)_{i\in\mathbb{N}}$ be a chain of balls such that $b_i \subseteq b$ and $\mathrm{rad}(b_i) = \mathrm{rad}(b) + i\mathrm{val}(p)$. Then for all $x, y \in \bigcap_i b_i$, $\mathrm{val}(x) = \mathrm{val}(y) = \gamma$ and $\mathrm{ac}_n(x) = \mathrm{ac}_n(y)$. Let $P = \sum_i a_i X^i \in \mathbb{Q}[X]$, let i_0 be minimal such that $a_i \neq 0$, then for all $i \neq i_0$, $\mathrm{val}(a_{i_0} x^{i_0}) = \mathrm{val}(a_{i_0} y^{i_0}) < \mathrm{val}(a_i x^i) = \mathrm{val}(a_i y^i)$. In fact, $\mathrm{val}(a_i x^i) - \mathrm{val}(a_{i_0} x_{i_0}) \geqslant n \cdot \mathrm{val}(p)$ for all $n \in \mathbb{N}$. Thus $\mathrm{val}(P(x)) = \mathrm{val}(P(y))$ and that $\mathrm{ac}_n(P(x)) = \mathrm{ac}_n(P(y))$. It now follows from field quantifier elimination that $x \equiv_\varnothing y$ and because any automorphism sending x to y must fix b they have the same type over b. Thus, there cannot be any ball in $\bigcap_i b_i$ algebraic over b and hence, by Proposition **(I.3.9)**, x and y have the same type over $\mathrm{acl}(b)$. Every type in b is of this form and none of them can be definable because $\bigcap_i b_i$ is a strict intersection.

Nevertheless, by Remark **(I.4.7)**, $\mathrm{Th}(\mathbb{Q}_p)$ has the invariant extension property and hence by Proposition **(III.7.4)**, invariant types are dense over algebraically closed sets.

In the following proposition, we show that the density of Δ-types invariant over real parameters for Δ finite suffices to prove weak elimination of imaginaries.

Proposition III.7.2:
Let T be an $\mathcal{L}$-theory and $\mathcal{R}$ a set of its sorts such that for all $N \vDash T$, all non empty $\mathcal{L}(N)$-definable sets X and all $\mathcal{L}$-formulas $\varphi(x;s)$, there exists $p \in \mathcal{S}^{\varphi}_{x}(N)$ which is consistent with X and $\mathrm{Aut}(N/\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)))$-invariant. Then T weakly eliminates imaginaries up to $\mathcal{R}$.

Proof. Let M be a saturated and homogeneous enough model of $\widetilde{T}$, E be any $\mathcal{L}$-definable equivalence relation, X be one of its classes in M, $\varphi(x,y)$ be an $\mathcal{L}$-formula defining E and $A = \mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner))$. By hypothesis, there exists an $\mathrm{Aut}(M/A)$-invariant type $p \in \mathcal{S}^{\varphi}_{x}(M)$ consistent with X. Because X is defined by an instance of φ, we have in fact $p(x) \vdash x \in X$. For all $\sigma \in \mathrm{Aut}(M/A)$, $\sigma(X)$ is another E-class and $\sigma(p) = p \vdash x \in X$. It follows that $X \cap \sigma(X) \neq \varnothing$ and hence $X = \sigma(X)$. We have just proved that $\ulcorner X \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(\mathcal{R}(\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\ulcorner X \urcorner)))$, i.e. X is weakly coded in $\mathcal{R}$. ∎

Corollary III.7.3:
*In the setting of Theorem **E** without the cardinality assumption on $\widetilde{\mathcal{L}}$, $\widetilde{T}$ eliminates imaginaries.*

Proof. By Corollary **(III.6.5)** and using similar techniques to those used in the proof of the theorem, we can prove that the assumption of Proposition **(III.7.2)** holds in $\widetilde{T}$ and hence that $\widetilde{T}$ weakly eliminates imaginaries up to the sorts $\mathcal{R}$. But because any finite sets in $\mathcal{R}$ are also definable in T and hence are coded in T, $\widetilde{T}$ eliminates imaginaries up to the sorts $\mathcal{R}$. ∎

Let us now consider the invariant extension property (cf. Definition **(0.4.13)**).

Proposition III.7.4:
Let T be an $\mathcal{L}$-theory, $A \subseteq M$ for some $M \vDash T$. The following are equivalent:

(i) *For all $\mathcal{L}(A)$-definable non empty sets X and $N \vDash T$, $N \subseteq A$, there exists $p \in \mathcal{S}(N)$ such that p is $\mathrm{Aut}(N/A)$-invariant and is consistent with X;*

(ii) *T has the invariant extension property over A.*

Proof. Let us first show that (ii) implies (i). Let $N \vDash T$, X be $\mathcal{L}(A)$-definable and $p \in \mathcal{S}(A)$ be any type containing X. Let $q \in \mathcal{S}(N)$ be an $\mathrm{Aut}(N/A)$-invariant extension of p. Then q is consistent with X.
Let us now assume (i) and let $\mathrm{Inv}(N/A) = \{p \in \mathcal{S}(N) : p \text{ is } \mathrm{Aut}(N/A)\text{-invariant}\}$.

Claim III.7.5: *The set $\mathrm{Inv}(N/A) \subseteq \mathcal{S}(N)$ is closed and hence compact.*

Proof. Let $p \in \mathcal{S}(N) \smallsetminus \mathrm{Inv}(N/A)$. There exists $\varphi(x;s)$, a tuple $m \in N$ and $\sigma \in \mathrm{Aut}(N/A)$ such that $\varphi(x;m) \in p$ and $\varphi(x;\sigma(m)) \notin p$. Then, the set $\{q \in \mathcal{S}(N) : \varphi(x;m) \in q$ and $\varphi(x;\sigma(m)) \notin q\} = \{q \in \mathcal{S}(N) : \varphi(x;m) \wedge \neg\varphi(x;\sigma(m)) \in q\}$ is open and has empty intersection with $\mathrm{Inv}(N/A)$. ♦

Let $p \in \mathcal{S}(A)$. By hypothesis, for all $\mathcal{L}(A)$-definable sets $X \neq \varnothing$, there exists $q_X \in \mathrm{Inv}(N/A)$ which is consistent with X. It follows that for all $\mathcal{L}(A)$-definable sets X, the closed set $F_X := \langle X \rangle \cap \mathrm{Inv}(N/A) \neq \varnothing$. Moreover, for any finite number of $X_i \in p$, $\cap_i F_{X_i} = F_{\bigwedge_i X_i}$ is non empty. As $\mathrm{Inv}(N/A)$ is compact, there exists $q \in \bigcap_{X \in p} F_X$. Then $q \in \mathrm{Inv}(N/A)$ and for all $X \in p$, $q \in F_X \subseteq \langle X \rangle$ so q does extend p. ■

To conclude:

Theorem III.7.6:

In the setting of Theorem ***E****, $\widetilde{T}$ eliminates imaginaries and has the invariant extension property.*

Proof. Elimination of imaginaries is proved in Corollary **(III.7.3)** and the invariant extension property then follows from Theorem **E** and Proposition **(III.7.4)**. ■

CHAPTER IV

Some model theory of valued differential fields

Le Vieux Monsieur
C'est vrai, j'ai un chat qui s'appelle Socrate.
E. Ionesco, *Rhinocéros*, Acte I

Building on the quantifier elimination result of [Sca00], this chapter aims at laying the foundations for a more sophisticated model theoretic study of $\mathrm{VDF}_{\mathcal{EC}}$ the model completion of valued differential fields with a valuation preserving derivation (the definitions can be found in Section IV.1.1). There are mainly two motivations behind looking at $\mathrm{VDF}_{\mathcal{EC}}$. The first reason is that it is a somewhat more complicated NIP theory than ACVF, for example. The analogy with stability theory is quite tempting: among stable fields, ACF is extremely well understood but is too tame (it is strongly minimal) for any of the more subtle behavior of stability to show. The theory DCF_0 on the other hand is still very reasonable (it is ω-stable) but some pathologies begin to show and in studying DCF_0 one gets a better understanding of stability. The theory $\mathrm{VDF}_{\mathcal{EC}}$ could play the similar role with respect to ACVF: it is a more complicated but still very tractable theory in which to experiment with NIP or metastability.

The other reason why $\mathrm{VDF}_{\mathcal{EC}}$ is interesting is much more application oriented. Valued fields appear naturally in algebraic geometry (among other things when studying singularities). Similarly, some diophantine questions have been shown to be closely related to differential fields. It only seems natural to think that the combination of the two should have natural applications (see for example [Sca97]). Furthermore, the model theory of difference valued fields also plays a role in Hrushovksi's work [Hrua] on "difference algebraic geometry". Difference valued fields also appear naturally in number theory and diophantine geometry, among them the Witt vectors over $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ equipped with the lifting of the Frobenius — although, to be completely honest, this field usual appears with its analytic structure (which is studied in Chapter II). We hope that the results in this chapter can also be seen as first steps in a more general program to study the model theory of valued fields with operators.

This chapter begins, in Section IV.1 with some definitions followed by the most important result in this chapter, Theorem **F**: the elimination of imaginaries and the invariant extension property for $\mathrm{VDF}_{\mathcal{EC}}$. In fact, this question was the motivation for the development of Chapter III and what we actually prove is the stronger result that types definable over geometric parameters are dense in $\mathrm{VDF}_{\mathcal{EC}}$.

In Section IV.2, we study the field of constants in $\mathrm{VDF}_{\mathcal{EC}}$ and show that it is a stably embedded pure model of ACVF. This section also contains the equivalent result for $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped with the lifting of the Frobenius automorphism.

Section IV.3 is concerned with the definable and algebraic closure in $\mathrm{VDF}_{\mathcal{EC}}$ and essentially shows that it is more complicated than what one could hope for. We do show though that in the value group, the residue field and the constant field it is what could be expected.
In Section IV.4, we explore more methodically the fact that there is a good notion of prolongation on the space of types in $\mathrm{VDF}_{\mathcal{EC}}$ and study some of its properties. This idea was already sketched, in a more abstract setting, in the proof of Theorem **E**.
Finally, Section IV.5 is about definable groups. Most of this section is an account, following [Hrub], of how to adapt the proofs of [Hru90] to the unstable setting as long as there is a definable generic around. We also generalize somewhat the notion of group chunk to be able to talk about non connected group more directly. The main goal is to produce an abstract version (cf. Theorem **(IV.5.42)**) of the classical proof that a group definable in DCF_0 (or in separably closed fields) definably embeds into an algebraic group so that we can apply it in other contexts. For example, we obtain an embedding theorem into groups interpretable in ACVF for a certain class of groups definable in $\mathrm{VDF}_{\mathcal{EC}}$.

IV.1. Imaginaries in $\mathrm{VDF}_{\mathcal{EC}}$

After defining $\mathrm{VDF}_{\mathcal{EC}}$ and recalling the known results about this theory, we will show that all of the work in Chapter III applies to $\mathrm{VDF}_{\mathcal{EC}}$ to obtain new results in Theorem **F**: elimination of imaginaries, invariant extension property and density of definable types.

IV.1.1. Some background

Let $\mathcal{L}_\partial$ be the three sorted language for valued fields with two new symbols $\partial : \mathbf{K} \to \mathbf{K}$ and $\partial : \mathbf{k} \to \mathbf{k}$. We define DVal to be the theory of $\mathcal{L}_\partial$-structures M such that $\mathbf{\Gamma}(M)$ is an abelian ordered group, $\mathbf{k}(M)$ is a differential field and $(\mathbf{K}(M), \mathrm{val}, \mathbf{k})$ is a valued field with a valuation preserving derivation — i.e. for all $x \in \mathbf{K}$, $\mathrm{val}(\partial(x)) \geqslant \mathrm{val}(x)$. Note that the residual map and the valuation map are not supposed to be onto in DVal.
The notion of ∂-Henselianity can take many equivalent forms but I will give the one considered in [Sca00], which is quite close to the one considered in Definition **(II.4.10)**. Recall that $\partial_\omega(x)$ is defined to be $(\partial^n(x))_{n\in\mathbb{N}}$.

Definition IV.1.1 (∂-Henselian)**:**
Let $(K, \mathrm{val}, \partial)$ *be a valued differential field.* K *is* ∂*-Henselian if for all* $P \in \mathcal{O}(K)[\overline{X}]$ *and* $a \in \mathcal{O}(K)$ *such that* $\mathrm{val}(P(\partial_\omega(a))) > 0$ *and* $\min_i\{\mathrm{val}(\frac{\partial}{\partial X_i}P(\partial_\omega(a)))\} = 0$*, there exists* $c \in \mathcal{O}$ *such that* $P(\partial_\omega(c)) = 0$ *and* $\mathrm{res}(c) = \mathrm{res}(a)$*.*

Definition IV.1.2 (Enough constants)**:**
Let $(K, \mathrm{val}, \partial)$ *be a valued differential field. We say that* K *has enough constants if* $\mathrm{val}(\mathrm{C}_K) = \mathrm{val}(K)$ *where* $\mathrm{C}_K := \{x \in K : \partial(x) = 0\}$ *denotes the field of constants.*

Now, let $\mathrm{VDF}_{\mathcal{EC}}$ be the $\mathcal{L}_\partial$-theory of valued fields with a valuation preserving derivation which are ∂-Henselian with enough constants, such that the residue field is differentially closed of characteristic zero and the value group is divisible — and val and $\mathbf{k}$ are onto.

Note that we call it $\mathrm{VDF}_{\mathcal{EC}}$ and not $\mathrm{VDF}_{\mathcal{EC},0}$ or $\mathrm{VDF}_{\mathcal{EC},0,0}$ because we will only work in equicharacteristic zero in this chapter. Similarly we will not specify the characteristic when speaking of ACVF, but it is understood that we are speaking of $\mathrm{ACVF}_{0,0}$.
As in the case of ACVF, $\mathrm{VDF}_{\mathcal{EC}}$ can be considered in the one sorted, two sorted or three sorted languages (cf section 0.2.2) that are enrichments of the valued field versions with symbols for the derivation. We will be denoting $\mathcal{L}_{\partial,\mathrm{div}}$ the one sorted language for $\mathrm{VDF}_{\mathcal{EC}}$.

Theorem IV.1.3 ([Sca00]):

(i) The theory $\mathrm{VDF}_{\mathcal{EC}}$ *is the model completion of* DVal*;*

(ii) The theory $\mathrm{VDF}_{\mathcal{EC}}$ *eliminates quantifiers (and is complete) in the one sorted language, the two sorted language and the three sorted language;*

(iii) The value group $\mathbf{\Gamma}$ *is stably embedded and a pure divisible ordered abelian group;*

(iv) The residue field $\mathbf{k}$ *is stably embedded and a pure model of* DCF_0*;*

(v) The theory $\mathrm{VDF}_{\mathcal{EC}}$ *is* NIP*.*

Proof. Item (i) follows from [Sca00, Theorem 7.1] and the fact that the theory of divisible ordered abelian groups is the model completion of abelian ordered groups and the theory of differentially closed fields is the model completion of differential fields.
By [Sca00, Theorem 7.1] we also have field quantifier elimination (in the three sorted language) relative to $\mathbf{k}$ and $\mathbf{\Gamma}$ (cf. Definition **(II.A.5)**). The stable embeddedness and purity results follow. Now, the theory induced on $\mathbf{k}$ and $\mathbf{\Gamma}$ are, on the one hand, differentially closed fields and, on the other, divisible ordered abelian groups that both eliminate quantifiers. Quantifier elimination in the three sorted language follows and so does qualifier elimination in the two other languages.
The fact that $\mathrm{VDF}_{\mathcal{EC}}$ is NIP is an easy consequence of elimination of quantifiers (in the one sorted language for example) because ACVF is NIP. ∎

Note that in $M \models \mathrm{VDF}_{\mathcal{EC}}$, the field of constants $\mathrm{C}_{\mathbf{K}}(M)$ is relatively algebraically closed inside $\mathbf{K}(M)$ which is algebraically closed, thus $\mathrm{C}_{\mathbf{K}}(M)$ is algebraically closed.

