Imaginaries in valued fields with operators

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Valued fields

Let (K, v) be a valued field:

- $\Gamma = \mathrm{v}(K)$ its value group;
- $\mathcal{O} = \{x \in K : \mathrm{v}(x) \geq 0\}$ its valuation ring;
- $\mathfrak{M} = \{x \in K : \mathrm{v}(x) > 0\}$ its maximal ideal;
- $k = \mathcal{O} \,/\, \mathfrak{M}$ its residue field.

Example (Hahn series field, Witt vectors)

- Let k be a field and Γ be an ordered Abelian group:

$$k((t^\Gamma)) = \{\sum_{\gamma \in \Gamma} c_\gamma t^\gamma : \text{ well-ordered support}\}.$$

- Let k be a perfect characteristic $p > 0$ field.

$$\mathrm{W}(k) = \{\sum_{i > i_0} a_i^{p^{-i}} p^i\}.$$

It is the unique complete, rank 1, mixed characteristic valued field whose residue field is k.

Operators

On a field K we consider:

- Automorphisms (of the field).
- Derivations: an additive morphism $\partial : K \to K$ that verifies the Leibniz rule:

$$\partial(xy) = \partial(x)y + x\partial(y).$$

- (Iterative) Hasse derivations: a collection $(\partial_n)_{n\leq 0}$ of additive morphisms $K \to K$ that verify
  - $D_0(x) = x$;
  - The generalised Leibniz rule:

$$\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y);$$

  - $D_n(D_m(x)) = \binom{m+n}{n}\partial_{m+n}(x)$

Operators

Example (Automorphisms)

- $(\overline{\mathbb{F}_p}^{\mathrm{alg}}, \mathrm{F}_p)$.
- Ultraproducts of the above.

Example (Derivations)

- Meromorphic functions on some open subset of $\mathbb{C}$.
- Germs at $+\infty$ of infinitely differentiable real functions.
- For (k, ∂) a differential field, $k((t^\Gamma))$ with $\partial(\sum_\gamma c_\gamma t^\gamma) = \sum_\gamma \partial(c_\gamma) t^\gamma$.

Example (Hasse Derivations)

- Let K be a characteristic $p > 0$ field and $(b_i)_{i \in I}$ a p-basis of K. There exists a a Hasse derivation ∂_i on K such that $\partial_{i,1}(b_i) = 1$ and $\partial_{i,n}(b_j) = 0$ otherwise.

Valued fields with operators

We want to consider fields with both structures.

- You can either not assume any interaction:
  - Separably closed valued fields (Delon,Hong,Hils-Kamensky-R.);
  - Differentially closed valued fields (Michaux,Guzy-Point);
- Or force some level of interaction:
  - Contractive derivations: $\mathrm{v}(D(x)) \geq \mathrm{v}(x)$ (Scanlon, R.);
  - Valued field automorphism: $\sigma(\mathcal{O}) = \mathcal{O}$ (Bélair-Macintyre-Scanlon, Durhan-van den Dries, Hrushovski, Pal, Durhan-Onay).

We will only consider existentially closed fields with operators.

- A priori, this rules out transseries.
- Actually, we will only need a certain form of quantifier elimination.

Contractive derivations

In $\mathcal{L}_{\partial,\mathrm{div}} := \{\mathbf{K}; 0, 1, +, -, \cdot, \partial, \mathrm{div}\}$:

Theorem (Scanlon, 2000)

The theory of equicharacteristic zero valued fields with a contractive derivation has a model completion $\mathrm{VDF}_{\mathcal{EC}}$ which is complete and eliminates quantifiers.

The theory $\mathrm{VDF}_{\mathcal{EC}}$ contains:

- The field is ∂-Henselian;
- $\mathrm{v}(\mathrm{C}_K) = \mathrm{v}(K)$ where $\mathrm{C}_K = \{x \in K : \partial(x) = 0\}$;
- The residue field is differentially closed;
- The value group is divisible.

Example

If (k, ∂) is differentially closed and Γ is divisible, then $k((t^\Gamma)) \models \mathrm{VDF}_{\mathcal{EC}}$.

