Model theory of valued fields*

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Ever since Ax, Kochen [1] and Ershov's [2] celebrated results on the model theory of non archimedean local fields, valued fields have been among the "classical" algebraic structures studied in model theory. This is in large part due to the ubiquity of valuations in number theory and algebraic geometry, making the study of valued fields an important bridge between model theory and these subjects and leading to numerous beautiful applications. Most of the subsequent work on the model theory of valued fields focused on the class of henselian valued fields — in particular in the large body of work on (uniform) p-adic, and eventually motivic, integration. However, the earlier seminal work of Robinson [8] focused on the smaller class of the algebraically closed valued fields; and this class eventually came back into the spotlight in the early 2000's when ideas and techniques imported from geometric stability theory allowed a deep understanding of the structure of algebraically closed valued fields, eventually leading to such masterpieces as the Hrushovski-Kazhdan theory of motivic integration [5] and the Hrushovski-Loeser theory of rigid analytic geometry [6]. In these notes, our goal is to give a self-contained exposition of two cornerstones of this geometric theory of algebraically closed valued fields. The first is a description of the definable sets in the guise of an elimination of quantifiers, essentially dating back to Robinson's aforementioned work [8]. The second is a description of all interpretable sets, in the guise of the Haskell-Hrushovski-Macpherson elimination of imaginaries [3].

1 Valuations

1.1 Definitions and examples

We start by recalling the definition of a valuation:

Definition 1.1.1. Let K be a field, a *valuation* on K is a (surjective) map $v : K \to \Gamma$, where $(\Gamma, +, 0, <,)$ is a totally ordered commutative monoid[1], such that for every $x, y \in K$:

1. $v(0) \neq v(1) = 0$;
2. $v(x + y) \geqslant \min\{v(x), v(y)\}$;
3. $v(xy) = v(x) + v(y)$.

This definition would also make sense if R were a commutative ring, and this notion is central to analytic geometry (be it Berkovich's or Huber's). But although model theory of valued fields is indeed related to these subjects, valuations on rings will mostly not appear in this text.

Before giving classical examples of valuations, we first prove a few easy consequences of the definition of a valuation:

Lemma 1.1.2. *Let (K, v) be a valued field. For every $x, y \in K$:*

1. *$v(-x) = v(x)$;*
2. *if $v(x) < v(y)$ then $v(x+y) = v(x)$ — that is, every triangle is isoceles;*
3. *$v(x) \leqslant v(0)$, and $v(x) = 0$ if and only if $x = 0$.*

Proof.

1. If $0 \square v(-1)$, with $\square \in \{\leqslant, \geqslant\}$, then $v(-1) \square v(-1) + v(-1) = v((-1)^2) = v(1) = 0$. So $v(-1) = 0$ and thus $v(-x) = v(-1) + v(x) = v(x)$.
2. Assume $v(x) < v(y)$. Since $v(x+y) \geqslant \min\{v(x), v(y)\} = v(x)$ it suffices to rule out that $v(x+y) > v(x)$. If so, we would have $v(x) = v(x+y-y) \geqslant \min\{v(x+y), v(y)\} > v(x)$, a contradiction.
3. If $v(x) > v(0)$, then $v(x) = v(x+0) = v(0)$, a contradiction. As for the second assertion, it follows from the fact that $\{x \in K \mid v(x) = v(0)\}$ is a proper (prime) ideal; which is thus equal to 0. Indeed, for any $x, y \in K$ with $v(y) = v(0)$, then $v(xy) = v(x) + v(0) = v(x \cdot 0) = v(0)$. Also, if $v(x) = v(y) = v(0)$, we have $v(x+y) \geqslant \min\{v(x), v(y)\} = v(0)$ and hence $v(x+y) = v(0)$ by the first assertion.

□

Remark 1.1.3.

1. From now on, we will write $\infty := v(0)$, which is both maximal and annihilating. We have $v(K) = v(K^\times) \sqcup \{\infty\}$, where $v(K^\times) \leqslant \Gamma$ is a subgroup.
2. Let $(\Gamma, +, 0, <, \infty) = (\mathbb{R}_{\geqslant 0}, \cdot, 1, >, 0)$. Then a valuation $|.| : K \to \mathbb{R}_{\geqslant 0}$ is exactly a multiplicative norm on K verifying the strong triangular inequality:

$$|x+y| \leqslant \max\{|x|, |y|\} \leqslant |x| + |y|$$

for every $x, y \in K$.

Note that, for historical reasons, the order convention is reversed between norms and valuations. Being close to zero is having a small norm but a large valuation.

Let us now give some classical examples of valuations. Fix K a field.

Example 1.1.4. The map $v(x) = \mathbb{1}_{x=0}$ is a valuation, called the trivial valuation.

Example 1.1.5.

1. The map $-\deg : K(x) \to \mathbb{Z} \cup \{\infty\}$[2] is a valuation.

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[1]That is, $<$ is a total order and for every $x, y, z \in \Gamma$:

1. $(x+y)+z = x+(y+z)$;
2. $x+y = y+x$;
3. $x+0 = x$;
4. if $x \leqslant y$ then $x+z \leqslant y+z$.

[2]By convention, $\deg(0) = -\deg(0) = \infty$.

2. For any $a \in K$, define $v_a : K(x) \to \mathbb{Z} \cup \{\infty\}$ by:

$$v_a(f) = \begin{cases} n & \text{if } f \text{ has a root of mutiplicity } n \text{ at } a; \\ -n & \text{if } f \text{ has a pole of mutiplicity } n \text{ at } a; \\ 0 & \text{if } f \text{ has neither a root or a pole at } a; \\ \infty & \text{if } f = 0. \end{cases}$$

   This is a valuation. Note that, for any $f \in K(x)^\times$, $v_a(f)$ is the unique $n \in \mathbb{Z}$ such that $f = (x-a)^n P/Q$ where $P, Q \in K[x]$ are prime to $x - a$.

Note that, for any $f \in K(x)$, $v_a(f) = v_0(f(x+a))$. We say that $f \mapsto f(x-a)$ is a valued field isomorphism $(K(x), v_a) \to (K(x), v_0)$. Similarly, for every $f \in K(x)$, we have $-\deg(f) = v_0(f(x^{-1}))$. Indeed, let $P, Q \in K[X] \smallsetminus \{0\}$. Then $P(x^{-1}) = x^{-\deg(P)} P_1(x)$ where $P_1(x) = x^{\deg(P)} P(x^{-1}) \in K[X] \smallsetminus (X)$ and hence $v_0(P(x^{-1})/Q(x^{-1})) = v_0(x^{\deg(Q)-\deg(P)}) + v_0(P_1(x)Q_1(x)) = -\deg(P/Q)$. So $f \mapsto f(x^{-1})$ is a valued field isomorphism $(K(x), -\deg) \to (K(x), v_0)$.

Therefore, all these examples are essentially the same. By a theorem of Ostrowski, they are the only example of valuations on $K(x)$ which are trivial on K. They have natural analogs on number fields, and in particular, on $\mathbb{Q}$:

Example 1.1.6. Fix $p \in \mathbb{Z}$ a prime. For every $x \in \mathbb{Q}^\times$, we define the p-adic valuation $v_p(x)$ to be the unique $n \in \mathbb{Z}$ such that $x = p^n a/b$ with $a, b \in \mathbb{Z}$ prime to p; and $v_p(0) = \infty$.

By another theorem of Ostrowski, they are the only non trivial valuations on $\mathbb{Q}$.

1.2 Some algebra

Fix (K, v) a valued field. Let us describe some of the natural algebraic objects associated to valuations:

Definition 1.2.1. We define:

- the *value group* $vK^\times := v(K^\times)$ and its associated monoid $vK := v(K) = vK^\times \sqcup \{\infty\}$;
- the *valuation ring* $\mathcal{O} = \mathcal{O}_v := \{x \mid v(x) \geqslant 0\} \subseteq K$, a local subring;
- its unique maximal ideal $\mathfrak{m} = \mathfrak{m}_v := \{x \mid v(x) > 0\} \subset \mathcal{O}$;
- the *residue field* $Kv := \mathcal{O}/\mathfrak{m}$ and $\operatorname{res} = \operatorname{res}_v : \mathcal{O} \to k$ the canonical projection.

Proof. We have $v(0) = \infty > 0$ and $v(1) = 0$, so $0, 1 \in \mathcal{O}$. Also, for every $x, y \in \mathcal{O}$, $v(x+y) \geqslant \min\{v(x), v(y)\} \geqslant 0$ and $v(xy) = v(x) + v(y) \geqslant 0 + 0 = 0$. So $\mathcal{O} \subseteq K$ is a subring. Similarly, if $x, y \in \mathfrak{m}$, $v(x+y) \geqslant \min\{v(x), v(y)\} > 0$ and if $x \in \mathcal{O}$ and $y \in \mathfrak{m}$, then $v(xy) = v(x) + v(y) > 0$ and hence $\mathfrak{m} \subseteq \mathcal{O}$ is a ideal. Note also that $x \in \mathcal{O}$ is invertible if and only if $x^{-1} \in \mathcal{O}$, i.e. $-v(x) = v(x^{-1}) \geqslant 0$, or, equivalently, $v(x) = 0$. So $\mathcal{O}^\times = \mathcal{O} \smallsetminus \mathfrak{m}$ and $\mathfrak{m}$ is indeed the unique maximal ideal in $\mathcal{O}$. ☐

Lemma 1.2.2. *The ring $\mathcal{O}$ is integrally closed — that is, for every $P = X^d + \sum_{i<d} a_i x^i \in \mathcal{O}[x]$ and $c \in K = \mathcal{O}_{(0)}$, if $P(c) = 0$, then $c \in \mathcal{O}$.*

Proof. Assume $v(c) < 0$. Then, for every $i < d$, $v(c^d) = d \cdot v(c) < v(a_i) + iv(c)$ and hence $v(P(c)) = v(c^d + \sum_{i<d} a_i c^i) = d \cdot v(c) \neq \infty$. ☐

If R is a ring and $\mathfrak{p} \subseteq R$ is a prime ideal, we denote by $R_{\mathfrak{p}}$ the localisation of R at $\mathfrak{p}$ — that is, the ring of fractions a/b with $a \in R$ and $b \in R \smallsetminus \mathfrak{p}$. In particular, if R is integral, $R_{(0)}$ denotes the usual field of fractions. Let us now describe the valuation rings and residue fields of the previous examples:

Example 1.2.3.

- Let $(K(x), v_a)$ be as defined in Example 1.1.5. Then

  $$\mathcal{O}_{v_a} = \{P/Q \mid P, Q \in K[x] \text{ and } Q \notin (x-a)\} = K[x]_{(x-a)}$$

  and

  $$\mathfrak{m}_{v_a} = (x-a)\mathcal{O}_{v_0}.$$

  It follows that $Kv_a = K[x]_{(x-a)}/(x-a) \simeq (K[x]/(x-a))_{(0)} \simeq K$. The isomorphism is induced by the map: $K[x] \to K$ sending P to $P(a)$.
- We have $\mathcal{O}_{-\deg} = \{f \in K(x) \mid \deg(f) \leqslant 0\}$, and since $f \mapsto f(x^{-1})$ is an isomorphism between $(K(x), -\deg)$ and $(K(x), v_0)$, the residue field of $-\deg$ is isomorphic to K, where the isomorphism is induced by the map $\mathcal{O}_{-\deg} \to K$ given by $\sum_{i=0}^{n} a_i x^i / \sum_{i=0}^{n} b_i x^i \mapsto a_n/b_n$ where $b_n \neq 0$. In a (very precise) sense, this is the map $f \mapsto f(\infty)$.
- If v_p be the p-adic valuation on $\mathbb{Q}$ defined in Example 1.1.6, then $\mathcal{O}_{v_p} = \mathbb{Z}_{(p)}$ and $\mathbb{Q}v_p \simeq \mathbb{F}_p$.

As we saw, a valuation comes with a lot more algebraic structure than the typical archimedean norm. Let us also show that valuation rings can be characterised in a purely algebraic manner — this will be useful in Corollary 1.2.8 to construct extensions of valuations.

Proposition 1.2.4. *Let K be a field and $R \subseteq K$ be a subring. The following are equivalent:*

1. *there exists a valuation v on K such that $R = \mathcal{O}_v$;*
2. *for some prime ideal $\mathfrak{p} \subset R$, $(R, \mathfrak{p})$ is maximal for domination*[3] *in K — in particular, R is local and $\mathfrak{p}$ is its maximal ideal;*
3. *for all $x \in K^\times$, either $x \in R$ or $x^{-1} \in R$;*
4. *principal ideals of R are totally ordered by inclusion and $K = R_{(0)}$;*
5. *sub-R-modules of K are totally ordered by inclusion;*
6. *the monoid $(K/R^\times, \cdot, \overline{1})$ is totally ordered, where $\overline{a} \leqslant \overline{b}$ if $b \in R \cdot a$, and $\pi : K \to K/R^\times$ is a valuation.*

We say that R is a *valuation ring* if these equivalent conditions hold.

Moreover, $R = R_{\mathfrak{p}}$ is local and $\mathfrak{p}$ is its maximal ideal, indeed, since $R \leqslant R_{\mathfrak{p}} \leqslant K$ and $R_{\mathfrak{p}} \cdot \mathfrak{p} \cap R = \mathfrak{p}$, $(R_{\mathfrak{p}}, R_{\mathfrak{p}} \cdot \mathfrak{p})$ dominates $(R, \mathfrak{p})$. By maximality, they are equal.

Proof. Let us first assume (i) and prove (ii). Let v be such that $R = \mathcal{O}_v$ and let us assume that $(R, \mathfrak{m}_v)$ is dominated by some $(S, \mathfrak{q})$ with $S \leqslant K$. We want to show that $S \leqslant R$. Fix some $s \in S$. If $v(s) < 0$, then $s^{-1} \in \mathfrak{p}$. So $1 = ss^{-1} \in S \cdot \mathfrak{p} \subseteq \mathfrak{q}$, a contradiction. It follows that $v(s) \geqslant 0$ and $s \in R = \mathcal{O}_v$.

[3] If $R_1, R_2 \leqslant K$ are subrings and $\mathfrak{p}_i \subset R_i$ are prime ideals, $(R_2, \mathfrak{p}_2)$ dominates $(R_1, \mathfrak{p}_1)$ if $R_1 \leqslant R_2$ and $\mathfrak{p}_2 \cap R_1 = \mathfrak{p}_1$.

Let us now assume (ii) and prove (iii). Fixing some $x \in K$, first assume that $\mathfrak{p}[x] \subset R[x]$ and let $\mathfrak{m} \supseteq \mathfrak{p}[x]$ be some maximal ideal of $R[x]$. Then $\mathfrak{m} \cap R$ is a ideal of R containing $\mathfrak{p}$ but that does not contain 1. So it is equal to $\mathfrak{p}$ and $(R[x], \mathfrak{m})$ dominates $(R, \mathfrak{p})$. By maximality, we have $R[x] = R$ and thus $x \in R$. Applying this to x^{-1}, we see that, if $\mathfrak{p}[x^{-1}] \subset R[x^{-1}]$, then $x^{-1} \in R$. So we may assume that $1 \in \mathfrak{p}[x]$ and $1 \in \mathfrak{p}[x^{-1}]$ to derive a contradiction. Let m and n be minimal such that $\sum_{i \leqslant m} a_i x^i = 1 = \sum_{j \leqslant m} b_j x^{-j}$, for some $a_i, b_j \in \mathfrak{p}$. We may assume that $m \leqslant n$. Since $1 - a_0 \in R^{\times}$, we have $c_i = a_i(1-a_0)^{-1} \in \mathfrak{p}$ and $\sum_{i=1}^{m} c_i x^i = 1$ it follows that $1 = \sum_{j<n} b_j x^{-j} + \sum_{i=1}^{m} b_m c_i x^{-(n-i)}$, contradicting the minimality of n.