IV.1.2. Imaginaries and invariant types in $\mathrm{VDF}_{\mathcal{EC}}$

Using the results of Chapter III, we can prove new results about $\mathrm{VDF}_{\mathcal{EC}}$. The question of elimination of imaginaries in $\mathrm{VDF}_{\mathcal{EC}}$ had remained open since it came up naturally after elimination of imaginaries to the geometric sorts was proved for ACVF. Apart from the general importance of describing interpretable sets, the question of the elimination of imaginaries in $\mathrm{VDF}_{\mathcal{EC}}$ was also linked to the question of the existence of invariant extensions in that theory. It is true that those two questions are quite distinct but a good knowledge of imaginaries usually helps to prove the invariant extension property (see for example Remark **(I.4.7)** in the case of $\mathbb{Q}_p$).
At the end of [HHM08], where it is shown that ACVF is metastable (see Theorem **(0.4.17)**) two other examples of metastable theories are given: the theory of $\mathbb{C}((t))$, and the theory

of $\mathrm{VDF}_{\mathcal{EC}}$. But the proof sketched there for the invariant extension property in $\mathrm{VDF}_{\mathcal{EC}}$ is incorrect. There are two problems: the first is that it relies on a problematic reduction to the field sort that is correct when proving the existence of metastability basis but is not in the case of the invariant extension property. The second is that even when both variables and parameters are in the valued field, the construction of the invariant type does not take in account the fact that the derivation in $\mathrm{VDF}_{\mathcal{EC}}$ has to preserve the valuation. To recover the fact that $\mathrm{VDF}_{\mathcal{EC}}$ is an example of metastability, it became necessary to find another proof of the invariant extension property.
The proof that we give here does not, per se, rely on the fact that we also prove elimination of imaginaries to the geometric sorts but, rather, both results follow from a stronger result about density of definable types over geometric parameters.
Let $\mathcal{L}^{\mathcal{G}}_{\partial}$ be the language $\mathcal{L}^{\mathcal{G}}$ enriched with a symbol for the derivation $\partial : \mathbf{K} \to \mathbf{K}$ and let $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ be the $\mathcal{L}^{\mathcal{G}}_{\partial}$-theory of models of $\mathrm{VDF}_{\mathcal{EC}}$.

Theorem F:
The theory $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ eliminates imaginaries and has the invariant extension property. Moreover, over algebraically closed parameters, definable types are dense.

Proof. We apply Theorems **E** and **(III.7.6)**.
Hypothesis **E.(i)** follows from Theorem **IV.1.3.(iii)** and **IV.1.3.(iv)** and the fact that, because the algebraic closure in DCF_0 is the field algebraic closure of the generated differential field and because DOAG is o-minimal, both of these theories eliminate $\exists^\infty$.
Hypothesis **E.(ii)** follows from the fact that if $k \vDash \mathrm{DCF}_0$ then the Hahn field $k((t^{\mathbb{R}}))$ with the derivation $\partial(\sum_i a_i t^i) = \sum_i \partial(a_i)t^i$, i.e. $\partial(t) = 0$, is a model of $\mathrm{VDF}_{\mathcal{EC}}$ and, by Corollary **(III.2.9)** the underlying valued field is uniformly stably embedded in every elementary extension.
Finally, Hypothesis **E.(iii)** is an easy consequence of elimination of quantifiers: let $\varphi(x;s)$ be an $\mathcal{L}_{\partial,\mathrm{div}}$-formula, then there exists an $\mathcal{L}_{\mathrm{div}}$-formula $\psi(u;t)$ and $n \in \mathbb{N}$ such that $\varphi(x;s)$ is equivalent modulo $\mathrm{VDF}_{\mathcal{EC}}$ to $\psi(\partial_n(x);\partial_n(s))$, i.e. for all $m \in \widetilde{N}$, ∂_n is an $\mathcal{L}_{\partial,\mathrm{div}}$-definable bijection between $\varphi(\widetilde{N};m)$ and $\psi(x,\partial_n(m)) \cap \partial_n(\mathbf{K}^{|x|})$. ∎

IV.2. Stable embeddedness of the field of constants

In this section we wish to study the stable embeddedness of the field of constants $\mathrm{C}_{\mathbf{K}}$ in $\mathrm{VDF}_{\mathcal{EC}}$. We will be using the results of Section III.2 and we will be working in the one sorted language $\mathcal{L}_{\partial,\mathrm{div}}$.

Proposition IV.2.1:
Any pair $L|K$ that is elementarily equivalent to a pair $L^\star|K^\star$, where $K^\star$ is maximally complete, is separated. In particular, in models of $\mathrm{VDF}_{\mathcal{EC}}$, the pair $\mathbf{K}|\mathrm{C}_{\mathbf{K}}$ is separated.

Proof. The first part of the corollary is an immediate consequence of Propositions **(III.2.2)** and the the fact that separation is preserved by elementary equivalence of the pair. The rest of the corollary then follows because, as $\mathrm{VDF}_{\mathcal{EC}}$ is complete, all the pairs $\mathbf{K}|\mathrm{C}_{\mathbf{K}}$ are elementary equivalent and, for any $K \vDash \mathrm{DCF}_0$ and $\Gamma \vDash \mathrm{DOAG}$, $K((t^\Gamma))$ with $D(\sum_{\gamma\in\Gamma} a_\gamma t^\gamma) =$

$\sum_{\gamma\in\Gamma} D(a_\gamma)t^\gamma$ is a model of $\mathrm{VDF}_{\mathcal{EC}}$ whose constant field is $\mathrm{C}_{\mathbf{K}}((t^\Gamma))$ which is maximally complete. ■

Proposition IV.2.2:
The field of constants $\mathrm{C}_{\mathbf{K}}$ *is stably embedded in models of* $\mathrm{VDF}_{\mathcal{EC}}$*. It follows that it is a pure model of* ACVF.

Proof. Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ and $\varphi(x,m)$ be a formula with parameters in $\mathbf{K}(M)$. By quantifier elimination, we may assume that φ is of the form $\psi(\partial_n(x),\partial_n(m))$ where ψ is an $\mathcal{L}_{\mathrm{div}}$-formula and $n\in\mathbb{N}$. For all $x\in\mathrm{C}_K$, $\varphi(x,m)$ is equivalent to $\psi(x,0,\ldots,0,\partial_n(m))$, i.e. an $\mathcal{L}_{\mathrm{div}}(\mathbf{K}(M))$-formula, hence it suffices to prove that $\mathrm{C}_{\mathbf{K}}(M)$ is stably embedded in $\mathbf{K}(M)$ as a valued field. But $\mathbf{K}(M)$ is algebraically closed, the pair $\mathbf{K}(M)|\mathrm{C}_{\mathbf{K}}(M)$is separated (cf. Proposition **(IV.2.1)**) and, because there are enough constants, $\mathrm{val}(\mathbf{K}(M)) = \mathrm{val}(\mathrm{C}_{\mathbf{K}}(M))$, hence we can apply Theorem **(III.2.7)**.
It now follows from quantifier elimination that any subset of $\mathrm{C}_{\mathbf{K}}$ definable in M is definable by a quantifier-free $\mathcal{L}_{\mathrm{div}}(\mathrm{C}_{\mathbf{K}}(M))$-formula and hence is defined in $\mathrm{C}_{\mathbf{K}}(M)$ by the same formula. ■

These result can be transposed easily to the Witt vectors over $\overline{\mathbb{F}_p}^{\mathrm{alg}}$ with the lifting of the Frobenius $\mathrm{W}(\mathrm{Frob}_p)$. Recall that in this field there are definable angular component maps ac_n for all the residue rings $\mathbf{R}_n := \mathcal{O}/p^n\mathfrak{M}$ that are compatible with the automorphism. Indeed, $\mathrm{Fix}(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})) = \mathbb{Q}_p$ where angular component maps ac_n (cf. section II.1) are definable and for all $x\in\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ define $\mathrm{ac}_n(x) := \mathrm{res}_n(xy^{-1})\mathrm{ac}_n(y)$ for any $y\in\mathbb{Q}_p$ such that $\mathrm{val}(x) = \mathrm{val}(y)$.
Thus we will consider the theory of this field in the three sorted language $\mathcal{L}^{\mathrm{ac}}_{\sigma,P}$ with the angular component maps described above and divisibility predicates P_n on the value group. Let WF_p denote $\mathrm{Th}_{\mathcal{L}^{\mathrm{ac}}_{\sigma,P}}(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}}),\mathrm{W}(\mathrm{Frob}_p),\mathrm{ac}_n)$.

Theorem IV.2.3 ([BMS07])**:**
The theory WF_p *eliminates quantifiers (and is complete).*

Proof. By [BMS07, Theorem 11.4], WF_p eliminates quantifiers in $\mathcal{L}^{\mathrm{ac}}_{\sigma,P}$ relative to $\mathbf{R}$ and $\mathbf{\Gamma}$. The theorem now follows from the fact that algebraically closed fields eliminate quantifiers and $\mathbb{Z}$-groups eliminate quantifiers once we add divisibility predicates. ■

Proposition IV.2.4:
The fixed field $\mathrm{Fix}(\mathbf{K})$ *is stably embedded in models of* WF_p *and is a pure valued field elementarily equivalent to* $\mathbb{Q}_p$

Proof. It is essentially the same proof as in Proposition **(IV.2.2)**. Let $M\models\mathrm{WF}_p$. By quantifier elimination, the intersection of any definable set in M with $\mathrm{Fix}(\mathbf{K})$ is the intersection with $\mathrm{Fix}(\mathbf{K})$ of a set definable in M as a valued field with angular components. Because $(\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}}),\mathrm{W}(\mathrm{Frob}_p))$ is a model of WF_p whose fixed field is $\mathbb{Q}_p$ (which is also maximally complete), by Proposition **(IV.2.1)**, in models of WF_p, the pair $\mathbf{K}|\mathrm{Fix}(\mathbf{K})$ is separated. We can now apply Theorem **(III.2.8)** to conclude. The fact that it is a pure valued field now follows by elimination of quantifiers (and the fact that the angular component maps we chose are definable in $\mathrm{Th}(\mathbb{Q}_p)$). ■

IV.3. Definable closure in $\mathrm{VDF}_{\mathcal{EC}}$

In this section, we investigate the definable and algebraic closure in $\mathrm{VDF}_{\mathcal{EC}}$ to show that, sadly, it is not as simple as one might hope. In DCF_0, the definable closure of a is exactly is field generated by $\partial_\omega(a)$. In $\mathrm{VDF}_{\mathcal{EC}}$, we have to, at least, take in account the Henselianization, but we show that the definable closure of a can be even larger than the Henselianization of the field generated by $\partial_\omega(a)$. The content of this section was already known to Ehud Hrushovski and Thomas Scanlon but was not written anywhere. The definable closure in $\mathrm{VDF}_{\mathcal{EC}}$ will be a major (as yet unsolved) issue in Section IV.5 when describing groups in $\mathrm{VDF}_{\mathcal{EC}}$.
We will be denoting by $\langle A\rangle_\partial$ the differential ring generated by A and $\langle A\rangle_{-1,\partial}$ the differential field generated by A.

Fact IV.3.1:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ be saturated enough. For all $C \subseteq M$, there exists $A \subseteq M$, $C \subseteq A$, such that $\mathrm{dcl}_{\mathcal{L}_{\mathrm{div}}}(\langle A\rangle_\partial) = \overline{\langle A\rangle_{-1,\partial}}^{\mathrm{h}} \subsetneq \mathrm{dcl}_{\mathcal{L}_{\partial,\mathrm{div}}}(A)$. In fact, there exists $a \in \mathrm{dcl}_{\mathcal{L}_{\partial,\mathrm{div}}}(A)$ that is transcendent over $\langle A\rangle_\partial$. In particular, we also have $\mathrm{acl}_{\mathcal{L}_{\mathrm{div}}}(\langle A\rangle_\partial) \subsetneq \mathrm{acl}_{\mathcal{L}_{\partial,\mathrm{div}}}(A)$.

Proof. Let $P(\overline{X}) \in \mathcal{O}(M)[\overline{X}]$, $a \in \mathcal{O}(M)$ and $\varepsilon \in \mathfrak{M}(M)$. Let $Q_a(\partial_\omega(x)) = x - a + \varepsilon P(\partial_\omega(x))$, then Q_a has a unique zero in M. Indeed $\mathrm{val}(Q_a(a)) > 0$, $\mathrm{val}(\frac{\partial Q_a}{\partial X_0}(a)) = \mathrm{val}(1) = 0$ and $\mathrm{val}(\frac{\partial Q_a}{\partial X_i}(a)) = \mathrm{val}(\varepsilon) + \mathrm{val}(\frac{\partial P}{\partial X_i}(a)) > 0$, hence σ-Henselianity applies. If $Q_a(x) = Q_a(y) = 0$, then $\mathrm{res}(x) = \mathrm{res}(a) = \mathrm{res}(y)$ and thus $\mathrm{val}(x-y) > 0$. Let $\eta := x - y$, we have

$$Q_a(y) = x + \eta - a + \varepsilon P(x+\eta) = x + \eta - \varepsilon(\sum_I P_I(a)\eta^I) = \eta + \varepsilon(\sum_{|I|>0} P_I(a)\eta^I).$$

But, if $\eta \neq 0$, $\mathrm{val}(\varepsilon P_I(a)\eta^I) > |I|\mathrm{val}(\eta) \geqslant \mathrm{val}(\eta)$ and hence $\mathrm{val}(Q_a(y)) = \mathrm{val}(\eta) \neq \infty$, a contradiction. I follows that differential equations that have infinitely many solutions in a differentially closed field have only one in a model of $\mathrm{VDF}_{\mathcal{EC}}$.
Let us now show that, in some cases, we do get new definable functions. We may assume that $C = \mathrm{dcl}(\mathbf{K}(C))$. Let k be a differential field, $\widetilde{a} \in k$ be differentially transcendental and let us embed $k[[\varepsilon]]$ with the usual valuation preserving derivation into M such that k and $\mathbf{k}(C)$ are independent and $\mathbf{K}(C)(\varepsilon)$ is a transcendental ramified extension of $\mathbf{K}(C)$. To avoid any confusion, let us denote by a the image of $\widetilde{a}$ by the embedding of k into $k[[\varepsilon]]$ and into M. One can check that for all $n \in \mathbb{N}$, $\mathrm{res}(\mathbf{K}(C)(\varepsilon, \partial_n(a))) = \mathbf{k}(C)(\partial_n(\widetilde{a}))$.
Let us now try to solve $x - a - \varepsilon\partial(x) = 0$ in $k[[\varepsilon]]$. Let $x = \sum x_i\varepsilon^i$ where $x_i \in k$, the equation can then be rewritten as:

$$\sum x_i\varepsilon^i = a\varepsilon^0 + \sum \partial(x_i)\varepsilon^{i+1},$$

and hence $x_0 = a$ and $x_{i+1} = \partial(x_i) = \partial^{i+1}(a)$. If $x \in \overline{\langle C, a, \varepsilon\rangle_\partial}^{\mathrm{alg}}$ then for some $n \in \mathbb{N}$, we must have $x \in \overline{\mathbf{K}(C)(\partial_n(a), \varepsilon)}^{\mathrm{alg}}$. Because $\partial^{n+1}(\widetilde{a})$ is transcendental over $\mathbf{k}(C)(\partial_n(\widetilde{a}))$, and any automorphism of $\sigma : k \cup \mathbf{k}(C)$ fixing $\mathbf{k}(C)$ can be lifted into an automorphism of $k[[\varepsilon]] \cup C$ fixing C and sending $\sum x_i\varepsilon^i \in k[[\varepsilon]]$ to $\sum \sigma(x_i)\varepsilon^i$, it follows that x has an infinite orbit over $\mathbf{K}(C)(\partial_n(a), \varepsilon)$, a contradiction. ∎

Nevertheless, by quantifier elimination, the definable closure of $\mathbf{K}$-generated sets in $\mathbf{\Gamma}$ and $\mathbf{k}$ is exactly what one could hope for. We will be working in the three sorted language.