Separably closed fields

- Let e be a positive integer.
- Let K be a characteristic $p > 0$ field with e commuting Hasse derivations ∂_i: $\partial_{i,n} \circ \partial_{j,m} = \partial_{j,m} \circ \partial_{i,n}$.
- The field K is strict if $C_K^1 := \{x \in K : \forall i,\ \partial_{i,1}(x) = 0\} = K^p$.

In $\mathcal{L}_{e,\mathrm{div}} := \{\mathbf{K}; 0, 1, +, -, \cdot, (\partial_{i,n})_{0 \leq i < e, 0 \leq n}, \mathrm{div}\}$:

Theorem

The theory $\mathrm{SCVH}_{p,e}$ of characteristic $p > 0$ strict separably closed valued fields with e commuting Hasse derivations such that $[K : K^p] = p^e$ is complete and eliminates quantifiers.

Let $K \models \mathrm{SCVH}_{p,e}$:

- $\mathrm{v}(K)$ is divisible and $k(K)$ is algebraically closed;
- K is dense in $\overline{K}^{\mathrm{alg}}$.

Imaginaries

An imaginary is an equivalent class of an $\varnothing$-definable equivalence relation.

Example

- Let $(X_y)_{y\in Y}$ be an $\varnothing$-definable family of sets.
  - Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$.
  - The set $Y/\equiv$ is a moduli space for the family $(X_y)_{y\in Y}$.
  - The imaginary $\ulcorner X_y \urcorner := y/\equiv$ is the canonical parameter of X_y.
- Let G be a definable group and $H \leqslant G$ be a subgroup. The group G/H is interpretable but *a priori* not definable.

Definition

A theory T eliminates imaginaries if for all $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f defined on D such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$

Shelah's eq construction

Definition

Let T be a theory. For all $\varnothing$-definable equivalence relation $E \subseteq \prod_i S_i$, let S_E be a new sort and $f_E : \prod S_i \to S_E$ be a new function symbol. Let

$$\mathcal{L}^{eq} := \mathcal{L} \cup \{S_E, f_E : E \text{ is an } \varnothing\text{-definable equivalence relation}\}$$

and

$$T^{eq} := T \cup \{f_E \text{ is onto and } \forall x, y\, (f_E(x) = f_E(y) \leftrightarrow xEy)\}.$$

Remark

- Let $M \models T$, then M can naturally be enriched into a model of T^{eq} that we denote M^{eq}.
- The theory T^{eq} eliminates imaginaries.

Imaginaries in fields

Theorem (Poizat, 1983)

The theory of algebraically closed fields in $\mathcal{L}_{\mathrm{rg}} := \{\mathbf{K}; 0, 1, +, -, \cdot\}$ and the theory of differentially closed fields in $\mathcal{L}_{\partial} := \mathcal{L}_{\mathrm{rg}} \cup \{\partial\}$ both eliminate imaginaries.

One cannot hope for such a theorem to hold for algebraically closed valued fields in $\mathcal{L}_{\mathrm{div}} := \mathcal{L}_{\mathrm{rg}} \cup \{\mathrm{div}\}$. Indeed,

- $K = \mathbb{C}((t^{\mathbb{Q}})) \models \mathrm{ACVF}$;
- $\mathbb{Q} = K^{\star} / \mathcal{O}^{\star}$ is both interpretable and countable;
- All definable set $X \subseteq K^n$ are either finite or have cardinality continuum.

Imaginaries in valued fields

Let (K, v) be a valued field, we define:

- $\mathbf{S}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$.
  - It is the moduli space of rank n free $\mathcal{O}$-submodules of K^n.
- $\mathbf{T}_n := \mathrm{GL}_n(K)/\mathrm{GL}_{n,n}(\mathcal{O})$
  - $\mathrm{GL}_{n,n}(\mathcal{O})$ consists of the matrices $M \in \mathrm{GL}_n(\mathcal{O})$ whose reduct modulo $\mathfrak{M}$ has only zeroes on the last column but for a 1 in the last entry.
  - It is the moduli space of $\bigcup_{s \in \mathbf{S}_n} s/\mathfrak{M}s = \{a + \mathfrak{M}s : s \in \mathbf{S}_n \text{ and } a \in s\}$.

