PSL UNIVERSITÉ PARIS

HABILITATION À DIRIGER DES RECHERCHES

DE L'UNIVERSITÉ PSL

Présentée à l'École normale supérieure

De la géométrie des valuations

Présentation des travaux par

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Le 6 mai 2025

Discipline

Mathématiques

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Cause we are living in a material world
And I am a material girl
Material Girl, *Madonna*

I'm not crazy about reality
but it's still the only place to get a decent meal.
Groucho Marx
(allegedly)

Ce fut donc sur la hache
que son choix se porta définitivement.
Crime et châtiment, *Fédor Dostoïevski*

Remerciements

Ce texte et plus largement ma vie mathématique dont il relate une part non négligeable doit son existence aux mathématiciennes et mathématiciens avec lesquels cela a été un plaisir constant de discuter et auprès desquels j'ai appris une grande partie de ce que je comprends aujourd'hui.

Je ne m'aventurerais surtout pas à tous les nommer, mais je tiens à remercier tout particulièrement mes collaborateurs les plus proches et dont les travaux apparaissent ici. Je pense en premier lieu à Martin Hils et Samaria Montenegro pour nos escapades qu'elles soient imaginaires ou pseudo closes (ou non mathématiques). Je remercie Immi Halupczok et Raf Cluckers pour m'avoir proposé de rejoindre la barque ?-minimale qui devint h-minimale après une longue errance lexicale. Je remercie Mariana Vicaría d'avoir œuvré avec moi pour éliminer les derniers imaginaires récalcitrants. Enfin, je n'oublie pas Pablo Cubides, Moshe Kamensky et Floris Vermeulen qui ont aussi participé de près à cette aventure !

Je tiens aussi à remercier Arthur Forey et Tom Scanlon pour toutes les mathématiques dont nous avons pu discuter au fil des années. S'ils n'apparaissent pas (encore) dans ma liste de co-auteurs ce n'est pas faute d'essayer.

Je voudrais remercier chaleureusement Matthias Aschenbrenner, Itaï Ben Yaacov et Antoine Ducros d'avoir accepté de lire le présent manuscript et d'en être les rapporteurs. Je suis aussi profondément reconnaissant à François Charles, Françoise Delon, Ehud Hrushovski, Franziska Janhke et François Loeser de compléter le jury de cette habilitation. J'ai une pensée émue pour Zoé Chatzidakis qui aurait du en faire partie si elle ne nous avait pas été enlevée trop tôt. Vous avez tous eu à des degrés divers une influence sur mes travaux et c'est un honneur que de soutenir cette habilitation devant vous.

Je remercie Élisabeth Bouscaren, Antoine Chambert-Loir et Martin Hils pour leurs commentaires sur les premières versions de ce texte qui en est sorti grandement amélioré.

Les quelques dix dernières années de ma vie mathématique ont été partagées entre le département de mathématiques de UC Berkeley, l'IMJ-PRG côté université de Paris 7 Diderot Cité, ainsi que le DMA de l'École normale supérieure. Je remercie tous mes nombreux collègues dans ces laboratoires ainsi que tous les personnels administratifs qui ont permis que l'on puisse faire des mathématiques dans de bonnes conditions.

Dans un registre un peu différent, mon travail aurait été bien morne sans la contribution de divers compositeurs qui dont la musique a accompagné mon écriture mathématique. Cela leur fait une belle jambe, mais les chapitres de ce mémoire sont dédiés à certaines de leurs œuvres, sans grande rime ni raison.

Je remercie ceux qui égayent ma vie hors du bureau (et parfois au bureau), tout particulièrement Max, Daphné, Rémi et Fathi. J'espère que l'on lancera encore de nombreux dés.

Last but not least, je remercie Catherine ; je soupçonne qu'elle sait pourquoi !

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Geometric model theory of valued fields

Étude n°6
Philip Glass

The topic of this text is the geometry and combinatorics of sets definable in fields with additional structure, mostly valuations. A valuation on a field K is essentially a multiplicative norm $|\cdot|$ on K with an important *caveat*.

- We require the norm to be ultrametric (we also say that it is non-archimedean): for every $x, y \in K$,
$$|x + y| \leqslant \max\{|x|, |y|\}.$$
- For mostly technical reasons, we allow the valuation group $|K^\times|$ not to be a subgroup of $\mathbb{R}_{>0}$ but any arbitrary ordered abelian group.

Valuations arise naturally in number theory or in algebraic geometry. Consider for example the p-adic norm on $\mathbb{Q}$ or the degree map, or rather its exponential, on any field of rational functions in one variable. Note that both of these examples do have a value group included in $\mathbb{R}_{>0}$ — it is actually isomorphic to $\mathbb{Z}$ — as do many natural examples.

The non-archimedean nature of a valuation has a number of interesting consequences underlining their highly algebraic nature. One of them is that the closed unit ball $\mathcal{O}$ is a local ring — called the valuation ring — whose maximal ideal $\mathfrak{m}$ is the open unit ball. The quotient $\mathrm{k} = \mathcal{O}/\mathfrak{m}$ is called the residue field of the valuation.

As mentioned above, the focus of this text will be the study of definable sets. Fix a field K and a collection of subsets $\mathcal{S}$[1] of K and its cartesian powers. We call this a field with additional structure. For example $\mathcal{S}$ could contain the valuation ring $\mathcal{O}$ of some valuation on K. By induction, a subset of K^n is said to be definable if it is:

- the solution set of a polynomial equation over K;
- the set $\{x \in K^n : P(x) \in C\}$, where $C \in \mathcal{S}$ and P is a tuple of polynomials over K;
- a boolean combination of definable sets;
- the projection to K^n of a definable subset of K^{n+m}.

In other words, a set is definable if is the set of tuples that verify a formula in the language of rings with additional symbols for the sets in $\mathcal{S}$. Such a formula is built from addition, multiplication, elements of K, equality, sets in $\mathcal{S}$, logical connectives and quantifiers running over K.

[1] Because $\mathcal{S}$ is for Shub-Niggurath [Fat; Bre24].

Given a field (potentially with additional structure), we can then attempt to describe the definable sets, what they look like, their geometric properties or the combinatorics of how they interact.

For example, if K is an algebraically closed field with no extra structure, by a theorem of Tarski (or its algebraic geometric formulation by Chevalley), the definable sets are exactly the constructible sets; that is, boolean combinations of solution sets for polynomial equations over K; and questions about the structure of definable sets essentially become questions of algebraic geometry. Note that, in that case, projections (equivalently quantification) turn out not to add any new definable sets.

In the field $\mathbb{R}$ with no additional structure, quantification does introduce new definable sets since the set of squares, *i.e.* positive reals, is definable. But this is essentially all the new definable sets you get and definable sets turn out to be (also by a theorem of Tarski) the semialgebraic sets; that is, boolean combinations of solution sets for polynomial inequalities over $\mathbb{R}$.

As those two examples show, the study of definable sets in a field (potentially with additional structure) often starts with finding an alternative description of definable sets that does not involve projections as they tend to introduce a lot of complexity — we often refer to those results as "quantifier elimination" results. In the case of valued fields, and particularly those of residue characteristic zero, such a classification of definable sets is well understood for henselian valued fields. Henselianity can be seen as an algebraic counterpart to completeness as it can be rephrased in terms of the inverse function theorem for étale morphisms.

The focus of this text is the description of the geometric properties of definable sets in these henselian valued fields — and in related structures — beyond their mere description as boolean combinations of certain sets.

In the first chapter, which exposes the work contained in [CHR22; Clu+23], our geometric considerations are informed by the behavior of real semialgebraic and subanalytic sets as well as their non-archimedean counterparts. But even more so, it is inspired by o-minimal geometry. A structure on the field $\mathbb{R}$ is said to be o-minimal if every definable subset of the affine line $\mathbb{A}^1$ is a finite union of points and intervals. This tameness condition on the combinatorics of definable sets prevents many wild phenomena and has many geometric consequences starting with cell decomposition results that describe not only the properties of definable sets but also of definable functions; that is, functions whose graph is definable.

For example definable functions are arbitrarily piecewise differentiable and the pieces can themselves be described as finite unions of products of open intervals and points, twisted by function that are themselves arbitrarily piecewise differentiable.

There have been multiple attempts at a similar axiomatic approach to non-archimedean geometry. In chapter 1, we describe such an approach called h-minimality which (contrary to previous attempts) accounts for all characteristic zero henselian fields. Although the required tameness condition on definable subsets of $\mathbb{A}^1$ is not as easily stated as in the case of o-minimality (see definition 1.1.8), it does hold in classical examples and it has similarly strong consequences. For example h-minimal fields enjoy cell decompositions with strong properties (see theorem 1.5.4).

In the second chapter, we look at a slightly different problem which is the focus of [MR23].

We are not concerned with the geometry of sets definable with one valuation, but with how multiple valuations can interact. As often in model theory, we consider multivalued fields that are sufficiently "closed" to eliminate complex arithmetic behavior. In that case, the closure property we consider takes the guise of a "local-global principle for the existence of points on algebraic varieties", where the localizations are controlled by a fixed theory of fields (see proposition 2.1.1).

We then underline two interesting phenomena. The first, previously observed in a number of examples, is that, under some technical assumption this "local-global principle" implies a stronger version where the points can be chosen arbitrarily close to points from the localizations (see proposition 2.1.2).

The second phenomena is that this stronger "local-global principle" has strong model theoretic consequences: the definable sets in K are controlled by definable sets in the localizations, and the sets definable in distinct localizations essentially do not interact. We illustrate this with two precise results: on the one hand computing the burden of definable sets (theorem 2.2.4), on the other characterizing forking (theorem 2.2.9).