We now prove that (iii) implies (iv). It follows from (iii) that $K = R_{(0)}$. Moreover, for every $a, b \in R$, If $a^{-1}b \in R$, then $(b) \subseteq (a)$. If not, by (iii), we must have $b^{-1}a \in R$ and thus $(a) \subseteq (b)$.

Let us now prove that (iv) implies (v). Let $\mathfrak{a}, \mathfrak{b} \leqslant K$ be sub-R-modules and let us assume that there is some $a \in \mathfrak{a} \smallsetminus \mathfrak{b}$. We want to show that $\mathfrak{b} \subseteq \mathfrak{a}$. Let $b \in \mathfrak{b}$ and write $a = a_0/a_1$ and $b = b_0/b_1$ with $a_0, a_1, b_0, b_1 \in R$. By (iv) we either have $a_0 b_1 \in (b_0 a_1)$, in which case $a \in R \cdot b \subseteq \mathfrak{b}$, a contradiction, or $b_0 a_1 \in (a_0 b_1)$, in which case, $b \in R \cdot a \subseteq \mathfrak{a}$.

To prove that (v) implies (vi), note that the ordered set $(K/R^{\times}, \leqslant)$ is isomorphic to the set of principal sub-R-modules of K which is totally ordered by (v). Let us show that it is a monoid order. If $b \in R \cdot a$, then for every $c \in K$, $bc \in R \cdot ac$. So $\overline{a} \leqslant \overline{b}$ does indeed imply $\overline{a} \cdot \overline{c} \leqslant \overline{b} \cdot \overline{c}$.

Let us now check that π is a valuation. We have $\pi(0) = \overline{0} \neq \overline{1} = \pi(1)$. For every $x, y \in K$ we have $x + y \in (x) \cap (y)$, so $\pi(x+y) \geqslant \min\{\pi(x), \pi(y)\}$, and, by definition, $\pi(xy) = \pi(x) \cdot \pi(y)$.

Finally, note that (vi) trivially implies (i). □

In fact, the valuation in condition (i) is essentially unique — and is thus the valuation constructed in (vi). In fact, we can read off various relations between valuations from their valuations rings:

Lemma 1.2.5. *Let v be a valuation on K, $f : K \to L$ a field morphism and w a valuation on L. The following are equivalent:*

1. *$(\mathcal{O}_w, \mathfrak{m}_w)$ dominates $(f(\mathcal{O}_v), f(\mathfrak{m}_w))$;*
2. *$\mathcal{O}_v = f^{-1}(\mathcal{O}_w)$;*
3. *there is a (unique) injective morphism $g : vK \to wL$ such that:*

$$\begin{tikzcd} K \arrow[r, "v"] \arrow[d, "f"'] & vK \arrow[d, dotted, "g"] \\ L \arrow[r, "w"'] & wL \end{tikzcd}$$

commutes — in other words, f is a valued field embedding.

Proof. We first prove that (i) implies (ii). If $(\mathcal{O}_w, \mathfrak{m}_w)$ dominates $(f(\mathcal{O}_v), f(\mathfrak{m}_w))$, by definition, we have $f(\mathcal{O}_v) \subseteq \mathcal{O}_w$ — so $\mathcal{O}_v \subseteq f^{-1}(\mathcal{O}_w)$ — and $f(\mathfrak{m}_v) \subseteq \mathfrak{m}_w$. Let now $x \in K \smallsetminus \mathcal{O}_v$. Then $x^{-1} \in \mathcal{O}_v \smallsetminus \mathcal{O}_v^{\times} = \mathfrak{m}_v$. So $f(x)^{-1} = f(x^{-1}) \in f(\mathfrak{m}_v) \subseteq \mathfrak{m}_w$ and hence $f(x) \notin \mathcal{O}_w$. So we do have $f^{-1}(\mathcal{O}_w) = \mathcal{O}_v$.

We now prove that (ii) implies (iii). For the above diagram to commute, for every $x \in K$, we must have

$$g(v(x)) = w(f(x)). \tag{1}$$

Note that, by (ii), for every $x, y \in K$, we have $v(x) \leqslant v(y)$ — that is $y \in x\mathcal{O}_v$ — if and only if $f(y) \in f(x)\mathcal{O}_w$ — that is $w(f(y)) \leqslant w(f(x))$. It follows that Equation (1) gives rise to a well-defined increasing function $g : vK \to wL$.

Finally, if (iii) holds, then, for every $x \in \mathcal{O}_v$, $w(f(x)) = g(v(x)) \geqslant 0$ and hence $f(\mathcal{O}_v) \subseteq \mathcal{O}_w$. Moreover, if $x \in \mathfrak{m}_v$, then $w(f(x)) = g(v(x)) > 0$ and hence $f(\mathfrak{m}_v) \subseteq \mathfrak{m}_w \cap f(\mathcal{O}_v)$, which is an ideal of $f(\mathcal{O}_v)$ whose unique maximal ideal is $f(\mathfrak{m}_v)$. So $f(\mathfrak{m}_v) = \mathfrak{m}_w \cap f(\mathcal{O}_v)$. □

Lemma 1.2.6. *Let $v_i : K \to \Gamma_i$, for $i = 1, 2$, be valuations. The following are equivalent:*

1. *$\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$;*
2. *there is a unique isomorphism $g : v_1K \to v_2K$ such that:*

$$\begin{tikzcd} & v_1K \arrow[dd, dotted, leftrightarrow, "g"] \\ K \arrow[ur, "v_1"] \arrow[dr, "v_2"'] & \\ & v_2K \end{tikzcd}$$

commutes.

We then say that v_1 and v_2 are equivalent valuations.

Let us now conclude this brief introduction to valuations by showing that they can always be extended to larger fields.

Proposition 1.2.7. *Let R be a ring, $\mathfrak{p} \subset R$ be prime and $K \geqslant R$ a field. Then there exists a valuation ring $\mathcal{O} \subseteq K = \mathcal{O}_{(0)}$ with maximal ideal $\mathfrak{m}$ such that $(\mathcal{O}, \mathfrak{m})$ dominates $(R, \mathfrak{p})$.*

Proof. The set of pairs $(S, \mathfrak{q})$, where $S \leqslant K$ and $\mathfrak{q} \subset S$ is prime, ordered by domination is inductive. Indeed, let $(S_i, \mathfrak{q}_i)_i$ be a chain and let $S = \bigcup_i S_i \leqslant K$, a subring, and $\mathfrak{q} = \bigcup_i \mathfrak{q}_i = \bigcup_i S \cdot \mathfrak{q}_i \subset S$ an ideal. If, for some $a, b \in S$, $ab \in \mathfrak{q}$, let i be sufficiently large such that $a, b \in S_i$ and $ab \in \mathfrak{q}_i$. Then one of a or b is in $\mathfrak{q}_i \subseteq \mathfrak{q}$ and hence $\mathfrak{q} \subset S$ is prime. Moverover, for any i, by construction $\mathfrak{q}_i \subseteq \mathfrak{q} \cap S_i$ and if $a \in \mathfrak{q} \cap S_i$, then, for some $j \geqslant i$, $a \in \mathfrak{q}_j \cap S_i = \mathfrak{q}_i$, so $(S, \mathfrak{q})$ dominates $(S_i, \mathfrak{q}_i)$.

By Zorn's lemma, $(R, \mathfrak{p})$ is contained in a maximal element $(\mathcal{O}, \mathfrak{m})$. By Proposition 1.2.4, $\mathcal{O}$ is a valuation ring and $\mathfrak{m}$ is its maximal ideal. □

Corollary 1.2.8 (Chevalley's extension theorem). *Let (K, v) be a valued field and $K \leqslant L$ be a field extension. Then there exists a valuation w on L extending v.*

Proof. Applying Proposition 1.2.7 to $(\mathcal{O}_v, \mathfrak{m}_v)$ in L, we find a valuation ring $\mathcal{O} \subseteq L = \mathcal{O}_{(0)}$, with maximal ideal $\mathfrak{m}$, that dominates $(\mathcal{O}_v, \mathfrak{m}_v)$. We conclude with Lemma 1.2.5. □

1.3 Some topology

Let us now also describe the natural topology associated to a valuation:

Definition 1.3.1. Fix $x \in K$ and $\gamma \in vK$. We define:

- the *closed ball* $\mathrm{B}_{\geqslant\gamma}(x) := \{y \in K \mid v(y - x) \geqslant \gamma\}$ of radius γ around x;
- the *open ball* $\mathrm{B}_{>\gamma}(x) := \{y \in K \mid v(y - x) > \gamma\}$ of radius γ around x.

By convention, K is considered to be the open ball of radius $-\infty$. Note that points are closed balls of radius ∞.

However, due to the ultrametric inequality, the combinatorics of balls is much simpler than in the Archimedean setting.

Lemma 1.3.2. *Let b be a ball in K. Then $I_b := \{x - y \mid x, y \in b\} \subseteq K$ is a sub-$\mathcal{O}$-module and $b = x + I_b$ for any $x \in b$.*

Proof. Let us first assume that $b = \mathrm{B}_{\geqslant v(d)}(c)$ with $c, d \in K$, then $\{x - c \mid x \in b\} = d\mathcal{O}$ is a sub-$\mathcal{O}$-module of K. Moreover, if $x, y \in b$, then $x - y = (x - c) - (y - c) \in d\mathcal{O}$ and hence $I_b \subseteq \{x - c \mid x \in b\} = d\mathcal{O} \subseteq I_b$. Similarly, if $b = \mathrm{B}_{>v(d)}(c)$, then $I_b = d\mathfrak{m}$ a sub-$\mathcal{O}$-module of K. By definition $b = c + I_b$ is an additive coset of I_b and hence $b = c + I_b = x + I_b$ for any $x \in b$. □

Lemma 1.3.3. *Let b_1, b_2 be balls of K, then at least one of the following holds:*

1. $b_1 \cap b_2 = \varnothing$;
2. $b_1 \subseteq b_2$;
3. $b_2 \subseteq b_1$.

Proof. Note that we either have $I_{b_1} \subseteq I_{b_2}$ or $I_{b_2} \subseteq I_{b_1}$. So we may assume that $b_1 \cap b_2 \neq \varnothing$ and $I_{b_1} \subseteq I_{b_2}$. Let $x \in b_1 \cap b_2$, we then have $b_1 = x + I_{b_1} \subseteq x + I_{b_2} = b_2$. □

But balls do generate a somewhat well-behaved topology:

Proposition 1.3.4. *Open balls generate a Hausdorff field topology. Moreover, open balls are closed in this topology.*

In other words, the ideals $\gamma\mathfrak{m} := \mathrm{B}_{>\gamma}(0) \subseteq \mathcal{O}$, for $\gamma \in vK^{\times}_{\geqslant 0}$, form a basis of neighbourhoods of 0 and we consider the (unique) additive group topology generated by this basis of neighbourhoods of 0.

Proof. Let $U \subset K$ be open and $a, b \in K$ be such that $a + b \in U$. Then there exists $\gamma \in vK^{\times}$ such that $U \supseteq (a + b) + \gamma\mathfrak{m} = (a + \gamma\mathfrak{m}) + (b + \gamma\mathfrak{m})$, *i.e.* (a, b) is in the interior of the preimage by $+$ of U in K^2. So $+$ is continuous. Similarly, if $-a \in U$, for some $\gamma \in vK^{\times}$ we have $U \supseteq -a + \gamma\mathfrak{m} = -(a + \gamma\mathfrak{m})$, So $-$ is continuous. Finally, if $ab \in U$, there exists $\gamma \in vK^{\times}$, that we may assume larger than 0, with $U \supseteq ab + \gamma\mathfrak{m} \supseteq ab + a\cdot\delta\mathfrak{m} + b\cdot\delta\mathfrak{m} + \delta\mathfrak{m}\cdot\delta\mathfrak{m} \supseteq (a + \delta\mathfrak{m})\cdot(b + \delta\mathfrak{m})$, provided $\gamma \leqslant \min\{\delta + v(a), \delta + v(b), \delta\}$.

Moreover, for every $a \in K^{\times}, \gamma \in vK^{\times}$ and $x \in (\max\{v(a), \gamma - 2\cdot v(a)\})\mathfrak{m}$, by Lemma 1.1.2.2, we have $v(a + x) = v(a)$, and thus $v((a + x)^{-1} - a^{-1}) = v(xa^{-1}(a + x)^{-1}) = v(x) - 2\cdot v(a) > \gamma$; so the inverse map is indeed continuous at a.

The topology is Hausdorff since, for every $a, b \in K$ distinct, $\mathrm{B}_{>v(a-b)}(a) \cap \mathrm{B}_{>v(a-b)}(b) = \varnothing$. Finally, for any $\gamma \in vK^{\times}$ and $c \notin \mathrm{B}_{>\gamma}(a)$, $\mathrm{B}_{>\gamma}(a) \cap \mathrm{B}_{>\gamma}(c) = \varnothing$, so any open ball is closed in that topology. □

Remark 1.3.5. However, every non trivial ball is both open and closed in this topology and the topology is also generated by the non trivial closed balls.

Let us conclude this section by introducing diverse notions of convergence adapted to the valued field setting,

Definition 1.3.6. Fix $\mathfrak{F}$ a (proper) filter[4] on K and $x \in K$.

1. The filter $\mathfrak{F}$ is *Cauchy* if for every $\gamma \in vK$, there is an open ball of radius γ in $\mathfrak{F}$ — equivalently if there is a ball of radius γ in $\mathfrak{F}$.
2. The filter $\mathfrak{F}$ *converges to* x, and we write $\lim \mathfrak{F} = x$, if for every $\gamma \in vK^\times$, $\mathrm{B}_{>\gamma}(x) \in \mathfrak{F}$ — equivalently, if $\mathfrak{F} \supseteq \mathfrak{N}_x$, the neighbourhood filter of x.
3. The field K is *complete* if every Cauchy filter on K converges to some $x \in K$.

Let us now show that every valued field admits a (unique) completion.

Definition 1.3.7 (Leading terms)**.** Fix $\gamma \in vK^\times_{\geqslant 0}$. We define the multiplicative monoid of γ-*leading terms* $\mathrm{RV}_\gamma = \mathrm{RV}_{\gamma,v} := K/(1+\gamma\mathfrak{m})$. Let $\mathrm{rv}_\gamma = \mathrm{rv}_{\gamma,v} : K \to \mathrm{RV}_\gamma$ denote the canonical projection.

Remark 1.3.8.

1. It is naturally a multiplicative monoid and we have the following short exact sequence :
$$1 \to R_\gamma^\times \to \mathrm{RV}_\gamma^\times \to vK^\times \to 0,$$
where $R_\gamma = \mathcal{O}/\gamma\mathfrak{m}$ and $\mathrm{RV}_\gamma^\times := \mathrm{RV}_\gamma \setminus \{0\}$. We also denote by v the natural map $\mathrm{RV}_\gamma \to vK$
2. There is also the trace of an additive structure on RV_γ. We will describe it later.
3. We usually simply denote RV_0 as RV and rv_0 as rv.

Definition 1.3.9. Let
$$\widehat{K} = \widehat{K}_v := \varprojlim_{\gamma \in vK^\times_{\geqslant 0}} \mathrm{RV}_\gamma$$
as multiplicative monoids, where the transition maps $\mathrm{rv}_{\gamma,\delta} : \mathrm{RV}_\gamma \to \mathrm{RV}_\delta$, for $\infty > \gamma > \delta \geqslant 0$, are the natural maps. We also define:

- $v : \widehat{K} \to vK$ by $v(x) = v(x_\gamma)$ for any $\gamma \in vK^\times_{\geqslant 0}$;
- $+ : \widehat{K}^2 \to \widehat{K}$ by $(x+y)_\gamma = \mathrm{rv}_\gamma(X_\varepsilon)$, where $X_\varepsilon := \mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)$ does not contain 0, for sufficiently large $\varepsilon \in vK^\times_{\geqslant 0}$. If $0 \in X_\varepsilon$ for all ε, then define $x + y = (0)_\gamma =: 0$.