Let $\mathcal{L}_{\mathcal{G}} := \{\mathbf{K}, (\mathbf{S}_n)_{n \in \mathbb{N}_{>0}}, (\mathbf{T}_n)_{n \in \mathbb{N}_{>0}}; \mathcal{L}_{\text{div}}, \sigma_n : \mathbf{K}^{n^2} \to \mathbf{S}_n, \tau_n : \mathbf{K}^{n^2} \to \mathbf{T}_n\}$.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

The $\mathcal{L}_{\mathcal{G}}$-theory of algebraically closed valued fields eliminates imaginaries.

Imaginaries and definable types

Proposition (Hrushovski, 2014)

Let T be a theory such that, for all $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$:

1. Any $\mathcal{L}^{\mathrm{eq}}(A)$-definable set is consistent with an $\mathcal{L}^{\mathrm{eq}}(A)$-definable type.
2. Any $\mathcal{L}^{\mathrm{eq}}(A)$-definable type p is $\mathcal{L}(A \cap M)$-definable.
3. Finite sets have canonical parameters.

Then T eliminates imaginaries.

Remark

It suffices to prove hypothesis 1 in dimension 1.

An aside: the invariant extension property

Definition

We say that T has the invariant extension property if for all $M \models T$ and $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, every type over A has a global A-invariant extension.

Proposition

The following are equivalent:

(i) The theory T has the invariant extension property.

(ii) For all $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$, any $\mathcal{L}^{\mathrm{eq}}(A)$-definable set is consistent with an $\mathcal{L}^{\mathrm{eq}}(A)$-definable type.

Remark

If T is NIP then the above are also equivalent to:

(iii)
- Forking equals dividing
- Lascar strong type, Kim-Pillay strong type and strong type coincide.

Differentially closed and separably closed fields

Let T be either the theory of characteristic zero differentially closed fields or the theory of strict characteristic $p > 0$ separably closed fields with e commuting Hasse derivations such that $[K : K^p] = p^e$.

- Hypothesis 1 is true by stability
- Hypothesis 3 is true because it is true in algebraically closed fields.
- As for Hypothesis 2:
  - Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$ and $p(x)$ be an $\mathcal{L}(A)$-definable type.
  - Let $\partial_\omega(x)$ denote either $(\partial^n(x))_{n \in \mathbb{Z}_{\geq 0}}$ or $(\partial_{0,i_0} \circ \ldots \partial_{e-1,i_{e-1}}(x))_{i_j \in \mathbb{Z}_{\geq 0}}$.
  - Let $a \models p$ and $q = \mathrm{tp}_{\mathcal{L}_{\mathrm{rg}}}(\partial_\omega(a)/M)$.
  - By elimination of imaginaries in ACF, q is $\mathcal{L}_{\mathrm{rg}}(A \cap M)$-definable.
  - So p is $\mathcal{L}(A \cap M)$-definable.

Prolongations

Let $\mathcal{L}$ be either $\mathcal{L}_{\partial,\mathrm{div}}$ or $\mathcal{L}_{e,\mathrm{div}}$ and T denote either $\mathrm{VDF}_{\mathcal{EC}}$ or $\mathrm{SCVH}_{p,e}$.

- Let $M \vDash T$. For all $p \in \mathcal{S}_x^{\mathcal{L}}(M)$, we define:

$$\nabla_\omega(p) := \{\phi(x_\omega; m) : \phi(\partial_\omega(x); m) \in p\} \in \mathcal{S}_{x_\omega}^{\mathcal{L}_{\mathrm{div}}}(M).$$

- By quantifier elimination, the map ∇_ω is injective.
- Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \vDash T^{\mathrm{eq}}$,

$$p \text{ is consistent with } X \iff \nabla_\omega(p) \text{ is consistent with } \partial_\omega(X);$$

$$p \text{ is } \mathcal{L}^{\mathrm{eq}}(A)\text{-definable} \iff \nabla_\omega(p) \text{ is } \mathcal{L}^{\mathrm{eq}}(A)\text{-definable.}$$

Hypothesis 1 and 2 (almost) reduce to questions about ACVF.

- The defining scheme of p consists of $\mathcal{L}(M)$-formulas and not $\mathcal{L}_{\mathrm{div}}(M)$ formulas.

Proving Hypothesis 1

It is proved by a technical construction.

