Transferring imaginaries

How to eliminate imaginaries in p-adic fields

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Contents

Codes and Quotients

Definition (Code)

In some structure M, a set X definable (with parameters) is said to be coded by some tuple a if there is a formula $\phi[x,y]$ such that

$$\phi[M,a'] = X(M) \iff a' = a.$$

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Definition (Representable quotient)

Let M be some structure, D be a definable set and E be a definable equivalence relation on D. The quotient D/E is said to be representable in M if there exists a definable function f with domain D such that

$$xEy \iff f(x) = f(y).$$

Eliminating imaginaries

Proposition

Let M be some structure with at least two constants, the following are equivalent:

(i) Any subset of M definable (with parameters) is coded,

(ii) Every quotient definable in M is representable.

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Example

- A non-example : infinite sets,
- An example : algebraically closed fields.

Shelah's construction

Definition

Let M be a $\mathcal{L}$-structure, we define a new language $\mathcal{L}^{\mathrm{eq}}$ and a $\mathcal{L}^{\mathrm{eq}}$-structure M^{eq} as follows:

- For any definable equivalence relation E on a product of $\mathcal{L}$-sorts $\prod_i S_i$, we add to $\mathcal{L}$ a sort S_E and a function $f_E : \prod_i S_i \to S_E$,
- In M^{eq}, S_E is interpreted as $\prod_i S_i(M)/E(M)$ and f_E as the canonical projection.

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Proposition

Let T be a complete theory. The language $\mathcal{L}^{\mathrm{eq}}$ and the theory $T^{\mathrm{eq}} = \mathrm{Th}(M^{\mathrm{eq}})$ does not depend on the choice of $M \vDash T$.

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Let T be a complete theory. The theory T^{eq} eliminates imaginaries.

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Proposition

Let T be a complete theory. The following are equivalent:

(i) T eliminates imaginaries,

(ii) For all, $M \models T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $d \in M$ such that:

$$d \in \mathrm{dcl}^{\mathrm{eq}}(e) \text{ and } e \in \mathrm{dcl}^{\mathrm{eq}}(d).$$

Finite sets

Definition (Weak elimination of imaginaries)

A complete theory T weakly eliminates imaginaries if for all $M \vDash T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $d \in M$ such that:

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Infinite sets weakly eliminate imaginaries.

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Proposition

Suppose T has weak elimination of imaginaries and every finite set in every model of T is coded, then T eliminates imaginaries.

Finite imaginaries

Definition (EI/UFI)

A complete theory T eliminates imaginaries up to uniform finite imaginaries if for all $M \models T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $d \in M$ such that:

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Proposition

Suppose T has EI/UFI and any finite quotient definable (with parameters) in any model of T is representable, then T eliminates imaginaries.

Contents

Some definitions

Definition

Let K be a field, a valuation on K is a map v from $K^\star$ to some abelian ordered group Γ that satisfies the following axioms:

(i) $\mathsf{v}(xy) = \mathsf{v}(x) + \mathsf{v}(y)$,

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- It has a unique maximal ideal $\mathfrak{M} = \{x \in K \mid \mathsf{v}(x) > 0\}$.
- We will be considering the group $\mathsf{RV} = K^\star/(1+\mathfrak{M})$.

Some examples

- Let p pe a prime number, then we can define the p-adic valuation on $\mathbb{Q}$ by taking $\mathsf{v}_p(p^n a/b) = n$ whenever $a \wedge b = a \wedge p = b \wedge p = 0$,

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- We will denote by $\mathbb{Q}_p$, the field of p-adic numbers, the completion of $\mathbb{Q}$ for the p-adic valuation. It is also a valued field,
- We will denote by ACVF the theory of algebraically closed valued field (in some language to be specified).

Imaginaries in valued fields

Remark

In the language of rings enriched with a predicate for $\mathsf{v}(x) \leq \mathsf{v}(y)$, the quotient $\Gamma = K^\star / \mathcal{O}^\star$ is not representable in any algebraically closed valued field nor in $\mathbb{Q}_p$

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However, in the case of *ACVF*, Haskell, Hrushovski and Macpherson have shown what imaginary sorts it suffices to add.

The geometric sorts

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- We can give an alternate definition of these sorts, for example $\mathsf{S}_n = \mathsf{GL}_n(K)/\mathsf{GL}_n(\mathcal{O})$,
- The geometric language $\mathcal{L}^{\mathcal{G}}$ is composed of the sorts K, S_n and T_n for all n, with the ring language on K and functions $\rho_n : \mathsf{GL}_n(K) \to \mathsf{S}_n$ and $\tau_n : \mathsf{S}_n \times K^n \to \mathsf{T}_n$,

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- S_1 can be identified with Γ and ρ_1 with v.