Proposition IV.3.2:
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq \mathbf{K}(M)$*, then*

$$\mathbf{\Gamma}(\mathrm{dcl}(A)) = \mathbf{\Gamma}(\mathrm{acl}(A)) = \mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial}) = \mathrm{val}(\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}}),$$

$$\mathbf{k}(\mathrm{dcl}(A)) = \mathrm{res}(\langle A\rangle_{-1,\partial}) \textit{ and } \mathbf{k}(\mathrm{acl}(A)) = \overline{\mathrm{res}(\langle A\rangle_{-1,\partial})}^{\mathrm{alg}} = \mathrm{res}(\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}}).$$

Proof. Let us first show that $\Gamma(\mathrm{acl}(A)) = \mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial})$. By quantifier elimination in the three sorted language, any formula with variables in $\mathbf{\Gamma}$ and parameters in A is of the form $\varphi(x, \mathrm{val}(a))$ where $a \in \langle A\rangle_\partial$ is a tuple. In particular, any $\gamma \in \mathbf{\Gamma}(M)$ algebraic over A is algebraic over $\mathrm{val}(\langle A\rangle_\partial)$ in $\mathbf{\Gamma}$ which is a pure divisible ordered abelian group. It follows immediately that $\gamma \in \mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial})$. The equality $\mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial}) = \mathrm{val}(\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}})$ is well-known. Finally, as $\mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial})$ is rigid over $\mathrm{val}(\langle A\rangle_{-1,\partial}) \subseteq \mathbf{\Gamma}(\mathrm{dcl}(A))$ the equality $\mathbf{\Gamma}(\mathrm{dcl}(A)) = \mathbf{\Gamma}(\mathrm{acl}(A))$ also holds.
As for the results concerning $\mathbf{k}$, they are proved similarly. Indeed, any formula with variables in $\mathbf{k}$ and parameters in A is of the form $\varphi(x, \mathrm{res}(a))$ where $a \in \langle A\rangle_{-1,\partial}$ is a tuple. As in DCF_0 the definable closure is just the differential field generated by the parameters and the algebraic closure is its field theoretic algebraic closure, the results follow. ∎

Concerning the definable closure and algebraic closure in the sort $\mathbf{K}$, although the situation is not ideal, we can nevertheless say something:

Corollary IV.3.3:
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq \mathbf{K}(M)$ *then* $\mathbf{K}(\mathrm{acl}(A))$ *is an immediate extension of* $\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}}$.

Proof. We have $\mathrm{val}(\mathbf{K}(\mathrm{acl}(A))) \subseteq \mathbf{\Gamma}(\mathrm{acl}(A)) = \mathrm{val}(\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}})$ where the second equality comes from Proposition **(IV.3.2)**. Similarly $\mathrm{res}(\mathbf{K}(\mathrm{acl}(A))) \subseteq \mathbf{k}(\mathrm{acl}(A)) = \mathrm{res}(\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}})$ and hence $\mathbf{K}(\mathrm{acl}(A))$ is an immediate extension of $\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{alg}}$. ∎

Corollary IV.3.4:
Let $M \vDash \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq \mathbf{K}(M)$ *then* $\mathbf{K}(\mathrm{dcl}(A))$ *is an immediate extension of* $\overline{\langle A\rangle_{-1,\partial}}^{\mathrm{h}}$ *and hence of* $\langle A\rangle_{-1,\partial}$.

Proof. By Proposition **(IV.3.2)**, we have that $\mathrm{res}(\mathbf{K}(\mathrm{dcl}(A))) \subseteq \mathbf{k}(\mathrm{dcl}(A)) = \mathrm{res}(\langle A\rangle_{-1,\partial})$ and $\mathrm{val}(\mathbf{K}(\mathrm{dcl}(A))) \subseteq \mathbf{\Gamma}(\mathrm{dcl}(A)) = \mathbb{Q} \otimes \mathrm{val}(\langle A\rangle_{-1,\partial})$. Let $L := \mathbf{K}(\mathrm{dcl}(A))$ and $F := \overline{\langle A\rangle_{-1,\partial}}^{\mathrm{h}}$. Let $c \in L$. We already know that $\mathrm{val}(c) \in \mathbb{Q} \otimes \mathrm{val}(F)$. Let n be minimal such that $n\mathrm{val}(c) = \mathrm{val}(a)$ for some $a \in F$. Then $\mathrm{res}(ac^{-n}) \in \mathrm{res}(L) = \mathrm{res}(F)$ and we can find $u \in F$ such that

$\operatorname{res}(ac^{-n}) = \operatorname{res}(u)$. As L must be henselian (indeed $\overline{L}^{\mathrm{h}} = \operatorname{dcl}_{\mathcal{L}_{\mathrm{div}}}(L) = L$), we can find $v \in L$ such that $v^n = ac^{-n}u^{-1}$, i.e. $(cv)^n = au^{-1} \in F$. Hence we may assume that c^n itself is in F. But derivations have a unique extension to algebraic extensions and, as F is henselian, the valuation also has a unique extension to the algebraic closure. It follows that any algebraic conjugate of c is also an $\mathcal{L}_{\partial,\mathrm{div}}$-conjugate of c. As $\mathbf{K}(M)$ is algebraically closed, it contains non trivial n-th roots of the unit and it follows that we must have $n = 1$. ∎

Remark IV.3.5:
It would be nice to know if there is an "algebraic" description of the definable closure, i.e. an equivalent of the Henselianisation in the valued case.

In the field of constants, though, we can describe both the definable closure and the algebraic closure.

Proposition IV.3.6:
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ *and* $A \subseteq \mathrm{C}_{\mathbf{K}}(M)$*, then* $\mathbf{K}(\operatorname{dcl}(A)) = \overline{\operatorname{Frac}(A)}^{\mathrm{h}}$ *and* $\mathbf{K}(\operatorname{acl}(A)) = \overline{\operatorname{Frac}(A)}^{\mathrm{alg}}$.

Proof. It follows from the fact that the pair $\mathbf{K}|\mathrm{C}_{\mathbf{K}}$ is separated (cf. Proposition **(IV.2.1)**) that $\mathrm{C}_{\mathbf{K}}$ does not have any immediate extension in $\mathbf{K}$. Hence $\operatorname{dcl}(A) \subseteq \mathrm{C}_{\mathbf{K}}(M)$ and $\operatorname{acl}(A) \subseteq \mathrm{C}_{\mathbf{K}}(M)$. The proposition now follows from the fact that $\mathrm{C}_{\mathbf{K}}$ is a pure model of ACVF. ∎

IV.4. Prolongations of the type space

We will be primarily working in the three sorted language for ACVF and $\mathrm{VDF}_{\mathcal{EC}}$ that will be denoted, respectively, $\mathcal{L}$ and $\mathcal{L}_\partial$. If $x \in \mathbf{K}$ or $x \in \mathbf{k}$ are elements, $\partial_\omega(x)$ will be $(\partial^n(x))_{n\in\mathbb{N}}$ and if $x \in \mathbf{\Gamma}$, $\partial_\omega(x)$ will be $(x)_{n\in\mathbb{N}}$. If x is a tuple of variables, we denote by x_∞ the tuple $(x^{(i)})_{i\in\mathbb{N}}$ where each $x^{(i)}$ is sorted like x. From time to time, we will need to talk about imaginary parameters and interpretable sets, I will then use the geometric languages $\mathcal{L}^{\mathcal{G}}$ and $\mathcal{L}^{\mathcal{G}}_\partial$ in which ACVF and $\mathrm{VDF}_{\mathcal{EC}}$ respectively eliminate imaginaries.
Let $M \models \mathrm{VDF}_{\mathcal{EC}}$ be sufficiently saturated and $A \leqslant M$ be a substructure.

Definition IV.4.1:
We define $\nabla_\omega : \mathcal{S}^{\mathcal{L}_\partial}_x(A) \to \mathcal{S}^{\mathcal{L}}_{x_\infty}(A)$ *to be the map that sends a type* p *to the (complete) type*

$$\nabla_\omega(p) \coloneqq \{\varphi(x_\infty, a) : \varphi \textit{ is an } \mathcal{L}\textit{-formula and } \varphi(\partial_\omega(x), a) \in p\}.$$

Proposition IV.4.2:
The function ∇_ω *is a homeomorphism onto its image (which is closed).*

Proof. As $\mathcal{S}^{\mathcal{L}_\partial}_x(A)$ is compact and $\mathcal{S}^{\mathcal{L}}_{x_\infty}(A)$ is Hausdorff, it suffices to show that ∇_ω is continuous and injective. Let us first show continuity. Let $U = \langle\varphi(x_\infty, a)\rangle \subseteq \mathcal{S}^{\mathcal{L}}_{x_\infty}(A)$, then $\nabla_\omega^{-1}(U) = \langle\varphi(\partial_\omega(x), a)\rangle \subseteq \mathcal{S}^{\mathcal{L}_\partial}_x(A)$. As for ∇_ω being injective, let p and $q \in \mathcal{S}^{\mathcal{L}_\partial}_x(A)$ and let $\varphi(x, a)$ be an $\mathcal{L}_\partial$-formula in $p \smallsetminus q$. By quantifier elimination, we can assume that φ is of the form $\theta(\partial_\omega(x), a)$ for some $\mathcal{L}$-formula θ. Then $\theta(x_\infty, a) \in \nabla_\omega(p) \smallsetminus \nabla_\omega(q)$. ∎

In fact, we can describe exactly what the closed image is. For every differential ring R let $R\{X_1, \ldots, X_m\}$ be the ring of differential polynomials in m variables — i.e. $R[X_i^{(j)} : j \in$

$\mathbb{N}\, 1 \leqslant i \leqslant m]$ — and let ∂ also denote the derivation on this ring. When $P \in R\{X_1, \ldots, X_m\}$, we will write $P^\star \in R[X_i^{(j)} : j \in \mathbb{N}$ and $1 \leqslant i \leqslant m]$ for the underlying polynomial.

Notation IV.4.3:
Let x be a tuple of variables in $\mathcal{L}$. We will denote by $x_{\mathbf{K}}$ the subtuple of variables in $\mathbf{K}$, $x_{\mathbf{k}}$ the subtuple of variables in $\mathbf{k}$ and $x_{\mathbf{\Gamma}}$ the subtuple of variables in $\mathbf{\Gamma}$.

Proposition IV.4.4:
Let $\mathcal{P}_x$ be the following set of $\mathcal{L}(A)$-formulas:

$$\begin{aligned} & \{\mathrm{val}(\partial(P)^\star(x_{\mathbf{K},\infty})) \geqslant \mathrm{val}(P^\star(x_{\mathbf{K},\infty})) : P \in \mathbf{K}(A)\{x_{\mathbf{K}}\}\} \\ \cup\ & \{P^\star(x_{\mathbf{k},\infty}) = 0 \Rightarrow \partial(P)^\star(x_{\mathbf{k},\infty}) = 0 : P \in \mathbf{k}(A)\{x_{\mathbf{k}}\}\} \\ \cup\ & \{x_j^{(i)} = \mathrm{res}((\tfrac{P}{Q})^\star(x_{\mathbf{K},\infty})) \Rightarrow x_j^{(i+1)} = \mathrm{res}(\partial(\tfrac{P}{Q})^\star(x_{\mathbf{K},\infty})) : i \in \mathbb{N} \textit{ and } x_j \in x_{\mathbf{k}}\} \\ \cup\ & \{x_j^{(i+1)} = x_j^{(i)} : x_j \textit{ is a } \mathbf{\Gamma}\textit{-variable}\} \end{aligned}$$

Let $\mathcal{P}_x(A) \subseteq \mathcal{S}^{\mathcal{L}}_{x_\infty}(A)$ be the set of types over A containing $\mathcal{P}_x$. Then $\nabla_\omega(\mathcal{S}^{\mathcal{L}_\partial}_x(A)) = \mathcal{P}_x(A)$.

Note that, in the three sorted language, res has two arguments that $\mathrm{res}(x, y) = \mathrm{res}(x/y)$. However, there is a slight abuse of notation in the above formulas especially in the expression $\mathrm{res}(\partial(\frac{P}{Q})^\star(x_{\mathbf{K},\infty}))$ which should read instead

$$\mathrm{res}(Q^\star(x_{\mathbf{K},\infty})\partial(P)^\star(x_{\mathbf{K},\infty}) - P^\star(x_{\mathbf{K},\infty})\partial(Q)^\star(x_{\mathbf{K},\infty}), (Q^\star(x_{\mathbf{K},\infty}))^2).$$

Proof. As, for all $P \in \mathbf{K}(A)\{\overline{X}\}$ and $c \in \mathbf{K}(M)$ of the right length, $\partial(P(c)) = \partial(P)(\partial_\omega(c))$ and similarly in $\mathbf{k}$, it is quite obvious that $\nabla_\omega(\mathcal{S}^{\mathcal{L}_\partial}_x(A)) \subseteq \mathcal{P}_x(A)$. Let us now show that this inclusion is an equality. Let $p \in \mathcal{P}_x(A)$ and $\overline{c} \models p$. Then, $\{P \in \mathbf{K}(A)\{\overline{X}\} : P^\star(\overline{c}_{\mathbf{K}}) = 0\}$ is a differential ideal as for all such P, $\mathrm{val}(\partial(P)^\star(\overline{c}_{\mathbf{K}})) \geqslant \mathrm{val}(P^\star(\overline{c}_{\mathbf{K}})) = \infty$. Hence $L = \mathbf{K}(A)(\overline{c}_{\mathbf{K}})$ can be endowed with a (unique) differential ring structure such that $\partial(c_{j,i}) = c_{j+1,i}$.
For any $P, Q \in \mathbf{K}(A)\{\overline{X}\}$, we have $\mathrm{val}(\partial(P)^\star(\overline{c}_{\mathbf{K}})) \geqslant \mathrm{val}(P^\star(\overline{c}_{\mathbf{K}}))$ and $\mathrm{val}(\partial(Q)^\star(\overline{c}_{\mathbf{K}})) \geqslant \mathrm{val}(Q^\star(\overline{c}_{\mathbf{K}}))$, and hence

$$\mathrm{val}\left(\partial\left(\frac{P^\star(\overline{c}_{\mathbf{K}})}{Q^\star(\overline{c}_{\mathbf{K}})}\right)\right) - \mathrm{val}\left(\frac{P^\star(\overline{c}_{\mathbf{K}})}{Q^\star(\overline{c}_{\mathbf{K}})}\right) = \mathrm{val}\left(\frac{\partial(P)^\star(\overline{c}_{\mathbf{K}})}{P^\star(\overline{c}_{\mathbf{K}})} - \frac{\partial(Q)^\star(\overline{c}_{\mathbf{K}})}{Q^\star(\overline{c}_{\mathbf{K}})}\right) \geqslant 0.$$

Similarly, there is a unique differential ring structure on $l := \mathbf{k}(A)(\overline{c}_{\mathbf{k}})$ such that $\partial(c_{j,i}) = c_{j+1,i}$ and these two differential structure commute with res. Finally let Λ be the group generated by $\mathbf{\Gamma}(A)$, $\mathrm{val}(L)$ and $\overline{c}_{\mathbf{\Gamma}}$. Then $C = L \cup l \cup \Lambda$ is a valued field with a valuation preserving derivation — in the broader sense where res and val may not be onto. As $\mathrm{VDF}_{\mathcal{EC}}$ is the model completion of such structures (cf. [Sca00, Theorem 7.1]), we can find an embedding $f : L \to M$ such that f fixes A. Let $q = \mathrm{tp}(f(c_0)/A)$, then $\nabla_\omega(q) = p$. ∎

We will now look at how ∇_ω and its inverse behave with respect to various properties of types. Let us begin by two very easy results. One must beware though that however innocent these question might seem to be, their converses actually present real challenges. A converse to Lemma **(IV.4.5)** is proved in Proposition **(IV.4.9)** and required the development of Section III.1 to be proved. I do not know if the converse of Lemma **(IV.4.6)** holds or not.

Lemma IV.4.5:
Let $p \in \mathcal{P}_x(M)$ be $\mathcal{L}(A)$-definable then $\nabla_\omega^{-1}(p)$ is $\mathcal{L}_\partial(A)$-definable.

Proof. Let $\varphi(x;s)$ be an $\mathcal{L}_\partial$-formula that we may assume is of the form $\psi(\partial_\omega(x),\partial_\omega(s))$ where $\psi(x_\infty;s_\infty)$ is an $\mathcal{L}$-formula. By definability of p, there exists an $\mathcal{L}(A)$-formula $\theta(s_\infty)$ such that for all tuples $m \in M$, $M \vDash \theta(\partial_\omega(m))$ if and only if $\psi(u,\partial_\omega(m)) \in p$, if and only if $\psi(\partial_\omega(x),\partial_\omega(m)) \in \nabla_\omega^{-1}(p)$. Hence for all $m \in M$, $\varphi(x;m) \in \nabla_\omega^{-1}(p)$ is equivalent to $M \vDash \theta(\partial_\omega(m))$. ∎

Lemma IV.4.6:
Let $p \in \mathcal{P}_x(M)$ be A-invariant then $\nabla_\omega^{-1}(p)$ is also A-invariant.

Proof. Let $\varphi(\partial_\omega(x),\partial_\omega(c)) \in \nabla_\omega^{-1}(p)$ be any $\mathcal{L}_\partial(M)$-formula, where $\psi(x_\infty;s_\infty)$ is an $\mathcal{L}$-formula. For all $\sigma \in \mathrm{Aut}_{\mathcal{L}_\partial}(M/A) \subseteq \mathrm{Aut}_{\mathcal{L}}(M/A)$, we have $\varphi(x_\infty;\sigma(\partial_\omega(c))) \in p$, i.e. $\varphi(\partial_\omega(x),\partial_\omega(\sigma(c))) \in \nabla_\omega^{-1}(p)$. ∎

Proposition IV.4.7:
Let $p \in \mathcal{P}_n(M)$ be stably dominated then $\nabla_\omega^{-1}(p)$ is also stably dominated.

