Toward an imaginary Ax-Kochen-Ershov principle

Work in progress with Martin Hils

Silvain Rideau

CNRS, IMJ-PRG, Université Paris Diderot

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A crash course on imaginaries

For all $\mathcal{L}$-theory T, we define:

- $\mathcal{L}^{\mathrm{eq}} = \mathcal{L} \cup \bigcup_{X \subseteq Y \times Z\ \emptyset\text{-definable}} \{E_X, f_X : Y \to E_X\}$.
- $T^{\mathrm{eq}} = T \cup \bigcup_X \{f_X$ induces an bijection $E_X \simeq Y/(X_{y_1} = X_{y_2})\}$.
- Every $M \models T$ has a unique $\mathcal{L}^{\mathrm{eq}}$-enrichment $M^{\mathrm{eq}} \models T^{\mathrm{eq}}$.
- If D is a collection of stably embedded A-definable set, D^{eq} denotes the collection of A-induced imaginary sorts of D.
- Any M-definable set X has a smallest definably closed set of definition $\ulcorner X \urcorner$ in M^{eq}.

Definition

Let T be a theory and D a collection of $\emptyset$-interpretable sets.

- T eliminates imaginaries up to D if, for all $e \in M^{\mathrm{eq}} \models T^{\mathrm{eq}}$, there exists $d \in D(\mathrm{dcl}(e))$ such that $e \in \mathrm{dcl}(d)$.
- T weakly eliminates imaginaries up to D if, for all $e \in M^{\mathrm{eq}} \models T^{\mathrm{eq}}$, there exists $d \in D(\mathrm{acl}(e))$ such that $e \in \mathrm{dcl}(d)$.

Imaginaries in valued fields

In $\mathrm{Hen}_{0,0}$, certain quotients cannot be eliminated:

- $\Gamma = K^\times/\mathcal{O}^\times$.
- $k = \mathcal{O}/\mathfrak{m}$.
- $S_n = \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$, the moduli space of lattices in K^n.
- For all $s \in S_n$, $V_s = \mathcal{O}s/\mathfrak{m}s$, a dimension n k-vector space.
- $T_n = \bigcup_{s \in S_n} V_s$.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

ACVF eliminates imaginaries up to $\mathcal{G} = K \cup \bigcup_n (S_n \cup T_n)$.

- k^{eq} and Γ^{eq}.

Unreasonable Hope (Imaginary AKE, first attempt)

$\mathrm{Hen}_{0,0}$ weakly eliminates imaginaries up to $\mathcal{G} \cup k^{\mathrm{eq}} \cup \Gamma^{\mathrm{eq}}$.

Some more imaginaries

Certain quotients cannot be eliminated in $\mathcal{G} \cup k^{\mathrm{eq}} \cup \Gamma^{\mathrm{eq}}$:

- K/K^n and, more generally, $(K/K^n)^{\mathrm{eq}}$.
  - Solved by considering $\mathrm{RV}^{\mathrm{eq}}$, where $\mathrm{RV} = K^\times/1 + \mathfrak{m} = T_1$.
- K/I for some $I \subseteq \mathcal{O}$ definable ideal which is not a multiple of $\mathcal{O}$ or $\mathfrak{m}$, and higher dimensional equivalent.
  - Prevented by requiring the value group to be definably complete, e.g ordered groups elementarily equivalent to $\mathbb{Z}$ or $\mathbb{Q}$.
- $R_b = \{b' \subseteq b \text{ maximal open subball}\}$ and, more generally, R_b^{eq}, if $R_b(\mathrm{dcl}(b)) = \emptyset$.
  - Solved by considering V_s^{eq} for some $s \in S_n(\mathrm{dcl}(b))$.

For all $M \models \mathrm{Hen}_{0,0}$ and $A = \mathrm{acl}(A) \subseteq \mathcal{G}(M)$, let $\mathrm{St}_A = \bigcup_{s \in S_n(A)} V_s$ and $D_A = A \cup \mathrm{RV} \cup \mathrm{St}_A$.

A New Hope (Imaginary AKE, second attempt)

Let $e \in M^{\mathrm{eq}} \models \mathrm{Hen}_{0,0}^{\mathrm{eq}}$ and $A = \mathcal{G}(\mathrm{acl}(e))$. Assume $\Gamma(M)$ is divisible or a $\mathbb{Z}$-group. Then e is weakly coded in D_A^{eq}.

A local look at imaginaries

Proposition

Let D be a collection of $\emptyset$-interpretable sets in T. Assume:

- For every definable X, there exists a $D(\mathrm{acl}(\ulcorner X \urcorner))$-invariant type $p(x)$ such that $p(x) \vdash x \in X$.

Then T weakly eliminates imaginaries up to D.