Burden is a notion of dimension for definable sets (see definition 2.2.1) and non forking (see definition 2.2.8) is a notion of independence between tuples in a structure. Both are defined purely from the combinatorics of definable sets. However, they turn out to capture meaningful geometric or algebraic notions in many structures. For example, in algebraically closed fields, burden coincides with Krull dimension and non forking with algebraic independence.

The third question that we will tackle is that of the "canonical" description of definable sets. Elimination of quantifiers provides a description of definable sets as boolean combinations of "basic" definable sets, but there might be numerous such descriptions and among those presentations, there might be some that involve no arbitrary choices.

For example, in an algebraically closed field with no extra structure, a definable set X can be entirely described by the following canonical data:

- consider the Zariski closure V of X;
- the definable set $V \setminus X$ has lower dimension and we can consider its Zariski closure;
- we iterate this process until we find a finite set.

This is still not entirely satisfactory as we still need to choose polynomial equations that define V and the other Zariski closed sets that we have constructed. But this can be done by considering their "smallest field of definition".

This question of the canonical presentation of definable sets is related to two other natural questions.

- The first is that of finding "moduli spaces" for families of uniformly definable sets: given definable sets $X \subseteq Y \times Z$ seen as the family of fibers $(X_z)_{z \in Z}$, we want to find a "canonical" parametrization $X' \subseteq Y \times Z'$ such that every fiber of X appears exactly once in X'.
- The second is that of describing quotients of definable sets by definable equivalence relations. Indeed, a "canonical" parametrization of X can easily be achieved by considering the quotient of Z by the definable equivalent relation "$X_{z_1} = X_{z_2}$". However, such a quotient might be much more complicated than definable sets.

Ideally, definable sets can be shown to be closed under taking quotients. This is the case in algebraically closed fields and in $\mathbb{R}$. In that case, for historical reasons, we say that the structure eliminates imaginaries.

In valued fields, however, this rarely happens. This is mostly due to the fact that, because a valuation is non archimedean, any point of a ball is its center and hence a ball is not canonically described by its center and its radius. In that case, the question naturally becomes to give a minimal collection of quotients that allow to describe all of the others. In algebraically closed valued fields, this was done by Haskell, Hrushovski and Macpherson in their seminal paper [HHM06] (see theorem 4.3.2). Chapters 3 and 4 present generalizations of this result to a large class of characteristic zero henselian fields (potentially with additional structure). The main results of [HR21; RV24] are exposed in chapter 3.

The last chapter is of a different nature as it aims to provide a new perspective on the proofs of these results and many others; one that is perhaps not entirely apparent in my published work. The main point is that, although the ultimate goal is the canonical description of definable sets and its geometric consequences, the core of the technical work is the study of non forking and related independence relations in valued fields. Indeed, as explained in proposition 4.1.8, elimination of imaginaries can be deduced from the existence of an independence relation with a few good properties. The rest of the chapter describes various natural independence relations in valued fields in a quest to eventually find one that does have those good properties (see corollary 4.9.1).

In that last chapter, we also describe the results in [HKR18; MR21; CRV24] as they are naturally related to this material.

1 Hensel minimality

Symphonie n°12, *L'année 1917*
Dimitri Chostakovich

Since the definition of o-minimality in the 1980s, it has become increasingly clear that this approach provides a possible framework for "tame" geometry à la Grothendieck. The main idea is that, by restricting the class of real functions we consider, and in particular by ruling out oscillation phenomena, we can obtain a class of geometric objects where "everything goes right":

- functions are arbitrarily piecewise differentiable;
- definable sets resemble, up to a controlled deformation, products of intervals;
- one can give explicit bounds on the number of rational points of a given height in a definable set — this is known as the Pila-Wilkie theorem [PW06].

These are but a few of the many consequences of this "axiomatic" approach to real geometry which make it an ideal setup in which to consider various questions from algebraic and arithmetic geometry.

However, the o-minimal approach is, by nature, restricted to the study of archimedean geometry. There has certainly been attempts to import the ideas of o-minimality into the non-archimedean world: C-minimality (for algebraically closed valued fields) and P-minimality (for p-adic and related fields)... But these notions have not enjoyed the same spectacular development as o-minimality has, and what's more, while the theory of models of all henselian fields of residue characteristic zero is well-studied and has good geometric properties, those notions are restricted to certain specific classes of henselian fields.

1.1 Residue characteristic zero

Hensel minimality (in its various guises) is an attempt to find a general axiomatic setup for non-archimedean geometry and provide a unified framework with strong geometric consequences. The starting point of Hensel minimality is the following notion.

Let K be a valued field, potentially with additional structure.

Definition 1.1.1. 1. Let $X \subseteq \mathrm{K}$ be definable and $C \subseteq \mathrm{K}$ be finite. We say that X is prepared by C if for any ball b disjoint from C, we either have

$$b \cap X = \varnothing \text{ or } b \subseteq X.$$

2. The structure K is said to be benign[1] if any definable subset of K, in any elementary extension, is prepared by some finite $C \subseteq \mathrm{K}$.

Remark 1.1.2. As fas as we know, contrary to what happens for o-minimality, preparation of definable sets in K does not imply preparation of definable sets in every elementary extension — an extension that preserves the interpretation of all first order formulas.

By compactness of first order logic, preparation in every elementary extension is equivalent to requiring that, in K, the size of C stays bounded as X varies in a definable family: for every $X \subseteq \mathrm{K} \times Z$ definable in K, there exists an integer N such that, for any $z \in Z(K)$, the set $X_z = \{y \in \mathrm{K} : (y, z) \in X\}$ is prepared by some $C \subseteq \mathrm{K}$ of size at most N.

Remark 1.1.3. Many "complete" valued fields are benign. Let p denotes the residue characteristic, it is the case if the following conditions hold:

- it is algebraically maximal — in any algebraic extension either the value group or the residue field must grow;
- if $p > 0$, the residue field does not have any degree p extension;
- if $p > 0$, any element of $[|p|, |p|^{-1}]$ is p-divisible.

In particular, any residue characteristic zero henselian valued field is benign.

Although this requires some reformulation to become apparent, this notion has very close ties to o-minimality. Recall that an ordered structure is said to be o-minimal if every definable set X in arity one is a finite union of points and open intervals. This can be rephrased as follows: there exists a finite set C such that for any interval I disjoint from C, we either have

$$I \cap X = \varnothing \text{ or } I \subseteq X.$$

As we will see in section 4.7, a field being benign does have remarkable consequences on definable sets in the pure henselian setting: we can compute "moduli space" for definable sets[2]. However, it does not seem, on its own, to imply an equivalent to o-minimal geometry. The main problem is, as always, the ultrametric nature of the valuation: even if we assume that the set C has minimal cardinality, it is not uniquely determined by X; whereas it is in the o-minimal case.

A solution to this problem is to specify, *a priori*, how C depends on X. Enter the leading terms $\mathrm{RV} = \mathrm{K}/(1 + \mathfrak{m})$. It is naturally a multiplicative monoid — actually $\mathrm{RV}^{\times} = \mathrm{K}^{\times}/1 + \mathfrak{m} = \mathrm{RV} \setminus \{0\}$ is a group. The valuation factorizes through $\mathrm{rv} : \mathrm{K} \to \mathrm{RV}$ and induces a group morphism $|\cdot| : \mathrm{RV}^{\times} \to |\mathrm{K}|$ with kernel $\mathrm{k}^{\times}$. In other words, we have a short exact sequence:

$$1 \to \mathrm{k}^{\times} \to \mathrm{RV}^{\times} \to |\mathrm{K}^{\times}| \to 1.$$

But this structure also retains a trace of the addition, albeit of a tropical nature (a hyperfield in Krasner's terminology [Kra59, §3]): for every $\zeta, \xi \in \mathrm{RV}$ define

$$\zeta \oplus \xi = \{\mathrm{rv}(z + x) : \mathrm{rv}(z) = \zeta \text{ and } \mathrm{rv}(z) = \xi\}.$$

[1] I borrow this terminology from [Tou], but somewhat backwards as Touchard uses it to describe a list of valued fields whereas we use it to capture a property that they all share.

[2] This is also easily seen to be the case in dense o-minimal groups.

This behaves as a "multivalued addition". To be precise, we have:

$$\zeta \oplus \xi = \begin{cases} \zeta & \text{if } |\zeta| > |\xi|; \\ \xi & \text{if } |\xi| > |\zeta|; \\ \zeta(1+\zeta^{-1}\xi) & \text{if } |\zeta| = |\xi| \text{ and } \zeta \neq -\xi; \\ \{\chi : |\chi| < |\zeta|\} & \text{if } \zeta = -\xi, \end{cases}$$

where, in the third case, the addition is computed in k.

The leading terms play a fundamental role in the model theory of valued fields as they encodes all the structure induced by quantification in algebraically maximal roughly Kaplansky fields (the valued fields described in remark 1.1.3).

Leading terms are intrinsically related to benign fields: if C is a finite set then maximal balls disjoint from C are exactly the fibers of the map $\mathrm{rv}_C : \mathrm{K} \to \mathrm{RV}^{|C|}$ defined by

$$x \mapsto (\mathrm{rv}(x-c))_{c \in C}.$$

In other words:

Lemma 1.1.4. *Any definable set $X \subseteq \mathrm{K}$ is prepared by a finite $C \subseteq \mathrm{K}$ if and only if*

$$X = \mathrm{rv}_C^{-1}(\mathrm{rv}_C(X)).$$

Let us now define the weakest form of Hensel minimality[3], which also holds in residue characteristic zero henselian valued fields (potentially with analytic structure) — see section 1.4 for precise statements.

Definition 1.1.5. A valued field K, potentially with additional structure, is said to be 0-h-minimal if, in any elementary extension, for every $A \subseteq \mathrm{K}$, any $A \cup \mathrm{RV}$-definable $X \subseteq \mathrm{K}$ is prepared by some A-definable finite set $C \subseteq \mathrm{K}$.