Proof. Since $v = v \circ \mathrm{rv}_{\gamma,\delta}$, v is indeed well-defined on $\widehat{K}$. As for $+$, let us fix $x, y \in \widehat{K}$. Note that X_ε is the open ball of radius $\delta = \varepsilon + \min\{v(x), v(y)\}$ around any of its elements. If $0 \notin X_\varepsilon$, then $v(X_\varepsilon)$ is a singleton $\{\gamma\}$ and $\mathrm{rv}_{\delta-\gamma}(X_\varepsilon)$ is also a singleton. Since the X_ε form a chain, $(x+y)_{\gamma-\delta}$ is well defined and γ is independent of ε, whereas δ increases as ε increases. □

Proposition 1.3.10. *The valued field* $(\widehat{K}, +, \cdot, v)$ *is complete.*

[4]That is, $\mathfrak{F} \subseteq \mathfrak{P}(K)$ such that:

1. $K \in \mathfrak{F}$, $\varnothing \notin \mathfrak{F}$;
2. for every $U, V \in \mathfrak{F}$, $U \cap V \in \mathfrak{F}$;
3. for every $U \subseteq V \subseteq K$, if $U \in \mathfrak{F}$ then $V \subseteq K$;

Proof. The fact the $(\widehat{K}, +, \cdot)$ is a field follows more or less directly from the definitions. Let us check distributivity, for example. Let $x, y, z \in \widehat{K}$. Then $\mathrm{rv}_\varepsilon^{-1}(z_\varepsilon)(\mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)) = \mathrm{rv}_\varepsilon^{-1}(z_\varepsilon) \cdot \mathrm{rv}_\varepsilon^{-1}(x_\varepsilon) + \mathrm{rv}_\varepsilon^{-1}(z_\varepsilon) \cdot \mathrm{rv}_\varepsilon^{-1}(y_\varepsilon)$. Since these sets characterise both $z(x+y)$ and $zx+zy$, they must be equal.

As for completeness, for every $x \in \widehat{K}$, and $\gamma \in v\widehat{K} = vK$, $\mathrm{B}_{>\gamma}(x) = \mathrm{rv}_{\gamma - v(x)}^{-1}(x_{\gamma - v(x)}) \subseteq \widehat{K}$. It follows that a Cauchy filter on $\widehat{K}$ is generated by sets of the form $\mathrm{rv}_\gamma^{-1}(\zeta_\gamma)$, for every $\gamma \in vK^\times_{\geqslant 0}$ and thus converges to $\zeta := (\zeta_\gamma)_\gamma \in \widehat{K}$. ☐

Remark 1.3.11. We have:

- $\iota : K \to \widehat{K}$ has dense image[5];
- $\mathcal{O}_{\widehat{K}} \simeq \varprojlim \mathcal{O}/\gamma\mathfrak{m}_K$ is the closure of $\mathcal{O}_K \subseteq \widehat{K}$;
- $\mathfrak{m}_{\widehat{K}} = \mathcal{O}_{\widehat{K}} \cdot \mathfrak{m}_K$ is the closure of $\mathfrak{m}_K \subseteq \widehat{K}$;
- $v\widehat{K} = vK$;
- $\widehat{K}v \simeq Kv$;
- $K/(1+\mathfrak{m}_K) \simeq \widehat{K}/(1+\mathfrak{m}_{\widehat{K}})$.

Let us now weaken the notion of Cauchy filter and convergence, to obtain the stronger notion of spherical completion, which will prove very important to study valued fields.

Definition 1.3.12. Fix $\mathfrak{F}$ a filter on K and $x \in K$.

1. The filter $\mathfrak{F}$ is *pseudo Cauchy* if it is generated by balls.
2. The filter $\mathfrak{F}$ accumulates at x, if any open ball around x meets any element of $\mathfrak{F}$ — equivalently, $x \in \bigcap_{U \in \mathfrak{F}} \overline{U} =: \overline{\mathfrak{F}}$.
3. The field K is *spherically complete* if every pseudo Cauchy filter on K accumulates at some in $x \in K$.

Remark 1.3.13.
- Usually, the accumulation points of a pseudo Cauchy filter $\mathfrak{F}$ are called its pseudo limits. Since balls are closed, in that case, we have $\overline{\mathfrak{F}} := \bigcap_{u \in \mathfrak{F}} U$.
- If it is not empty, the set $I = \{x - y \mid x, y \in \overline{\mathfrak{F}}\}$ is an $\mathcal{O}$-submodule of K called the width of $\mathfrak{F}$. As for balls, we have $\mathfrak{F} = x + I$ for any $x \in \overline{\mathfrak{F}}$.

The (potential) uniqueness of spherical completions is a much harder question. We conclude this section with the useful notion of the pushforward of a filter:

Definition 1.3.14. Let $\mathfrak{F}$ be a filter on some set X and $f : X \to Y$. We denote by $f_\star\mathfrak{F}$ the filter generated by $\{f(U) \mid U \in \mathfrak{F}\}$ — that is, the filter $\{V \subseteq Y \mid f(U) \subseteq V, \text{ for some } U \in \mathfrak{F}\}$.

Proof. Let $U, V \in \mathfrak{F}$. Note that, since $U \neq \varnothing$, $f(U) \neq \varnothing$. and since $f(U \cap V) \subseteq f(U) \cap f(V)$, the set $\{f(U) \mid U \in \mathfrak{F}\}$ has the finite intersection property and thus generates a filter. ☐

[5] In fact, $\widehat{K}$ is the unique, up to unique K-isomorphism, complete dense valued field extension of K. It also has the following universal properties: for every $f : K \to L$ with L complete and $g : K \to F$ with dense image, we have:

$$
\begin{tikzcd}
& \widehat{K} \arrow[dd, dotted, "\exists!"] \\
K \arrow[ur] \arrow[dr] & \\
& L
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widehat{K} \\
K \arrow[ur] \arrow[dr] & \\
& F \arrow[uu, dotted, "\exists!"']
\end{tikzcd}
$$

2 Algebraically closed valued fields

We now move on the more model theoretic part of these notes and focus on algebraically closed (non trivially) valued fields. The following language is one of the classically considered languages for valued fields. As we will see later, elimination of quantifiers will yield some more information (on the structure of the residue field and valued field) than we would typically get by working with a single sort for the field.

Definition 2.0.1. Let $\mathfrak{L}_{\mathrm{k},\Gamma}$ be the three sorted language with:

- a sort K with the ring language $\mathfrak{L}_{\mathrm{rg}} = (+,-,0,\cdot,1)$;
- a sort Γ with the ordered group language $\mathfrak{L}_{\mathrm{og}} = (+,-,0,<)$ and a constant ∞;
- a sort k with the ring language;
- a map $\mathrm{v} : \mathrm{K} \to \Gamma$;
- a map $\rho : \mathrm{K}^2 \to \mathrm{k}$.

Any valued field (K, v) can be made into a $\mathfrak{L}_{\mathrm{k},\Gamma}$-structure by interpreting K as the field K, k as the field k and Γ as vK, with its ordered monoid structure, $0 = v(1)$, $\infty = v(0)$ and $-$ is interpreted as the inverse on $vK^{\times}$ and $-\infty = \infty$. The map v is interpreted as the valuation and $\rho(a,b)$ is interpreted as $\mathrm{res}(a/b)$ if $b \neq 0$ and $a/b \in \mathcal{O}_K$, and 0 otherwise.

Definition 2.0.2. Let VF denote the $\mathfrak{L}_{\mathrm{k},\Gamma}$-theory of valued fields and ACVF denote the $\mathfrak{L}_{\mathrm{k},\Gamma}$-theory of algebraically closed non trivially valued fields.

Remark 2.0.3.

1. For all $n \in \mathbb{Z}_{>0}$, ACVF $\models (\forall x : \Gamma)(\exists y : \Gamma)\; n \cdot y = x$ — that is, the group $\Gamma^{\times} := \Gamma \smallsetminus \{\infty\}$ is divisible.
2. For every $P \in \mathbb{Z}[x, y]$ where y is a tuple, ACVF $\models (\forall y : \mathrm{k})(\exists x : \mathrm{k})\; P(x, y) = 0$ — that is, the residue field k is algebraically closed.
3. Any $M \models$ VF embeds into $N \models$ ACVF. If M is not trivially valued, we may assume $\mathrm{K}(N) = \mathrm{K}(M)^{\mathrm{a}}$.

In other words, VF $\models$ ACVF$_{\forall}$, the set of universal consequences of ACVF.

Proof.

1. Let $c \in K$ be such that $\mathrm{v}(c) = x$ and $a \in K$ such that $a^n = c$. We have $n \cdot \mathrm{v}(a) = \mathrm{v}(a^n) = \mathrm{v}(c)$.
2. Let $c \in \mathcal{O}$ be such that $\mathrm{res}(c) = y$ and $a \in K$ such that $P(c, a) = 0$. By Lemma 1.2.2, $a \in \mathcal{O}$ and we have $P(\mathrm{res}(c), \mathrm{res}(a)) = \mathrm{res}(P(c, a)) = 0$.
3. If M is trivially valued, then $(\mathrm{K}(M), \mathrm{v})$ embeds in $(\mathrm{K}(M)(x), v_0)$ which is non trivially valued. So we may assume M is not trivially valued. This is then an immediate consequence of Corollary 1.2.8.

$\square$

2.1 Elimination of quantifiers

In this section, we prove the elimination of quantifiers in ACVF, Theorem 2.1.8. To do so, we prove an embedding lemma, Proposition 2.1.6, of which quantifier elimination will follow formally. In fact, we will first prove three specific cases of the embedding lemma that, put together yield Proposition 2.1.6.

Let $M \vDash \mathrm{ACVF}$ and $A \leqslant M$. Assume that $\mathrm{K}(A)$ and $\mathrm{k}(A)$ are fields. In Propositions 2.1.2, 2.1.3 and 2.1.5, we describe various extensions by one valued field element. We start by extensions lifting a given residue element.

Definition 2.1.1. An exact lift of $Q \in \mathrm{k}[x]$ is $P \in \mathcal{O}[x]$ with $\mathrm{res}(P) = Q$ and $\deg(P) = \deg(Q)$.

Proposition 2.1.2 (Purely residual 1-types). *Fix any $\alpha \in \mathrm{k}(M)$. Let $P \in \mathcal{O}(A)[x]$ be an exact lift of its minimal polynomial*[6] *over $\mathrm{res}(\mathcal{O}(A))$ and $Q \in \mathrm{k}(A)[x]$ be its minimal polynomial over $\mathrm{k}(A)$.*

1. *For every $R = \sum_i r_i x^i \in \mathrm{K}(A)[x]$, every non zero $S = \sum_j s_j x^j \in \mathrm{K}(A)[x]$, both with degree smaller than P, and every $a \in \mathrm{res}^{-1}(\alpha)$:*
   - *$\mathrm{v}(R(a)) = \min_i \mathrm{v}(r_i) \neq \infty$;*
   - *$\rho(R(a), S(b)) = \rho(r_{i_0}, s_{j_0})\mathrm{res}(R_0)(\alpha)/\mathrm{res}(S_0)(\alpha)$, where $R = r_{i_0}R_0$, $S = s_{j_0}S_0$ and $\mathrm{v}(r_{i_0})$ and $\mathrm{v}(s_{j_0})$ are minimal.*
2. *There exists $a \in \mathrm{K}(M)$ with $\mathrm{res}(a) = \alpha$ and $P(a) = 0$.*
3. *Such an a is uniquely determined, up to $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-isomorphism, by P and Q: for every $N \vDash \mathrm{ACVF}$, every embedding $f : A \to N$, every $a \in \mathrm{K}(M)$ and $b \in \mathrm{K}(N)$, if:*
   - *$P(a) = 0$ and $\mathrm{res}(a) = \alpha$;*
   - *$f(P)(b) = 0$ and $f(Q)$ is the minimal polynomial of $\mathrm{res}(b)$ over $\mathrm{k}(f(A))$;*

   then f can be extended by sending a to b.

Proof.

1. Let i_0 be such that $\mathrm{v}(r_{i_0})$ is minimal. Let $R_0 := r_{i_0}^{-1}R$. Then $\mathrm{res}(R_0) \neq 0$. By minimality of $\mathrm{res}(P)$, $\mathrm{res}(R_0(a)) = \mathrm{res}(R_0)(\alpha) \neq 0$ and hence $\mathrm{v}(R_0(a)) = 0$. It follows that $\mathrm{v}(R(a)) = \mathrm{v}(r_{i_0}) = \min_i \mathrm{v}(r_i)$. Similarly $\mathrm{res}(S_0(a)) = \mathrm{res}(S_0)(\alpha) \neq 0$. It follows that $R(a)/S(a) \in \mathcal{O}$ if and only if $r_{i_0}/s_{j_0} \in \mathcal{O}$ and, in that case, $\mathrm{res}(R(a)/S(a)) = \mathrm{res}(r_{i_0}/s_{j_0})\mathrm{res}(R_0(a))/\mathrm{res}(S_0(a))$.
2. Let $P = c\prod_j(x - e_j)$, where $c \in \mathcal{O}_A^{\times}$. Since $\mathcal{O}$ is integrally closed, cf. Lemma 1.2.2, we have $e_j \in \mathcal{O}$ for all j. For any $a \in \mathrm{res}^{-1}(\alpha)$, $\mathrm{res}(P)(\alpha) = \mathrm{res}(P(a)) = \mathrm{res}(c)\prod_j \mathrm{res}(a) - \mathrm{res}(e_j) = 0$. It follows that there exists a j such that $\mathrm{res}(e_j) = \mathrm{res}(a) = \alpha$.
3. Let C be the structure generated by Aa. By 1, P is the minimal polynomial of a over $\mathrm{K}(A)$. So we have $\mathrm{K}(C) = \mathrm{K}(A)[a] \simeq \mathrm{K}(A)[x]/P$. Also by 1, $\Gamma(C) = \Gamma(A)$ and $\mathrm{k}(C) \subseteq \mathrm{k}(A)(\alpha) \simeq (\mathrm{k}(A)[x]/Q)_{(0)}$. Since $f(Q)$ is the minimal polynomial of β over $\mathrm{k}(f(A))$, we also have a ring embedding $g|_{\mathrm{k}} : \mathrm{k}(A)(\alpha) \to \mathrm{k}(N)$ extending $f|_{\mathrm{k}}$ and sending α to β. In particular, $\mathrm{k}(f(P))$ is the minimal polynomial of β over $\mathrm{k}(\mathcal{O}(f(A)))$; and $f(P)$ is an exact lift. Applying 1 to $\beta := \mathrm{res}(b)$, we see that $f(P)$ is the minimal polynomial of b over $\mathrm{K}(f(A))$ and thus $f|_{\mathrm{K}}$ extends to a ring embedding $g|_{\mathrm{K}} : \mathrm{K}(C) \to \mathrm{K}(N)$ sending a to b. Finally, let $g|_{\Gamma} = f|_{\Gamma}$.