Proof. We will need the following result:

Claim IV.4.8: *Let D be $\mathcal{L}^{\mathcal{G}}(M)$-definable. If D is stable and stably embedded in* ACVF*, then it is also stable and stably embedded in* $\mathrm{VDF}_{\mathcal{EC}}$*.*

Proof. It follows from [HHM06, Lemma 2.6.2 and Remark 2.6.3], cf. Proposition **(0.4.3)**, that $D \subseteq \mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(E \cup \mathbf{k})$ for some finite $E \subseteq D$. Because $\mathbf{k}$ also eliminates imaginaries, is stable and stably embedded in $\mathrm{VDF}_{\mathcal{EC}}$, it immediately follows that D is stably embedded and stable in $\mathrm{VDF}_{\mathcal{EC}}$ too. ♦

Therefore, the stable part over M in $\mathrm{VDF}_{\mathcal{EC}}^{\mathcal{G}}$, $\mathrm{St}_M^{\mathcal{L}_\partial^{\mathcal{G}}}$, is an enrichment of the stable part over M in $\mathrm{ACVF}^{\mathcal{G}}$, $\mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}$. Now let $c \vDash \nabla_\omega^{-1}(p)$ and $B \subseteq \mathbf{K}$ be such that

$$\mathrm{St}_M^{\mathcal{L}_\partial^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(Mc)) \downarrow_M^{\mathcal{L}_\partial^{\mathcal{G}}} \mathrm{St}_M^{\mathcal{L}_\partial^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(MB))$$

where $\downarrow_M^{\mathcal{L}_\partial^{\mathcal{G}}}$ denotes independence in $\mathrm{St}_M^{\mathcal{L}_\partial^{\mathcal{G}}}$ over M. In particular,

$$\mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(M\partial_\omega(c))) \downarrow_M^{\mathcal{L}_\partial^{\mathcal{G}}} \mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(M\partial_\omega(B))).$$

As M is a model of $\mathrm{VDF}_{\mathcal{EC}}$, it follows that

$$\mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(M\partial_\omega(c))) \downarrow_M^{\mathcal{L}^{\mathcal{G}}} \mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(M\partial_\omega(B)))$$

where $\downarrow_M^{\mathcal{L}^{\mathcal{G}}}$ denotes independence in $\mathrm{St}_M^{\mathcal{L}^{\mathcal{G}}}$ over M. As $\partial_\omega(c) \vDash p$ and p is stably dominated,

$$\begin{aligned}
\mathrm{tp}_{\mathcal{L}_\partial^{\mathcal{G}}}(B/\mathrm{St}_M^{\mathcal{L}_\partial}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(Mc))) &\vdash \mathrm{tp}_{\mathcal{L}^{\mathcal{G}}}(\partial_\omega(B)/\mathrm{St}_M^{\mathcal{L}}(\mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(M\partial_\omega(c))))\\
&\vdash \mathrm{tp}_{\mathcal{L}}(M\partial_\omega(B)/M\partial_\omega(c))\\
&\vdash \mathrm{tp}_{\mathcal{L}_\partial}(B/Mc).
\end{aligned}$$

Here the last implication comes from the fact that ∇_ω is one to one on the space of types. It now follows from Lemma **(0.4.7)** [4.7 prelim], that $\nabla_\omega^{-1}(p)$ is stably dominated. ∎

Let me now give some converses.

Proposition IV.4.9:
Let $p \in \mathcal{S}_x^{\mathcal{L}_\partial}(M)$ *be* $\mathcal{L}_\partial(A)$*-definable, then* $\nabla_\omega(p)$ *is* $\mathcal{L}^{\mathcal{G}}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(A))$*-definable.*

Proof. Let $\varphi(x_\infty; s)$ be an $\mathcal{L}$-formula. Then exists some $\mathcal{L}_\partial(A)$-formula $\theta(s)$ such that for all tuple $m \in M$, $\varphi(x_\infty; m) \in \nabla_\omega(p)$ — i.e. $\varphi(\partial_\omega(x); m) \in p$ — if and only if $M \models \theta(m)$. It follows that $\nabla_\omega(p) \in \mathcal{S}_{x_\infty}^{\mathcal{L}}(M)$ is $\mathcal{L}_\partial(A)$-definable. By Corollary **(III.1.5)**, it is in fact $\mathcal{L}^{\mathcal{G}}(\mathrm{dcl}_{\mathcal{L}_\partial^{\mathcal{G}}}(A))$-definable. ∎

Lemma IV.4.10:
Let $p \in \mathcal{S}_x^{\mathcal{L}_\partial}(M)$ *be finitely satisfiable in* A*, then* $\nabla_\omega(p)$ *is finitely satisfiable in* A*.*

Proof. Let $\varphi(x_\infty; s)$ be any $\mathcal{L}$-formula and let $m \in M$ be such that $\varphi(x_\infty; m) \in \nabla_\omega(p)$, i.e. $\varphi(\partial_\omega(x), m) \in p$. Then there exists $a \in A$ such that $M \models \varphi(\partial_\omega(a), m)$, i.e. $\nabla_\omega(p)$ is finitely satisfiable in A. ∎

Proposition IV.4.11:
Assume M *sufficiently saturated and homogeneous and let* $p \in \mathcal{S}_x^{\mathcal{L}_\partial}(M)$ *be* $\mathcal{L}_\partial(M)$*-definable. The following are equivalent:*

(i) p *is stably dominated;*

(ii) p *is generically stable;*

(iii) p *is orthogonal to* Γ*;*

(iv) $\nabla_\omega(p)$ *is stably dominated;*

(v) $\nabla_\omega(p)$ *is generically stable;*

(vi) $\nabla_\omega(p)$ *is orthogonal to* Γ*;*

Proof. The equivalence of (iv), (v) and (vi) is well known in ACVF (cf. Proposition **(0.4.20)**). It also follows from Proposition **(0.4.19)**, that (i) ⇒ (ii) ⇒ (iii) also holds in any NIP theory. We proved in Proposition **(IV.4.7)** that (iv) implies (i). Let us now prove that (iii) implies (iv). By Proposition **(IV.4.9)**, $\nabla_\omega(p)$ is $\mathcal{L}(A)$-definable for some $A \subseteq M$. Let $C \models \mathrm{VDF}_{\mathcal{EC}}$ be maximally complete, and contain A and let $c \models p|_C$. By Proposition **(IV.4.9)**, $\nabla_\omega(p)$ is $\mathcal{L}(C)$-definable. As p is orthogonal to $\mathbf{\Gamma}$, we also have $\mathbf{\Gamma}(C) \subseteq \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}}(Cc)) \subseteq \mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}_\partial}(Cc)) = \mathbf{\Gamma}(C)$. By Theorem **(0.4.17)**, because C is maximally complete, $\mathrm{tp}(\partial_\omega(c)/C\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}}(Cc)))$ is stably dominated. But $\mathrm{tp}(\partial_\omega(c)/C\mathbf{\Gamma}(\mathrm{dcl}_{\mathcal{L}}(Cc))) = \mathrm{tp}(\partial_\omega(c)/C) = \nabla_\omega(p)|_C$ and hence $\nabla_\omega(p)$ is also stably dominated. ∎

IV.5. Groups in $\mathrm{VDF}_{\mathcal{EC}}$

The inspiration for this section is drawn from [Hrub, Section 3] and [KP02]. In [Hrub], Hrushovski shows that most of the tools developed in [Hru90] for stable groups can be generalized to an unstable setting as long as we have a definable generic around. In this section we give an account of this work, adding the case of $\star$-definable groups (cf. Proposition **(IV.5.24)**) that is missing from [Hrub] and generalizing the groups chunk theorem somewhat so as to be able to talk about non connected groups more directly. We then use those tools to give an abstract analog (in a non stable setting, using definable generics) of Pillay's result that groups definable in DCF_0 can be definably embedded in algebraic groups; or of the similar result of [BD01] for the groups definable in separably closed fields (SCF) of finite imperfection degree. We then apply this abstract group construction in $\mathrm{VDF}_{\mathcal{EC}}$. The result one could hope for — in analogy to DCF_0 — is the following:

Question IV.5.1: Is every group interpretable in $\mathrm{VDF}_{\mathcal{EC}}$ definably isomorphic to a subgroup of a group interpretable in ACVF?

But we are far from answering it. We can only cope with groups with d-generics (see Definition **(IV.5.8)** below) and we cannot cope with complications related to the — quite ugly — definable closure in $\mathrm{VDF}_{\mathcal{EC}}$. This issue with definable closure has already presented itself in the case of ACFA (algebraically closed fields with a generic automorphism), but in that case, the definable closure is still inside the algebraic closure of the generated difference field and so (both in [CH99] and [KP02]), replacing the group chunk by a group configuration (and using the supersimplicity of ACFA) a similar proof scheme still worked. In the case of $\mathrm{VDF}_{\mathcal{EC}}$, coping with the definable closure seems to require new methods.
Nevertheless, the abstract construction presented in this section can probably be used to describe all definable groups with d-generics in other enriched valued fields where the definable closure would be better understood.
We defined in Definition **(III.6.6)** the notion of $(\mathcal{L},\star)$-definable set, let us now do the same with functions.

Definition IV.5.2 ($\star$-definable functions)**:**
Let $\mathcal{L}$ be a language, M an $\mathcal{L}$-structure and $x=(x_i)_{i\in I}$ and $y=(y_j)_{j\in J}$ be (potentially infinite) tuples of variables. Let S_i be the sort in which the variable x_i lives, and similarly for S_j. A partial function $f:\prod_{i\in I}S_i(M)\to\prod_{j\in J}S_j(M)$ is said to be $(\mathcal{L},x,y)$-definable — or simply an $(\mathcal{L},\star)$-definable function, if we do not want to specify x or y — if for all $j\in J$ there exists an $\mathcal{L}$-definable function $f_j:\prod_{i\in I}S_i\to S_j$ such that $f=\prod_{j\in J}f_j$.

Although the definition above seems too restrictive, it is a classical result (cf. [Poi87, Section 4.5] for the ∞-def case) that considering functions whose graph is $\star$-definable does not actually allow any new function:

Proposition IV.5.3:
Let M be a saturated enough $\mathcal{L}$-structure. Any function whose graph is $(\mathcal{L},\star)$-definable in M is in fact the restriction to an $(\mathcal{L},\star)$-definable set of an $(\mathcal{L},\star)$-definable function.

Proof. Let us assume the graph of f is given by $\bigcap_{\varphi\in P}\varphi(x,y)$. We may assume that P is closed under finite intersections. As $\bigcap_{\varphi\in P}\varphi(x,y)$ is the graph of a function we have:

$$\bigwedge_{\varphi\in P}\varphi(x,y)\wedge\bigwedge_{\varphi\in P}\varphi(x,z)\Rightarrow\bigwedge_j y_j=z_j.$$

By compactness (and closure of P under intersection) for fixed j, there exists a formula $\varphi_j\in P$ such that $\varphi_j(x,y)\wedge\varphi_j(x,z)\Rightarrow y_j=z_j$, i.e. the formula $\theta(x,y_j):=\exists y_{\neq j}\,\varphi_j(x,y)$ is the graph of an $\mathcal{L}$-definable function f_j. It is also clear that the domain is $(\mathcal{L},\star)$-definable as it is given by $\bigwedge_{\varphi\in P}\exists y\,\varphi(x,y)$. ∎

As one might guess, an $(\mathcal{L},\star)$-definable group will be a group in the category of $(\mathcal{L},\star)$-definable sets: an $(\mathcal{L},\star)$-definable set equipped with an $(\mathcal{L},\star)$-definable group law.

Example IV.5.4:
Let $(G_\alpha)_{\alpha\in\mathrm{A}}$ be a projective system of $\mathcal{L}$-definable groups, then $G := \varprojlim G_\alpha = \{g \in \prod_\alpha G_\alpha :$ for all $f : G_\alpha \to G_\beta$ in the projective system, $f(g_\alpha) = g_\beta\}$ is an $(\mathcal{L},\star)$-definable set. Moreover, for all $g, h \in G$, $(g \cdot h)_\alpha = g_\alpha \cdot h_\alpha$ and the group law is also $(\mathcal{L},\star)$-definable.

If T is stable, it is shown in [Hru90] that any $(\mathcal{L},\star)$-definable group is a prolimit of $\mathcal{L}$-definable groups as in the above Example **(IV.5.4)**. This proof generalizes (cf. Proposition **(IV.5.24)**), in the unstable context, to groups with a d-generic (cf. Definition **(IV.5.9)**).
Let T be any $\mathcal{L}$-theory that eliminates imaginaries and $M \vDash T$ be sufficiently saturated and homogeneous. Let $(G,\cdot)$ be an $(\mathcal{L}(M),\star)$-definable group. Note that there is a slight abuse of notation here: what I really mean when I say $(\mathcal{L}(M),\star)$-definable is $(\mathcal{L}(A),\star)$-definable for some small set of parameters $A \subseteq M$. Let $\Delta(x;s)$ be a (small) set of $\mathcal{L}(M)$-formulas where x is sorted like the points of G — as for G what I really should say is a set of $\mathcal{L}(A)$-formulas for some small $A \subseteq M$. Note that contrary to what was done in the previous chapter the formulas in Δ can have fixed parameters, to, potentially, take in account the fact that G is defined with parameters.
For all $\varphi(x;s) \in \Delta$, let j_φ be the set of indices from x that actually appear in φ and $\psi_\varphi(x,y,z)$ be an $\mathcal{L}(M)$-formula that defines the graph of $(x,y) \mapsto (x\cdot y)_{j_\varphi}$. We say that Δ is closed under (left) action of G if for all $\varphi(x;s) \in \Delta$, tuple $m \in M$ and $g \in G(M)$, there exists and instance $\psi(x)$ of Δ such that for all $x \in G$, $\exists z\, \psi_\varphi(g,x,z) \wedge \varphi(z;m) \iff \psi(x)$.

Lemma IV.5.5:
Assume G is a definable group. Let $\Delta(x;s)$ be a finite set of $\mathcal{L}$-formulas and $A \subseteq M$ be such that G is $\mathcal{L}(A)$-definable. Then there exists a finite set $\Theta(x;s,t) \supseteq \Delta(x;s)$ of $\mathcal{L}(A)$-formulas which is closed under left action of G.

Proof. The finite set $\Theta(x;s,t) := \Delta(x;s) \cup \{t \in G \wedge \exists z\,(\psi_\varphi(t,x,z) \wedge \varphi(z;s)) : \varphi \in \Delta\}$ is closed under left action of G by associativity of the group law. ∎

Remark IV.5.6:
If G is not defined over $\varnothing$ but one wants Θ to contain $\mathcal{L}$-formulas, it is possible at the cost of making Θ infinite. Indeed, for all $\varphi \in \Delta$, let us rewrite the formula $\psi_\varphi(y,x,z)$ above as $\psi_\varphi(y,x,z;m)$ where $m \in M$ and ψ_φ is an $\mathcal{L}$-formula. Similarly , rewrite θ_φ as $\theta_\varphi(x;m)$. Let $\Delta_G(x;s,t,u) := \{\theta_\varphi(x;u) \wedge \theta_\varphi(t;u) \wedge \exists z\,(\psi_\varphi(t,x,z;u) \wedge \varphi(z;s)), \Delta^0 := \Delta$, $\Delta^{n+1} := \Delta^n_G$ and $\Theta := \bigcup_{n\in\mathbb{N}} \Delta^n$. Then Θ is closed under left action of G.

From now on, let $\Delta(x;s)$ be a set of $\mathcal{L}$-formulas closed under left action by G. At the cost of adding new constants to $\mathcal{L}$, this also covers the case where Δ is a set of $\mathcal{L}(A)$-formulas.

Definition IV.5.7 ($^g p$)**:**
Let $p \in \mathcal{S}^\Delta_x(M)$ be a type consistent with G and $g \in G(M)$. We define $^g p := \mathrm{tp}_\Delta(g\cdot a/M)$ for any $a \vDash p$ such that $a \in G$.