Remark 1.1.6.
- A set $X \subseteq \mathrm{K}$ is said to be $A \cup \mathrm{RV}$-definable if it is defined by a formula whose parameters are in $A \cup \mathrm{RV}$; in other words if there is $Y \subseteq \mathrm{K}^{n+1}$ such that $X = Y_z = \{x \in \mathrm{K} : (x,z) \in Y\}$ for some $z \in \mathrm{K}^n$, that this fiber only depends on the tuple $\mathrm{rv}(z)$ and that Y is defined by a formula that only involves polynomials with coefficients in the ring generated by A (as well as symbols of the language).
- Again, we can avoid any reference to elementary extensions and parameters for definable sets, if we require instead that preparation in K holds uniformly (see [CHR22, definition 2.6.2.1] for a definition of uniform preparation).

However, the only consequence of this notion that we know of is that K must be definably spherically complete ([CHR22, lemma 2.7.1]): any definable chain of balls has a non empty intersection. In particular, K must be algebraically maximal. As we will see in section 4.5, this "analytic" property is helpful in understanding the geometry of sets definable in valued fields; but we are still far from the stronger consequences of o-minimality.

As it turns out, we also need more local information which can be expressed using coarsenings of the initial valuation — this is reminiscent of non-standard real analysis. It is then useful to introduce the following additional parameter spaces:

[3]Second weakest if we also count "benign".

Definition 1.1.7. For any $\lambda \in |\mathrm{K}^\times|_{\leqslant 1}$, define

$$\mathrm{RV}_\lambda = \mathrm{K}^\times/(1+\lambda\mathfrak{m})$$

where $\lambda\mathfrak{m} = \{x \in \mathrm{K} : |x| < \lambda\}$.

This is once again a "hyperfield" with multiplicative structure and the trace of additive structure with a tropical behavior. We can now define a range of minimality notions. They all hold in in residue characteristic zero henselian valued fields (potentially with analytic structure) – see section 1.4.

Definition 1.1.8. Let K be a valued field, potentially with additional structure.

- We say that a definable set $X \subseteq \mathrm{K}$ is λ-prepared by some finite set $C \subseteq \mathrm{K}$ if
$$X = \mathrm{rv}_{\lambda,C}^{-1}(\mathrm{rv}_{\lambda,C}(X)),$$
where $\mathrm{rv}_{\lambda,C}^{-1}(x) = (\mathrm{rv}_\lambda(x-c))_{c\in C}$.
- Let $\ell \geqslant 0$ be either an integer or ω. The structure K is said to be ℓ-h-minimal if, in every elementary extension, for every $\lambda \in |K^\times|_{\leqslant 1}$, every $A \subset \mathrm{K}$ and every $A' \subset \mathrm{RV}_\lambda$ of cardinality $|A'| \leqslant \ell$, any $(A \cup \mathrm{RV} \cup A')$-definable set $X \subset \mathrm{K}$ can be λ-prepared by a finite A-definable set $C \subset \mathrm{K}$.

Remark 1.1.9.

- As before, the explicit reference to elementary extensions and parameters can be avoided if we require, *a priori*, uniform preparation in K.
- We can interpret λ-preparation in the following geometric way: for any open ball b of radius $\mathrm{rad}(b)$ with $|b - C| = \min_{x\in b, c\in C} |x-c| \geqslant \lambda^{-1} \cdot \mathrm{rad}(b)$ we either have
$$X \cap b = \varnothing \text{ or } b \subseteq X.$$

The most relevant values of ℓ are 0, 1 and ω.

1.2 Differentiation

The case of $\ell = 1$ might seem somewhat ad-hoc, but it is sufficient to imply the existence of a rich geometry. In fact, it is equivalent to some basic tame geometric behavior.

In this section, let K be a residue characteristic zero valued field, potentially with additional structure.

Theorem 1.2.1 ([CHR22, theorem 2.9.1])**.** *The structure K is 1-h-minimal if and only if, in every elementary extension, for every $A \subset \mathrm{K} \cup \mathrm{RV}$ and for every A-definable $f : \mathrm{K} \to \mathrm{K}$, the following two conditions hold:*

a. There exists a finite A-definable $C \subset \mathrm{K}$, and for every ball $b \subset \mathrm{K}$ disjoint from C, some $\mu_b \in |\mathrm{K}|$ such that
$$|f(x) - f(y)| = \mu_B \cdot |x-y|,$$
for every $x, y \in b$.

b. The set $\{d \in \mathrm{K} : f^{-1}(d)$ is infinite$\}$ is finite.

The first condition expresses that, locally, a definable function preserves balls and that the ratio between the radii in the domain and the image is constant. The second condition is necessary for the existence of a well behaved notion of dimension for definable sets.

We also show that 1-h-minimality implies much stronger geometric tameness than those two conditions. First of all, the first condition in theorem 1.2.1 can be improved to the so called "Jacobian property", central to the development of motivic integration.

Assume K is 1-h-minimal.

Theorem 1.2.2 ([CHR22, corollary 3.2.7]). *For any definable $f : \mathrm{K} \to \mathrm{K}$, there exists a finite set $C \subset \mathrm{K}$ such that for any ball b disjoint from C and any $x, y \in b$,*

1. *The derivative f' exists on b and $\mathrm{rv} \circ f'$ is constant on b.*
2. $\mathrm{rv}(f(x) - f(y)) = \mathrm{rv}(f'(b))\mathrm{rv}(x - y)$.
3. *For every open ball $b' \subset b$, $f(b')$ is either a point or an open ball.*

Remark 1.2.3.
- For any $\lambda \in |\mathrm{K}^\times|_{\leqslant 1}$, the second statement actually also holds of rv_λ: if b is an open ball with $|b - C| \geqslant \lambda^{-1} \cdot \mathrm{rad}(b)$, for any $x, y \in b$,

$$\mathrm{rv}_\lambda(f(x) - f(y)) = \mathrm{rv}_\lambda(f'(b))\mathrm{rv}_\lambda(x - y).$$

- In [CHR22, theorem 5.4.10], we also prove a higher dimensional version of the jacobian property.

Theorem 1.2.2 gives a very very tight control on the approximation of f by its first derivative. Our strongest result in that direction is that definable functions are well approximated by their Taylor polynomials:

Theorem 1.2.4 ([CHR22, theorem 3.2.2]). *Let $r \geqslant 0$ be an integer. For every definable $f : \mathrm{K} \to \mathrm{K}$, there exists a finite set C such that for any ball b disjoint from C and any $x, y \in b$, we have:*

$$\left| f(y) - \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(y - x)^i \right| \leqslant \left| f^{r+1}(x) \cdot (y - x)^{r+1} \right|.$$

Remark 1.2.5. If f is $A \cup \mathrm{RV}$-definable with $A \subseteq \mathrm{K}$, then C can be chosen A-definable.

1.3 Mixed characteristic

Henselian valued fields in mixed characteristic often fail to be 0-h-minimal, let alone ℓ-h-minimal. For example the set P_p of p-th powers in $\mathbb{Q}_p$ cannot be prepared by any finite set. Indeed, for any $n \geqslant 0$, the open balls of radius $|p|^{pn}$ around p^{pn} are all disjoint and intersect P_p non trivially. However P_p is $|p|$-prepared by $\{0\}$.

There are two natural ways to get around this issue:
- work with a residue characteristic zero coarsening;
- take into account the valuation of integers in the definition of ℓ-h-minimality.

This leads to the following two definitions.

Definition 1.3.1. Let K be a (sufficiently saturated) characteristic zero valued field, potentially with additional structure, and let $\ell \geqslant 0$ be either an integer or ω.

- Let $|\cdot|_0$ denote the finest residue characteristic zero coarsening of $|\cdot|$ — whose valuation ring is $\mathcal{O}[p^{-1}]$. We say that K is ℓ-h^{ecc}-minimal if the valued field $(K, |\cdot|_0)$, with $|\cdot|$ as additional structure, is ℓ-h-minimal.
- We say that K is ℓ-h^{mix}-minimal if for every $\lambda \in |\mathrm{K}^\times|_{\leqslant 1}$, every integer $n \geqslant 1$, every $A \subset \mathrm{K}$ and every $A' \subset \mathrm{RV}_\lambda$ of cardinality $|A'| \leqslant \ell$ and every $(A \cup \mathrm{RV}_n \cup A')$-definable set $X \subset \mathrm{K}$, there exists an integer $m \geqslant 1$ and an A-definable finite set $C \subset \mathrm{K}$ such that X is $|m| \cdot \lambda$-prepared by C.

For $\ell \in \{0, 1, \omega\}$ these two notions are in fact equivalent:

Theorem 1.3.2. *Let K be a (sufficiently saturated) characteristic zero valued field, potentially with additional structure, and let $\ell \in \{1, \omega\}$. The following are equivalent:*

1. *K is ℓ-h^{ecc}-minimal;*
2. *K is ℓ-h^{mix}-minimal.*

Moreover, if $\ell = 1$ both of these conditions hold if and only if for every integer n, every $A \subset \mathrm{K} \cup \mathrm{RV}_n$ and every A-definable $f : \mathrm{K} \to \mathrm{K}$, the following two conditions hold:

a. *There exists a finite A-definable $C \subset \mathrm{K}$ and a positive integer m, and, for every open ball b with $|b - C| \geqslant |m|^{-1} \cdot \mathrm{rad}(b)$, some $\mu_b \in |\mathrm{K}|$ such that*

$$|m| \cdot \mu_b \cdot |x - y| \leqslant |f(x) - f(y)| \leqslant |m|^{-1} \cdot \mu_b \cdot |x - y|,$$

for every $x, y \in b$.

b. *The set $\{d \in \mathrm{K} : f^{-1}(d)$ is infinite$\}$ is finite.*

Condition 2 follows from condition 1 by compactness and the converse implication is a consequence of the resplendency results presented in section section 1.5. If $\ell = 1$, the equivalence with conditions a and b then follow from theorem 1.2.1. Note that this is a slight misrepresentation of the proof as, when $\ell = 1$, resplendency is proved using conditions a and b, making the proof of the equivalence more convoluted.