   For any $R = \sum_i r_i x^i \in \mathrm{K}(A)[x]$ with degree smaller than P, by 1, we have $g(\mathrm{v}(R(a))) = f(\min_i \mathrm{v}(r_i)) = \min_i \mathrm{v}(f(r_i)) = \mathrm{v}(f(R)(b)) = \mathrm{v}(g(R(a)))$. Similarly, for any non

[6] We allow P to be 0; in which case, α is transcendental over $\mathrm{res}(\mathcal{O}(A))$

zero $S = \sum_i s_i x^i \in \mathrm{K}(A)[x]$ with degree smaller than P,

$$\begin{aligned} g(\rho(R(a),S(a))) &= g(\rho(r_{i_0},s_{j_0})\mathrm{res}(R_0)(\alpha)/\mathrm{res}(S_0)(\alpha)) \\ &= \rho(f(r_{i_0}),f(s_{j_0}))f(\mathrm{res}(R_0))(\beta)/f(\mathrm{res}(S_0))(\beta) \\ &= \rho(f(R)(b),f(S)(b)) \\ &= \rho(g(R(a)),g(S(a))), \end{aligned}$$

with i_0, j_0, R_0 and S_0 defined as in 1. So $g : C \to N$ is indeed an $\mathfrak{L}_{\mathrm{k},\Gamma}$-embedding sending a to b.

□

We now consider extensions lifting a given value group element:

Proposition 2.1.3 (Purely ramified 1-types). *Fix any $\gamma \in \Gamma^\times(M)$. Let n (respectively m) be its order in $\Gamma^\times(M)/\mathrm{v}(\mathrm{K}^\times(A))$ (respectively $\Gamma^\times(M)/\Gamma^\times(A)$) and $c \in \mathrm{K}(A)$ be such that $n \cdot \gamma = \mathrm{v}(c)$*[7]*.*

1. *For every $R = \sum_i r_i x^i \in \mathrm{K}(A)[x]$, every non zero $S = \sum_j s_j x^j \in \mathrm{K}(A)[X]$, both with degree smaller than n and every $a \in \mathrm{v}^{-1}(\gamma)$:*
   - *$\mathrm{v}(R(a)) = \min_i(\mathrm{v}(r_i) + i\gamma)$ and the minimum is attained exactly once;*
   - *$\rho(R(a),S(a)) = \mathbb{1}_{i_0=j_0}\rho(r_{i_0},s_{j_0})$, where $\mathrm{v}(r_{i_0})+i_0\gamma$ and $\mathrm{v}(s_{j_0})+j_0\gamma$ are minimal and $\mathbb{1}_{i_0=j_0} = 1$ if $i_0 = j_0$ and 0 otherwise.*
2. *There exists $a \in \mathrm{K}(M)$ with $\mathrm{v}(a) = \gamma$ and $a^n = c$*[8]*.*
3. *Such an a is uniquely determined, up to $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-isomorphism, by n, c, m, $m\cdot\gamma \in \Gamma(A)$ and, when $m = \infty$, by the set $\{\varepsilon \in \mathbb{Q}\cdot\Gamma(A) \mid \varepsilon < \gamma\}$. That is, for every $N \models \mathrm{ACVF}$, embedding $f : A \to N$, $a \in \mathrm{K}(M)$ and $b \in \mathrm{K}(N)$, if:*
   - *$a^n = c$ and $\mathrm{v}(a) = \gamma$;*
   - *$b^n = f(c)$;*
   - *$\mathrm{v}(b)$ is order m in $\Gamma^\times(N)/\Gamma^\times(f(A))$ and $m\cdot\mathrm{v}(b) = f(m\cdot\gamma)$;*
   - *when $m = \infty$, for all $\varepsilon \in \Gamma(A)$ and $r \in \mathbb{Q}$, $\mathrm{v}(b) > r\cdot f(\varepsilon)$ if and only if $\gamma > r\cdot\varepsilon$;*

   then f can be extended by sending a to b.

Proof. 1. We always have $\mathrm{v}(R(a)) = \mathrm{v}(\sum_i(r_i a^i)) \geq \min_i \mathrm{v}(r_i a^i) = \min_i(\mathrm{v}(r_i) + i\cdot\gamma)$. If the inequality were strict, there would exist $i < j < n$ such that $\mathrm{v}(r_i a^i) = \mathrm{v}(r_j a^j)$, i.e. $(j-i)\mathrm{v}(a) = \mathrm{v}(r_i) - \mathrm{v}(r_j) \in \mathrm{v}(\mathrm{K}(A))$, contradicting the minimality of n. We have also proved that all the $\mathrm{v}(r_i a^i) = \mathrm{v}(r_i) + i\cdot\gamma$ are distinct — in particular the minimum is unique. It follows that $\mathrm{res}(R(a)/r_{i_0}a^{i_0}) = \mathrm{res}(\sum_i r_i) = 1$. Note also that $\mathrm{v}(R(a)) - \mathrm{v}(S(a)) = 0$ if and only if $\mathrm{v}(r_{i_0}) - \mathrm{v}(s_{j_0}) + (i_0 - j_0)\gamma = 0$. Since $|i_0 - j_0| < n$, this is, in turn, equivalent to $i_0 = j_0$ and $\mathrm{v}(r_{i_0}) = \mathrm{v}(s_{j_0})$. In that case, $\mathrm{res}(R(a)/S(a)) = \rho(r_{i_0}a^{i_0}, s_{j_0}a^{j_0}) = \rho(r_{i_0}, s_{j_0})$.

2. Assume $n < \infty$. For any a with $a^n = c$, we have $n\mathrm{v}(a) = \mathrm{v}(c) = n\gamma$ and hence $\mathrm{v}(a) = \gamma$. If $n = \infty$, pick $c = 0$.

[7] By convention, $\infty\cdot\gamma = \infty$

[8] By convention $a^\infty = 0$

3. Let C be the structure generated by Aa. By 1, the minimal polynomial of a over $\mathrm{K}(A)$ is $x^n - c$. So we have $\mathrm{K}(C) = \mathrm{K}(A)[a] \simeq \mathrm{K}(A)[x]/(x^n - c)$. Also by 1, $\Gamma(C) = \Gamma(A) + \mathbb{Z}\cdot\gamma \simeq (\Gamma(A) \oplus \mathbb{Z}\cdot x)/(m \cdot x - m \cdot \gamma)$ and $\mathrm{k}(C) = \mathrm{k}(A)$. Since, by hypothesis, $\delta := \mathrm{v}(b)$ also has order m in $\Gamma^\times(N)/\Gamma^\times(f(A))$ and $m \cdot \delta = f(m \cdot \gamma)$, we have a group embedding $g|_\Gamma : \Gamma(C) \to \Gamma(N)$ sending γ to δ. In particular, δ is order n in $\Gamma^\times(M)/\mathrm{v}(\mathrm{K}^\times(f(A)))$ and $n \cdot \delta = \mathrm{v}(f(c))$. Applying 1 to δ, we see that $x^n - f(c)$ is the minimal polynomial of b over $f(\mathrm{K}(A))$ and hence $f|_\mathrm{K}$ extends to a ring embedding $g|_\mathrm{K} : \mathrm{K}(C) \to \mathrm{K}(N)$ sending a to b. Finally, let $g|_\mathrm{k} = f|_\mathrm{k}$.
If $m \neq \infty$, for every $r \in \mathbb{Z}$ and $\varepsilon \in \Gamma(A)$, we have $r \cdot \gamma > \varepsilon$ if and only if $r \cdot m \cdot \gamma > m \cdot \varepsilon$, if and only if $f(r \cdot m \cdot \gamma) > f(m \cdot \varepsilon)$, if and only if $r \cdot \delta > f(\varepsilon)$. If $m = \infty$ and $r \neq 0$, $r \cdot \gamma > \varepsilon$ if and only if $\gamma > r^{-1} \cdot \varepsilon$, if and only if, by hypothesis, $\delta > r^{-1} \cdot f(\varepsilon)$, if and only if $r \cdot \delta > f(\varepsilon)$. So $g|_\Gamma$ preserves $<$. Also, for any $R = \sum_i r_i x^i \in \mathrm{K}(A)[x]$ with degree smaller than P, by 1, we have $g(\mathrm{v}(R(a))) = g(\min_i(\mathrm{v}(r_i) + i \cdot \gamma)) = \min_i(\mathrm{v}(f(r_i)) + i \cdot \delta) = \mathrm{v}(f(R)(b)) = \mathrm{v}(g(R(a)))$. And, for any non zero $S = \sum_i s_i x^i \in \mathrm{K}(A)[x]$ with degree smaller than P, $g(\rho(R(a), S(a))) = g(\mathbb{1}_{i_0=j_0}\rho(r_{i_0}, s_{j_0})) = \mathbb{1}_{i_0=j_0}\rho(f(r_{i_0}), f(s_{j_0})) = \rho(f(R)(b), f(S)(b)) = \rho(g(R(a)), g((a)))$, with i_0, j_0 as in 1. So $g : C \to N$ is indeed an $\mathfrak{L}_{\mathrm{k},\Gamma}$-embedding sending a to b.

□

Finally, we consider extensions that add a pseudo-limit to a pseudo-Cauchy filter. It will be practical to use the morphism rv of Definition 1.3.7 since it computes both valuative and residual information at the same time.

Lemma 2.1.4. *Let $\mathfrak{B}$ be a pseudo Cauchy filter on $\mathrm{K}(A)$, without a minimal element, and $P \in \mathrm{K}(A)[x]$. Then one of the following holds:*
- *there is a ball $b \in \mathfrak{B}$ such that $\mathrm{rv}(P)|_b$ is constant.*
- *there is a root of P in $\overline{\mathfrak{B}}$ and any ball $b \in \mathfrak{B}$ contains two points $a_1, a_2 \in \mathrm{K}(A)$ such that $\mathrm{rv}(P(a_1)) \neq \mathrm{rv}(P(a_2))$.*

Proof. Let $b \in B$ contain the minimal number of roots of P. Then for every $x, y \in b$ and every root a a root of P outside b, we have $\mathrm{v}(x - a) < \mathrm{v}(y - x)$ and thus $y - a = y - x + x - a \in x - a + (x - a)\mathfrak{m} = (x - a)(1 + \mathfrak{m})$. So $\mathrm{rv}(x - a) = \mathrm{rv}(y - a)$. If b contains no root of P, then $\mathrm{rv}(P(x)) = \mathrm{rv}(c\prod_{P(a)=0}(x - a)^{m_a}) = \mathrm{rv}(c)\prod_{P(a)=0}\mathrm{rv}(x - a)^{m_a} = \mathrm{rv}(c)\prod_{P(a)=0}\mathrm{rv}(y - a)^{m_a} = \mathrm{rv}(P(y))$. So $\mathrm{rv}(P)|_b$ is contant.

If b does contain some root of P, since b is not minimal in $\mathfrak{B}$, we find a ball b_0 of $\mathrm{K}(A)$ in $\mathfrak{B}$ with $b_0 \subset b$. By choice of b, all the roots of P in b are in b_0. Also, we find $a_1 \in b_0(A)$ and $a_2 \in b(A) \smallsetminus b_0$. Then for every root a of P in b (and hence b_0), we have $\mathrm{v}(a_1 - a) > \mathrm{v}(a_2 - a)$ and hence $\mathrm{v}(P(a_1)/P(a_2)) = \mathrm{rv}(\prod_{a \in b, P(a)=0}(a_1 - a)/(a_2 - a)) > 0$. In particular, $\mathrm{rv}(P(a_1)) \neq \mathrm{rv}(P(a_2))$. □

Proposition 2.1.5 (Immediate 1-types). *Fix a pseudo Cauchy filter $\mathfrak{B}$ over $\mathrm{K}(A)$. Let $P \in \mathrm{K}(A)[x]$ have minimal degree among those polynomials such that $\mathrm{rv}(P)$ is not eventually constant on $\mathfrak{B}$.*

1. *For every $R \in \mathrm{K}(A)[x]$ with degree smaller than P, there exists $U \in \mathfrak{B}$ with $\mathrm{rv} \circ R|_U$ constant, equal to an element of $\mathrm{rv}(\mathrm{K}(A))$.*

2. *If $P \neq 0$, there exist $a \in \mathrm{K}(M)$ with $a \in \overline{\mathfrak{B}}$ and $P(a) = 0$.*
3. *Such an a is uniquely determined, up to $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-isomorphism, by $\mathfrak{B}$ and P. That is, for every $N \models \mathrm{ACVF}$, embedding $f : A \to N$, $a \in \mathrm{K}(M)$ and $b \in \mathrm{K}(N)$, if:*
   - *$P(a) = 0$ and $a \in \overline{\mathfrak{B}}$;*
   - *$f(P)(b) = 0$ and $b \in \overline{f_\star \mathfrak{B}}$;*

   then f can be extended by sending a to b.

Proof. 1. By minimality of P, by Lemma 2.1.4, $\mathrm{rv}(R)$ is constant on some $b \in \mathfrak{B}$. Since we may assume that b is a ball of A, $b \cap A \neq \varnothing$ and hence $\mathrm{rv}(R(b)) \in \mathrm{rv}(\mathrm{K}(A))$.

2. If there is no root of P in $\overline{\mathfrak{B}}$, then, by Lemma 2.1.4, $\mathrm{rv} \circ P$ is eventually constant on $\mathfrak{B}$. Since $0 \in \overline{P_\star \mathfrak{B}}$, we must have that P is eventually equal to 0 on $\mathfrak{B}$. If $P \neq 0$, $\mathfrak{B}$ contains the finite set of roots of P; in particular, $\overline{\mathfrak{B}} \neq \varnothing$.
3. Let C be the structure generated by Aa. By 1, P is the minimal polynomial of a over $\mathrm{K}(A)$. So we have $\mathrm{K}(C) = \mathrm{K}(A)[a] \simeq \mathrm{K}(A)[x]/P$. Also, by 1, we have $\Gamma(C) = \Gamma(A)$ and $\mathrm{k}(C) = \mathrm{k}(A)$. By 1 applied at $f_\star \mathfrak{B}$, the minimal polynomial of b over $\mathrm{K}(f(A))$ is $f(P)$, so $f|_{\mathrm{K}}$ extends to a ring embedding $g|_{\mathrm{K}} : \mathrm{K}(C) \to \mathrm{K}(N)$ sending a to b. Let also $g|_{\Gamma} = f|_{\Gamma}$ and $g|_{\mathrm{k}} = f|_{\mathrm{k}}$. Note that, since $\mathrm{rv}(\mathrm{K}(A)) = \mathrm{rv}(\mathrm{K}(C))$, the natural maps induced by $f|_{\mathrm{K}}$ and $g|_{\mathrm{K}}$ on RV coincide — we denote it as f.
   For any $R \in \mathrm{K}(A)[x]$ with degree smaller than P, by 1, we have
   $$\begin{aligned} g(\mathrm{rv}(R(a))) &= f(\mathrm{rv}(R)(U)) \\ &= \mathrm{rv}(f(R))(f(U)) \\ &= \mathrm{rv}(f(R))(b) \\ &= \mathrm{rv}(g(R(a))). \end{aligned}$$
   Since $\mathrm{v}(c) = \mathrm{v}(\mathrm{rv}(c))$ and $\rho(c,d) = \mathbb{1}_{\mathrm{v}(c)=\mathrm{v}(d)}\mathrm{rv}(c)\mathrm{rv}(d)^{-1}$, we see that g is compatible with both v and ρ.

$\square$

Proposition 2.1.6 (ACVF embedding lemma). *Let $M, N \models \mathrm{ACVF}$, $A \leqslant M$ and $f : A \to N$. There exists an elementary map $h : N \to N^\star$ and an embedding $g : M \to N$ such that:*

$$\begin{tikzcd} M \arrow[r, "g"] & N^\star \\ A \arrow[u, dash] \arrow[r, "f"'] & N \arrow[u, "h"'] \end{tikzcd}$$

commutes.