Because Δ is closed under left action of G, one can check that $^g p$ is well defined and the map $(g,p) \mapsto {}^g p$ defines an action of G on Δ-types consistent with G. For all $\varphi \in \Delta$ and $m \in M$, we have $\varphi(x;m) \in {}^g p$ if and only if $\varphi(g\cdot x;m) \in p$ where $\varphi(g\cdot x;m)$ is a (slightly abusive) notation for $\exists z\,\psi_\varphi(g,x,z) \wedge \varphi(z;m)$. In particular, if $A \subseteq M$ is such that G is

$(\mathcal{L}(A), \star)$-definable and p is $\mathcal{L}(A)$-definable, then ${}^g p$ is $\mathcal{L}(Ag)$-definable and we can take $d_{{}^g p} x\, \varphi(x; s) = d_p x\, \varphi(g \cdot x; s)$.
There are many definitions of generic types in the unstable context, among them one that is related to forking (see [NP06; HP11]):
Recall that, whenever we consider $A \subseteq M$, it is supposed to be small.

Definition IV.5.8 (f-generic)**:**
Let $A \subseteq M$. We say that $p \in \mathcal{S}^{\Delta}_x(M)$ consistent with G is an f-generic Δ-type in G over A if for all $g \in G(M)$, ${}^g p$ does not fork over A.

Although f-genericity is the more usual notion of genericity, what we will be needing here is a definable version (which appears in [Hrub]):

Definition IV.5.9 (d-generic)**:**
Let $A \subseteq M$ and $p \in \mathcal{S}^{\Delta}_x(M)$ be consistent with G. We say that p is a d-generic type in G over A if for all $g \in G(M)$, ${}^g p$ is $\mathcal{L}(A)$-definable.

When we do not want to specify the (small) set of parameters A, we will simply say that p is a d-generic in G.

Proposition IV.5.10:
Assume T is NIP *and $A = \mathrm{acl}(A)$. Let $p \in \mathcal{S}^{\Delta}_x(M)$. The following are equivalent:*

(i) *p is d-generic over A;*

(ii) *p is $\mathcal{L}(M)$-definable and f-generic over A.*

We define $\mathrm{Aut}(M/\mathrm{Lstp}(A))$ to be the group generated by $\mathrm{Aut}(M/N)$ for all $N \preccurlyeq M$ containing A. It is the group of automorphisms fixing A that preserve Lascar strong type, where a and c have the same Lascar strong type if there exists $N \preccurlyeq M$ containing A such that $a \equiv_{\mathcal{L}(N)} c$. Note that if $(x_i)_{i<\omega}$ is an $\mathcal{L}(A)$-indiscernible sequence, x_0 and x_1 have the same Lascar strong type. For more on the matter, see [Sim, Chapter 5].

Proof. We will be needing the following easy result, which has nothing to do with NIP:

Claim IV.5.11: *Let X be an $\mathcal{L}(M)$-definable set.*

$$X \text{ is } \mathcal{L}(A)\text{-definable} \iff X \text{ is } \mathrm{Aut}(M/\mathrm{Lstp}(A))\text{-invariant}$$

Proof. The left to right implication follows from $\mathrm{Aut}(M/\mathrm{Lstp}(A)) \subseteq \mathrm{Aut}(M/A)$. Let us now assume that X is $\mathrm{Aut}(M/\mathrm{Lstp}(A))$-invariant and let $(e_i)_{i<\omega}$ be an $\mathcal{L}(A)$-indiscernible sequence such that $e_0 = \ulcorner X \urcorner$. Then e_0 and e_1 have the same Lascar strong type and hence there exists $\sigma \in \mathrm{Aut}(M/N)$ for some N, $A \subseteq N \preccurlyeq M$, such that $\sigma(e_0) = e_1$, but because $\mathrm{Aut}(M/N) \subseteq \mathrm{Aut}(M/\mathrm{Lstp}(A))$, it follows that $e_1 = \sigma(e_0) = e_0$ and hence that $e_0 = \ulcorner X \urcorner \in \mathrm{acl}(A) = A$. ♦

Recall now that, in an NIP theory, a type $p \in \mathcal{S}^{\Delta}_x(M)$ does not fork over A if and only if it is $\mathrm{Aut}(M/\mathrm{Lstp}(A))$-invariant (cf. [Sim, Proposition 5.21] which clearly works for Δ-types). Let us now assume that p is d-generic over A. In particular, $p = {}^1 p$ is $\mathcal{L}(A)$-definable and for all $g \in G(M)$, ${}^g p$ is A-invariant and hence $\mathrm{Aut}(M/\mathrm{Lstp}(A))$-invariant. On the other hand, if p is $\mathcal{L}(M)$-definable and f-generic over A, then for all $g \in G(M)$, ${}^g p$ is $\mathcal{L}(M)$-definable and $\mathrm{Aut}(M/\mathrm{Lstp}(A))$-invariant. By Claim **(IV.5.11)**, ${}^g p$ is in fact $\mathcal{L}(A)$-definable. ■

Proposition IV.5.12:
Let $p \in \mathcal{S}_x^{\Delta}(M)$ be a d-generic Δ-type in G. Then for all $\varphi \in \Delta$, the set $\{{}^g p|_{\varphi} : g \in G\}$ is finite.

Proof. There exists some (small) $A \subseteq M$ such that for all $g \in G(M)$, ${}^g p$ is $\mathcal{L}(A)$-definable and hence for all $\varphi(x;s) \in \Delta$, there exists an $\mathcal{L}(A)$-formula $\psi_g(s)$, depending on g, such that $M \models \forall s\, d_p\varphi(g\cdot x;s) \iff \psi_g(s)$. By compactness, there exist a finite number of $\mathcal{L}(A)$-formulas $(\psi_i)_{0\leqslant i<k}$ such that for all $g \in G(M)$,

$$M \models \bigvee_{0\leqslant i<k} (\forall s\, d_p\varphi(g\cdot x;s) \iff \psi_i(s)).$$

For all i, let q_i be the φ-type defined by ψ_i. Then $\{{}^g p|_{\varphi} : g \in G(M)\} \subseteq \{q_i : 0 \leqslant i < k\}$ which is indeed finite. ■

Definition IV.5.13 (Locally finite)**:**
A set $P \subseteq \mathcal{S}_x^{\Delta}(M)$ is said to be locally finite if for all $\varphi(x;s) \in \Delta$, $\{p|_{\varphi} : p \in P\}$ is finite.

We have just shown that for all $p \in \mathcal{S}^{\Delta}(M)$ d-generic in G, $\{{}^g p : g \in G\}$ is locally finite.
For the rest of this section, let $\mathfrak{A} \leqslant \mathrm{Aut}(M)$ be a group. Considering $\mathfrak{A}$-invariant sets where $\mathfrak{A}$ is not of the form $\mathrm{Aut}(M/A)$ for some $A \subseteq M$ may seem like an unnecessary generalization, but we will need it to prove the finer considerations about the parameters in Theorem **(IV.5.42)**. In the proof of that theorem there are two problems: construct a group definable in some language $\mathcal{L}$ from a group definable in some enrichment $\widetilde{\mathcal{L}}$ and control the parameters over which this new group is defined. But we only control the parameters in $\widetilde{\mathcal{L}}$ and hence we show that everything we construct is invariant under the group of $\widetilde{\mathcal{L}}$-automorphisms over A (which will be $\mathfrak{A}$ in the proof). Thus, in the end, we obtain that the group we constructed is definable in $\mathcal{L}$ over $\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(A)$.
For simpler statements, one can take $\mathfrak{A} = \mathrm{Aut}(M/A)$ and replace any instance of (relatively) $\mathcal{L}(M)$-definable $\mathfrak{A}$-invariant sets by (relatively) $\mathcal{L}(A)$-definable and any instance of $(\mathcal{L}(M),\star)$-definable $\mathfrak{A}$-invariant by $(\mathcal{L}(A),\star)$-definable.

Definition IV.5.14 (relatively $\mathcal{L}$-definable)**:**
Let X be $(\mathcal{L},\star)$-definable and $Y \subseteq X(M)$. We say that Y is relatively $\mathcal{L}$-definable in X if there exists an $\mathcal{L}$-definable set Z such that $Y = X(M) \cap Z(M)$.

Proposition IV.5.15:
Assume G is $\mathfrak{A}$-invariant and Δ is finite. Let $P \subseteq \mathcal{S}_x^{\Delta}(M)$ be a locally finite set of Δ-types d-generic in G which is $\mathfrak{A}$-invariant — i.e. for all $p \in P$ and $\sigma \in \mathfrak{A}$, $\sigma(p) \in P$. Then $\mathrm{Stab}_G(P) := \{g \in G :$ for all $p \in P, {}^g p \in P\}$ is a relatively $\mathcal{L}(M)$-definable $\mathfrak{A}$-invariant subgroup of G of finite index.

It follows that if Δ is infinite, $\mathrm{Stab}_G(P)$ is an $(\mathcal{L}(M),\star)$-definable $\mathfrak{A}$-invariant subgroup with bounded index.

Proof. As noted earlier $(g,p) \mapsto {}^g p$ is a group action of G on the Δ-types consistent with G. Hence for all such P, $\mathrm{Stab}_G(P)$ is indeed a subgroup. It is defined by

$$\bigwedge_{\varphi\in\Delta} \bigwedge_{p\in P} \bigvee_{q\in P} \forall s\,(d_q\varphi(x;s) \iff d_p\varphi(g\cdot x;s)).$$

As P is locally finite, the above seemingly infinite formula is in fact an $\mathcal{L}(M)$-formula and as P and G are $\mathfrak{A}$-invariant, so is $\mathrm{Stab}_G(P)$.
Pick any $p \in P$, $\mathrm{Stab}_G(p) \leqslant \mathrm{Stab}_G(P)$ and the index of $\mathrm{Stab}_G(p)$ in G is equal to $|\{{}^g p : g \in G\}|$ which is finite by Proposition **(IV.5.12)**. ∎

Remark IV.5.16:
If G is $(\mathcal{L}(A),x)$-definable for some $A \subseteq M$ and $p \in \mathcal{S}(M)$ is d-generic in G over $\mathrm{acl}(A)$, then the set $P := \{\sigma(p) : \sigma \in \mathrm{Aut}(M/A)\}$ is an $\mathrm{Aut}(M/A)$-invariant locally finite set of types d-generic in G.

If we take Δ to be the set of all formulas and P to be a singleton, we recover what is usually called the connected component of G. We will not be using this result afterwards.

Proposition IV.5.17:
Let $p \in \mathcal{S}(M)$ be d-generic in G, then $\mathrm{Stab}_G(p)$ does not contain any proper $(\mathcal{L}(M),\star)$-definable subgroup of bounded index in G.
In particular, for any $A \subseteq M$ such that G is $(\mathcal{L}(A),\star)$-definable and $p \in \mathcal{S}(M)$ is d-generic in G over A, $\mathrm{Stab}_G(p)$ is the intersection of all relatively $\mathcal{L}(A)$-definable subgroups of finite index in G — equivalently the intersection of all $(\mathcal{L}(A),\star)$-definable subgroups of bounded index in G. Moreover $\mathrm{Stab}_G(p)$ is normal in G and does not depend on the choice of p.

The following proof is taken almost word for word from [HP11, Proposition 5.6].

Proof. Let us begin with the following claim:

Claim IV.5.18: *Let $A \subseteq M$ be such that G is $(\mathcal{L}(A),\star)$-definable and p is d-generic in G over A, then $\mathrm{Stab}_G(p) = \{a_1^{-1}\cdot a_2 : a_i \in G$ and $a_1 \equiv_{\mathcal{L}(A)} a_2\}$.*

Proof. Let $a_1 \equiv_{\mathcal{L}(A)} a_2$. Then there exists $\sigma \in \mathrm{Aut}(M/A)$ such that $\sigma(a_1) = a_2$. Moreover, as both p and ${}^{a_1}p$ are $\mathrm{Aut}(M/A)$-invariants and G is $(\mathcal{L}(A),\star)$-definable we have:

$${}^{a_1}p = \sigma({}^{a_1}p) = {}^{\sigma(a_1)}\sigma(p) = {}^{a_2}p.$$

It follows that ${}^{a_1^{-1}\cdot a_2}p = p$. Let now $h \in \mathrm{Stab}_G(p)$, $a \models p|_{Ah}$ and $c = h\cdot a$. Then $c \models p|_{Ah}$, in particular $a^{-1} \equiv_{\mathcal{L}(A)} c^{-1}$ and $h = c\cdot a^{-1} = (c^{-1})^{-1}\cdot a^{-1}$. ◆

Thus, $\mathrm{Stab}_G(p)$ does not depend on p and is normal. Moreover, for any $(a,b) \models p\otimes p|_M$, $a^{-1}\cdot b \in \mathrm{Stab}_G(p)$, i.e. ${}^{a^{-1}}p$ is consistent with $\mathrm{Stab}_G(p)$. Therefore, we may assume that p itself is consistent with $\mathrm{Stab}_G(p)$. Moreover, if $a \in G(M)$ is such that ${}^a p \in \mathrm{Stab}_G(p)$ then for all $g \models p|_M$, $a\cdot g \models {}^a p$ and hence ${}^a p = {}^{a\cdot g}p = p$ and p is the only type in its orbit that is consistent with $\mathrm{Stab}_G(p)$.
Let $H \leqslant \mathrm{Stab}_G(p)$ be an $(\mathcal{L}(M),\star)$-definable subgroup of bounded index in G. Then there is a coset of $a\cdot H$ such that $p(x) \vdash x \in a\cdot H$. Take any $b \in \mathrm{Stab}_G(p)$, then $p = {}^b p \vdash x \in b\cdot a\cdot H$. It follows that $b\cdot a\cdot H = a\cdot H$, i.e. $b \in a\cdot H\cdot a^{-1}$. It follows that $\mathrm{Stab}_G(p) = a\cdot H\cdot a^{-1} = H$.

The proposition now follows from the fact that $\mathrm{Stab}_G(p)$ is the intersection of relatively $\mathcal{L}(A)$-definable subgroups of finite index by Proposition **(IV.5.15)**. ∎

One can find the first version of the following proposition, in the stable context, in [Hru90, Theorem 2]. The proof given here follows [Hrub, Proposition 3.14] in a slightly different context. Recall the notations of Definitions **(III.6.6)** and **(IV.5.2)**

Proposition IV.5.19:
Let x be a finite tuple of variables and $(G,\cdot)$ an $\mathfrak{A}$-invariant $(\mathcal{L}(M),x)$-definable group — hence an ∞-definable group. Assume there exists $P \subseteq \mathcal{S}(M)$ an $\mathfrak{A}$-invariant locally finite set of types d-generic in G. Then there exists an $\mathcal{L}(M)$-definable $\mathfrak{A}$-invariant group H such that $G \leqslant H$ and G is the intersection of $\mathcal{L}(M)$-definable $\mathfrak{A}$-invariant subgroups of H.

Proof. Replacing P by $\{{}^g p : g \in G(M) \text{ and } p \in P\}$, we may assume that $\mathrm{Stab}_G(P) = G$. Let $(X_\alpha)_{\alpha\in\mathrm{A}}$ be $\mathcal{L}(M)$-definable sets such that $G = \bigcap_\alpha X_\alpha$, m be an $\mathcal{L}(M)$-definable function such that for all $a, b \in G$, $a \cdot b = m(a,b)$ and i an $\mathcal{L}(M)$-definable function such that for all $a \in G$, $a^{-1} = i(a)$. We may assume that $(X_\alpha)_{\alpha\in\mathrm{A}}$ is closed under finite intersections. By compactness there exists α_0 such that x, y and $z \in X_{\alpha_0}$ implies that $m(m(x,y),z)$ and $m(x,m(y,z))$ are both defined and equal, that $m(x,1) = m(1,x) = x$ and that $m(i(x),x)$ and $m(x,i(x))$ are also both defined and equal to 1. We may also assume that for all $\alpha \in \mathrm{A}$ $X_{\alpha_0} \subseteq X_\alpha$.
For all $\alpha \in \mathrm{A}$, let $Y_\alpha = \{a \in X_{\alpha_0} : \text{ for all } p \in P, m(a, X_\alpha) \in p\}$ which is $\mathcal{L}(M)$-definable and $\mathfrak{A}$-invariant.

Claim IV.5.20: $G = \bigcap_{\alpha\in\mathrm{A}} Y_\alpha$.