Sometimes, it is easier to look for a definable p. One can then proceed in two steps:

- For every definable X, find a $\mathrm{acl}(\ulcorner X \urcorner)$-definable type $p(x)$ such that $p(x) \vdash x \in X$.
- For any $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ show that any A-definable type p is $D(A)$-definable.

Density of quantifier free definable types $\mathrm{Hen}_{0,0}$

Let $T \supseteq \mathrm{Hen}_{0,0}$ be a complete theory in an RV-enrichment of $\mathcal{L}_{\mathrm{div}}$.

(Almost) Theorem

Assume k and Γ are stably embedded and algebraically bounded. Assume also that Γ is definably complete.

- For all $A \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$ and quantifier free A-definable $\mathcal{L}_{\mathrm{div}}$-type p, then p is $\mathcal{G}(\mathrm{dcl}(A))$-definable.
- Let X be definable in $M \models T$. There exists a quantifier free $\mathrm{acl}(\ulcorner X \urcorner)$-definable $\mathcal{L}_{\mathrm{div}}$-type p consistent with X.

- The first statement is essentially proved by Johnson in his account of elimination of imaginaries in ACVF.
- The proof of the second statement is a mix of existing arguments.

Completing quantifier free types

Let $M \preccurlyeq \mathfrak{C} \models T$ and $a \in K$ be a tuple.

An alternative formulation of field quantifier elimination

Assume $\mathrm{rv}(M(a))) \subseteq \mathrm{dcl}_0(M\rho(a))$, where $\rho(a) \in \mathrm{RV}(\mathrm{dcl}_0(Ma))$. Then

$$\mathrm{tp}_0(a/M) \cup \mathrm{tp}(\rho(a)/\mathrm{rv}(M)) \vdash \mathrm{tp}(a/M).$$

- If a is generic in some ball b over M and $c \in b(M)$, then

$$\mathrm{rv}(M(a)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M)\mathrm{rv}(a-c)).$$

- Moreover, if b is open, $\rho(a) = \mathrm{rv}(a-c)$ does not depend on the choice of $c \in b(M)$.
- So $[\rho]_q$, the germ of ρ over the b-definable type $q = \mathrm{tp}_0(a/M)$, is in $\mathrm{dcl}(b)$.
- It follows that $\mathrm{tp}(a/M)$ is $b\mathrm{RV}(M)$-invariant.

Computing $\mathrm{rv}(M(a))$

Proposition

Assume $\mathrm{tp}_0(a/M)$ is N-definable for some $N \preccurlyeq M$, then there exists $\rho(a) \in \mathrm{dcl}_0(Na)$ such that $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M)\rho(a))$.

Let $c \in K$ be such that $p = \mathrm{tp}_0(ac/M)$ is A-definable for some $A \subseteq M^{\mathrm{eq}}$ and $q = \mathrm{tp}_0(a/M)$. Assume one of the following holds:

- c is generic in an open ball or a strict intersection of balls over $M(a)$;
- c is generic in a closed ball b over $M(a)$ and there exists $g(a) \in R_b(\mathrm{dcl}_0(Ma))$ with $[q]_g \in \mathrm{dcl}(A)$;
- $c \in M(a)^{\mathrm{alg}}$.

Then there exists $\rho(a) \in \mathrm{RV}(\mathrm{dcl}_0(Ma))$ with $[\rho]_p \in \mathrm{dcl}(A)$ and

$$\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M(a))\rho(ac)).$$

Finding invariant types

Corollary

Assume $\mathrm{tp}_0(a/M)$ is N-definable for some $N \preccurlyeq M$, then $\mathrm{tp}(a/M)$ is $N\mathrm{RV}(M)$-invariant.

Assume k and Γ are stably embedded and algebraically bounded. Assume also that Γ is definably complete.

- Pick any $e \in M^{\mathrm{eq}}$ and let $A = \mathcal{G}(\mathrm{acl}(e))$. Let f be $\emptyset$-definable and $a \in K^n$ such that $e = f(a)$.
- We find a quantifier free A-definable $\mathcal{L}_{\mathrm{div}}$-type p consistent with $f^{-1}(e)$. So we may assume $\mathrm{tp}_0(a/M)$ is A-definable.
- So $\mathrm{tp}(a/M)$ — and hence $\mathrm{tp}(e/M)$ — is $N\mathrm{RV}(M)$-invariant, for any $A \subseteq N \preccurlyeq M$.
- Since RV is stably embedded, $e \in \mathrm{dcl}(N\mathrm{RV}(M))$.
- It follows that there exists some $\mathcal{G}(\mathrm{acl}(e))$-definable set E which is internal to RV.