Proof. The family of pairs of embeddings (g,h), with $g : C \to N^\star$, $C \leqslant M$ and $h : N \to N^\star$ elementary, is inductive — where (g_1, h_1) is smaller than (g_2, h_2) if $C_1 \leqslant C_2$ and there exists $i : N_1^\star \to N_2^\star$ elementary such that

$$\begin{tikzcd} C_2 \arrow[r, "g_2"] & N_2^\star & \\ C_1 \arrow[u, dash] \arrow[r, "g_1"'] & N_1^\star \arrow[u, "i"] & N \arrow[l, "h_1"] \arrow[ul, "h_2"'] \end{tikzcd}$$

commutes — and contains (f, id). By Zorn's lemma, it contains a maximal element (g, h) — with domain $C \leqslant M$ — larger than (f, id). There remains to show that $M = C$. It suffices to show that $\mathrm{K}(M) = \mathrm{K}(C)$.

First, we note that, by the universal property of localisations g has a (unique) extension g_0 to $\mathrm{K}(C)_{(0)} \cup \mathrm{k}(C)_{(0)} \cup \Gamma(C)$. Indeed, g_0 is a ring morphism on the sorts K and k, and it is equal to g on Γ. For any $c \in \mathrm{K}(C)$ and non zero $d \in \mathrm{K}(C)$, $g_0(\mathrm{v}(c/d)) = g(\mathrm{v}(c) - \mathrm{v}(d)) = \mathrm{v}(g(c)) - \mathrm{v}(g(d)) = \mathrm{v}(g_0(c/d))$ and, for any $a \in \mathrm{K}(C)$ and non zero $b \in \mathrm{K}(C)$, $g_0(\rho(a/b, c/d)) = g(\rho(ad, cb)) = \rho(g(a)g(d), g(c)g(b)) = \rho(g_0(a/b), g_0(c/d))$. By maximality of g, both $\mathrm{K}(C)$ and $\mathrm{k}(C)$ are fields.

Claim 2.1.6.1. $\mathrm{k}(M) = \mathrm{res}(\mathrm{K}(C))$.

Proof. Pick any $\alpha \in \mathrm{k}(M)$. Let $P \in \mathcal{O}(C)[x]$ be an exact lift of its minimal polynomial over $\mathrm{res}(\mathcal{O}(C))$ and $Q \in \mathrm{k}(C)[x]$ its minimal polynomial over $\mathrm{k}(C)$. By compactness, we find $i : N^\star \to N^\dagger$ and $\beta \in \mathrm{k}(N^\dagger)$ whose minimal polynomial over C is $i \circ g(P)$ — if $Q \neq 0$, any root of $g(Q)$ in $N^\star$ works and we can take $i = \mathrm{id}$; if $Q = 0$, then the set $\{P(x) \neq 0 \mid P \in \mathrm{k}(g(C))\}$ is finitely satisfiable in N since $\mathrm{k}(N)$ is infinite. By Proposition 2.1.2.2, we can also find $a \in \mathrm{K}(M)$ and $b \in \mathrm{K}(N^\dagger)$ such that $P(a) = 0 = g(P)(b)$, $\mathrm{res}(a) = \alpha$ and $\mathrm{res}(b) = \beta$. Applying Proposition 2.1.2.3, we find a pair $(g', i \circ h)$ larger than (g, h) with g' defined at a. By maximality, $a \in \mathrm{K}(C)$ and hence $\alpha \in \mathrm{res}(\mathrm{K}(C))$. ◊

Claim 2.1.6.2. $\Gamma(M) = \mathrm{v}(\mathrm{K}(C))$.

Proof. Pick any $\gamma \in \Gamma(M)$. Let n (respectively m) its order in $\Gamma^\times(M)/\mathrm{v}(\mathrm{K}^\times(C))$ (respectively in $\Gamma^\times(M)/\Gamma^\times(C)$) and $c \in \mathrm{K}(C)$ such that $n \cdot \gamma = \mathrm{v}(c)$. By compactness, we find $i : N^\star \to N^\dagger$ and $\delta \in \Gamma^\times(N^\dagger)$ of order m in $\Gamma^\times(N^\dagger)/\Gamma^\times(i \circ g(C))$, $m \cdot \delta = i \circ g(m \cdot \gamma)$ and such that, for every $\varepsilon \in \Gamma(C)$ and $r \in \mathbb{Q}$, $\delta > r \cdot i(g(\varepsilon))$ if and only if $\gamma > r \cdot \varepsilon$ — if $m \neq \infty$, any element of $\Gamma(N^\star)$ with $m \cdot \delta = g(m \cdot \gamma)$ works and we can take $i = \mathrm{id}$; if $m = \infty$, the set $\{x > r \cdot g(\varepsilon) \mid r \in \mathbb{Q}, \varepsilon \in \Gamma(C)$ and $\delta > r \cdot \varepsilon\} \cup \{x < r \cdot g(\varepsilon) \mid r \in \mathbb{Q}, \varepsilon \in \Gamma(C)$ and $\delta < r \cdot \varepsilon\}$ is finitely satisfiable in N since $\Gamma^\times(N)$ is divisible and hense dense. By Proposition 2.1.3.2, we can also find $a \in \mathrm{K}(M)$ such that $a^n = c$ and $\mathrm{v}(a) = \gamma$, and $b \in \mathrm{K}(N^\dagger)$ such that $b^n = g(c)$ and $\mathrm{v}(b) = \delta$. Applying Proposition 2.1.3.3, we find a pair $(g', i \circ h)$ larger than (g, h) with g' defined at a. By maximality, $a \in \mathrm{K}(C)$ and hence $\alpha \in \mathrm{v}(\mathrm{K}(C))$. ◊

Claim 2.1.6.3. *Any pseudo Cauchy filter $\mathfrak{B}$ over $\mathrm{K}(C)$ that accumulates at some $a \in M$ also accumulates at some $c \in \mathrm{K}(C)$.*

Proof. Let $P \in K(C)[x]$ have minimal degree such that $\mathrm{rv}(P)$ is not eventually constant on $\mathfrak{B}$. If $P \neq 0$, by Proposition 2.1.5.2, there exists $c \in \mathrm{K}(M)$ such that $P(c) = 0$ and $c \in \overline{\mathfrak{B}}$. If $P = 0$, $c := a$ satisfies those same requirements. Since, by Lemma 2.1.4, the eventual non constancy of the rv of a polynomial on $\mathfrak{B}$ is witnessed by points of A, $g_\star(P)$ is also minimal such that $\mathrm{rv}(g_\star P)$ is not eventually constant on $g_\star\mathfrak{B}$. By compactness and Proposition 2.1.5.2, we also find $i : N^\star \to N^\dagger$ and $b \in \mathrm{K}(N^\dagger)$ with $b \in \overline{g_\star\mathfrak{B}}$ and $g(P)(b) = 0$ — if $P \neq 0$, some root of $g(P)$ in $N^\star$ works and we can take $i = \mathrm{id}$; if $P = 0$, the set of balls in $g_\star\mathfrak{B}$ is finitely satisfiable in N since it is a filter. Applying Proposition 2.1.5.3, we find a pair $(g', i \circ h)$ larger than (g, h) with g' defined at c. By maximality, $c \in \mathrm{K}(C)$ and, by construction, we do have $c \in \overline{\mathfrak{B}}$. ◊

Fix $a \in \mathrm{K}(M)$ and let $\mathfrak{B}$ be the maximal pseudo Cauchy filter over $\mathrm{K}(C)$ that accumulates at a — i.e. the filter generated by the balls of $\mathrm{K}(C)$ containing a. By Claim 2.1.6.3, $\mathfrak{B}$ accumulates at some $c \in \mathrm{K}(C)$. Let us assume that $a \neq c$. By Claim 2.1.6.2, there is some $d \in \mathrm{K}(C)$ such that $\mathrm{v}(a-c) = \mathrm{v}(d) \in \Gamma^{\times}(M)$. Then $(a-c)/d \in \mathcal{O}$ and, by Claim 2.1.6.1, there is some $e \in \mathcal{O}^{\times}(C)$ such that $\mathrm{res}((a-c)/d) = \mathrm{res}(e)$; equivalently $(a-c)/d - e \in \mathfrak{m}$ and hence $a \in c + de + d\mathfrak{m} =: b$, the open ball of radius $\mathrm{v}(d)$ around $c + de$. By construction, $b \in \mathfrak{B}$, but $\mathrm{v}(c + de - c) = \mathrm{v}(d) + \mathrm{v}(e) = \mathrm{v}(d)$, so $c \notin b$, a contradiction. It follows that $a = c \in \mathrm{K}(C)$, and hence $\mathrm{K}(M) = \mathrm{K}(C)$. □

Remark 2.1.7.

1. Proposition 2.1.6 does not hold if we replace ρ by the more straightforward map res. Let $\mathfrak{L}'_{\mathrm{k},\Gamma}$ be $\mathfrak{L}_{\mathrm{k},\Gamma}$ without the map ρ and with a map $\mathrm{res} : \mathrm{K} \to \mathrm{k}$ instead. Let $K := \mathbb{Q}(a,b)(t)$ and v be the t-adic valuation. Let A be the $\mathfrak{L}'_{\mathrm{k},\Gamma}$-substructure of K with $\mathrm{K}(A) = \mathbb{Z}[at, t]$, $\mathrm{k}(A) = \mathbb{Q}(a,b)$, $\Gamma(A) = \mathbb{Z} \cup \{\infty\}$. Let $f : A \to K$ send at to bt, fix t and be the identity on res and Γ. This is an $\mathfrak{L}'_{\mathrm{k},\Gamma}$-embedding. If g extends f and is defined at t^{-1}, we would have $a = f(a) = f(\mathrm{res}(a)) = \mathrm{res}(g(att^{-1})) = \mathrm{res}(f(at)f(t)^{-1}) = \mathrm{res}(btt^{-1}) = b$, a contradiction. So Proposition 2.1.6 fails for $\mathfrak{L}'_{\mathrm{k},\Gamma}$.
   In fact, the formula $\exists x\ \mathrm{res}(x) = a \wedge xt = at$ is not preserved by f and is therefore not equivalent to a quantifier free formula. It is, however, equivalent to $\rho(at, t) = a$.
2. In the last paragraph of the proof, we have, in fact, shown that a spherically complete field does not have any proper immediate extension. It follows that if K is spherically complete with $vK^{\times}$ divisible and Kv algebraically closed, then any algebraic extension of K is immediate and hence trivial. So K is algebraically closed.

We can now deduce our main theorem of this section:

Theorem 2.1.8 (Robinson, 1956). *The $\mathfrak{L}_{\mathrm{k},\Gamma}$-theory* ACVF *eliminates quantifiers.*

Proof. By [9, Theorem 3.2.5], this is an immediate consequence of Proposition 2.1.6. □

Corollary 2.1.9. *The class of existentially closed models of* VF *coincides with* ACVF.

Proof. By Remark 2.0.3.3, any model of VF embeds in a model of ACVF, so it suffices to check that embeddings between models of ACVF are existentially closed. But this is an immediate consequence of elimination of quantifiers, *cf.* Theorem 2.1.8. □

Corollary 2.1.10. *The completions of* ACVF *are the theories* $\mathrm{ACVF}_{p,q}$*, with p, q prime or zero, of non trivially valued algebraically closed valued fields of characteristic p and residue characteristic q.*

Note that if $p > 0$, then $q = p$.

Proof. By [9, Theorem 3.2.5], it suffices to find a common substructure to any two models of $\mathrm{ACVF}_{p,q}$. If $q = p > 0$, the trivially valued field $\mathbb{F}_p$ embeds (uniquely) in any model of $\mathrm{ACVF}_{p,p}$. If $q = p = 0$, the trivially valued field $\mathbb{Q}$ embeds (uniquely) in any model of $\mathrm{ACVF}_{0,0}$. Finally, the field $\mathbb{Q}$ with the p-adic valuation embeds (uniquely) in every model of $\mathrm{ACVF}_{0,p}$. □

Note that, as a corollary, we get the classical "conjugation theorem" of valuation theory:

Corollary 2.1.11. *Let (K, v) be a valued field, L be a normal extension and w_1, w_2 be two valuations on L extending v, then there exists $\sigma \in \operatorname{aut}(L/K)$ such that $w_2 \circ \sigma$ is equivalent to w_1.*

Proof. Let L_i be the $\mathfrak{L}_{\mathrm{k},\Gamma}$-structure associated to (L, w_i). By hypothesis, the identity on K is an $\mathfrak{L}_{\mathrm{k},\Gamma}$-embedding from L_1 to L_2. By Remark 2.0.3.3, $L_i \leqslant M_i \vDash \mathrm{ACVF}$. By elimination of quantifiers (Theorem 2.1.8), there exists an $\mathfrak{L}_{\mathrm{k},\Gamma}(K)$-embedding $\sigma : L_1 \to M_2$. Since L is normal, $\sigma(L_1) = L_2$. So we do have $\sigma \in \operatorname{aut}(L/K)$ and $w_2 \circ \sigma$ is equivalent to w_1. □

2.2 Properties of definable sets

We now give a complete description of types concentrating on K over algebraically closed subsets of K in ACVF. Fix $M \vDash \mathrm{ACVF}$ and $A = A^{\mathrm{a}} \leqslant \mathrm{K}(M)$.

Definition 2.2.1. Let $\mathfrak{B}$ be a pseudo Cauchy filter over A. An element $a \in \mathrm{K}(M)$ is said to be *generic in $\mathfrak{B}$ over A* if, for every ball b of A:

$$a \in b \text{ if and only if } b \in \mathfrak{B}.$$

We write $\eta_{\mathfrak{B}}|_A$ for the — *a priori* partial — type of generics of $\mathfrak{B}$ over A.

Proposition 2.2.2. *Let $a \in \mathrm{K}(M)$. Let $\mathfrak{B}$ be generated by $\{b \text{ ball in } A \mid a \in b\}$. Then:*

$$\eta_{\mathfrak{B}}|_A \vDash \operatorname{tp}(a/A).$$

In particular $\eta_{\mathfrak{B}}|_A$ is a complete type over A.

Proof. Note that, by construction, $a \vDash \eta_{\mathfrak{B}}|_A$. Let us first assume that $\mathfrak{B}$ is generated by some closed ball $b_0 = \mathrm{B}_{\geqslant \mathrm{v}(d)}(c)$, with $c, d \in A$. If $d = 0$, then $\eta_{\mathfrak{B}}|_A(x) \vDash x = c$ which generates a complete type. Otherwise, let $a' \vDash \eta_B|_A$ and $\alpha' = \operatorname{res}((a' - c)/d)$. If $\alpha' \in \operatorname{res}(A)^{\mathrm{a}} = \operatorname{res}(A)$ — by Proposition 2.1.2.2 — then we find $e \in \mathrm{K}(A)$ with $\operatorname{res}(e) = \alpha'$, *i.e.* $\mathrm{v}(a' - (c + de)) > \mathrm{v}(d)$, so $a' \in \mathrm{B}_{>\mathrm{v}(d)}(c + de)$, a contradiction to $a' \vDash \eta_B|_A$. So $\alpha' \notin \operatorname{res}(A)^{\mathrm{a}}$ and, by Proposition 2.1.2.3, $(a' - c)/d \equiv_A (a - c)/d$, and hence $a' \equiv_A a$.

Let us now assume that $\mathfrak{B}$ is not principal — *i.e.* it is not generated by a single ball — and that $\overline{\mathfrak{B}} \cap A = \varnothing$. If the minimal P such that $0 \in \overline{P_\star \mathfrak{B}}$ is not 0, by 2.1.5.2, $\overline{\mathfrak{B}} \cap A = \overline{\mathfrak{B}} \cap A^{\mathrm{a}} \neq \varnothing$, a contradiction. So $P = 0$ and, since any $a' \vDash \eta_B|_A$ is in $\overline{\mathfrak{B}}$, by 2.1.5.3, we have $a' \equiv_A a$.