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Theorem (Haskell, Hrushovski and Macpherson, 2006)

The $\mathcal{L}^{\mathcal{G}}$-theory ACVF eliminates imaginaries.

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Question

Are all imaginaries in $\mathbb{Q}_p$ coded in the geometric sorts or are there new imaginaries in this theory?

Contents

A first example : real-closed fields

Example (Square roots)

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- However, the 1-to-2 correspondance $F = \{(x, y) \mid y^2 = x - a\}$ is definable both in K and $\overline{K}^{\mathrm{alg}}$,
- F is the Zariski closure of the graph of f and $f(x)$ can be defined (in K) as the greatest y such that $(x, y) \in F$,
- In fact, f can be coded by the code of F in $\overline{K}^{\mathrm{alg}}$ (which is K).

The general setting

- Let $\widetilde{\mathcal{L}} \subseteq \mathcal{L}$ be two languages,
- Let $\widetilde{T}$ be a $\widetilde{\mathcal{L}}$ theory that eliminates quantifiers and imaginaries,
- Let T be a $\mathcal{L}$-theory such that $\widetilde{T}_\forall \subseteq T$.

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Under what hypotheses can we deduce that T eliminates imaginaries?

Let $\widetilde{M} \models \widetilde{T}$ and $M \models T$ such that $M \subseteq \widetilde{M}$. Let us fix some notations:

- Let $A \subseteq \widetilde{M}$, we will write $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(A)$ for the (quantifier-free) $\widetilde{\mathcal{L}}$-definable closure in $\widetilde{M}$,
- Let $A \subseteq M^{\mathrm{eq}}$, we will write $\mathrm{dcl}^{\mathrm{eq}}_{\mathcal{L}}(A)$ for the $\mathcal{L}^{\mathrm{eq}}$-definable closure in M^{eq}.

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Similarly for acl, tp and TP.

Dominant sorts

Definition

In a theory, a set of sorts $\mathbf{S}$ will be called dominant if for any other sort S of the language, there is a surjective $\varnothing$-definable function $f : \prod_i S_i \to S$ where the S_i are in $\mathbf{S}$.

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Example

- The sorts of "real" elements (i.e. the original sorts from M) are dominant in M^{eq},
- In a valued field in the geometric language, the sort K is dominant.

We will write $\mathrm{dom}(M)$ for the union of the dominant sorts in M.

Algebraic boundedness

Hypothesis (i)

For all $M_1 \preccurlyeq \widetilde{M}$ and $c \in \mathrm{dom}(M)$, $\mathrm{dcl}^{\mathrm{eq}}_{\mathcal{L}}(M_1 c) \cap M \subseteq \mathrm{acl}_{\widetilde{\mathcal{L}}}(M_1 c)$.

Coping with $\mathrm{dcl}_{\widetilde{\mathcal{L}}}(M)$

Hypothesis (ii)

For all $e \in \mathrm{dcl}_{\widetilde{\mathcal{L}}}(M)$, there exists a tuple $e' \in M$ such that for all $\sigma \in \mathrm{Aut}(\widetilde{M})$ with $\sigma(M) = M$, σ fixes e if and only if it fixes e'.

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Proposition

Hypothesis (ii) implies that finite sets are coded in T.

Unary imaginaries

Hypothesis (iii)

Any $\mathcal{L}(M)$-definable unary set $X \subseteq \mathrm{dom}(M)$ is coded.

Germs

Definition

Let $A \subseteq \widetilde{M}$, r and s be A-definable functions and $p \in \mathsf{TP}_{\widetilde{\mathcal{L}}}(A)$ that contains the domain of both r and s. The functions r and s are said to have the same p-germ if for some $c \vDash p$, $r(c) = s(c)$.

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Remark

- If r and s have the same p-germ, then for any $c \vDash p$, $r(c) = s(c)$,
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- If p is a definable type and we only consider the germs of a family of uniformely defined functions r_b, $\partial_p r_b$ is an imaginary,

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Remark

- If r and s have the same p-germ, then for any $c \vDash p$, $r(c) = s(c)$,
- "Having the same p-germ" is an equivalence relation on A-definable functions. We will write $\partial_p r$ for the class of all A-definable functions having the same p-germ as r,
- If p is a definable type and we only consider the germs of a family of uniformely defined functions r_b, $\partial_p r_b$ is an imaginary,
- In any case, if p is $\mathsf{Aut}(\widetilde{M}/A)$-invariant, then the action of $\mathsf{Aut}(\widetilde{M}/A)$ on $\widetilde{\mathcal{L}}(\widetilde{M})$-definable functions induces an action on p-germs.