Imaginaries in $\mathrm{Hen}_{0,0}$, take one

(Almost) Theorem

Assume k and Γ are stably embedded and algebraically bounded, Γ is definably complete and for all $A \subseteq M^{\mathrm{eq}}$ and any A-definable ball b, either b isolates a complete type or $R_b(\mathrm{dcl}(A)) \neq \emptyset$.

- for all $A \subseteq M^{\mathrm{eq}}$, there exists $N \supseteq \mathcal{G}(A)$ such that $\mathrm{tp}(N/\mathcal{G}(A)) \vdash \mathrm{tp}(N/A)$;
- T weakly eliminates imaginaries up to $\mathcal{G} \cup \mathrm{RV}^{\mathrm{eq}}$.

- Let k be a characteristic zero bounded PAC field, then $k((t))$ and $k((t^{\mathbb{Q}}))$ eliminate imaginaries up to $\mathcal{G}$, provided certain constants are added to the residue field.
- The above result still holds if one adds angular components; i.e. a section of $1 \to k^{\times} \to \mathrm{RV} \to \Gamma \to 0$.
- With some tweaking, similar results should hold for k elementarily equivalent to a finite extension of $\mathbb{Q}_p$.

Imaginaries in $\mathrm{Hen}_{0,0}$, take two

Assume that for all $A \subseteq \mathcal{G}(M)$ and $\epsilon \in \mathrm{St}_A(\mathrm{dcl}_0(\mathfrak{C}))$, there is $\eta \in \mathrm{St}_A(\mathfrak{C})$ with $\epsilon \in \mathrm{dcl}_0(A\eta)$ and η is definable over $A\epsilon$ in $(\mathfrak{C}^{\mathrm{alg}}, \mathfrak{C})$.

- If $\mathrm{tp}_0(a/M)$ is stably dominated over A and c is generic, over $M(a)$, in a closed ball $b \in \mathrm{dcl}_0(Aa)$, then

$$\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M(a))\mathrm{St}_A(M)ac).$$

- For all $A \subseteq \mathcal{G}(M)$, there exists $N \supseteq \mathcal{G}(A)$ such that $\mathrm{tp}(N/M)$ is $AD_A(M)$-invariant.

Theorem

If $\mathrm{tp}_0(a/M)$ is A-definable then $\mathrm{tp}(a/M)$ is $AD_A(M)$-invariant.

(Almost) Theorem

Assume that k is stably embedded and algebraically bounded and Γ is a pure ordered group which is either divisible or a $\mathbb{Z}$-group. Then any $e \in M^{\mathrm{eq}}$ is weakly coded in D_A^{eq}, where $A = \mathcal{G}(\mathrm{acl}(e))$.

Valued fields with operators

Let $\delta = \{\delta_i : K \to K \mid i \in I\}$, $\mathcal{L}_\delta = \mathcal{L} \cup \delta$. Let $T_\delta \supseteq T \supseteq \mathrm{ACVF}_{0,0}$ and $M \preccurlyeq \mathfrak{C} \models T_\delta$. Assume that for all tuples $a \in K$, $\mathrm{tp}(\delta(a)/M) \vdash \mathrm{tp}_\delta(a/M)$.

Corollary

If $\mathrm{tp}_0(\delta(a)/M)$ is A-definable, for some $A \subseteq \mathcal{G}(M)$, then $\mathrm{tp}_\delta(a/M)$ is $AD_A(M)$-invariant.

Theorem (R.,R.-Simon)

Assume that k, Γ are stably embedded and k^{eq}, Γ^{eq} eliminate $\exists^\infty$.

- For any $\mathcal{L}_\delta(M)$-definable X, there exists $a \in X$ such that $\mathrm{tp}_0(a/M)$ is $\mathcal{L}_\delta(\mathrm{acl}_{\mathcal{L}_\delta}(\ulcorner X \urcorner))$-definable.
- Assume, moreover that any externally $\mathcal{L}$-definable subset of $\Gamma^n(M)$ which is $\mathcal{L}_\delta(M)$-definable is $\mathcal{L}(M)$-definable. Then, for every $A = \mathrm{dcl}_\delta(A) \subseteq M^{\mathrm{eq}}$, any $\mathcal{L}_\delta(A)$-definable quantifier free $\mathcal{L}_{\mathrm{div}}$-type is $\mathcal{L}(\mathcal{G}(A))$-definable.

The asymptotic theory of $(\mathbb{F}_p(t)^{\mathrm{alg}}, \Phi_p)$

Let VFA_0 be the theory of equicharacteristic zero existentially closed σ-Henselian fields with an ω-increasing automorphism:

- $\sigma(\mathcal{O}) = \mathcal{O}$;
- if $x \in \mathfrak{m}$, for all $n \in \mathbb{Z}_{>0}$, $\mathrm{v}(\sigma(c)) > \mathrm{v}(c)$.