Let us now deal with the remaining case. We may therefore assume that $\mathfrak{B}$ is not generated by a single closed ball and that there exists some $c \in \overline{\mathfrak{B}} \cap A$. Let $a' \vDash \eta_B|_A$. For every $\gamma \in \mathrm{v}(A)$, we have $\mathrm{v}(a' - c) \geqslant \gamma$ if and only if $\mathrm{B}_{\geqslant\gamma}(c) \in \mathfrak{B}$, which is equivalent, since $\mathfrak{B}$ is not generated by $\mathrm{B}_{\geqslant\gamma}(c)$, to $\mathrm{B}_{>\gamma}(c) \in \mathfrak{B}$, *i.e.* $\mathrm{v}(a' - c) > \gamma$. It follows that $\mathrm{v}(a' - c) \notin \mathrm{v}(A) = \mathbb{Q} \cdot \mathrm{v}(A)$ — the equality follows from Proposition 2.1.3.2 — and that, by Proposition 2.1.3.3, $(a' - c) \equiv_A (a - c)$. So $a' \equiv_A a$. □

Remark 2.2.3. We see from the proof that there is a correspondence between the descriptions of the types over A concentrating on K in algebraic terms and in terms of generics of pseudo Cauchy filters:

- generics of closed balls correspond, up to translation and scaling, to residual extensions;

- generics of open balls correspond, up to translation, to ramified extensions where the cut is of the form γ^+;
- generics of non principal pseudo Cauchy filters with an accumulation point in A correspond, up to translation, to the other ramified extensions;
- generics of non principal pseudo Cauchy filters without accumulation points in A correspond to immediate extensions.

Proposition 2.2.2 states, among other thing that types in K are entirely determined by their restriction to the Boolean algebra generated by balls. By some abstract non-sense, every definable subset of K is, up to equivalence, in said algebra.

Definition 2.2.4.
- An A-Swiss cheese is a set of the form $b \smallsetminus \bigcup_{i<n} b_i$ where b is a ball in A and the $b_i \subset b$ are (disjoint) subballs, in A.
- A Swiss cheese $b \smallsetminus \bigcup_i b_i$ is nested inside some other Swiss cheese $d \smallsetminus \bigcup_j d_j$ if there exists a j such that $b = d_j$.

Theorem 2.2.5 (Holly, 1995). *Any $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-definable subset of* K *has a unique decomposition as a finite disjoint union of non-nested A-Swiss cheeses.*

Proof. Let $\Delta(x)$ be the set of finite unions of A-Swiss cheeses.

Claim 2.2.5.1. $\Delta(x)$ *is stable under Boolean combinations.*

Proof. It suffices to show that the intersection of two A-Swiss cheeses is an A-Swiss cheese and that the complement of an A-Swiss cheese is a finite union of A-Swiss cheese. Let $B = b \smallsetminus \bigcup_i b_i$ and $D = d \smallsetminus \bigcup_i d_i$ be two Swiss cheeses. We have $B \cap D = (b \cap d) \smallsetminus (\bigcup_i (d \cap b_i) \cup \bigcup_j (b \cap d_j))$ where some of the intersections might be empty. Similarly $\mathrm{K} \smallsetminus B = (\mathrm{K} \smallsetminus b) \cup \bigcup_i b_i$. $\lozenge$

Proposition 2.2.2 implies that for all $p \in \mathcal{S}_x(A)$, $p|_\Delta := p \cap \Delta \vDash p$. In other terms, for any $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-formula $\varphi(x)$, $a \in \varphi(M)$ and $c \in \mathrm{K}(M)$, if $c \vDash \mathrm{tp}_\Delta(a)$, then $M \vDash \varphi(c)$. By compactness (see, for example [9, Lemma 3.1.1]), any $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-definable subset of K is equivalent to a formula in $\Delta(x)$, that is a finite union of A-Swiss cheeses. Since the union of two non disjoint A-Swiss cheeses is an A-Swiss cheese — $(b \smallsetminus \bigcup_i b_i) \cup (d \smallsetminus \bigcup_i d_i) = b \smallsetminus (\bigcup_i (b_i \smallsetminus d) \cup \bigcup_{i,j} (b_i \cap d_j))$, where $d \subseteq b$ — and the union of two nested swiss cheeses is a swiss cheese — $b \smallsetminus \bigcup_i b_i \cup (d \smallsetminus \bigcup_i d_i) = b \smallsetminus (\bigcup_{i>0} b_i \cup \bigcup_j d_j)$, where $d = b_0$ — we may assume that the finite union is a disjoint union of non-nested A-Swiss cheeses.

Uniqueness now follows from:

Claim 2.2.5.2. *Let $(D_i)_{i<n}$ be disjoint non nested Swiss cheeses and B be some Swiss cheese such that $B \subseteq \bigcup_i D_i$, then there is some i such that $B \subseteq D_i$.*

Proof. We may assume that the D_i form a minimal cover. In particular, we then have that, for every i, $B \cap D_i \neq \varnothing$. If $n = 1$ then the claim is proved. So, let us assume that $n \geqslant 2$.

Let us first assume that the D_i and B are balls. Pick $c_i \in B \cap D_i$. Let b be the smallest ball containing all the c_i. It is a closed ball of radius $\gamma := \min_{i \neq j} \mathrm{v}(c_i - c_j)$. We have $b \subseteq B \subseteq \bigcup_i D_i$. If $b \subseteq D_i$, for some i, then every c_j is in $b \subseteq D_i$, contradicting that $D_i \cap D_j = \varnothing$. So we must have that, for every i, $D_i \subset b$. Let $d_i := \mathrm{B}_{>\gamma}(c_i)$, it is the maximal strict subball of b

containing c_i, in particular $D_i \subseteq d_i$. So $b \subseteq \bigcup_i d_i$. Let d be some maximal open subball of b, then d intersects some d_i and by maximality $d = d_i$. It follows that $\mathrm{R}_b := \{d \subset b \mid \text{maximal open subball}\} = \{c + \gamma\mathfrak{m} \mid c \in b\}$ is finite. However, if we choose $a \in b$ and $e \in \mathrm{v}^{-1}(\gamma)$, $c \mapsto (c-a)/e$ is a bijection $b \to \mathcal{O}$ sending elements of R_b to elements of k which is infinite; a contradiction.

Let us now come back to the general case $B = b \smallsetminus \bigcup_j b_j$, $D_i = d_i \smallsetminus \bigcup_\ell d_{i,\ell}$, where the $d_{i,\ell}$ are disjoint. Since $B \subseteq \bigcup_i D_i$, we have $b \subseteq \bigcup_i d_i \cup \bigcup_j b_j$. It might happen that the d_i and b_j are not disjoint, but a subset of them is and covers b. So, by the previous case, and since $b_j \subset b$, there is some i such that $b \subseteq d_i$. If $b \cap d_{i,\ell} = \varnothing$, for all i, we indeed have $B \subseteq D_i$ and that concludes the proof. So we may assume that there is some ℓ such that $b \cap d_{i,\ell} \neq \varnothing$. If $b \subseteq d_{i,\ell}$, then $B \cap D_i = \varnothing$; a contradiction. Hence $d_i \subset b$. It follows that $d_{i,\ell} \smallsetminus \bigcup_j \subseteq B \cap d_{i,\ell} \subseteq \bigcup_{i'} D'_i \cap d_{i,\ell} \subseteq \bigcup_{i' \neq i} D_{i'}$, since $D_i \cap d_{i,\ell} = \varnothing$. In other words, $d_{i,\ell} \subseteq \bigcup_{i' \neq i} d_{i'} \cup \bigcup_j b_j$. So, by the case for balls proved above, either $d_{i,\ell} \subseteq b_j$, for some j, or $d_{i,\ell} \subseteq d_{i'}$, for some $i' \neq i$. In the latter case, since $D_i \cap D'_i = \varnothing$, $d'_i \subseteq \bigcup_{\ell'} d_{i,\ell'}$ and hence it is covered by one of the $d_{i,\ell'}$. Recall that the $d_{i,\ell'}$ are disjoint and $d_{i,\ell} \cap d_{i'} \neq \varnothing$ and hence $d_{i,\ell} = d_{i'}$, contradicting the assumption that the D_i are not nested. It follows that any $d_{i,\ell}$ that intersects b is contained in some b_j, that is $B \subseteq D_i$. ◊

Uniqueness of the decomposition follows from the fact that whenever a Swiss cheese is included in a finite disjoint union of non-nested Swiss cheeses, then it is included in one of those Swiss cheeses. □

Remark 2.2.6. Note that we have also proved the following very useful fact: if $A = A^{\mathrm{a}} \subseteq \mathrm{K}(M)$, then any A-definable ball is an A-ball; that is, it has a center in A and its valuation is in $\mathrm{v}A$. If A is non trivially valued, by model completeness, $A \succcurlyeq M$, so the above statement is clear. But if A is trivially valued, it requires a little work.

We will now describe the structure induced on the residue field and the value group. But let us introduce some terminology first:

Definition 2.2.7. Let T be an $\mathfrak{L}$-theory and D be a $\mathfrak{L}$-definable set. We say that D is stably embedded if for every $M \vDash T$ and every $\mathfrak{L}(M)$-definable $X \subseteq D^n$, X is $\mathfrak{L}(D(M))$-definable.

Definition 2.2.8. Let T be an $\mathfrak{L}$-theory and D be some $\mathfrak{L}'$-structure interpretable[9] in T. We say that D is a pure $\mathfrak{L}'$-structure if any $\mathfrak{L}$-definable $X \subseteq D^n$ is $\mathfrak{L}'$-definable.

In particular, if D is also stably embedded, any $\mathfrak{L}(M)$-definable subset of D^n is $\mathfrak{L}'(D(M))$-definable.

Definition 2.2.9. Let T be an $\mathfrak{L}$-theory, two $\mathfrak{L}$-definable sets D_1 and D_2 are orthogonal if for every $M \vDash T$, any $\mathfrak{L}(M)$-definable set $X \subseteq D_1^{n_1} \times D_2^{n_2}$ is a finite union of boxes of the form $Y_1 \times Y_2$ where $Y_i \subseteq D_i^{n_i}$ is $\mathfrak{L}(M)$-definable.

Let us now fix some $M \vDash \mathrm{ACVF}$ and some $A \leqslant M$.

Proposition 2.2.10. *If $X \subseteq \mathrm{k}^n$ is $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-definable, then it is $\mathfrak{L}_{\mathrm{rg}}(\mathrm{k}(A))$-definable. In particular, the residue field k is a stably embedded pure ring.*

[9] That is for every sort X of $\mathfrak{L}$ a choice of an $\mathfrak{L}$-definable set X^T, for every function symbol $f : X \to Y$ of $\mathfrak{L}$, an $\mathfrak{L}$-definable function $f^T : \prod_i X_i^T \to Y^T$ and for every relation symbol $R \subseteq X$ of $\mathfrak{L}$, an $\mathfrak{L}$-definable subset of $\prod_i X_i^T$.

Proof. By elimination of quantifiers (Theorem 2.1.8), and since $\mathfrak{L}_{\mathrm{rg}}(\mathrm{k}(A))$-definable sets are closed under Boolean combinations, it suffices to consider atomic formulas. So we may assume X is defined by $R(x, \rho(P(a), Q(a)), \alpha) = 0$ where x, y, z are tuples, $R \in \mathbb{Z}[x, y, z]$, $P, Q \in \mathbb{Z}[t]$ are y-tuples, $a \in \mathrm{K}(A)^t$ and $\alpha \in \mathrm{k}(A)^z$ is a tuple. Since $\rho(P(a), Q(a)) \in \mathrm{k}(A)$, this is indeed an $\mathfrak{L}_{\mathrm{rg}}(\mathrm{k}(A))$-formula. □

Proposition 2.2.11. *If $X \subseteq \Gamma^n$ is $\mathfrak{L}_{\mathrm{k},\Gamma}(A)$-definable, then it is $\mathfrak{L}_{\mathrm{og}}(\Gamma(A))$-definable. In particular, the value group Γ is a stably embedded pure ordered monoid.*

Proof. As above, it suffices to consider atomic formulas. So we may assume X is defined by $L(x, \mathrm{v}(P(a)), \gamma) < 0$ where L is a $\mathbb{Z}$-linear function, $P \in \mathbb{Z}[t]$ is a tuple, $a \in \mathrm{K}(M)^t$ and $\gamma \in \Gamma(M)$ is a tuple. This is indeed an $\mathfrak{L}_{\mathrm{og}}(\Gamma(A))$-formula. □

Proposition 2.2.12. *The value group Γ and the residue field k are orthogonal.*

Proof. Since finite unions of boxes are closed under Boolean combinations, it suffices to consider atomic formulas. But variables from Γ and k cannot both occur in the same atomic $\mathfrak{L}_{\mathrm{k},\Gamma}$-formula. So subsets of $\Gamma^n \times \mathrm{k}^m$ defined by atomic formulas are either of the form $\Gamma^n \times Y_2$ for some $Y_2 \subseteq \mathrm{k}^m$ or $Y_1 \times \mathrm{k}^m$ for some $Y_1 \subseteq \Gamma^n$. □

We conclude this section by describing the algebraic closure in ACVF:

Proposition 2.2.13. *We have* $\mathrm{acl}(A) = \mathrm{K}(A)^{\mathrm{a}} \cup \mathrm{k}(A)^{\mathrm{a}} \cup \mathbb{Q} \cdot \Gamma(A)$.

Proof. Fix $\alpha \in \mathrm{k}(M)$ and let P be its minimal polynomial over $\mathrm{k}(A)$. Then, either $P \neq 0$, in which case, since P has finitely many roots, we have $\alpha \in \mathrm{acl}(A)$, or $P = 0$, in which case, by Proposition 2.1.2 and Theorem 2.1.8, there is an $\mathfrak{L}(A)$-elementary embedding sending α to any $\beta \in M^\star \succcurlyeq M$ transcendental over $\mathrm{k}(A)$ and hence $\alpha \notin \mathrm{acl}(A)$. So $\mathrm{k}(\mathrm{acl}(A)) = \mathrm{k}(A)^{\mathrm{a}}$.

Fix $\gamma \in \Gamma^\times(M)$ and let n be its order in $\Gamma^\times(M)/\Gamma^\times(A)$. Then, if $n \neq \infty$, since $\Gamma^\times(M)$ is torsion free, $\gamma \in \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$. If $n = \infty$, by Proposition 2.1.2 and Theorem 2.1.8, there is an $\mathfrak{L}(A)$-elementary embedding sending γ to any $\delta \in M^\star \succcurlyeq M$ realising the same cut. So $\gamma \notin \mathrm{acl}(A)$. It follows that $\Gamma(\mathrm{acl}(A)) = \mathbb{Q} \cdot \Gamma(A)$.

Now, fix $a \in \mathrm{K}(M)$. By Proposition 2.2.2, $\mathrm{tp}(a/\mathrm{K}(A)^{\mathrm{a}})$ is the generic of some pseudo Cauchy filter $\mathfrak{B}$. Since non trivial balls are infinite, such a generic is algebraic if and only if $\mathfrak{B}$ contains a singleton, in which case $a \in K(A)^{\mathrm{a}}$. So $\mathrm{acl}(\mathrm{K}(A)) = \mathrm{K}(A)^{\mathrm{a}}$.