The equivalence in theorem 1.3.2 (and compactness) allows us to translate many results, for example those in section 1.2, from residue characteristic zero to mixed characteristic. One can, for example deduce the following from theorem 1.2.4:

Theorem 1.3.3 ([Clu+23, theorem 3.1.2]). *Assume K is a characteristic zero 1-h^{mix}-minimal valued field, potentially with additional structure, and let $r \geqslant 0$ be an integer. For every definable $f : \mathrm{K} \to \mathrm{K}$, there exists an integer $m \geqslant 1$ and a finite set C such that for every ball b with $|b - C| \geqslant |m|^{-1} \cdot \mathrm{rad}(n)$ and any $x, y \in b$, we have:*

$$\left| f(y) - \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(y - x)^i \right| \leqslant \left| \frac{1}{(r+1)!} f^{r+1}(x) \cdot (y - x)^{r+1} \right|.$$

1.4 Examples

The main family of examples comes from valued fields with (or without) analytic structure as studied in [CL11].

Let A be a ring and $I \subset A$ a proper ideal. A (separated) Weierstrass system is a collection of rings of formal powers series $\mathcal{A}_{m,n} \subseteq A[[x_1 \dots, x_n, y_1, \dots y_m]]$ that verify a certain number of properties chief among which the Weierstrass division (and preparation) theorems (see [CL11, definition 4.1.5]). The asymmetry between x and y is due to the fact that we require the coefficients to converge I-adically as the degree in x grows.

The typical example of a Weierstrass system is $\mathcal{A}_{m,n} = A\langle x_1 \dots, x_n\rangle[[y_1, \dots y_m]]$ where A is assumed noetherian, separated complete in its I-adic topology.

A valued field with $\mathcal{A}$-structure is a valued field K with compatible interpretations of every $f \in \mathcal{A}_{m,n}$ as functions $f^K : \mathcal{O}^n \times \mathfrak{m}^m \to \mathcal{O}$ (see [CL11, definition 4.1.6] for the exact compatibility conditions).

Let K be a characteristic zero valued field with either

- no further structure;
- an $\mathcal{A}$-structure.

Theorem 1.4.1 ([CHR22, theorem 6.2.1]). *1. If K has residue characteristic zero, then K is ω-h-minimal.*

2. If K has residue characteristic $p > 0$, then K is ω-h^{mix}-minimal.

Amusingly, another family of examples comes from *o*-minimality itself. Let k be an o-minimal expansion of a real closed field. Let $k \preccurlyeq K$ be an elementary extension of k and $\mathcal{O} \subseteq K$ be the convex hull of k. Then O is a valuation ring and we denote its associated valuation by $|\cdot|$. We assume that it is non trivial.

Theorem 1.4.2 ([CHR22, theorem 6.3.4]). *If k is power bounded then $(K, |\cdot|)$ with its o-minimal structure is 1-h-minimal.*

If k is an expansion of $\mathbb{R}$, it is power bounded if and only if any definable function is eventually bounded by a polynomial — equivalently if the exponential function is not definable. For example, the expansion of $\mathbb{R}$ by analytic functions restricted to products of bounded closed intervals is a power bounded o-minimal structure.

1.5 Resplendency and cell decomposition

Let us conclude this chapter by stating our cell decompositions results. This will be made easier if we know an important technical feature of Hensel minimality: it is preserved by naming subsets of the leading terms.

Theorem 1.5.1 ([CHR22, theorem 4.1.19]). *Fix $\ell \in \{0, 1, \omega\}$ and let K be an ℓ-h-minimal valued field. Let K' be an RV-expansion of K — that is, an expansion obtained by naming subsets of cartesian powers of RV or, to be precise, their pullbacks by rv. Then K' is ℓ-h-minimal*

The equivalent result holds for h^{mix}-minimal fields (see [Clu+23, propostion 2.6.5]).

From this result (and some further work) we can deduce that h-minimality is preserved under coarsening; this is a crucial part of the proof of theorem 1.3.2.

Theorem 1.5.2. *Fix $\ell \in \{1,\omega\}$ and let $(K, |\cdot|)$ be an ℓ-h-minimal valued field. Let $|\cdot|'$ be a coarsening of $|\cdot|$. Then $(K, |\cdot|')$, with its original structure as well, is ℓ-h-minimal.*

If $\ell = \omega$, this is [Clu+23, corollary 4.2.4]. If $\ell = 1$, it is a consequence (and a crucial step of the proof of) [Clu+23, theorem 2.2.8].

One other important consequence of theorem 1.5.1 is that any 1-h-minimal valued field K has an RV-expansion (and hence a 1-h-minimal expansion) in which algebraic closure is equal to definable closure in the field: for every $A \subseteq \mathrm{K}$, every finite A-definable $C \subseteq \mathrm{K}$ and every $c \in C$, the singleton $\{c\}$ is A-definable.

In that case the notion of cell is easier to define.

Definition 1.5.3. Let $A \subseteq K$, let $(c_i)_{0<i\leqslant n} : \mathrm{K}^{i-1} \to \mathrm{K}$ and $R \subseteq \mathrm{RV}^n$ be A-definable. Then

$$\{x \in \mathrm{K}^n : \mathrm{rv}(x_i - c_i(x_{<i})) \in R\}$$

is called an A-definable cell.

Theorem 1.5.4 ([CHR22, theorem 5.2.3 and addenda])**.** *Let K be a 1-h-minimal valued field where algebraic closure is equal to definable closure and let $\mathrm{A} \subseteq \mathrm{K}$. Then any A-definable subset of K^n can be partitioned into finitely many A-definable cells.*

Moreover:

- *finitely many A-definable functions $X \to \mathrm{K}$ can be assumed to be continuous on the cells (in particular the functions c_i themselves);*
- *some definable family $X \subseteq \mathrm{RV}^m \times \mathrm{K}^n$ can be assumed to have constant fibers on cells;*
- *up to a permutation of the coordinates, the c_i can be chosen to be 1-Lipshitz on cells.*

If algebraic closure is not equal to definable closure in K, we can first go to a 1-h-minimal expansion where it is and find a cell decomposition there. We can recover a partition in the original language by applying [CHR22, lemma 4.3.4], but the elements of the partition are parametrized in a more complicated way and are not cells anymore in the naive sense of definition 1.5.3.

As before, there is an equivalent statement for 1-h^{mix}-minimal valued fields (see [Clu+23, theorem 3.3.3] and addend[4], 1-h-minimality refers to what we defined as 1-h^{mix}-minimality here.). In [Clu+23, section 4], we explain how to deduce counting results for transcendental curves using these results following [CCL15; CFL].

[4]Note that in [Clu+23]

2 Pseudo closed fields

Le sacre du printemps
Igor Stravinski

The work presented until now has a local flavor. We consider "complete" fields for a given valuation. However, in this section, we will consider a more global situation. The fields we consider come with finitely many V-topologies (*i.e.* topologies arising from valuations or orders) and are assumed to verify a local-global principle for the existence of points in geometrically integral varieties.

These local-global principles go back to Ax's definition of pseudo algebraically closed fields in order to axiomatize the theory of finite fields [Ax68]. A field K is said to be pseudo algebraically closed if every geometrically integral algebraic variety over K has a K-point. Prestel's generalization of this notion, meant to take into account the orders on the field, gives it more clearly the appearance of a local-global principle. A field K is said to be pseudo-real closed if any geometrically integral algebraic variety over K, with a smooth point in every real closure, has a K-point. The equivalent notion for p-adic valuations then emerged under the name of pseudo p-adically closed fields [Gro87]. Various generalizations of these two notions were then developed with the aim of unifying the two approaches by not specifying the type (p-adic or ordered) of the topologies at play. Examples include Ershov's regularly closed fields and Pop's pseudo classically closed fields.

All these definitions gave rise to the first developments of a model theory for each of these classes (elementarity, decidability, etc.). However, with the notable exception of pseudo algebraically closed fields which played a central role in the development of simplicity theory and continue to be important examples for the understanding of Shelah's hierarchy, the development of a thorough model theory of these structures had to wait for S. Montenegro's thesis [Mon17b; Mon17a] in which she classifies (in Shelah's sense) pseudo real closed and pseudo p-adically fields, describes deviation and proves elimination of imaginaries for pseudo real closed fields.

In this chapter, we describe our attempt at providing a general framework to study those two examples (and their generalizations) as well as new examples. What sets our work apart from previous attempts is that we aim for a more model theoretic flavor both in our definitions and our goals, with no particular focus on more arithmetic notions such as orders or p-adic topologies. As a particular example, we aim at describing pseudo algebraically closed fields with (independent) valuations, a family of structures that had so far escaped scrutiny.

2.1 Pseudo T-closed fields

Let T be a theory of large fields, potentially with additional structure given by some language $\mathfrak{L}$. A field K is said to be large if every irreducible variety (of nonnegative dimension) over K with a smooth K-point has infinitely many — equivalently Zariski dense many.