Proof. Let $a \in G$, then $i(a) = a^{-1} \in G$ too. Fix some $\alpha \in \mathrm{A}$. By compactness, there exists β such that $m(X_\beta^2) \subseteq X_\alpha$. Then we have $i(a) \in X_\beta$ and hence $m(i(a), X_\beta) \subseteq X_\alpha$. For all $x \in X_\beta$ we have $m(i(a),x) \in X_\alpha$ and hence $m(a, m(i(a),x)) \in m(a, X_\alpha)$ but as a, $i(a)$ and $x \in X_{\alpha_0}$, $m(a,m(i(a),x)) = x$ and it follows that $X_\beta \subseteq m(a, X_\alpha)$. As for all $p \in P$, $X_\beta \in p$, we also have $m(a, X_\alpha) \in p$ and hence $a \in Y_\alpha$.
Conversely, let $a \in \bigcap_{i\alpha} Y_\alpha$ and pick any $p \in P$. For all $\alpha \in \mathrm{A}$, $m(a, X_\alpha) \in p$, i.e. $\bigcap_i m(a, X_\alpha) \in p$. Let $b \models p$. It follows that there exists $c \in \bigcap_\alpha X_\alpha = G$ such that $b = m(a,c)$. As a, b and $c \in X_{\alpha_0}$, $a = m(a,1) = m(a, m(c, i(c))) = m(m(a,c), i(c)) = b \cdot c^{-1} \in G$. ♦

Note that if $X_\alpha \cap X_\beta = X_\gamma$, then $Y_\alpha \cap Y_\beta = Y_\gamma$, i.e. $(Y_\alpha)_{\alpha\in\mathrm{A}}$ is also closed under finite intersections. By compactness there exists α_1 such that for all a and $b \in Y_{\alpha_1}$, $m(a,b)$ and $i(a) \in X_{\alpha_0}$. Let $\mathrm{A}_1 := \{\alpha \in \mathrm{A} : Y_\alpha \subseteq Y_{\alpha_1}\}$. We also have $G = \bigcap_{\alpha\in\mathrm{A}_1} Y_\alpha$.
Let $S_\alpha = \{a \in X_{\alpha_0} : m(a, Y_\alpha) \subseteq Y_\alpha\}$.

Claim IV.5.21: *For all $\alpha \in \mathrm{A}_1$, (S_α, m) is a semi-group.*

Proof. It suffices to prove that if a and $b \in S_\alpha$, then $m(a,b) \in S_\alpha$. For all $x \in Y_\alpha$, $m(a,x) \in Y_\alpha$ and $m(b,x) \in Y_\alpha$, hence $m(m(a,b),x) = m(a, m(b,x)) \in Y_\alpha$. Furthermore, as a and $b \in Y_\alpha \subseteq Y_{\alpha_1}$, $m(a,b) \in X_{\alpha_0}$. ♦

Claim IV.5.22: *For all $\alpha \in \mathrm{A}_1$, $G \subseteq S_\alpha \subseteq Y_\alpha$.*

Proof. Let $a \in G$, then, obviously $a \in X_{\alpha_0}$. Let $b \in Y_\alpha$ and $p \in P$, then ${}^{a^{-1}}p \in P$ and hence $m(b, X_\alpha) \in {}^{a^{-1}}p$. It follows that $m(m(a,b), X_\alpha) = m(a, m(b, X_\alpha)) \in p$, so $m(a,b) \in Y_\alpha$ and $a \in S_\alpha$. Let now $a \in S_\alpha$, then $a \in X_{\alpha_0}$ and as $1 \in Y_\alpha$, $a = m(a,1) \in Y_\alpha$. ♦

Let $G_\alpha := \{a \in S_\alpha : i(a) \in S_\alpha\}$. Then for all $\alpha \in \mathrm{A}_1$, G_α is a group and $G \subseteq G_\alpha \subseteq S_\alpha \subseteq Y_\alpha$. Therefore we also have $G = \bigcap_{\alpha \in \mathrm{A}_1} G_\alpha$. ■

Remark IV.5.23:
There is a classic counter-example to Proposition **(IV.5.19)** when the group does not have a d-generic. Let M be an $\aleph_0$-saturated real closed field. The group $\mathcal{I}$ of infinitesimal elements $\{x \in M : \text{ for all} n \in \mathbb{N}, -n < x < n\}$ is an ∞-definable subgroup of the additive group $\mathbb{G}_a(M)$ but there is no proper definable subgroup of $\mathbb{G}_a$ containing $\mathcal{I}$.

Proposition IV.5.24:
Let $(G, \cdot)$ be an $(\mathcal{L}(M), \star)$-definable $\mathfrak{A}$-invariant group. Assume there exists $P \subseteq \mathcal{S}(M)$ an $\mathfrak{A}$-invariant locally finite set of types d-generic in G. Then there exists a projective system of $\mathcal{L}(M)$-definable $\mathfrak{A}$-invariant groups $(H_\alpha)_{\alpha \in \mathrm{A}}$ and an $(\mathcal{L}(M), \star)$-definable $\mathfrak{A}$-invariant group isomorphism $f : G \to H := \varprojlim H_\alpha$.

This proof is adapted from the proof of [Hru90, Proposition 3.4] in the stable context. Recall that if p and $q \in \mathcal{S}(M)$ are two $\mathrm{Aut}(M/C)$-invariant types for some $C \subseteq M$, for all $D \subseteq M$ we define $p \otimes q|_{CD}$ to be the type of tuples (a,b) such that $a \models p|_{CD}$ and $b \models p|_{CDa}$. Note that if p and q are $\mathcal{L}(C)$-definable then $p \otimes q$ is also $\mathcal{L}(C)$-definable and we can take $d_{p\otimes q}(x,y)\, \varphi(x,y,s) = d_p x\, (d_q y\, \varphi(x,y,s))$.

Proof. As in the previous proof, we may assume that $\mathrm{Stab}_G(P) = G$. Let $x = (x_i)_{i\in I}$ be a tuple of variables sorted as G and $J := \{j \subseteq I : |j| < \infty\}$. For all $j \in J$ let $c_1 E_j c_2$ hold if for all p_1 and $p_2 \in P$,

$$p_1 \otimes p_2(x,y) \vdash (x \cdot c_1 \cdot y)_j = (x \cdot c_2 \cdot y)_j \wedge (y \cdot c_1^{-1} \cdot x)_j = (y \cdot c_2^{-1} \cdot x)_j$$

where for any J-tuple a, a_j is the tuple of elements from c with indices in j. Let $\varphi(y; s, t, x)$ be the $\mathcal{L}(M)$-formula $\bigwedge_{i\in j} (x\cdot s\cdot y)_i = (x\cdot t\cdot y)_i \wedge (x\cdot s^{-1}\cdot y)_i = (x\cdot t^{-1}\cdot y)_i$. As P is locally finite, $\{d_p\varphi : p \in P\}$ is finite — up to equivalence. Let $(\psi_i(x; s,t))_{0\leqslant i<k}$ be the $\mathcal{L}(M)$-formulas in this set. For the same reason $\{d_p\psi_i : p \in P \text{ and } 0 \leqslant i < k\} = \{\theta_j(s,t) : 0 \leqslant j < l\}$. It is now easy to see that for all s and $t \in G$, sE_jt is defined by $\bigwedge_{0\leqslant j<l} \theta_j(s,t)$, i.e. E_j is $\mathcal{L}(M)$-definable. As P and G are $\mathfrak{A}$-invariant, it is quite easy to see that E_j is also $\mathfrak{A}$-invariant.

Claim IV.5.25: *For all $c_1, c_2 \in G$, if $c_1 E_j c_2$ then $1E_j(c_1 \cdot c_2^{-1})$ and $1E_j(c_1^{-1} \cdot c_2)$.*

Proof. Let p_1 and $p_2 \in P$ and $(a,b) \models p_1 \otimes p_2|_{Cc_1c_2}$. Then $c_2 \cdot b \models {}^{c_2}p_2|_{Cc_1c_2a}$ and $c_2^{-1} \cdot b \models {}^{c_2^{-1}\cdot g_2}p|_{Cc_1c_2a}$. Hence $(a \cdot c_1 \cdot (c_2^{-1} \cdot b))_j = (a \cdot c_2 \cdot (c_2^{-1} \cdot b))_j = (a \cdot 1 \cdot b)_j$ and $(a \cdot c_1^{-1} \cdot (c_2 \cdot b))_j = (a \cdot c_2^{-1} \cdot (c_2 \cdot b))_j = (a \cdot 1 \cdot b)_j$. As $(c_1 \cdot c_2^{-1})^{-1} = c_2 \cdot c_1^{-1}$ and $(c_1^{-1} \cdot c_2)^{-1} = c_2^{-1} \cdot c_1$ the symmetric argument allows us to conclude. ♦

It follows immediately that the E_j-class of 1 is a normal subgroup G_j of G and that the E_j-classes are the cosets of G_j. Let H_j be the ∞-definable $\mathfrak{A}$-invariant set $G/E_j = G/G_j$ and $\pi_j : G \to H_j$ be the canonical projection — which is an $\mathfrak{A}$-invariant definable map.

The graph of the group law on H_j is the image under π_j of the graph of G's group law. It is therefore ∞-definable $\mathfrak{A}$-invariant and hence, by Proposition **(IV.5.3)** it is in fact a definable function. Moreover if $j_1 \subseteq j_2$, $G_{j_2} \subseteq G_{j_1}$ and hence there is a canonical morphism $\pi_{j_1,j_2} : H_{j_2} \to H_{j_1}$ which is definable and $\mathfrak{A}$-invariant for the same reason as the group laws. Let $H := \varprojlim H_j$.

Claim IV.5.26: *For all c_1 and $c_2 \in G$, if $\pi_j(c_1) = \pi_j(c_2)$ for all $j \in J$, then $c_1 = c_2$.*

Proof. Pick any $p \in P$ and let $(a, b) \vDash p^{\otimes 2}|_{Cc_1c_2}$. For all $j \in J$, $(a \cdot c_1 \cdot b)_j = (a \cdot c_2 \cdot b)_j$ — $a \cdot c_1 \cdot b = a \cdot c_2 \cdot b$ — and hence $c_1 = c_2$. ♦

Claim IV.5.27: *For all $a \in H$, there exists $c \in G$ such that for all $j \in J$, $\pi_j(c) = a_j$.*

Proof. For all $(j_l)_{0\leqslant l<k} \in J$, let $j := \bigcup_l j_l \in J$. Then there exists $c \in G$ such that $\pi_j(c) = a_j$. It follows that for all $l < k$, $\pi_{j_l}(c) = \pi_{j_l,j}(\pi_j(c)) = a_{j_l}$. We now conclude by compactness. ♦

It follows that $(\pi_j)_{j\in J}$ is an isomorphism between G and H. As H is only a prolimit of ∞-definable groups, we are not quite done yet. It is obvious that $(\pi_j)_\star P := \{(\pi_j)_\star p : p \in P\}$ is an $\mathfrak{A}$-invariant locally finite set of types d-generics in H_j and by Proposition **(IV.5.19)**, for all $j \in J$, $H_j = \bigcap_{L_j} H_{j,l}$ for some $\mathfrak{A}$-invariant $\mathcal{L}(M)$-definable subgroups $H_{j,l}$ of a common $\mathfrak{A}$-invariant $\mathcal{L}(M)$-definable group K_j. As usual we may assume that the $H_{j,l}$ are closed under intersection — for fixed j — and we order L_j by $l_1 \subseteq_j l_2$ if $H_{j,l_2} \subseteq H_{j,l_1}$. Let $L := \{(j, l) : j \in L_j\}$. For all $j \in J$ and $l_1 \leqslant_j l_2 \in L_j$, let $\rho_{(j,l_1),(j,l_2)}$ be the inclusion morphism $H_{j,l_2} \to H_{j,l_1}$ and for all $j_1 \subset j_2 \in J$, let $\rho_{(j_1,l_1),(j_2,l_2)}$ be $\pi_{j_1,j_2}|_{H_{j_2,l_2}}$ whenever $\pi_{j_1,j_2}(H_{j_2,l_2}) \subseteq H_{j_1,l_1}$. By compactness, for all $l_1 \in L_{j_1}$, there exists $l_2 \in L_{j_2}$ such that $\pi_{j_1,j_2}(H_{j_2,l_2}) \subseteq H_{j_1,l_1}$ and it follows that there is an $(\mathcal{L}(M), \star)$-definable $\mathfrak{A}$-invariant isomorphism from H onto $\varprojlim H_{j,l}$. ■

Let us now consider group chunks, a tool central to the construction of groups in model theory. The initial idea is due to Weil [Wei55] in the setting of algebraic groups. It was then transposed to a more general topological setting in [Dri90] and to the stable setting in [Hru90]. The notion of group chunk on a $\star$-type appeared when trying to prove the aforementioned result that groups definable in DCF_0 are subgroups of algebraic groups (see for example [Pil97]).

Recall that if f is an $\mathcal{L}(M)$-definable function and $p \in \mathcal{S}(M)$, we define $f_\star p$ to be the $\mathcal{L}$-type over M of any $f(a)$ where $a \vDash p$.

The following definition is a generalization of the standard notion: if one takes $P = \{p\}$, one gets a definition equivalent to the one given in [Hrub, p. 3.10]. Note that in the definition below there is an explicit reference to both the left inverse and the right inverse, which does not appear as explicitly in [Hru90, Theorem 1].

Definition IV.5.28 ($\star$-definable group chunk)**:**
Let $x = (x_i)_{i\in I}$ be a tuple of variables and $P \subseteq \mathcal{S}^{\mathcal{L}}_x(M)$ be a locally finite set of $\mathcal{L}(M)$-definable types. An $(\mathcal{L}, x)$-definable group chunk over P is a tuple of $(\mathcal{L}(M), x^2, x)$-definable functions (F, G, H) such that for all p_1, p_2 and $p_3 \in P$:

(i) *For all $a \vDash p_1|_M$, $(F_a)_\star p_2 \in P$ where $F_a(x) := F(a, x)$;*

(ii) *For all* $a \models p_1|_M$*, there exists* $p \in P$ *such that* $(F_a)_\star p = p_2$*;*

(iii) $p_1 \otimes p_2 \vdash G(x, F(x,y)) = y \wedge H(F(x,y),y) = x$*;*

(iv) $p_1 \otimes p_2 \otimes p_3(x,y,z) \vdash F(x, F(y,z)) = F(F(x,y),z)$*.*

Example IV.5.29:
Let G be an $(\mathcal{L}(M),\star)$-definable group with a d-generic p, then the multiplication of G induces a natural $(\mathcal{L}(M),\star)$-definable group chunk over $\{{}^g p : g \in G\}$.

Let us now show the converse: any $\star$-definable group chunk is isomorphic to the group chunk of a $\star$-definable group. The following propositions are improvements to [Hrub, Paragraph 3.11]. The first improvement is that we consider $\star$-definable group chunks and not just definable group chunks (but that changes nothing to the proof) the other change is that we consider a group chunk over a locally finite set of types and not a singleton; that does not change the essence of the proof either but it does make it trickier.

Proposition IV.5.30:
Let $P \subseteq \mathcal{S}^{\mathcal{L}}_x(M)$ *be a locally finite,* $\mathfrak{A}$*-invariant set of* $\mathcal{L}(M)$*-definable types and* (F,G,H) *be an* $\mathfrak{A}$*-invariant* $(\mathcal{L}(M),x)$*-definable group chunk over* P*. Then there is some* $\mathfrak{A}$*-invariant* $(\mathcal{L}(M),\star)$*-definable one to one function* f *and some* $\mathfrak{A}$*-invariant* $(\mathcal{L}(M),\star)$*-definable group* $(G,\cdot)$ *such that* $f_\star P := \{f_\star p : p \in P\}$ *is an* $\mathfrak{A}$*-invariant locally finite set of types* d*-generic in* $G = \mathrm{Stab}_G(f_\star P)$ *and for all* p_1 *and* $p_2 \in P$*,* $p_1 \otimes p_2(x,y) \vdash f(G(x,y)) = f(x)^{-1} \cdot f(y)$*.*

Before getting into this technical proof, let me just say that the idea is the same as always: the group we construct is the group generated by the germs g_a of multiplication by realizations of types in P and we show that this group consists of the elements of the form $g_a \cdot g_b^{-1}$.

Proof. Let $P_1 := \{q \in \mathcal{S}^{\mathcal{L}}_x(M) : \text{for all } \mathcal{L}\text{-formulas } \varphi(x;s),\ q|_\varphi \in \{p|_\varphi : p \in P\}\}$. Then P_1 is also a locally finite $\mathfrak{A}$-invariant set of $\mathcal{L}(M)$-definable types. Moreover, for all q_1, q_2 and $q_3 \in P_1$, $a \models q_1|_M$, $(F_a)_\star q_2 = q_3$ if and only if for all $\mathcal{L}$-formulas φ,

$$M \models d_{q_1}x\, \forall s\, (d_{q_2}y\, \varphi(F(a,y);s) \iff d_{q_3}z\, \varphi(z;s)).$$

But by definition of P_1, there exists p_1, p_2 and p_3 such that this formula is equivalent to $M \models d_{p_1}x\, \forall s\, (d_{p_2}y\, \varphi(F(x,y);s) \iff d_{p_3}z\, \varphi(z;s))$. It follows that **IV.5.28.(i)** and **IV.5.28.(ii)** also hold for P_1. Similarly **IV.5.28.(iii)** and **IV.5.28.(iv)** also hold for P_1. In particular, we may assume that $P_1 = P$.