Controling germs

Hypothesis (iv)

For any $A = \mathsf{acl}^{\mathsf{eq}}_{\mathcal{L}}(A) \cap M$ and $c \in \mathsf{dom}(M)$, there exists an $Aut(\widetilde{M}/A)$-invariant type $\widetilde{p} \in \mathsf{TP}_{\widetilde{\mathcal{L}}}(\widetilde{M})$ such that $\widetilde{p}|M$ is consistent with $\mathsf{tp}_{\mathcal{L}}(c/A)$.
Moreover, for any $\widetilde{\mathcal{L}}(B)$-definable function r:

($\star$) There exists a sequence $(\varepsilon_i)_{i \in \kappa}$, with $\varepsilon_i \in \mathsf{dcl}_{\widetilde{\mathcal{L}}}(AB)$ such that any $\sigma \in \mathsf{Aut}(\widetilde{M}/A)$ fixes $\partial_{\widetilde{p}} r$ iff σ fixes almost every ε_i.

Rigidity of finite sets

Hypothesis (v)

For all $A = \mathsf{acl}^{\mathsf{eq}}_{\mathcal{L}}(A) \cap M$ and $c \in \mathsf{dom}(M)$, $\mathsf{acl}^{\mathsf{eq}}_{\mathcal{L}}(Ac) \cap M = \mathsf{dcl}^{\mathsf{eq}}_{\mathcal{L}}(Ac) \cap M$.

The theorem

Theorem (EI/UFI Criterion)

If the hypotheses (i) to (iv) are true, then T eliminates imaginaries up to uniform finite imaginaries.

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Corollary (EI Criterion)

If the hypotheses (i) to (v) are true, then T eliminates imaginaries.

Contents

p-adic imaginaries

Theorem

Let F be a finite extension of $\mathbb{Q}_p$, then the theory of F in the language $\mathcal{L}^{\mathcal{G}}$ with a constant added for a generator of $F \cap \overline{\mathbb{Q}}^{\mathsf{alg}}$ over $\mathbb{Q}$ eliminates imaginaries.

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Proof.

It follows from the EI criterion. $\square$

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Let $\mathcal{L}^{\mathcal{G}^-}$ be the language $\mathcal{L}^{\mathcal{G}}$ restricted to the sorts K and S_n.

p-adic imaginaries

Theorem

Let F be a finite extension of $\mathbb{Q}_p$, then the theory of F in the language $\mathcal{L}^{\mathcal{G}}$ with a constant added for a generator of $F \cap \overline{\mathbb{Q}}^{\mathsf{alg}}$ over $\mathbb{Q}$ eliminates imaginaries.

Let $\mathcal{L}^{\mathcal{G}^-}$ be the language $\mathcal{L}^{\mathcal{G}}$ restricted to the sorts K and S_n.

Corollary

Let F be a finite extension of $\mathbb{Q}_p$, then the theory of F in the language $\mathcal{L}^{\mathcal{G}^-}$ with a constant added for a generator of $F \cap \overline{\mathbb{Q}}^{\mathsf{alg}}$ over $\mathbb{Q}$ eliminates imaginaries.

Uniformity

Theorem

Let $L = \prod L_p/\mathcal{U}$ be an ultraproduct of finite extensions L_p of $\mathbb{Q}_p$. The theory of L in the language $\mathcal{L}^{\mathcal{G}^-}$ with some added constants eliminates imaginaries.

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Let $L = \prod L_p/\mathcal{U}$ be an ultraproduct of finite extensions L_p of $\mathbb{Q}_p$. The theory of L in the language $\mathcal{L}^{\mathcal{G}^-}$ with some added constants eliminates imaginaries.

Proof.

The EI/UFI criterion applies in $\mathcal{L}^{\mathcal{G}}$ (and we reduce to $\mathcal{L}^{\mathcal{G}^-}$ in the same manner). It remains to show that definable finite quotient are represented, but one can show that they are internal to RV and that the induced theory on RV eliminates imaginaries. $\square$

Uniformity

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Corollary

For any equivalence relation E on a set D definable in L_p uniformly in p, there exists uniformly definable non-empty set X and function $f : X \times D \to S_m \times K^l$ such that for any prime p, and any $a \in X(L_p)$, for all $x, y \in D(L_p)$, we have: $f(a, x) = f(a, y)$ iff xEy.