Given a field K with an $\mathfrak{L}$-structure — an $\mathfrak{L}$-field — we can consider models of T containing K as its "completions". This naturally leads to the following local-global principle, and its model theoretic reformulation:

Proposition 2.1.1 ([MR23, proposition 3.3]). *The following are equivalent:*

1. *for every geometrically integral smooth variety V with an L-point for every $L \vDash T$ in which K embeds — we say that V is totally T — then $V(K)$ is non empty (equivalently Zariski dense in V);*
2. *for any regular extension $K \leqslant F$ such that F can be embedded over K in an elementary extension of any $L \vDash T$ in which K embeds — we say that the extension $K \leqslant F$ is totally T — then K is existentially closed in L as a ring.*

Any such K is said to be pseudo T-closed.

Varying T, we recover some of the previously mentioned examples (and many others). For example:

- If T is the theory of algebraically closed fields (in the ring language) then pseudo T-closed fields are pseudo algebraically closed fields.
- If T is the theory of real closed fields, in the ring language, then pseudo T-closed fields are pseudo real closed fields.
- If T is the theory of real closed fields in the ordered ring language then pseudo T-closed fields are pseudo real closed fields whose only order is the one named in the language.

An oddity in the definition of a pseudo T-closed field, especially if T is a theory of fields with additional structure, is that we only ask for existential closure as a ring.

If T is the theory of real closed fields (in the ring language), if $K \leqslant L$ is a totally real regular extension then, by definition, all orders of K can be extended to L and Prestel [Pre81, theorem 7.1] shows that K is actually existentially closed in L when naming all the orders[1]. This result also has a geometric reformulation; see proposition 2.1.2.

And this phenomenon can be witnessed in multiple other examples. A result of Kollar [Kol07, theorem 2] implies that any regular extension of a pseudo algebraically closed field is also existentially closed as a valued field if the valuation is non trivial. This result is particularly striking because, contrary to orders in the pseudo real closed case, a non trivial valuation on a pseudo algebraically closed field is not encoded into the arithmetic of the field.

This leads to the following natural strengthening of the notion of pseudo T-closed fields. Let I be a finite set of theories of relational expansions of fields that eliminate quantifiers. We denote by $(T_i)_{i\in I}$ the elements of I and by $\mathfrak{L}_i$ the language of T_i. We assume that any model of T_i can be made into an $\mathfrak{L}$-structure which is a model of T. The choice of the T_i amounts to choosing certain "completions" whose structure we wish to take in account in the local-global principle.

[1] Equivalently finitely many of them

Let $T_I = \bigcup_i T_{i\forall}$, where $T_{i\forall}$ is the set of the universal consequences of T_i (*i.e.* the theory of substructures of its models) and let $K \models T_I$ be an $\mathfrak{L}$-field.

Proposition 2.1.2 ([MR23, proposition 4.3]). *The following are equivalent:*

1. *any regular totally T extension $K \leqslant F \models T_I$ is $\mathfrak{L}_I$-existentially closed;*
2. *for every geometrically integral smooth totally T variety V over K and for every $\mathfrak{L}_i(K)$-definable K-Zariski dense $X_i(M_i) \subseteq V$, for all $i \in I$, $\bigcap_i X_i(K)$ is non empty (equivalently, is Zariski dense in V).*

Such a field is said to be I-pseudo T-closed.

As we discussed above, if T is the theory of real closed fields and the T_i are theories of real closed fields with a named ordering, if K is pseudo T-closed and the named orders are distinct on K, then K is I-pseudo T-closed; this is also the case if T is the theory of algebraically closed fields and I is the singleton consisting of the theory of algebraically closed (non trivially) valued fields.

In both of those cases, the T_i are theories of topological fields. This turns out to be an important ingredient of the general phenomenon at play here.

Let us now assume that:

(H) the T_i are theories of fields with a (non discrete) definable henselian V-topology (that is, a topology induced by a real closed order or a henselian valuation for which one bounded open set is definable) and that, moreover, in models of T_i, any definable set has non empty V-topological interior in the rational points of its Zariski closure.

Each theory T_i then induces a V-topology on K generated by the K-points of the $\mathfrak{L}_i(K)$-definable open sets.

Theorem 2.1.3 ([MR23, theorem 5.11]). *Assume that*

(F) for any $M \models T$ containing K there exists an $i \in I$ and $M_i \models T_i$ containing K which embeds over K (as a field) in an elementary extension of M.

Remark 2.1.4. Hypothesis **(F)** essentially expresses that K only has finitely many "completions" in the sense of T — and that all these completions are among the T_i. This finiteness hypothesis is crucial for the proof but, as illustrated by the case of pseudo real closed fields, it is not necessary.

It would be interesting to have a proof that covers both cases (and more !). One test case would be to show that a pseudo real closed field with infinitely many orders and one valuation independent from those orders is existentially closed as a valued field in a regular totally real extension — in other words, the equivalent of Kollar's result [Kol07, theorem 2] for a pseudo real closed field with infinitely many orders.

Then K is I-pseudo T-closed if and only if the following conditions hold:

1. *K is pseudo T-closed;*
2. *the topologies induced by each of the T_i on K are distinct;*
3. *for all $i \in I$, $\mathrm{dcl}_i(\varnothing) \subseteq K^{\mathrm{a}}$, where dcl_i denotes the definable closure in T_i;*
4. *for all $i \in I$, any non-empty open $\mathcal{L}_i(K)$-definable $X \subseteq \mathbb{A}^1$ contains a K-point.*

Remark 2.1.5. In particular, given finitely many independent valuations on a pseudo algebraically closed field, it is existentially closed in any regular extension as a ring with all valuations named.

2.2 Burden and forking

The second half of [MR23] describes some of the model theoretic properties of I-pseudo T-closed fields in the bounded perfect case. Recall that T is a theory of large fields, potentially with additional structure, and the T_i, for $i \in I$, are theories of relational expansions of fields that eliminate quantifiers.

A field is said to be bounded if, for every d, it has finitely many extensions of degree d. Conjecturally boundedness and perfection are needed to ensure any sort of finite dimensional behavior. For technical reasons, we will now assume that we have added constants for a subfield with the same Galois group.

We mostly focus on two related questions: computing the burden and characterizing forking. Burden is a combinatorial notion of dimension defined as follows. Let X be a definable set. Recall that families of subsets of X are said to be uniformly definable if they are all fibers of a definable set $Y \subseteq X \times Z$.

Definition 2.2.1. Let X be definable in some structure M and let κ be a cardinal.

1. An *inp-pattern* of depth κ in X consists, for each $l < \kappa$, of a family $(Y_{lj})_{j<\omega}$ of uniformly definable of subsets of X, such that:
   - for every $l < \kappa$, for some $k_l \geqslant 1$, any intersection of k_l elements of the family $(Y_{lj})_{j<\omega}$ is empty.
   - for every $f : \kappa \to \omega$, we have
   $$\bigcap_{l<\kappa} Y_{lf(l)} \neq \varnothing.$$
2. The *burden* of X, denoted $\mathrm{bdn}(X)$ is the supremum[2] (if it exists) of all cardinals κ such that there exists an inp-pattern of depth κ in X; otherwise, we say that the burden of X is infinite.

Remark 2.2.2. A structure M (in a language $\mathfrak{L}$) is said to be NTP_2 if no definable set has infinite burden; in other words, if there is a bound on the size of inp-patterns in any definable set.

Remark 2.2.3. There are (at least) two natural sources of burden. Let M be a structure and let X be some infinite definable set.

1. We have $\mathrm{bdn}(X^n) \geqslant n$. Indeed, if $(a_j)_{j<\omega} \in X(M)$ are pairwise distinct, define
   $$Y_{l,j} = \{x \in X^n : x_l = a_j\},$$
   for every $l < n$ and $j < \omega$. They form an inp-pattern of depth n on X^n. Indeed, for any l and any distinct j_1 and j_2, $Y_{l,j_1} \cap Y_{l,j_2} = \varnothing$; and for any choice of j_l, for all $l < n$,
   $$\bigcap_l Y_{lj_l} = \{(a_{j_l})_{l<n}\}.$$
   This example illustrates how burden does compute some form of dimension.

[2] To keep track of finer estimates, burden can be computed in $\mathrm{Card}^\star$ [Tou, Definition 1.27] where we distinguish between a maximum and a proper supremum.

2. Let X be a definable set that admits n independent definable Hausdorff topologies $(\tau_i)_{i<n}$ — topologies are said to be independent if open sets for distinct topologies always intersect. Then $\mathrm{bdn}(X) \geqslant n$. Indeed for every $i < n$, let $Y_{i,j} \subseteq X$ be disjoint uniformly definable open sets for τ_i. Since the topologies are independent, they form an inp-pattern of depth n on X.

Our first result is a computation of burden in terms of the burden of the completions. Let bdn_i denote burden computed in the theory T_i. Let K be a I-pseudo T-closed field which is both perfect and bounded. We also assume both hypothesis **(H)** and **(F)** of theorem 2.1.3. We consider K as an $\bigcup_i \mathfrak{L}_i$-structure.

The dimension of a definable set X is the (Krull) dimension of its Zariski closure.

Theorem 2.2.4 ([MR23, theorem 7.9]). *Let X be definable in K with parameters. Assume that $\dim(X) = d$. We have*

$$\mathrm{bdn}(X) = \mathrm{bdn}(\mathbb{A}^d) = \sum_i \mathrm{bdn}_i(\mathbb{A}^d).$$

Remark 2.2.5. In particular, K is NTP_2 if and only if all the T_i are.

Coming back to the example of multivalued pseudo algebraically closed fields, we get:

Corollary 2.2.6. *Let $(K, |\cdot|_1, \ldots, |\cdot|_n)$ be a bounded perfect pseudo algebraically closed field and the $|\cdot|_i$ be independent valuations. Then*

$$\mathrm{bdn}(\mathbb{A}^m) = mn.$$

In particular, $(K, |\cdot|_1, \ldots, |\cdot|_n)$ is NTP_2 and burden is additive.

Let us now consider forking. Let M be a (sufficiently saturated and homogeneous) structure and let $A \subseteq M$.