Claim IV.5.31: *For all* p_1*,* $p_2 \in P$ *and* $a \models p_1|_M$*,* $(G_a)_\star p_2 \in P$ *where* $G_a(x) := G(a,x)$*.*

Proof. By **IV.5.28.(ii)**, for all $a \models p_1|_M$, there exists $p_3 \in P$ such that $(F_a)_\star p_3 = p_2$. Let $C \subseteq M$ be such that all types in P are defined over C and G is $(\mathcal{L}(C),\star)$-definable, $a \models p_1|_C$ and $b \models p_3|_{Ca}$, then $c := F(a,b) \models p_2|_{\mathrm{acl}(A)Ca}$ and by **IV.5.28.(iii)**, $G(a,c) = G(a,F(a,b)) = b \models p_3|_{Ca}$. ♦

Claim IV.5.32: *For all* p_1 *and* $p_2 \in P$*,* $p_1 \otimes p_2(x,y) \vdash F(x, G(x,y)) = y$*.*

Proof. Let $a \vDash p_1|_M$. By **IV.5.28.(ii)**, there exists $p_3 \in P$ such that $(F_a)_\star p_3 = p_2$. Let $b \vDash p_3|_{Ma}$, then $c := F(a,b) \vDash p_2|_{Ma}$ and $F(a, G(a,c)) = F(a, G(a, F(a,b))) = F(a,b) = c$. ♦

The goal now is to find an $(\mathcal{L}(M), \star)$-definable set S that contains all realizations of all $p \in P$ but such that it is not too big so that all its points still have similar properties with respect to F as the realizations of the types in P.
For all $\mathcal{L}$-formula φ, let $\theta_\varphi(x) := \bigwedge_{p\in P} \bigvee_{q\in P} \forall s\, (d_q y\, \varphi(y;s) \iff d_p y\, \varphi(F(x,y);s))$. As P is locally finite the disjunction and conjunction are finite, and this is an $\mathcal{L}(M)$-formula. Because P is $\mathfrak{A}$-invariant, θ_φ is also $\mathfrak{A}$-invariant. Let $\chi_\varphi(x) := \bigwedge_{p\in P} \bigvee_{q\in P} \forall s\, (d_q y\, \varphi(y;s) \iff d_p y\, \varphi(G(x,y);s))$. Similarly, there are $\mathfrak{A}$-invariant $\mathcal{L}(M)$-formulas $\alpha(x)$ and $\zeta(x)$ equivalent respectively to

$$\bigwedge_{p\in P} \bigwedge_{q\in P} d_q y\, d_p z\, F(x, F(y,z)) = F(F(x,y), z)$$

and to

$$\bigwedge_{p\in P} d_p y\, G(x, F(x,y)) = y = F(x, G(x,y)) \wedge H(F(x,y), y) = x.$$

Let

$$S(x) := \{\alpha(x), \zeta(x)\} \cup \{\theta_\varphi(x), \chi_\varphi(x) : \varphi \text{ an } \mathcal{L}\text{-formula}\}.$$

Claim IV.5.33: *For all $p \in P$, $S \subseteq p$.*

Proof. Let $a \vDash p|_M$ and $q \in P$. For all $\mathcal{L}$-formulas φ and all $m \in M$, $d_{(F_a)_\star q} y\, \varphi(y;m) \iff d_q y\, \varphi(F(a,y);m)$ and $d_{(G_a)_\star q} y\, \varphi(y;m) \iff d_q y\, \varphi(G(a,y);m)$. Moreover, by **IV.5.28.(i)** and Claim **(IV.5.31)**, we have $(F_a)_\star q$ and $(G_a)_\star q \in P$. Thus, we have proved that $M \vDash \theta_\varphi(a) \wedge \chi_\varphi(a)$.
For all q_1 and $q_2 \in P$, by **IV.5.28.(iv)**, $q_1 \otimes q_2(y,z) \vdash F(a, F(y,z)) = F(F(a,y),z)$ we have $M \vDash d_{q_1} y\, d_{q_2} z\, F(a, F(y,z)) = F(F(a,y),z)$. The rest of the proof follows by **IV.5.28.(iii)**, Claim **(IV.5.32)** and similar considerations. ♦

Claim IV.5.34: *For all $a \vDash S$ and $p \in P$, $(F_a)_\star p$ and $(G_a)_\star p \in P$.*

Proof. Let φ be an $\mathcal{L}$-formula, as $M \vDash \theta_\varphi(a) \wedge \chi_\varphi(a)$, there exists q_1 and $q_2 \in P$ such that $d_{q_1} y\, \varphi(y;s) \iff d_p y\, \varphi(F(a,y);s)$ — i.e. $(F_a)_\star p|_\varphi = q_1|_\varphi$ — and $d_{q_2} y\, \varphi(y;s) \iff d_p y\, \varphi(G(a,y);s)$ — $(G_a)_\star p|_\varphi = q_2|_\varphi$. It follows that $(F_a)_\star p$ and $(G_a)_\star p \in P_1 = P$. ♦

Claim IV.5.35: *For all types p_1 and $p_2 \in P$, $a \vDash S$ and $(b,c) \vDash p_1 \otimes p_2|_{Ma}$, $G(a, F(b,c)) = F(G(a,b),c)$.*

Proof. Let $d := G(a,b)$, then $F(a,d) = F(a, G(a,b)) = b$ and $d \vDash p_3|_{Ma}$ and $F(d,c) \vDash p_4|_{Mab}$ for some p_3 and $p_4 \in P$ and $c \vDash p_2|_{Mad}$. Moreover, $G(a, F(b,c)) = G(a, F(F(a,d),c)) = G(a, F(a, F(d,c))) = F(d,c) = F(G(a,b),c)$. ♦

Claim IV.5.36: *For all a, b, c and $d \vDash S$, there exists e and $l \vDash S$ such that for all $p \in P$,*

$$p(x) \vdash F(a, G(b, F(c, G(d,x)))) = F(e, G(l,x)).$$

Proof. Pick any $p_1 \in P$ and let $e_1 \models p_1|_{Mabcd}$. For all $p \in P$, $p(x) \vdash G(d, F(e_1, x)) = F(G(d, e_1), x)$ and $e_2 := G(d, e_1) \models p_2|_{Mabcd}$ for some $p_2 \in P$. We also have p_3, p_4 and $p_5 \in P$ such that $e_3 := F(c, e_2) \models p_3|_{Mabcd}$, $e_4 := G(b, e_3) \models p_4|_{Mabcd}$ and $e_5 := F(a, e_4) \models p_5|_{Mabcd}$. Moreover, $p(x) \vdash F(c, F(e_2, x)) = F(F(c, e_2), x)$, $p(x) \vdash G(b, F(e_3, x)) = F(G(b, e_3), x)$ and $p(x) \vdash F(a, F(e_4, x)) = F(F(a, e_4), x)$. It follows that for all $p \in P$ and $x \models p|_{Mabcde_1}$, as $G(e_1, x) \models q|_{Mabcde_1}$ for some $q \in P$,

$$\begin{aligned} F(a, G(b, F(c, G(d, x)))) &= F(a, G(b, F(c, G(d, F(e_1, G(e_1, x)))))) \\ &= F(F(a, G(b, F(c, G(d, e_1)))), G(e_1, x)) \\ &= F(e_5, G(e_1, x)) \end{aligned}$$

and by Claim **(IV.5.33)**, we do have e_1 and $e_5 \in S$. ♦

For a and b, let $R_{a,b}(x) = F(a, G(b, x))$. For all $i \in I$, the i-th component of $R_{a,b}$ (denoted $R_{a,b,i}$) is an $\mathfrak{A}$-invariant $\mathcal{L}(M)$-definable function. Let $(a, b)E_i(c, d)$ if and only if $\bigwedge_p d_p x\, R_{a,b}(x) = R_{c,d}(x)$. As P is locally finite, E_i is $\mathcal{L}(M)$-definable and it is clearly $\mathfrak{A}$-invariant. Let $h_i(a, b)$ be an $\mathfrak{A}$-invariant $\mathcal{L}(M)$-definable function such that $h_i(a, b) = h_i(c, d)$ if and only $(a, b)E_i(c, d)$. Then $h(a, b) := (h_i(a, b))_{i\in I}$ is an $\mathfrak{A}$-invariant $(\mathcal{L}(M), \star)$-definable map. Let $G := h(S \times S)$ and $m(h(a, b), h(c, d)) := h(e, l)$ for some e and l are as in Claim **(IV.5.36)**. Then m is well defined. Indeed, for all $i \in \{1, 2\}$, let a_i, b_i, c_i, d_i, e_i and $l_i \models S$ be such that $h(a_1, b_1) = h(a_2, b_2)$ and $h(c_1, d_1) = h(c_2, d_2)$ and for all $p \in P$,

$$p(x) \vdash F(a_i, G(b_i, F(c_i, G(d_i, x)))) = F(e_i, G(l_i, x)).$$

Let $x \models p|_{M(a_ib_ic_id_ie_il_i)_{i\in\{1,2\}}}$ for some $p \in P$, then $F(c_1, G(d_1, x)) \models q|_{Ma_1b_1}$, for some $q \in P$. Hence

$$\begin{aligned} F(e_1, G(l_1, x)) &= F(a_1, G(b_1, F(c_1, G(d_1, x)))) \\ &= F(a_2, G(b_2, F(c_1, G(d_1, x)))) && \text{as } h(a_1, b_1) = h(a_2, b_2) \\ &= F(a_2, G(b_2, F(c_2, G(d_2, x)))) && \text{as } h(c_1, d_1) = h(c_2, d_2) \\ &= F(e_2, G(l_2, x)), \end{aligned}$$

i.e. $h(e_1, l_1) = h(e_2, l_2)$. Moreover, as G and the graph of m are the image under an $\mathfrak{A}$-invariant $(\mathcal{L}(M), \star)$-definable map of an $\mathfrak{A}$-invariant $(\mathcal{L}(M), \star)$-definable set, they are both $\mathfrak{A}$-invariant $(\mathcal{L}(M), \star)$-definable.

Claim IV.5.37: *(G, m) is a group.*

Proof. The function m is obviously associative, as germ composition is. Let us show that for all $a \in S$, $h(a, a)$ is the unit and for all $b \in S$, $h(a, b)$ is the inverse of $h(b, a)$. Let $c \in S$, $p \in P$ and $x \models p|_{Mabc}$, $F(b, G(c, x)) \models q_1|_{Mabc}$ for some $q_1 \in P$ and

$$F(a, G(a, F(b, G(c, x)))) = F(b, G(c, x)) = F(b, G(c, F(a, G(a, x))))$$

and as $G(a, x) \models q_2|_{Mab}$ for some $q_2 \in P$,

$$F(a, G(b, F(b, G(a, x)))) = F(a, G(a, x)).$$

This concludes the proof of the claim. ♦

Let $a \in S$, $q \in P$, $b \vDash q|_{Ma}$ and $c := F(a,b)$, then b and $c \in S$ and for all $p \in P$ and $x \vDash p|_{Mab}$, $F(c, G(b,x)) = F(F(a,b), G(b,x)) = F(a, F(b, G(b,x))) = F(a,x)$. Let $f(a) = h(b,c)$ where b and c are as above. Then f is well defined for the same reasons m was and it is $\mathfrak{A}$-invariant $(\mathcal{L}(M),\star)$-definable. Let a and $b \in S$ be such that $f(a) = f(b)$, then for any $p \in P$ and $x \vDash p|_{Mab}$, $a = H(F(a,x),x) = H(F(b,x),x) = b$, i.e. f is one to one.

Claim IV.5.38: *For all $p \in P$ and a, $b \in S$, there exists $q \in P$ such that ${}^{h(a,b)}(f_\star p) = f_\star q$.*

Proof. Let $q := (F_a)_\star(G_b)_\star p \in P$. For all $C \subseteq M$ such that all the types in P are $\mathcal{L}(C)$-definable and G and f are $(\mathcal{L}(C),\star)$-definable and all $c \vDash p|_{Cab}$, $F(a, G(b, F(c,x))) = F(a, F(G(b,c),x)) = F(F(a, G(b,c)),x) = F(e,x)$ where $e := F(a, G(b,c)) \vDash q|_{Cab}$. It follows that $m(h(a,b), f(c)) = f(e) \vDash f_\star q|_{Cab}$. ♦

As $f_\star q$ is $\mathcal{L}(M)$-definable for all $q \in P$, we have just proved that for all $p \in P$, $f_\star p$ is d-generic in G and that $G = \mathrm{Stab}_G(f_\star P)$. Finally, let p_1, $p_2 \in P$ and $(a,b) \vDash p_1 \otimes p_2|_M$. For all $p \in P$ and $x \vDash p|_{Mab}$, $F(b,x) = F(a, G(a, F(b,x))) = F(a, F(G(a,b)),x)$, i.e. $f(b) = m(f(a), f(G(a,b)))$. ∎

Remark IV.5.39:
The conclusion that for all p_1 and $p_2 \in P$, $p_1 \otimes p_2(x,y) \vdash f(G(x,y)) = f(x)^{-1} \cdot f(y)$ might be a little surprising, but let us show that it implies what one would think to be the more reasonable conclusion: for all p_1 and $p_2 \in P$, $p_1 \otimes p_2(x,y) \vdash f(F(x,y)) = f(x) \cdot f(y)$. Indeed, let $a \vDash p_1|_M$ and $p_3 \in P$ be such that $(G_a)_\star p_3 = p_2$. Let $b \vDash p_3|_{Ma}$ and $c := G(a,b) \vDash p_2|_{Ma}$, then $F(a,c) = F(a, G(a,b)) = b$ and $f_0(c) = f_0(G(a,b)) = f_0(a)^{-1} \cdot f_0(b) = f_0(a)^{-1} \cdot f_0(F(a,c))$.

In fact, there is an equivalence of categories between groups with d-generics and group chunks.

Proposition IV.5.40:
Let $(G,\cdot)$ and $(H,\cdot)$ be two $\mathfrak{A}$-invariant $(\mathcal{L}(M),\star)$-definable groups, $P \subseteq \mathcal{S}^{\mathcal{L}}_x(M)$ be an $\mathfrak{A}$-invariant locally finite set of types d-generic in G and f_0 be an $\mathfrak{A}$-invariant $(\mathcal{L}(M),\star)$-definable function. If for all p_1, $(f_0)_\star p_1(y) \vdash y \in H$ and for all $p_2 \in P$, $p_1 \otimes p_2(x,y) \vdash f_0(x^{-1} \cdot y) = f(x)^{-1} \cdot f(y)$, then there exists a unique $\mathfrak{A}$-invariant $(\mathcal{L}(M),\star)$-definable group morphism $f : \mathrm{Stab}_G(P) \to H$ such that for all $p \in P$, $p(x) \vdash f(x) = f_0(x)$.
Moreover, if G and H are both $\mathcal{L}(M)$-definable, f can be extended to some $\mathfrak{A}$-invariant $\mathcal{L}(M)$-definable subgroup of G with finite index — of the form $\mathrm{Stab}_G(P|_\Delta)$ for some finite set of $\mathcal{L}(M)$-formulas Δ, where $P|_\Delta := \{p|_\Delta : p \in P\}$.
Finally, if f_0 is one to one, so is f.

Proof. Let $g \in \mathrm{Stab}_G(P)$ and $p_1 \in P$. Then $p_2 := {}^g p_1 \in P$ and $g = (g \cdot a) \cdot a^{-1}$ where $g \cdot a \vDash p_2|_M$ and $a \vDash p_1$. Therefore, we must have $f(g) = f(g \cdot a) \cdot f(a^{-1}) = f_0(g \cdot a) \cdot f_0(a)^{-1}$. We have just proved uniqueness, but let us now prove that this indeed defines a group morphism.