Claim 2.2.13.1. *For every $\zeta \in \mathrm{k}(M) \cup \Gamma(M)$, $C := \mathrm{K}(\mathrm{acl}(\mathrm{K}(A)\zeta)) \subseteq \mathrm{K}(A)^{\mathrm{a}}$.*

Proof. Let us first assume that $\zeta \in \mathrm{k}(M)$. Since $\mathrm{res}^{-1}(\zeta)$ is infinite, there exists $c \in M^\star \succcurlyeq M$ transcendental over C, with $\mathrm{res}(c) = \zeta$. If $\zeta \in \Gamma^\times(M)$, we find such a c with $\mathrm{v}(c) = \zeta$. In both cases, we have $\zeta \in \mathrm{acl}(\mathrm{K}(A)c)$ and hence $C \subseteq \mathrm{K}(\mathrm{acl}(\mathrm{K}(A)c)) = \mathrm{K}(A)(c)^{\mathrm{a}}$. Now $1 = \mathrm{trdeg}(\mathrm{K}(A)(c)/\mathrm{K}(A)) = \mathrm{trdeg}(C(c)/\mathrm{K}(A)) = \mathrm{trdeg}(C/\mathrm{K}(A)) + \mathrm{trdeg}(c/C) = \mathrm{trdeg}(C/\mathrm{K}(A)) + 1$, so $C \subseteq \mathrm{K}(A)^{\mathrm{a}}$. ◊

By induction on an enumeration of $\mathrm{k}(A) \cup \Gamma(A)$, we have $\mathrm{K}(\mathrm{acl}(A)) = \mathrm{K}(A)^{\mathrm{a}}$. □

Corollary 2.2.14. *Any $\mathfrak{L}_{\mathrm{k},\Gamma}(M)$-definable function $f : \mathrm{k}^n \times \Gamma^m \to \mathrm{K}$ has finite image.*

Proof. Let $Y = f(\mathrm{k}^n \times \Gamma^m)$. Then, for every elementary $h : M \to M^\star$, $h_\star Y(M^\star) \subseteq \mathrm{acl}(\mathrm{K}(h(M)) \cup \mathrm{k}(M^\star) \cup \Gamma(M^\star)) = \mathrm{K}(h(M))$, by Proposition 2.2.13. If Y is infinite, then, by compactness, there exists an elementary $h : M \to M^\star$ such that $h_\star Y(M^\star) \setminus \mathrm{K}(h(M)) \neq \varnothing$. It follows that Y is finite. □

Remark 2.2.15. Describing the definable closure in all three sorts is more complicated. If $A \subseteq \mathrm{K}(M)$, then, by Proposition 2.2.10, $\mathrm{k}(\mathrm{dcl}(A))$ is the perfect hull of the fraction field of $\mathrm{k}(A)$. Similarly, by Proposition 2.2.11, $\Gamma^\times(\mathrm{dcl}(A))$ is the divisible hull of the group generated by $\mathrm{v}A^\times$. We can also prove that $\mathrm{K}(\mathrm{dcl}(A))$ is the perfect hull of the henselianization of $\mathrm{K}(A)$, but this goes slightly beyond the scope of these notes.

3 Imaginaries

In this last section, we wish to consider three related questions:

- What do quotients of definable sets by definable equivalence relations look like?
- Do definable families admit moduli spaces: is there a definable set whose points are in definable bijection with the definable sets that appear in the family?
- Does a definable set have a smallest set of definition?

Before answering these three questions in ACVF, let us introduce some general background and terminology. Let $\mathfrak{L}$ be some language, M be an $\mathfrak{L}$-structure and $X \subseteq M^x$ be $\mathfrak{L}(M)$-definable.

Proposition 3.0.1 (Poizat, 1983). *Let T be a theory such that for every finite tuples of sorts X and Y, there exists a finite tuple of sorts Z and $\mathfrak{L}$-definable injective maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$ with disjoint images*[10]*. The following are equivalent:*

1. *T eliminates imaginaries: for every $M \models T$ and $\mathfrak{L}(M)$-definable X set $X \subseteq M^x$, there exists an $\mathfrak{L}$-formula $\psi(xy)$ and $a \in M^y$ such that, for all $c \in M^y$, $\psi(M, c) = X$ if and only if $c = a$ (we say that a is a canonical parameter of X via ψ);*
2. *T uniformly eliminates imaginaries : for every $\mathfrak{L}$-definable $V \subseteq X \times Y$, there exists an $\mathfrak{L}$-definable $W \subseteq X \times Z$ such that for every $M \models T$ and $a \in Y(M)$, there exists a unique $c \in Z(M)$ such that $V_a := \{x \in X \mid (x, a) \in V\} = W_c$.*
3. *Any interpretable set is represented, in T, by a definable set: for every $M \models T$, $A \leqslant M$ and $\mathfrak{L}(A)$-definable equivalence relation $E \subseteq X \times X$, there exists an $\mathfrak{L}(A)$-definable map $f : X \to Z$ such that for all $x_1, x_2 \in X$, $x_1 E x_2$ if and only if $f(x_1) = f(x_2)$.*

Sketch of proof. Assuming (i), (ii) follows by compactness. We use the representability of disjoint unions to go from the finitely many definable families given by compactness to just one.

Now assuming (ii) and given X and E, as in (iii), let W be as in (ii) applied to $E \subseteq X \times X$. We now define $f(x)$ to be the unique z such that $E_x = W_z$.

Finally, assuming (iii) and given some M-definable set $X = \varphi(M, a)$, we apply (ii) to the equivalence relation $y_1 E y_2$ defined by $\forall x, \varphi(x, y_1) \Leftrightarrow \varphi(x, y_2)$, to get some $f : Y \to Z$ as in (ii). Then $\psi(x, z) := \exists y\ f(y) = z \wedge \varphi(x, y)$ and $f(a)$ satisfy (i). □

[10] In other words, the category of $\mathfrak{L}$-definable sets has finite coproducts.

Note that if a is a canonical parameter of X, via some φ, then $\mathrm{dcl}(a)$ is the smallest dcl-closed set of definition of X — in particular, it does not depend on the choice of φ. If M is sufficiently large, this leads to a purely semantic, Galois-type, characterisation of canonical parameters:

Lemma 3.0.2. *Let M be an $\aleph_1$-saturated and strongly $\aleph_1$-homogeneous structure, X an M definable set and $a \in M$ a tuple. The following are equivalent:*

1. *a is a canonical parameter of X — via some φ;*
2. $\mathrm{aut}_X(M) = \{\sigma \in \mathrm{aut}(M) \mid \sigma(X) = X\} = \mathrm{aut}(M/a)$.

We will thus simply say that the tuple a codes X.

Proof. Let us first assume that a is a canonical parameter of X. Then any automorphism of M that stabilizes X fixes a. Conversely, since X is a-definable, any automorphism fixing a stabilizes X. So (ii) holds.

If (ii) holds, write X as $\varphi(M, m)$ for some tuple $m \in M$. Then, the set $Y = \{m \in M \mid \varphi(M, m) = X\}$ is stabilized by $\mathrm{aut}_(M/a)$ and hence is a-definable — by some formula $\psi(z, a)$. We may assume implies that for all $c, m_1, m_2 \in M$, $\psi(m_1, c) \wedge \psi(m_2, c)$ implies that $\varphi(M, m_1) = \varphi(M, m_2)$. Also, given any $m \in Y$, every automorphism fixing m stabilizes X and hence fixes a. By compactness, there exists an $\mathfrak{L}$-definable map f such that for every $m \in Y$, $f(m) = a$. We now consider the formula $\theta(x, y) := \exists z,\ f(z) = y \wedge \psi(z, y) \wedge \varphi(x, z)$. Then $\theta(M, a) = X$ and, if $c, m \in M$ are such that $\varphi(M, m) = X$, $\psi(m, c)$ and $f(m) = c$, then $m \in Y$, so $c = f(m) = a$. So a is a canonical parameter of X via φ. □

Before we move on to the more complicated case of algebraically closed valued fields, let us consider the imaginaries in algebraically closed fields:

Theorem 3.0.3 (Poizat, 1983). *The $\mathfrak{L}_{\mathrm{rg}}$-theory of algebraically closed fields (uniformly) eliminates imaginaries.*

Sketch of proof. Let K be an algebraically closed field and $X \subseteq \mathrm{K}^n$ be $\mathfrak{L}_{\mathrm{rg}}(K)$-definable. We may assume that X is a class of an $\mathfrak{L}_{\mathrm{rg}}$-definable equivalence relation. Let p be the generic type of an irreducible components of the Zariski closure $\overline{X}^{\mathrm{z}} \subseteq K^n$. The type p is entirely determined by $I(p) := \{P \in K[x] \mid p \vDash P = 0\} = \bigcup_d V_d(p)$, where $V_d(p) := \{P \in K[x] \mid p \vDash P = 0 \text{ and } \deg(P) \leqslant d\} \leqslant K^{m_d}$ is a sub-K-vector space of dimension r_d. Note that $\bigwedge^{r_d} V_d(p) \leqslant \bigwedge^{r_d} K^{m_d} \simeq K^{n_d}$ is dimension 1 — it is therefore an element $a_d \in \mathbb{P}(K^{n_d})$ which can be $\mathfrak{L}_{\mathrm{rg}}$-definably identified with a subset of K^{n_d+1}.

Then any automorphism σ of K fixing the $(a_d)_{d\geqslant 0}$ fixes the type p. Moreover, since $p \vDash X$ and since X is a class of an $\mathfrak{L}_{\mathrm{rg}}$-definable equivalence relation, σ stabilizes X. By compactness, X is $\mathfrak{L}_{\mathrm{rg}}(a_d)$-definable for some sufficiently large d. Note that the orbit of p under $\mathrm{aut}_X(M)$ is contained in the set of generic types of irreducible components of the Zariski closure $\overline{X}^{\mathrm{z}} \subseteq K^n$, which is finite. Thus so is the orbit C of a_d.

Since for any element of $c \in C$, X is (uniformly) c-definable, it suffices to show that the finite set C has a canonical parameter. Let $P(x, y) = \prod_{c \in C}(x - \sum_i c_i y^i)$ and b be the tuple of its coefficients. Then C is $\mathfrak{L}_{\mathrm{rg}}(d)$-definable and d is fixed by $\mathrm{aut}_C(M)$; so d is a canonical parameter of C and hence X, concluding the proof. □

Let us now consider imaginaries in algebraically closed valued fields.

Remark 3.0.4. There are balls that do not have a canonical basis in the three sorted language. Take, for example, an open ball of radius $\gamma \notin \mathrm{dcl}(\varnothing)$ at distance γ from 0. Then the fixed points of $\mathrm{aut}_b(M)$ are included in $\mathrm{acl}(\gamma)$, but b is not $\mathrm{acl}(\gamma)$ definable.

So ACVF does not eliminate imaginaries in the three sorted language. We can, however, describe a — minimal — set of imaginaries which is sufficient to eliminate all imaginaries:

- Let S_n be the set lattices in K^n: rank n free $\mathcal{O}$-submodules of K^n. Note that any such module is determined by its basis in K^n, and two bases of K^n, seen as elements of $\mathrm{GL}_n(\mathrm{K})$ generate the same $\mathcal{O}$-module exactly when they are in the same coset for $\mathrm{GL}_n(\mathcal{O})$. This yields an identification $\mathrm{S}_n \simeq \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O})$.
- Let $\mathrm{T}_n := \bigsqcup_{s \in \mathrm{S}_n} s/\mathfrak{m}s$. It is a collection of dimension n k-vector spaces. To any $M \in \mathrm{GL}_n(\mathrm{K})$, we can associate an element of T_n by considering the coset $a + s/\mathfrak{m}s$ where s is the lattice with basis M and a is its last column. Then two elements of $\mathrm{GL}_n(K)$ give rise to the same element in s_n if and only if they are in the same coset for the subgroup $\mathrm{GL}_{n,n}(\mathcal{O})$ of matrices whose coordinate-wise reduction modulo $\mathfrak{m}$ has zeros on the first column, except for a 1 on the diagonal. Note however, that we only get non zero elements in any $s/\mathfrak{m}s$ in this manner. So T_n can be identified with $\mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_{n,n}(\mathcal{O}) \sqcup \mathrm{S}_n(\mathrm{K})$, where any $s \in \mathrm{S}_n(\mathrm{K})$ is identified with the zero of $s/\mathfrak{m}s$.

Remark 3.0.5. We have:

- $\mathrm{S}_1 \simeq \mathrm{K}^\times/\mathcal{O}^\times = \Gamma$.
- $\mathrm{T}_1 \simeq \mathrm{K}^\times/(1+\mathfrak{m}) = RV$.
- The set of closed balls with finite radius can be $\varnothing$-definably embedded in S_2;
- The set of open balls with finite radius can be $\varnothing$-definably embedded in T_2;

So these interpretable sets do take care of the obstruction to elimination of imaginaries raised in Remark 3.0.4. In fact, they also take care of similar higher dimensional obstructions. The main result whose proof we will now sketch is that these new sorts are sufficient.

Definition 3.0.6 (Geometric language)**.** Let $\mathfrak{L}_{\mathcal{G}}$ be the language with:

- a sort K with the ring language;
- sorts S_n, for all $n \in \mathbb{Z}_{>0}$;
- a map $\mathrm{s}_n : K^{n^2} \to \mathrm{S}_n$;
- sorts T_n, for all $n \in \mathbb{Z}_{>0}$;
- a map $\mathrm{t}_n : K^{n^2} \to \mathrm{T}_n$.

We denote by $\mathcal{G}$ the set of $\mathfrak{L}_{\mathcal{G}}$-sorts. Any valued field has a natural $\mathfrak{L}_{\mathcal{G}}$-structure given by interpreting s_n and t_n as the canonical projections on $\mathrm{GL}_n(\mathrm{K})$; and $\mathcal{O}^n$, respectively $\mathfrak{m}^n$, outside.

Theorem 3.0.7 (Haskell–Hrushovksi–Macpherson, 2006)**.** *The $\mathfrak{L}_{\mathcal{G}}$-theory of (non trivially) valued algebraically closed fields eliminates imaginaries.*

The proof we give below is a proof of Johnson [7], building up on a proof of Hrushovski [4]. The notion of a definable type is central:

Definition 3.0.8. Let M be some $\mathfrak{L}$-structure. A type $p \in \mathcal{S}_x(K)$ is said to be $\mathfrak{L}(M)$-definable if for every formula $\varphi(x, y)$, there exists an $\mathfrak{L}(M)$-formula $\theta(y)$ such that, for all $a \in M$,

$$p \vDash \varphi(x, a) \text{ if and only if } M \vDash \theta(a).$$

The other important notion is that of a (definable) valued vector space:

Definition 3.0.9. Let $K \vDash$ ACVF and V be a K-vector space. A valuation on V is a map $w : V \to \Sigma \sqcup \{\infty\}$, where $(\Sigma, \leqslant)$ is totally ordered and $\mathrm{v}K^\times$ acts strictly increasingly on Σ, such that:

1. $w(0) = \infty > \Sigma$;
2. for all $x, y \in V$, $w(x + y) \geqslant \min\{w(x), v(y)\}$;
3. for all $a \in K$, $w(ax) = \mathrm{v}(a) + w(x)$.

We say that a valuation w on K^n is definable if $w(x) \leqslant w(y)$ is.

Sketch of proof for Theorem 3.0.7. Let $K \vDash$ ACVF be sufficiently large and $X \subseteq \mathrm{K}^n$ be some $\mathfrak{L}_{\mathcal{G}}(K)$-definable set.

Claim 3.0.9.1. *There exists an $\mathfrak{L}_{\mathcal{G}}(K)$-definable type $p \in \mathcal{S}_X(K)$ with a finite $\mathrm{aut}_X(K)$-orbit.*

Proof. Assume first that $X \subseteq \mathrm{K}$. Then, by Theorem 2.2.5, X has a unique decomposition $\bigcup_{i<n} b_i \smallsetminus b_{i,j}$ of disjoint non nested Swiss cheeses. Then $p := \eta_{b_0}$ has the required properties. If $X \subseteq \mathrm{K}^{n+1}$, by induction, we find an $\mathfrak{L}_{\mathcal{G}}(K)$-definable $q(x) \in \mathcal{S}_{\pi(X)}(K)$ with a finite $\mathrm{aut}_X(K)$-orbit, where $\pi : \mathrm{K}^{n+1} \to \mathrm{K}^n$ is the projection on the first n coordinates. Let $a \in K^\star \succcurlyeq K$ realise q. By the dimension 1 case, we find an $\mathfrak{L}_{\mathcal{G}}(K^\star)$-definable $r_a \in \mathcal{S}_{X_a}(K^\star)$ with a bounded $\mathrm{aut}_{X_a}(K^\star)$-orbit — in particular a finite $\mathrm{aut}_X(K^\star/a)$-orbit.