Definition 2.2.7. An M-definable set X is said to divide over A if there exist $\sigma_j \in \mathrm{aut}(M/A)$, for $j < \omega$, and an integer k such that for every $J \subseteq \omega$ of size k, $\bigcap_{j\in J} \sigma_j(X) = \varnothing$.

This is meant as a combinatorial characterization of "smallness". For example, the measure of $\varphi(M, b)$ would be zero for any probability measure invariant under the action of $\mathrm{aut}(M/A)$. However, formulas that divide over A do not, in general, form an ideal. Formulas in the ideal that they generate are said to fork over A:

Definition 2.2.8.
1. An M-definable set forks over A if it is a finite union of M-definable sets that divide over A.
2. A (partial) type over M forks over A if it contains an M-definable set that forks over A.
3. Given tuples $a, b \in M$ and a set $C \subseteq M$, we say that a is forking independent from b over C if $\mathrm{tp}(a/Cb)$ does not fork over C. We write $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_C b$.

This purely combinatorial notion of independence has a few properties expected of an "independence relation" — it is (left) transitive in towers : if $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{Cd} b$ and $d \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{C} b$ then $ad \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{C} b$. But it is also lacking a number of other expected properties — it might not be symmetric[3] nor even satisfy existence: there might exist a tuple a and a set $A \subseteq M$ such that $a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{A} A$. This is quite pathological !

Essentially by construction, forking satisfies existence if and only if dividing formulas form an ideal, *i.e.* "forking equals dividing". The question of existence for forking is one of the staples of understanding and using the forking machinery in a given structure.

Despite its obvious combinatorial nature, in many algebraic cases, forking captures a natural notion of independence: in algebraically closed fields (or more generally in bounded perfect pseudo algebraically closed fields), forking independence coincides with algebraic independence.

Let $\mathop{\smile\!\!\!\!\!\mid}^{i}$ denote forking independence in any $M_i \vDash T_i$ containing K and let τ be the topology generated by the open sets of all the topologies τ_i, for $i \in I$.

Theorem 2.2.9 ([MR23, proposition 8.1 and corollary 8.3]). *Let $A \subseteq K$.*

1. *Let $X \subseteq \mathbb{A}^d$ be definable in K and such that X is τ-dense in $\bigcap_i X_i$, where X_i is τ_i-open and definable in M_i over K. Then X does not divide over A if and only if, for all $i \in I$, X_i does not divide over A in M_i.*
2. *Suppose that, for every $i \in I$, T_i is NTP_2 and forking equals dividing in T_i, then if $A \subseteq B \subseteq K$ and $a \in K$ is a tuple,*

$$a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{A} B \text{ if and only if, for all } i,\ a \mathop{\smile\!\!\!\!\!\mid}^{i}_{A} B.$$

In the case of a pseudo algebraically closed field $(K, |\cdot|_1, \ldots, |\cdot|_n)$ with independent valuations, given the description of forking in algebraically closed valued fields (see corollary 4.3.5), we obtain a complete characterization of forking in terms of "quantifier free invariant" types. Consider K as a structure in the language $\mathfrak{L}$ which is the language of rings with a additional predicate $x|_i y$, for all $i \leqslant n$, interpreted as $|x|_i \geqslant |y|_i$.

Corollary 2.2.10. *Let $a, b \in K$ be tuples and let $A \subseteq K$. We have $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_{A} b$ if and only if $\mathrm{tp}(a/Ab)$ is with an A-invariant quantifier free type over some sufficiently large elementary extension of K.*

[3] By work of Kim and Pillay [KP97], it is symmetric if and only if (the theory of) M is simple; and it then enjoys a wealth of good properties like 3-existence over models.

3 Imaginaries in valued fields

Quatuor pour la fin du temps
Olivier Messiaen

A recurrent question in various mathematical fields is the existence, or at least the description, of moduli spaces for the objects we are studying: in other words, an object, in a well understood category, whose points are in bijection with the objects we wish to classify.

Model theory is no exception to this rule. It has gradually become apparent, particularly since Shelah's work, that the category of definable sets is not rich enough to describe the combinatorics of definable sets in a satisfactory way. The natural category to consider is therefore the one in which we have added the quotients of sets definable by a definable equivalence relation, usually called interpretable sets, and whose elements are called imaginaries. For examples, "moduli spaces" for families of uniformly definable sets always exist in the category of interpretable sets.

Since the work of Poizat [Poi83], the question of describing those interpretable sets has become, under the name of elimination of imaginaries, a natural step of the model theoretical study of any given structure. This question is also at the heart of some recent applications of model theory. A concrete example is the one developed in [HMR18] where the classification of imaginaries in p-adic fields and their ultraproducts allows us to extend Denef's rationality results for integrals of definable functions [Den84, theorem 3.2] to new classes of examples. The elimination of imaginaries in algebraically closed valued fields is also, in a more subtle way, a central ingredient of the theory of rigid analytic geometry à la Hrushovski-Loeser [HL16].

While valued fields have had a central place in model theory since the early development of the discipline with the work of A. Robinson [Rob77] and those of Ax and Kochen and independently Ershov [AK65; Ers65], and while the theoretical foundations of the elimination of imaginaries were laid in the early 1980s, the classification of imaginaries in valued fields only really got underway in the early 2000s. One of the reasons for this late development is that the ultrametric nature of valuations makes it a more complex question than in other algebraic structures usually considered. In fact, unlike what happens in an algebraically closed or real closed field, there are interpretable sets that are not definably isomorphic to definable sets. The question then becomes to describe those new quotients.

The deeply influential result of Haskell, Hrushovski and Macpherson [HHM06] changes the situation radically by giving an explicit description of interpretable sets in an algebraically closed valued field $(K, |.|)$: all interpretable sets are (up to a definable bijection) subsets of the space Mod_n of all definable $\mathcal{O}$-submodules of K^n, for some n. In fact, they prove a refinement of this classification by showing that it suffices to consider products of the valued field K, the spaces $\mathrm{Gr}_n = \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O})$ of all free rank n $\mathcal{O}$-submodules of K^n and the spaces $\mathrm{Lin}_n = \bigcup_{s\in\mathrm{Gr}_m} s/\mathfrak{m}s$ of their reduction modulo the maximal ideal $\mathfrak{m}$.

In this chapter we will describe generalizations of this result to non algebraically closed valued fields and valued fields with additional structure.

3.1 Definitions and notation

An important technical tool for the classification of interpretable sets is a construction due to Shelah that allows us, given any structure M in a language $\mathfrak{L}$ to expand it to a structure M^{eq} where the points are now all the imaginary points of M. This allows us to apply the tools of model theory (chief among which compactness of first order logic) to imaginary elements.

Definition 3.1.1.
- The language $\mathfrak{L}^{\mathrm{eq}}$ is the language containing $\mathfrak{L}$ with an additional sort S_X for every set $X \subseteq Y \times Z$ definable without parameters and an additional symbol $f_X : Z \to S_X$.
- The structure M^{eq} is the expansion of M to an $\mathfrak{L}^{\mathrm{eq}}$-structure given by considering

$$S_X(M^{\mathrm{eq}}) = Z(M)/\sim_X,$$

where $z_1 \sim_X z_2$ if $X_{z_1} = \{y \in Y : (y, z_1) \in X\} = X_{z_2}$, and interpreting f_X as the canonical projection.

From now on, when considering a structure M, we will implicitly be working in M^{eq} and all notions (like definable closure dcl, algebraic closure acl, etc.) will be computed in M^{eq}.

Definition 3.1.2. Given a definable set X, we denote by $\ulcorner X \urcorner \subseteq M^{\mathrm{eq}}$ the intersection of all $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ such that X is A-definable.

Any dcl-generating subset of $\ulcorner X \urcorner$ is called a code of X.

We can check that X is $\ulcorner X \urcorner$-definable, so it is the smallest dcl-closed set of definition for X. Let $\mathcal{D}$ be a collection of sets interpretable without parameters.

Definition 3.1.3.
- A definable set X is coded in $\mathcal{D}$ if it is definable with parameters in $\mathcal{D}(\ulcorner X \urcorner) = \mathcal{D} \cap \ulcorner X \urcorner$ — *i.e.*, it admits a code in $\mathcal{D}$.
- The structure M eliminates imaginaries down to $\mathcal{D}$ if, in every elementary extension, every definable set is coded in $\mathcal{D}$.

Remark 3.1.4.
1. Elimination of imaginaries down to $\mathcal{D}$ is equivalent to the fact that for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{dcl}(e))$ such that $e \in \mathrm{dcl}(d)$.
2. By compactness, provided $\mathrm{dcl}(\varnothing)$ is not too small, elimination of imaginaries down to $\mathcal{D}$ implies that every interpretable set is (up to a definable bijection) a subset of a finite product of sets in $\mathcal{D}$.

Definition 3.1.5. The structure M weakly eliminates imaginaries down to $\mathcal{D}$ if, for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{acl}(e))$ such that $e \in \mathrm{dcl}(d)$.

If M weakly eliminates imaginaries down to $\mathcal{D}$, it might not be the case that finite sets are coded in $\mathcal{D}$. This is the main difference between the two notions: if M weakly eliminates imaginaries down to $\mathcal{D}$ and finite sets of tuples in $\mathcal{D}$ are coded in $\mathcal{D}$, then M eliminates imaginaries down to $\mathcal{D}$.

3.2 Generalized geometric imaginaries

The main issue to generalize [HHM06] is that, in an arbitrary Henselian valued field, we do not expect their result to hold since the residue field k and the value group Γ might come with their own imaginaries. In fact, there are more subtle obstructions coming from the imaginaries of interpretable finite dimensional k-vector space. We have to introduce new spaces to take those imaginaries into account too.