- Given an enrichment T of ACVF in a language $\mathcal{L}$, such that k and Γ eliminate imaginaries,
- A set $A = \operatorname{acl}^{eq}(A) \subseteq M^{eq} \models T^{eq}$,
- An $\mathcal{L}^{eq}(A)$-definable set X,
- A finite set Δ of $\mathcal{L}_{div}$-formulas,
- We find an $\mathcal{L}^{eq}(A)$-definable Δ-type p consistent with X.

Definable types in enrichments of NIP theories

Definition (Uniform stable embeddedness)

Let M be some structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formula $\phi(x;y)$ there exists a formula $\psi(x;z)$ such that for all tuple $c \in M$,

$$\phi(A;c) = \psi(A;a)$$

for some tuple $a \in A$.

Proposition (Simon-R.)

Let T be an NIP be an $\mathcal{L}$-theory and $\widetilde{T}$ be a complete enrichment of T in a language $\widetilde{\mathcal{L}}$. Assume that there exits $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.
Let X be a set that is both externally $\mathcal{L}$-definable and $\widetilde{\mathcal{L}}$-definable, then X is $\mathcal{L}$-definable.

In particular, any $\mathcal{L}$-type which is $\widetilde{\mathcal{L}}$-definable is in fact $\mathcal{L}$-definable.

Externally definable sets in NIP theories

Proposition (Simon-R.)

Let T be an NIP $\mathcal{L}$-theory, $U(x)$ be a new predicate and $\phi(x;t) \in \mathcal{L}$. There exists $\psi(x;s) \in \mathcal{L}$ and $\theta \in \mathcal{L}_U$ a sentence such that for all $M \vDash T$ and $U \subseteq M^{|x|}$ we have:

$$U \text{ is externally } \phi\text{-definable} \Rightarrow M_U \vDash \theta_U \Rightarrow U \text{ is externally } \psi\text{-definable.}$$

- It follows that (a uniform version of) the previous proposition's conclusion is a first order statement.
- Hence it suffices to find one model of T where it holds (uniformly enough); for example, a model where all externally $\mathcal{L}$-definable sets are $\mathcal{L}$-definable.
- If we are looking at $T = \text{ACVF}$, then models of the form $k((t^{\mathbb{R}}))$ have this property.

Imaginaries in $\mathrm{VDF}_{\mathcal{EC}}$

Let $\mathcal{L}_{\mathcal{G},\partial} = \mathcal{L}_{\mathcal{G}} \cup \{\partial\}$ and $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ the enrichment of $\mathrm{VDF}_{\mathcal{EC}}$ to $\mathcal{L}_{\mathcal{G}}$.

Theorem

The theory $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ eliminates imaginaries.

Proof.

Apply the criterion. □

Imaginaries in $\mathrm{SCVH}_{p,e}$

Let $\mathcal{L}_{\mathcal{G},p,e} = \mathcal{L}_{\mathcal{G}} \cup \{\partial_{i,n} : 0 \leq i < e \text{ and } n \geq 0\}$ and $\mathrm{SCVH}^{\mathcal{G}}_{p,e}$ the enrichment of $\mathrm{SCVH}_{p,e}$ to $\mathcal{L}_{\mathcal{G}}$.

Theorem

The theory $\mathrm{SCVH}^{\mathcal{G}}_{p,e}$ eliminates imaginaries.

Proof.

Applying the criterion requires to understand the pair $(\overline{K}^{\mathrm{alg}}, K)$ where $K \vDash \mathrm{SCVH}_{p,e}$. One can prove a quantifier elimination result for this structure by improving certain results of Delon. □

Other examples

- Let $\mathcal{F}_{p,q} := (\overline{\mathbb{F}_p((t))}^{\mathrm{alg}}, \mathrm{F}_p)$. We can consider

$$\prod_p \mathcal{F}_{p,p}/\mathfrak{U} \text{ or } \prod_q \mathcal{F}_{p,q}/\mathfrak{U}.$$

- $\mathrm{W}_{p,q} := (\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}}), \mathrm{W}(\mathrm{F}_q))$ and their ultraproducts.

The main issue is that definable types are not dense in these structure, so one has to find another approach, probably using invariant types.

Thanks!