Let $c \vDash r_a$. Then $p(xy) = \mathrm{tp}(ac/K)$ has a finite $\mathrm{aut}_X(K)$-orbit. There remains to prove that it is definable. Fix a $\mathfrak{L}$-formula $\varphi(x, y, z)$ and $c \in K$ some tuple. We have $\varphi(x, y, c) \in r$ if and only if $\varphi(x, a, c) \in r_a$. Let $Y_a := \{c \in K \mid \varphi(x, a, c) \in r_a\}$. This set is $K^\star$-definable and it has a finite $\mathrm{aut}(K^\star/Ka)$-orbit $(Y_i)_{i<n}$. Then, the relation $c_1 E_a c_2$ (in $K^\star$) defined by $\bigwedge_{i<n} c_1 \in Y_i \leftrightarrow c_2 \in Y_i$ is Ka-definable and has finitely many classes. So the relation (in K) $c_1 E c_2$ defined by $c_1 E_x c_2 \in q(x)$ is K-definable and also has finitely many orbits. By construction, if $c_1 E c_2$ and $\varphi(x, y, c_1) \in r$, then $\varphi(x, y, c_2) \in r$. In other words, the set $\{c_1 \in K \mid \varphi(x, y, c_1) \in r\}$ is a (finite) union of E-classes and is, therefore, K-definable. ◊

Let now $p \in \mathcal{S}_{\mathrm{K}^\ell}(K)$ be any $\mathfrak{L}_{\mathcal{G}}(K)$-definable type — in particular it could be the type obtained in Claim 3.0.9.1, but it does not have to be. We wish to find some $a \in \mathcal{G}(K)$ such that $\mathrm{aut}(K/p) = \mathrm{aut}(K/a)$. For very $d \in \mathbb{Z}_{\geqslant 0}$, let w_d be the (definable) valuation on $\{P \in \mathrm{K}[x] : \deg(P) \leqslant d\} \simeq \mathrm{K}^{m_d}$ such that $w_d(P) \leqslant w_d(Q)$ if and only if $p(x) \vDash \mathrm{v}(P(x)) \leqslant \mathrm{v}(Q(x))$. Note that w_p is $\mathrm{aut}(K/p)$-invariant, where $\mathrm{aut}(K/p) = \{\sigma \in \mathrm{aut}(K) : \sigma(p) = p\}$. By quantifier elimination, the $(w_d)_{d\geqslant 0}$ characterise p.

We also define $\gamma \trianglelefteq \delta$, for every $\gamma, \delta \in w_p\mathrm{K}^{m_d}$, if there exists $a \in \mathrm{K}$ such that $\gamma \leqslant \delta + \mathrm{v}(a)$. This is a — definable $\mathrm{aut}(K/p)$-invariant — preorder whose associated equivalence relation $\equiv$ refines the $\mathrm{vK}^\times$-orbits. Finally, let $\gamma \simeq \delta$ if $\delta - \mathrm{v}(a) \leqslant \gamma \leqslant \delta + \mathrm{v}(a)$, for every $a \in \mathcal{O} \smallsetminus \{0\}$. This — definable $\mathrm{aut}(K/p)$-invariant — equivalence relation refines $\equiv$ and its classes have at most one element in each $\mathrm{vK}^\times$-orbit.

Claim 3.0.9.2. *There exists a basis* $(P_i)_{i<m_d}$ *of* K^{m_d} *such that for all* $a_i \in K$, $w_p(\sum_i a_i P_i) = \min_i \mathrm{v}(a_i) + w_p(P_i)$. *Moreover, the* P_i *can be chosen such that* $\gamma_i := w_p(P_i)$ *is* $\mathrm{aut}(K/p)$*-invariant and, for all i and j, if* $\gamma_i \equiv \gamma_j$*, then* $\gamma_i \sim \gamma_j$.

Proof. Any (finite dimensional) valued vector space (V, w) over a spherically complete field K has a separated basis; that is a basis b such that for every $a_i \in K$, $w(\sum_i a_i b_i) = \min_i \mathrm{v}(a_i) + w(b_i)$. Since any completion of ACVF admits spherically complete models, any definable valuation on K^n admits a separable basis.

Now, for every $\gamma \in W_p K^{m_d}$, let $V_{\trianglelefteq\gamma} := \{x \in K^{m_d} : \gamma \trianglelefteq w_p(x)\}$. This an $\mathrm{aut}(K/p)$-invariant subspace of K^{m_d}. It has a basis of $\mathrm{aut}(K/p)$-invariant elements — given by considering the lift of the canonical basis of any injective coordinate projection to some K^n. In particular, it contains an $\mathrm{aut}(K/p)$-invariant element which is not in the proper subspace $V_{\triangleright\gamma}\{x \in K^{m_d} : \gamma \triangleleft w_p(x)\}$. In other words, every $\equiv$-class contains an $\mathrm{aut}(M/p)$-invariant element.

Moreover, by definable completeness of $\mathrm{vK}^\times \vDash \mathrm{DOAG}$, its $\sim$-class intersects any $\mathrm{vK}^\times$-orbit in its $\equiv$-class. It follows that every $\mathrm{vK}^\times$ contains an $\mathrm{aut}(M/p)$-invariant element and that they can be chosen in the same $\sim$-class whenever they are in the same $\equiv$-class. By scaling, we may assume that valuations of the elements of a separated basis of (K^{m_d}, w_p) are those $\mathrm{aut}(M/p)$-invariant elements. ◊

Claim 3.0.9.3. *There exists a tuple* $a \in \mathcal{G}(K)$ *such that* $\mathrm{aut}(K/p) = \mathrm{aut}(K/a)$.

Proof. Fix some i. Let $R_{d,i} := \{P \in \mathrm{K}^{m_d} \mid w_p(P) \geqslant \gamma_i\}$. They are $\mathrm{aut}(K/p)$-invariant $\mathcal{O}$-submodules of K^{m_d} and p is entirely characterised by them. Let $V_{d,i} := \{P \in \mathrm{K}^{m_d} \mid w_p(P) \trianglelefteq \gamma_i\}$ be its K-span, $W_{d,i} := \{P \in \mathrm{K}^{m_d} \mid w_p(P) \triangleright \gamma_i\}$ and $S_{d,i}$ be the $\mathcal{O}$-submodule generated by the P_j with $\gamma_j \trianglelefteq \gamma_i$. Then $S_{d,i}/\mathrm{W}_{d_i} \subseteq V_{d,i}/W_{d,i}$ is a lattice and, since $\mathfrak{m}S_{d,i} \subseteq R_{d,i}$, is determined by its image $L_{d,i}$ in $S_{d,i}/\mathfrak{m}S_{d,i}$, which is a k-subspace.

Choosing an $\mathrm{aut}(K/p)$-invariant basis of $V_{d,i}$ and $W_{d,i}$, we may assume that $V_{d,i} = \mathrm{K}^n$ and $W_{d,i} = \{0\}$. Then $S_{d,i} \in \mathrm{S}_n(K)$ and there remains to code $L_{d,i}$. Taking exterior powers, we may assume $\dim_{\mathrm{k}}(L_{d,i}) = 1$. Let $\pi : \mathrm{K}^n \to \mathrm{K}$ be the projection on the first coordinate. Then π induces a short exact sequence $0 \to S'_{d,i} \to S_{d,i} \to \pi(S_{d,i}) \to 0$ of lattices. If $L_{d,i} \subseteq S'_{d,i}/\mathfrak{m}S'_{d,i}$, we conclude by induction on the rank of $S'_{d,i}$. Otherwise, $\pi|_{L_{d,i}}$ is injective and it suffices to code the function $\pi(S_{d_i})/\mathfrak{m}\pi(S_{d_i}) \to S_{d_i}/\mathfrak{m}S_{d,i}$ induced by its inverse. But it can be identified with an element of $H_{d,i}/\mathfrak{m}H_{d,i}$, where $H_{d,i}$ is the lattice $\mathrm{Hom}_{\mathcal{O}}(\pi(S_{d,i}), S_{d,i}) \subseteq \mathrm{Hom}_K(K, K^n) \simeq K^n$. ◊

Let us come back to the proof of Theorem 3.0.7. If X is a class of an $\mathfrak{L}_{\mathcal{G}}$-definable equivalence relation, p is as in Claim 3.0.9.1, and a is as in Claim 3.0.9.3, then a has a finite $\mathrm{aut}_X(K)$-orbit C. Moreover, for any $e \in C$, both p and X are $\mathrm{aut}(K/e)$-invariant. It follows that X is $\mathrm{aut}_C(K)$-invariant. So there only remains to code the (finite) set C:

Claim 3.0.9.4. *Let* C *be a finite subset of sorts of* $\mathfrak{L}_{\mathcal{G}}$*. Then, there exists a definable type* p *concentrating on some* K^ℓ *such that* $\mathrm{aut}_C(K) = \mathrm{aut}(K/p)$ *— in particular, by Claim 3.0.9.3,* C *is coded by some* $a \in \mathcal{G}(K)$.

For every definable types $p(x)$ and $q(y)$, let $p(x) \otimes q(y)$ denote the definable type whose restriction to any (small) $A \subseteq K$ is the type of pair ab with $a \vDash p|A$ and $b \vDash q|Aa$.

Proof. We may assume that the elements of C are identically sorted. We first assume that $C \subseteq \mathrm{K}^n \times \mathrm{k}^m$. Let $C_k \subseteq \mathrm{k}$ be the (finite) set of elements of k that appear in C. Let $P(x) = \prod_{c \in C_k} x - c$, α its tuple of coefficients and $a \vDash \eta_{\mathrm{res}^{-1}(\alpha)} = \bigotimes_i \eta_{\mathrm{res}^{-1}}(\alpha_i)$ — which depends on a choice of enumeration of α. Then the roots of $Q(x) = \sum_i a_i x^i$ are in one to one correspondance by res with the elements of C_{k}. So C is in $\mathfrak{L}_{\mathcal{G}}(a)$-definable bijection with a finite subset $D \subseteq \mathrm{K}^{n+m}$, which is coded by some $d \in \mathrm{K}^{\ell}$ by elimination of imaginaries in the theory of algebraically closed fields. Then $p = \mathrm{tp}(d/K)$ has the required properties.

We now consider the general case. Since tensor product induces an $\mathfrak{L}_{\mathcal{G}}$-definable injection $\mathrm{S}_n \times \mathrm{S}_m \to \mathrm{S}_{n+m}$ and $\mathrm{T}_n \times \mathrm{T}_m \to \mathrm{T}_{n+m}$, we may assume that $C \subseteq \mathrm{K}^n \times \mathrm{S}_m \times \mathrm{T}_{\ell}$. Lengthening the tuples in C, we can further assume that $\ell = m$ and if $(a, s, t) \in C$ with $c \in \mathrm{K}^n$, $s \in \mathrm{S}_n$ and $t \in \mathrm{T}_n$, we have $t \in s/\mathfrak{m}s$.

For every $m \geqslant 0$, let $\eta_{\mathrm{GL}_m(\mathcal{O})}$ be the type of elements in $a \in \mathcal{O}^{m^2}$ such that $\mathrm{res}(a)$ is generic in $\mathrm{GL}_m(\mathrm{k})$. It is a $\mathfrak{L}_{\mathcal{G}}$-definable type and for every $A \vDash \eta_{\mathrm{GL}_m}$ and $B \in \mathrm{GL}_m(\mathcal{O}(K))$, $B \cdot A \vDash \eta_{\mathrm{GL}_m(\mathcal{O})}$. Then, for every $s \in \mathrm{S}_m(K)$ and every basis $B \in \mathrm{GL}_n(K)$ of s, the type η_s of $B \cdot A$ does not depend on the choice of B. It is $\mathfrak{L}_{\mathcal{G}}(s)$-definable.

For every $c = (a, s, t) \in C$, let p_c be the type of a, b, e where $b \vDash \eta_s$ and $e \in \mathrm{k}^m$ is the coordinates of t in the basis $b/\mathfrak{m}s$ of $s/\mathfrak{m}s$. Then, for any two $c_1, c_2 \in C$, $p_{c_1} \otimes p_{c_2}(x, y) = p_{c_2} \otimes p_{c_1}(y, x)$ — the types p_{c_i} are said to be generically stable. So $p_C := \bigotimes_{c \in C} p_c$ does not depend on a choice of C. So it is a definable $\mathrm{aut}_C(K)$-invariant type and C is $\mathrm{aut}(K/p)$-invariant.

Let $E \subseteq \mathrm{K}^{n+m^2} \times \mathrm{k}^m$ be the set enumerated by some realisation of p_C. By the first paragraph, E is coded by some tuple $e \in \mathcal{G}(K)$. Then, $p = \mathrm{tp}(e/K)$ has the required properties. ◊

As explained above Claim 3.0.9.4, this concludes the proof of Theorem 3.0.7. □

Remark 3.0.10. In the previous proof, we implicitly used the following criterion given by Hrushovski [4] for elimination of imaginaries. If T is an $\mathfrak{L}$-theory, with dominant sorts[11] S, such that, for every $M \vDash T$:

- for every M-definable X of a sort in S, there exists an M-definable type $p \vDash X$ with a finite $\mathrm{aut}_X(M)$-orbit;
- any M-definable type p, is a-definable for some (infinite) tuple $a \in M$ with finite $\mathrm{aut}(M/a)$-orbit;
- any finite set of tuples $C \subseteq M$ has a code in M.

Then T eliminates imaginaries.

References

[1] James Ax and Simon Kochen. Diophantine problems over local fields I. *Amer. J. Math.*, 87(3):605–630, 1965.

[2] Ju. L. Ershov. On the elementary theory of maximal normed fields. *Dokl. Akad. Nauk SSSR*, 165:21–23, 1965.

[11] That is, for all $M \vDash T$, $M \subseteq \mathrm{dcl}(S(M))$.

[3] Deirdre Haskell, Ehud Hrushovski, and Dugald Macpherson. Definable sets in algebraically closed valued fields: Elimination of imaginaries. *J. Reine Angew. Math.*, 597:175–236, 2006.

[4] Ehud Hrushovski. Imaginaries and definable types in algebraically closed valued fields. In Antonio Campillo Lopez, Franz-Viktor Kuhlmann, and Bernard Teissier, editors, *Valuation Theory in Interaction*, number 10 in EMS Ser. Congr. Rep., pages 297–319. Eur. Math. Soc., Zürich, 2014.

[5] Ehud Hrushovski and David Kazhdan. Integration in valued fields. In *Algebraic Geometry and Number Theory*, number 253 in Progr. Math., pages 261–405. Birkhäuser Boston, 2006.

[6] Ehud Hrushovski and François Loeser. *Non-archimedean tame topology and stably dominated types*, volume 192 of *Ann. of Math. Stud.* Princeton University Press, Princeton, NJ, 2016.

[7] Will Johnson. On the proof of elimination of imaginaries in algebraically closed valued fields. arXiv: 1406.3654.

[8] Abraham Robinson. *Complete Theories*. Number 46 in Stud. Logic Found. Math. North-Holland Publishing Co., Amsterdam, second edition, 1977.

[9] Katrin Tent and Martin Ziegler. *A Course in Model Theory*. Number 40 in Lect. Notes Log. Cambridge Univ. Press, Cambridge, Assoc. Symbol. Logic, Chicago, IL edition, 2012.