Claim IV.5.41: *For $i \in \{1,2,3,4\}$, let $p_i \in P$ and $a_i \vDash p_i|_M$ such that $a_1 \cdot a_2^{-1} = a_3 \cdot a_4^{-1}$, then $f_0(a_1) \cdot f_0(a_2)^{-1} = f_0(a_3) \cdot f_0(a_4)^{-1}$.*

Proof. Pick any $p \in P$ and let $e \vDash p|_{M(a_i)_{0<i\leqslant 4}}$. By Remark **(IV.5.39)** and the fact that $a_2 \cdot e \vDash {}^{a_2}p|_{Ma_1a_2}$ and ${}^{a_2}p \in P$, $f_0(a_1 \cdot a_2^{-1} \cdot e) = f_0(a_1) \cdot f_0(a_2^{-1} \cdot e) = f_0(a_1) \cdot f_0(a_2)^{-1} \cdot f_0(e)$. Similarly, $f_0(a_1 \cdot a_2^{-1} \cdot e) = f_0(a_3 \cdot a_4^{-1} \cdot e) = f_0(a_3) \cdot f_0(a_4)^{-1} \cdot f_0(e)$. ♦

It follows that f defined as above is well defined. Note that for all $a \in \mathrm{Stab}_G(p)$, $f(a)$ is defined by $d_p x\, f_0(a \cdot x) = f(a) \cdot f_0(x)$ for any $p \in P$ and hence f is $\mathfrak{A}$-invariant and $(\mathcal{L}(M), \star)$-definable. Moreover, for all a_1 and $a_2 \in \mathrm{Stab}_G(P)$, $p \in P$ and $c \vDash p|_{Ma_1a_2}$, $a_1 \cdot c \vDash {}^{a_1}p|_{Ma_1a_2}$, ${}^{a_1}p \in P$ and $f_0(a_1 \cdot a_2 \cdot c) = f(a_1) \cdot f_0(a_2 \cdot c) = f(a_1) \cdot f(a_2) \cdot f_0(c)$ — i.e. $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$ — and if $q \in P$ and $(a, c) \vDash q \otimes p$, then $f_0(a \cdot c) = f_0(a) \cdot f_0(c)$ — i.e. $f(a) = f_0(a)$.
If f_0 is one to one and $a, b \in \mathrm{Stab}_G(P)$ are such that $f(a) = f(b)$, then for any $p \in P$, $p(x) \vdash f_0(a \cdot x) = f(a) \cdot f_0(x) = f(b) \cdot f_0(x) = f_0(b \cdot x)$. As f_0 is one to one, for all $c \vDash p|_{Mab}$, $a \cdot c = b \cdot c$ and hence $a = b$.
Now assume that G and H are $\mathcal{L}(M)$-definable and let $A \subseteq M$ be a set of codes for these groups (and their multiplicative law). Let $\varphi(x) = \bigwedge_{p \in P} d_p y\, (f_0(x \cdot y) = f_0(x) \cdot f_0(y) \wedge f_0(x^{-1} \cdot y) = f_0(x)^{-1} \cdot f_0(y))$ which is indeed an $\mathcal{L}(M)$-formula as P is locally finite. Let us rewrite φ as $\varphi(x; u)$ where φ is now an $\mathcal{L}$-formula. Similarly let $\theta(x; s, t, u)$ be an $\mathcal{L}$-formula of which $f_0(s \cdot x) = t \cdot f_0(x)$ is an instance, $\xi(x; u)$ be an $\mathcal{L}$-formula of which $x \in G$ is an instance and $\zeta(x; u)$ be an $\mathcal{L}$-formula of which $f_0(x) \in H$ is an instance. Let $\Delta(x; s, t, u) := \{\varphi(x; u), \theta(x; s, t, u), \xi(x; u), \zeta(x; u)$ and $\Theta(x; s, t, u, v)$ be a finite set of $\mathcal{L}(A)$-formulas closed under left action of G as in Lemma **(IV.5.5)**. Note that because G and H are $\mathfrak{A}$-invariant, A is fixed point-wise by $\mathfrak{A}$-invariant and hence so is Θ.
In the proof, we can replace P by $P|_\Theta$ which is still $\mathfrak{A}$-invariant. It follows that we can replace $\mathrm{Stab}_G(P)$ by $\mathrm{Stab}_G(P|_\Theta)$ — which, by Proposition **(IV.5.15)** is $\mathfrak{A}$-invariant, $\mathcal{L}(M)$-definable and has finite index in G. ∎

Let $\mathcal{L} \subseteq \widetilde{\mathcal{L}}$ be two languages, $\mathcal{R}$ be the set of $\mathcal{L}$-sorts, T be an $\mathcal{L}$-theory which is NIP and eliminates imaginaries, $\widetilde{T} \supseteq T$ be an $\widetilde{\mathcal{L}}$-theory, $\widetilde{N} \vDash \widetilde{T}$ be saturated and homogeneous enough and $N := \widetilde{N}|_{\mathcal{L}}$.

Theorem IV.5.42:

Assume there exists $\widetilde{M} \vDash \widetilde{T}$ with $M|_{\mathcal{L}}$ uniformly stably embedded in every elementary extension. Let $\widetilde{A} \subseteq \widetilde{N}$ be such that $\mathcal{R}(\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})) = \mathcal{R}(\widetilde{A})$, $(G, \cdot)$ be an $\widetilde{\mathcal{L}}(\widetilde{A})$-definable group with a d-generic type p over $\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})$. Assume that there exists an $(\widetilde{\mathcal{L}}(\widetilde{A}), \star)$-definable one to one function f and $(\mathcal{L}(\mathcal{R}(\widetilde{A})), \star)$-definable functions m and i such that for all g_1 and $g_2 \in G$, $f(g_1 \cdot g_2) = m(f(g_1), f(g_2))$ and $f(g_1^{-1}) = i(f(g_1))$. Then there exists an $\mathcal{L}(\mathcal{R}(\widetilde{A}))$-definable group H and an $\widetilde{\mathcal{L}}(\widetilde{A})$-definable one to one group morphism $h : G \to H$.

Proof. Let $A := \mathcal{R}(\widetilde{A})$, $\mathfrak{A} = \mathrm{Aut}_{\widetilde{\mathcal{L}}}(M/\widetilde{A})$, $P := \{\sigma({}^g p) : g \in G(M)$ and $\sigma \in \mathfrak{A}\} = \{{}^g(\sigma(p)) : g \in G(M)$ and $\sigma \in \mathfrak{A}\}$ and $Q := \{(f_\star p)|_{\mathcal{L}} : p \in P\}$. Then $\mathrm{Stab}_G(P) = G$ and $Q \subseteq \mathcal{S}^{\mathcal{L}}(N)$ is an $\mathfrak{A}$-invariant locally finite set of $\widetilde{\mathcal{L}}(\widetilde{N})$-definable $\mathcal{L}$-types. By Corollary **(III.1.5)**, each $q \in Q_0$ is in fact $\mathcal{L}(N)$-definable. Let $m_1(x, y) := m(i(x), y)$ and $m_2(x, y) := m(x, i(y))$.

Claim IV.5.43: *The tuple (m, m_1, m_2) is an $\mathfrak{A}$-invariant $(\mathcal{L}(M), \star)$-definable group chunk over Q.*

Proof. Let q_1 and $q_2 \in Q$. There exists p_1 and $p_2 \in P$ such that $q_i = (f_\star p_i)|_{\mathcal{L}}$, for $i \in \{1,2\}$. Let $a \models p_1|_{\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})}$, $\widetilde{C} \subseteq \widetilde{N}$ and $c \models p_2|_{\mathrm{acl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})\widetilde{C}a}$. Then $m(f(a), f(c)) = f(a \cdot c)$ and hence $(m_{f(a)})_\star f_\star p_2 = f_\star {}^a p$. It follows that $(m_{f(a)})_\star q_2 = (m_{f(a)})_\star((f_\star p_2)|_{\mathcal{L}}) = (f_\star {}^a p_2)|_{\mathcal{L}} \in Q$. Let also $q_3 := (f_\star {}^{a^{-1}} p_2)\big|_{\mathcal{L}} \in Q$. For the same reason as above $(m_{f(a)})_\star q_3 = q_2$. Thus **IV.5.28.(i)** and **IV.5.28.(ii)** hold.
For all x, y and $z \in G$,

$$m(f(x), m(f(y), f(z))) = f(x \cdot y \cdot z) = m(m(f(x), f(y)), f(z)),$$

$$\begin{aligned} m_1(f(x), m(f(x), f(y))) &= m(i(f(x)), m(f(x), f(y))) \\ &= m(f(x^{-1}), f(x \cdot y)) \\ &= f(x^{-1} \cdot x \cdot y) \\ &= f(y) \end{aligned}$$

and, similarly, $m_2(m(f(x), f(y)), f(y)) = f(x)$. It follows that **IV.5.28.(iii)** and **IV.5.28.(iv)** also hold. ♦

By Proposition **(IV.5.30)**, there exists an $\mathfrak{A}$-invariant $(\mathcal{L}(N), \star)$-definable group $(L, \cdot)$ and an $\mathfrak{A}$-invariant $(\mathcal{L}(N), \star)$-definable one to one function l such that $l_\star Q$ is an $\mathfrak{A}$-invariant locally finite set of d-generics of L and for all q_1 and $q_2 \in Q$, $q_1 \otimes q_2(x,y) \vdash l(G(x,y)) = l(m(i(x), y)) = l(x)^{-1} \cdot l(y)$. By Proposition **(IV.5.24)**, there exists a projective system of $\mathfrak{A}$-invariant $\mathcal{L}(N)$-definable groups $(H_\beta, \cdot)_{\beta \in \mathrm{B}}$ and an $\mathfrak{A}$-invariant $(\mathcal{L}, \star)$-definable groups isomorphism j between L and $\varprojlim H_\beta$. For all $\beta \in \mathrm{B}$, let $\pi_\beta : \varprojlim H_\beta \to H_\beta$ be the canonical projection and $h_\beta = \pi_\beta \circ j \circ l \circ f$. Then h_β is $\widetilde{\mathcal{L}}(\widetilde{A})$-definable and for all p_1 and $p_2 \in P$, $p_1 \otimes p_2(x,y) \vdash h_\beta(x^{-1} \cdot y) = h_\beta(x)^{-1} \cdot h_\beta(y)$.
Moreover, $\varprojlim h_\beta$ is a one to one function and by compactness there exists $\mathrm{B}_0 \subseteq \mathrm{B}$ finite such that $h_0 := \varprojlim_{\beta \in \mathrm{B}_0} h_\beta$ is already one to one. Let $H := \varprojlim_{\beta \in \mathrm{B}_0} H_\beta$. Because B_0 is finite, H is $\mathfrak{A}$-invariant $\mathcal{L}(N)$-definable and hence we have $\ulcorner H \urcorner^{\mathcal{L}} \in \mathcal{R}(\mathrm{dcl}^{\mathrm{eq}}_{\widetilde{\mathcal{L}}}(\widetilde{A})) = \mathrm{dcl}_{\mathcal{L}}(A)$, i.e. H is $\mathcal{L}(A)$-definable. Similarly, h_0 is $\widetilde{\mathcal{L}}(\widetilde{A})$-definable and we still have for all p_1 and $p_2 \in P$, $p_1 \otimes p_2(x,y) \vdash h_0(x^{-1} \cdot y) = h_0(x)^{-1} \cdot h_0(y)$. By Proposition **(IV.5.40)**, there exists an $\widetilde{\mathcal{L}}(\widetilde{A})$-definable one to one group morphism $G = \mathrm{Stab}_G(P) \to H$. ■

As a first corollary, let me reprove a well known result about groups definable in DCF_0. Recall that $\mathcal{L}_{\mathrm{rg}}$ is the language of rings and $\mathcal{L}_{\mathrm{rg},\partial} := \mathcal{L}_{\mathrm{rg}} \cup \{\partial\}$ is the language of differential rings.

Corollary IV.5.44:
Let $K \models \mathrm{DCF}_0$, $k \leqslant K$ a differential field and G an $\mathcal{L}_{\mathrm{rg},\partial}(k)$-definable group, then G embeds $\mathcal{L}_{\mathrm{rg},\partial}(k)$-definably into an $\mathcal{L}_{\mathrm{rg}}(k)$-definable group.

Proof. Note that for all $C \subseteq K \models \mathrm{DCF}_0$, $\mathrm{dcl}_{\mathcal{L}_{\mathrm{rg},\partial}}(C) = \mathrm{dcl}_{\mathcal{L}_{\mathrm{rg}}}(\partial_\omega(C))$. In particular, we have that $\mathrm{dcl}_{\mathcal{L}_{\mathrm{rg},\partial}}(k) = k$ and the multiplication and inverse in G are of the right form to apply Theorem **(IV.5.42)**. As DCF_0 and ACF_0 are stable, G has a d-generic over $\mathrm{acl}_{\mathcal{L}_{\mathrm{rg},\partial}}(k)$ — the generic type in the connected component of the unit and any model of ACF_0 is uniformly stably embedded in any elementary extension. Applying Theorem **(IV.5.42)**, we find an $\mathcal{L}_{\mathrm{rg},\partial}(k)$-definable embedding of G into an $\mathcal{L}_{\mathrm{rg}}(k)$-definable group H. ■

Remark IV.5.45:

1. It follows from [Art70, Théorème 3.7.(iii) and corollaire 3.13.(i)] that every $\mathcal{L}_{\mathrm{rg}}(k)$-definable group chunk is $\mathcal{L}_{\mathrm{rg}}(k)$-definably isomorphic to the group chunk of an algebraic group over k. In particular the group G in Corollary **(IV.5.44)** is $\mathcal{L}_{\mathrm{rg},\partial}(k)$-definably embedded in an algebraic group over k.

   It is very possible that there is a model theoretic equivalent of the proof that any constructible group is an algebraic group over the same parameters in the non connected case using a group chunk over all the generic types of the group and not just the one in the connected component (as is the case in [Poi87, Section 4.5]).

2. The usual proof of this result proceeds first by the connected case (which only needs group chunks over a singleton) and then introduces new parameters in some model of DCF_0 containing k to embed the whole group in an algebraic group (cf. [BD01; Poi87] for similar proofs in different settings). So it might be possible that control of the parameters over any differential field in the non connected case is new.

Finally, let me give a first very partial answer to Question **(IV.5.1)**. Note that there are three serious restrictions in this result compared to the result in DCF_0 (see Corollary **(IV.5.44)**): we have to assume there is a d-generic in G, we have to assume the group law is reasonable and we do not consider all interpretable groups. Let $M \vDash \mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ be saturated and homogeneous enough and $A = \mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}_{\partial}}(\mathbf{K}(A)) \subseteq M$.

Corollary IV.5.46:
Let $(G,\cdot)$ be an $\mathcal{L}^{\mathcal{G}}_{\partial}(A)$-definable group, with a d-generic, inside the sorts **K**, **k** *and* **Γ**. *If for all g and $h \in G(M)$, $\partial_\omega(g\cdot h) \in \mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(A, \partial_\omega(g), \partial_\omega(h))$ and $\partial_\omega(g^{-1}) \in \mathrm{dcl}_{\mathcal{L}^{\mathcal{G}}}(A, \partial_\omega(g))$, then there exists an $\mathcal{L}^{\mathcal{G}}(A)$-definable group H, with d-generic, and an $\mathcal{L}^{\mathcal{G}}(A)$-definable embedding $G \to H$.*

Proof. For all g, $h \in G(M)$ and $n \in \mathbb{N}$, there exists, by hypothesis an $\mathcal{L}^{\mathcal{G}}(A)$-definable function m_n such that $\partial^n(g\cdot h) = m_n(\partial_\omega(g), \partial_\omega(h))$. By compactness, there are finitely many $\mathcal{L}^{\mathcal{G}}(A)$-definable functions $m_{n,i}$ such that for all g, $h \in G(M)$, there exists an i such that $\partial^n(g\cdot m) = m_{n,i}(\partial_\omega(g), \partial_\omega(h))$. By quantifier elimination in the three sorted language, the set "$\partial^n(x,y) = m_{n,i}(\partial_\omega(g), \partial_\omega(h))$" is equivalent to $\theta_{n,i}(x,y)$ for some $\mathcal{L}^{\mathcal{G}}$-formula θ. Hence we can glue the different $m_{n,i}$ to obtain an $\mathcal{L}^{\mathcal{G}}(A)$-definable function m_n. Then the $((\mathcal{L}^{\mathcal{G}}(A)),\star)$-definable function $m = \prod_{i\in\mathbb{N}} m_{n,i}$ is such that for all g, $h \in G(M)$, $\partial_\omega(g\cdot h) = m(\partial_\omega(g), \partial_\omega(h))$. Similarly, we find an $((\mathcal{L}^{\mathcal{G}}(A)),\star)$-definable function i such that for all $g \in G(M)$, $\partial_\omega(g^{-1}) = i(\partial_\omega(g))$. We now apply Theorem **(IV.5.42)**. ∎

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List of Notations

Le Vieux Monsieur, au Logicien*.*
C'est très beau, la logique.

Le Logicien, au Vieux Monsieur*.*
À condition de ne pas en abuser.

E. Ionesco, *Rhinocéros*, Acte I