Let K be a valued field, potentially with additional structure. For every integer n, we write Mod_n for the (ind-interpretable) space of all definable $\mathcal{O}$-submodules of K^n. Let $\mathrm{Mod} = \bigcup_n \mathrm{Mod}_n$. For every $R \in \mathrm{Mod}_n$, the quotient $R/\mathfrak{m}R$ is a k-vector space of dimension at most n.

Consider $(\mathrm{k}, R/\mathfrak{m}R)$ in the (two sorted) language of k-vector spaces — with the ring structure on k, the group structure on V and scalar multiplication. For every $d \geqslant 1$, for every interpretable set $X = V^n/E$, where E is definable (without parameters) in the theory of dimension d vector spaces, and for every $R \in \mathrm{Mod}$, with $\dim_{\mathrm{k}}(R/\mathfrak{m}R) = d$, we define X^R to be the interpretation of X in the structure $(\mathrm{k}, R/\mathfrak{m}R)$. We define

$$\mathrm{k}^{\mathrm{leq}} = \coprod_{X,R} X^R.$$

This is the space of imaginaries in all the $R/\mathfrak{m}R$ as R varies in Mod. We call them the k-linear imaginaries.

In the spirit of the Ax-Kochen-Ershov principle, in [HR21; RV24] we show under various technical assumptions that, along with imaginaries from Γ, these are the only new imaginaries.

We say that an ordered group Γ has property **(D)** if for any finite set $\Delta(x, y)$ of formulas (where x is a single variable in Γ) containing $x < y_0$ and any $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable Δ-type $p(x)$ is contained in an E-definable complete type $q(x)$.

An ordered group Γ is said to have bounded regular rank if it has at most countably many definable convex subgroups. All subgroups of $\mathbb{R}^n$ for any $n > 0$ have bounded regular rank. It follows from work of Vicaría [Vic23] that they all have property **(D)**.

Theorem 3.2.1 ([RV24, theorems 6.5 and 6.6]). *Let K be a residue characteristic zero henselian valued field[1], potentially with additional structure on k and Γ, separately. Assume that*

- *the value group has property **(D)**, or;*
- *if the value group is discrete of bounded regular rank and it has no further structure beyond adding a constant for a uniformizer $\pi \in \mathcal{O}$ — an element with maximal valuation < 1.*

Further assume that either:

- *an angular component map is added to the language — an angular component is a retraction of $\mathrm{k}^\times \to \mathrm{RV}^\times$, or ;*
- *the group $\mathrm{k}^\times$ is divisible, or;*

[1] The choice of the language does not matter much here as we will be working in K^{eq}. But for the sake of completeness, let us say we consider K in the language with three sorts for the valued field, the value group and RV.

- *for every $n \in \mathbb{Z}$, $\Gamma/n\Gamma$ is finite (and we add constants for lifts of its elements to* RV*).*

Then K weakly eliminates imaginaries down to $\mathrm{K} \cup \Gamma^{\mathrm{eq}} \cup \mathrm{k}^{\mathrm{leq}}$.

Weaker versions of this result appear in work with Hils [HR21, theorem A]. Building on this work, we later proved the result above with Vicaria. The general proof scheme of these results is expanded on in chapter 4.

Remark 3.2.2.
1. In [HR21], we also show that the conclusion of theorem 3.2.1 also holds in finitely ramified mixed characteristic, with perfect infinite residue field that eliminates $\exists^\infty$ and definably complete value group, at the cost of enlarging $\mathrm{k}^{\mathrm{leq}}$ to contain the imaginaries of all the $R/n\mathfrak{m}R$, for $n \geqslant 1$ and $R \in \mathrm{Mod}$. Note that these new sets are more complicated as they are finite rank free modules over finite length (ramified) Witt vectors of k.
2. It seems probable that, at the cost of expanding $\mathrm{k}^{\mathrm{leq}}$ in a similar manner, the conclusion of theorem 3.2.1 holds in any finitely ramified mixed characteristic valued field.

3.3 Multiplicative difference valued fields

In [HR21], we also classify imaginaries in valued fields with operators. An important example, isolated in [Pal12], is that of valued fields with a multiplicative automorphism. An automorphism is said to be multiplicative when Γ is an ordered $\mathbb{Z}[\sigma]$-module for some choice of ring order on $\mathbb{Z}[\sigma]$. There are two particularly interesting multiplicative behaviors for σ:

- The ω-increasing case: $|\sigma(x)| < |x|^{\mathbb{Z}}$, for all $x \in \mathfrak{m}$. One gets the asymptotic theory of $(\mathbb{F}_p((t))^{\mathrm{a}}, |\cdot|_t, \phi_p)$ as p grows, where $|\cdot|_t$ is the t-adic valuation and ϕ_p is the Frobenius automorphism.
- the isometric case: $|\sigma(x)| = |x|$ for all $x \in \mathrm{K}$. This is the asymptotic theory of $(\mathbb{C}_p, |\cdot|_p, \sigma_p)$ as p grows, where σ_p is a lift of the Frobenius automorphism on k.

Let $\mathcal{G}$ contain K, Gr_n and Lin_n, for all $n \geqslant 1$.

Theorem 3.3.1 ([HR21, theorem B]). *Let K be an existentially closed residue characteristic zero multiplicative difference valued field, potentially with a σ-equivariant angular component map. Then K eliminates imaginaries down to $\mathcal{G}$.*

In particular, any set uniformly interpretable in $(\mathbb{F}_p((t))^{\mathrm{a}}, |\cdot|_t, \phi_p)$ as p varies is uniformly definably embedded in $\mathcal{G}$. In the case of $(\mathbb{C}_p, |\cdot|_p, \sigma_p)$, the situation is more complicated as geometric elimination is not known (or expected) for fixed p. However, it does hold uniformly for large p.

Chapter 4 also describes the main ingredients in the proof of the that theorem. Section 4.10 describes similar results that can be obtained with the same methods for other "valued fields with operators".

4 Imaginaries and independence

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Imaginaries are naturally related to the geometric question of finding "moduli spaces" for families of definable sets. But they initially appear in Shelah's work as a way to describe forking and obstructions to (independent) amalgamation in stable theories: to be precise, obstructions to uniqueness of 2-amalgamation, or equivalently, existence of 3-amalgamation. Allow us a small detour into this stable picture.

4.1 Independence and amalgamation

Let us now restrict our attention to the stable case. Let M be a structure in some language $\mathfrak{L}$. Recall the definition of forking (definition 2.2.8).

Definition 4.1.1. A type $p(x)$ over M is said to be A-definable if for every formula $\varphi(x,y)$, the set $\{a \in M^y : \varphi(x,a) \in p\}$ is A-definable.

Theorem 4.1.2. *The following are equivalent:*

(i) 2-uniqueness over models: *for every $C \preccurlyeq M$ and every tuples $a, b_1, b_2 \in M$ such that $a \downfree^{\mathrm{f}}_C b_1$ and $a \downfree^{\mathrm{f}}_C b_2$, if $b_1 \equiv_C b_2$ then $b_1 \equiv_{Ca} b_2$;*

(ii) every type over M is M-definable.

In that case, (the theory of) M is said to be stable. These are but two of the many conditions equivalent to stability.

In a stable theory, 2-uniqueness does not hold over arbitrary sets, but the "finite equivalence theorem" states that it does over bases that are algebraically closed in M^{eq}; in other words, the obstruction to 2-uniqueness (and hence 3-existence) is given by algebraic imaginaries — that is, finite A-definable equivalence relations.

Proposition 4.1.3. *Assume M is stable. Let $E \subseteq M^{\mathrm{eq}}$. The following are equivalent:*

(i) $\mathrm{acl}(E) \subseteq \mathrm{dcl}(E)$*;*

(ii) 2-uniqueness over E: *for every tuples $a, b_1, b_2 \in M$ such that $a \downfree^{\mathrm{f}}_E b_1$ and $a \downfree^{\mathrm{f}}_E b_2$, if $b_1 \equiv_E b_2$ then $b_1 \equiv_{Ea} b_2$;*

(iii) 3-existence over E: *for every tuples $a, b, c_1, c_2 \in M$ such that $b \downfree^{\mathrm{f}}_E a$, $c_1 \downfree^{\mathrm{f}}_E a$, $c_2 \downfree^{\mathrm{f}}_E b$ and $c_1 \equiv_E c_2$, there exists $c^\star \in M$ such that $c^\star \downfree^{\mathrm{f}}_E ab$, $ac^\star \equiv_E ac_1$ and $bc^\star \equiv_E bc_2$.*

(iv) for every tuples $a, b, c \in M$ such that $a \equiv_E b$, $b \downfree^{\mathrm{f}}_E a$ and $c \downfree^{\mathrm{f}}_E ab$, there exists $c^\star \in M$ such that $ac^\star \equiv_E bc^\star \equiv_E ac$.

Proof. (i)⇒(ii) This non trivial implication is the “finite equivalence theorem” [She78, ch. III, theorem 2.8].

(ii)⇒(iii) The fact that 3-existence follows from 2-uniqueness is immediate: given a, b, c_1, c_2 as in (iii), by 2-uniqueness, any realization of $\mathrm{tp}(c_1/E) = \mathrm{tp}(c_2/E)$ independent from ab satisfies the conclusion of (iii).

(iii)⇒(iv) Condition (iv) is a particular case of condition (iii): take $c_1 = c$ and c_2 such that $bc_2 \equiv_F ac$, then by (iii) there is a $c^\star$ such that $ac^\star \equiv_F ac$ and $bc^\star \equiv_F bc' \equiv_F ac$.