We work in $\mathcal{L}_\sigma^{\mathrm{RV}}$ with sorts K and RV, the ring language on both K and RV, and maps $\mathrm{rv} : K \to \mathrm{RV}$, $\sigma : K \to K$ and $\sigma_{\mathrm{RV}} : \mathrm{RV} \to \mathrm{RV}$. By results of Hrushovski, Durhan and Pal:

- For all $(k, \sigma_k) \models \mathrm{ACFA}_0$ and $(\Gamma, \sigma_\Gamma) \models \omega\mathrm{DOAG}$, $(k((\Gamma)), \sigma) \models \mathrm{VFA}_0$ where $\sigma(\sum_\gamma a_\gamma t^\gamma) = \sum_\gamma \sigma_k(a_\gamma) t^{\sigma(\gamma)}$.
- For every non-principal ultrafilter $\mathfrak{U}$ on the set of primes, $\prod_{p \to \mathfrak{U}} (\mathbb{F}_p(t)^{\mathrm{alg}}, \Phi_p) \models \mathrm{VFA}_0$.
- VFA_0 eliminates field quantifiers.
- k is stably embedded and a pure model of ACFA_0.
- Γ is stably embedded and a pure model of $\omega\mathrm{DOAG}$. In particular, it is o-minimal.

Imaginaries in VFA_0

Let $\mathcal{L} = \mathcal{L}_\sigma^{\mathrm{RV}} \setminus \{\sigma\}$, $T = \mathrm{VFA}_0|_{\mathcal{L}}$ and $\delta = \{\sigma^i \mid i \in \mathbb{Z}_{\leq 0}\}$.

- By field quantifier elimination, for all $M \models \mathrm{VFA}_0$ and tuple $a \in K$, $\mathrm{tp}(\delta(a/M)) \vdash \mathrm{tp}_\delta(a/M)$.

Proposition

Let $T_0 \subseteq T_1$ two o-minimal theories (in $\mathcal{L}_0 \subseteq \mathcal{L}_1$) and $M_1 \models T_1$. Then, any externally $\mathcal{L}_0$-definable subset of M_1^n which is $\mathcal{L}_1(M_1)$-definable is $\mathcal{L}_0(M_1)$-definable.

- For every $M \models \mathrm{VFA}_0$, any externally $\mathcal{L}$-definable subset of $\Gamma^n(M)$ which is $\mathcal{L}_\sigma^{\mathrm{RV}}(M)$-definable is $\mathcal{L}(M)$-definable.

Theorem

Any $e \in M^{\mathrm{eq}} \models \mathrm{VFA}_0^{\mathrm{eq}}$ is weakly coded in D_A^{eq}, where $A = \mathcal{G}(\mathrm{acl}(e))$.

Imaginaries in $\mathcal{D}_A$

Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}} \models \mathrm{VFA}^{\mathrm{eq}}$. $\mathrm{St}_A = \bigcup_{s \in S_n(\mathrm{acl}(A))} V_s$ with its $\mathrm{acl}(A)$-induced structure is a collection of $k = V_{\mathcal{O}^\times}$-vector spaces (with flags and roots) and for all $s \in \mathrm{acl}(A)$, an isomorphism $\sigma_a : V_s \to V_{\sigma(s)}$.

Proposition (adapted from Hrushovski, 2012)

St_A is supersimple and eliminates imaginaries.

(Almost) Theorem

- $\mathcal{D}_A = \mathrm{RV} \cup \mathrm{St}_A$ eliminates imaginaries.
- VFA_0 eliminates imaginaries up to $\mathcal{G}$.

Mixed characteristic

Most of what we did can be transported to mixed characteristic by consider the first equicharacteristic zero coarsening.
Let $M \equiv W(\mathbb{F}_p^{\mathrm{alg}})$ and $\mathrm{RV}_n = K/1 + p^n\mathfrak{m}$.

(Almost) Theorem

- For any M-definable X, there exists $a \in K$ such that $\mathrm{tp}(a/M)$ is $\mathcal{G}(\mathrm{acl}(\ulcorner X \urcorner)) \bigcup_n \mathrm{RV}_n(M)$-invariant.
- $W(\mathbb{F}_p^{\mathrm{alg}})$ weakly eliminates imaginaries up to $\mathcal{G} \cup (\bigcup_n \mathrm{RV}_n)^{\mathrm{eq}}$.

Conjecture

- $W(\mathbb{F}_p^{\mathrm{alg}})$ eliminates imaginaries up to $\mathcal{G}$.
- $(W(\mathbb{F}_p^{\mathrm{alg}}), W(\Phi_p))$ eliminates imaginaries up to $\mathcal{G}$.