(iv)⇒(i) This implication is a particular case of a more general phenomenon highlighted in proposition 4.1.8. □

Note that proposition 4.1.3 implies a characterization of forking in stable theories in terms of definable types:

Corollary 4.1.4. *Assume M is stable. Let $a \in M$ be a tuple and $E \subseteq B \subseteq M^{\mathrm{eq}}$. Then $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_E B$ if and only if there exists an* $\mathrm{acl}(E)$*-definable type containing* $\mathrm{tp}(a/B)$.

This follows from the fact that definable types have a “canonical basis”:

Definition 4.1.5. If $p(x)$ is an M-definable type, the canonical basis of p, noted $\mathrm{cb}(p)$, is the smallest $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ such that p is A-definable. It consists of $\ulcorner\{a \in M^y : \varphi(x,a) \in p\}\urcorner$ for every formula $\varphi(x,y)$.

There is also a Galois interpretation of the canonical basis: provided M is sufficiently saturated and homogeneous, we have

$$\{\sigma \in \mathrm{aut}(M/A) : \sigma(p) = p\} = \mathrm{aut}(M/\mathrm{cb}(p)).$$

Let us come back to corollary 4.1.4. If $a \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_E B$, then $\mathrm{tp}(a/B)$ is included in a type p over M that does not fork over E, and hence over $\mathrm{acl}(E)$. By 2-uniqueness, this type p is uniquely determined by its restriction to $\mathrm{acl}(E)$. It follows that it is fixed by $\mathrm{aut}(M/\mathrm{acl}(E))$ and hence that $\mathrm{cb}(p) \subseteq \mathrm{acl}(E)$. The converse holds in all generality.

In turn, elimination of imaginaries is reflected in the properties of (forking) independence:

Proposition 4.1.6. *Let M be a sufficiently saturated stable $\mathfrak{L}$-structure and let $F \subseteq E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. Then*

$$E \subseteq \mathrm{dcl}(F)$$

if and only the two following conditions hold:

(a) for every tuples $a, b \in M$, there exists $a^\star \equiv_E a$ with $a^\star \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_F b$;

(b) for every tuples $a, b, c \in M$ such that $a \equiv_F b$, that $b \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_F a$ and that $c \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_F ab$, there exists $c^\star \in M$ such that $ac^\star \equiv_F bc^\star \equiv_F ac$.

Remark 4.1.7. 1. “2-existence for $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}$” states an there does exists an $a^\star \equiv_E a$ with $a^\star \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{f}}_E b$. Condition (a) is a stronger statement which is equivalent to a form of “geometric elimination”: $E \subseteq \mathrm{acl}(F)$.

2. As we saw in proposition 4.1.3, condition (b) is equivalent to $\mathrm{acl}(F) \subseteq \mathrm{dcl}(F)$.

Beyond the stable realm, there is no reason for imaginaries to be the only obstructions to 3-existence — or for 3-existence to hold at all[1]. But the converse does hold in full generality and, as it turns out, it is entirely unrelated to forking.

Let M be a sufficiently saturated and homogeneous $\mathfrak{L}$-structure. By an (abstract) independence relation, we mean a relation on triples (A, B, C) of (small) subsets of M^{eq}, which we usually denote

$$A \downarrow_C B.$$

Proposition 4.1.8. *Let $F \subseteq E \subseteq M^{\mathrm{eq}}$. Assume the following:*

(a) for every tuples $a, b \in M$, there exists $a^\star \equiv_E a$ with $a^\star \downarrow_F b$;

(b) for every tuples $a, b, c \in M$ such that $a \equiv_F b$, that $b \downarrow_F a$ and that $c \downarrow_F ab$, there exists $c^\star \in M$ such that $ac^\star \equiv_F bc^\star \equiv_F ac$.

Then $E \subseteq \mathrm{dcl}(F)$.

Proof. Fix some $e \in E$. Let $a \in M$ be a tuple such that $e \in \mathrm{dcl}(a)$ and f an $\mathfrak{L}$-definable map such that $e = f(a)$. We have to show that e is the sole realization of $\mathrm{tp}(e/F)$. Let $e' \equiv_F e$ and $b \equiv_F a$ be such that $e' = f(b)$. By hypothesis (a), we find $b^\star \equiv_E b$ such that $b^\star \downarrow_F a$. Take any $c \equiv_E a$ — in particular, we have $f(c) = e = f(a)$. By hypothesis (a), we may assume that $c \downarrow_F ab^\star$. Then by hypothesis (iii), $c^\star$ such that $ac^\star \equiv_F b^\star c^\star \equiv_F ac$ — in particular, $f(b^\star) = f(c^\star) = f(a) = e$, and hence $e' = f(b) = f(b^\star) = e$. □

Remark 4.1.9. There is a trade-off between conditions (a) and (b). Weaker independence notions — that is notions that have more independent sets — will more easily satisfy hypothesis (a), whereas stronger independence notions will more easily satisfy hypothesis (b).

4.2 Invariant and definable independence

When studying imaginaries in concrete structures (say, in a valued field !), we often rely on independence relations that are stronger than forking and more obviously geometric in nature. One such natural choice is "invariant independence".

Again, let M be a sufficiently saturated and homogeneous $\mathfrak{L}$-structure.

Definition 4.2.1. Fix some $C \subseteq M^{\mathrm{eq}}$.

1. A type $p(x)$ over M is said to be C-invariant if for every formula $\varphi(x, y)$, the set $\{a \in M^y : \varphi(x, a) \in p\}$ is $\mathrm{aut}(M/C)$-invariant.
2. For every tuples $a, b \in M$, we write $a \downarrow_C^{\mathrm{inv}} b$ if $\mathrm{tp}(a/Cb)$ is contained in a C-invariant type.

Remark 4.2.2. 1. Note that $\downarrow^{\mathrm{inv}}$ is left transitive: for every tuples $a, b, d \in M$ and $C \subseteq M$, if $a \downarrow_C^{\mathrm{inv}} b$ and $d \downarrow_{Ca}^{\mathrm{inv}} b$ then $ad \downarrow_C^{\mathrm{inv}} b$.

2. We could have chosen to consider invariance under the group $\mathrm{aut}(M/C)$ which preserves all types over C. There are other natural choices of (smaller) automorphism

[1] Indeed, 3-existence is incompatible with an order and, even in simple theories, it is is a long-standing open question whether imaginaries (as opposed to hyperimaginaries) are the only obstructions to 3-existence.

groups that we could have considered : namely the group $\mathrm{aut}(M/\mathrm{acl}(C))$ of automorphisms preserving "Shelah strong types", the group $\mathrm{aut}(M/\mathrm{bdd}(C))$ of automorphisms preserving "Kim-Pillay strong types" (equivalently, every bounded C-type-definable equivalence relation) or the group of automorphisms preserving "Lascar strong types" (equivalently, every bounded C-invariant equivalence relation).
Although these notions do play a fundamental role in the model theory of potentially unstable structures, they are meant to account for the existence of "hyperimaginaries" which, in the present text, we would rather eliminate !

3. Invariance independence is a very strong notion of independence, so much so that hypothesis (b) of proposition 4.1.8 always holds: if p is an F-invariant type containing $\mathrm{tp}(c/Fab)$ and $a \equiv_F b$, then, for any $\sigma \in \mathrm{aut}(M/F)$ with $b = \sigma(a)$, we have $\sigma(p) = p$ and hence $ac \equiv_F bc$; so taking $c^\star = c$ works.

Since F-invariant types form a closed set of the space of types over M, hypothesis (a) of proposition 4.1.8 can be reformulated as the density of invariant types among E-definable sets:

Proposition 4.2.3. *Let $F \subseteq E \subseteq M^{\mathrm{eq}}$. The following are equivalent:*

(i) for every tuples $a, b \in M$, there exists $a^\star \equiv_E a$ with $a^\star \downarrow_F^{\mathrm{inv}} b$;

(ii) any E-definable set is contained an F-invariant type;

(iii) any type over E is is contained in an F-invariant type — that is, for any tuple $a \in M$, we have

$$a \downarrow_F^{\mathrm{inv}} E;$$

(iv) there exists $N \equiv_E M$ (in some common elementary extension) such that $\mathrm{tp}(N/M)$ is F-invariant.

Remark 4.2.4. If either of these conditions hold, then, by proposition 4.1.8, we have $E \subseteq \mathrm{dcl}(F)$ — in other words types over E are determined by their restriction to F. But we actually have much more: if N is as in condition (iv) and $a, b \in M$ are such that $a \equiv_F b$, then $a \equiv_N b$ — that is, the type over F implies the Lascar strong type over E.

The equivalent conditions of proposition 4.2.3 therefore not only provide a criterion for elimination of imaginaries, but also for the triviality of the Lascar group and hence of the Kim-Pillay and Shelah groups.

Proving the equivalent condition of proposition 4.2.3 involves two, a priori distinct problems:

- given an E-definable set X, to find an E-invariant type containing X;
- showing that a given E-invariant type is in fact F-invariant.

Since invariant types do not, in general, have a canonical basis in M^{eq} (or rather, their "canonical basis" consists of somewhat complex hyperimaginaries), the second item can be quite daunting unless we know how the type was constructed. This is one of the main reasons why we would rather work with definable types, since they do have a canonical basis in M^{eq}.

Given that definable types are more tractable than invariant types, we might want to consider the stronger "definable independence" instead:

Definition 4.2.5. For any tuples $a, b \in M$ and $C \subseteq M^{\mathrm{eq}}$, we write $a \downfree^{\mathrm{def}}_C b$ if $\mathrm{tp}(a/Cb)$ is contained in a C-definable type over M.

Remark 4.2.6. An important feature of $\downfree^{\mathrm{def}}$ is its good transitivity properties on the left. Let $a, b, d \in M$ be tuples and let $C \subseteq M^{\mathrm{eq}}$.

1. If $a \downfree^{\mathrm{def}}_C b$ and $d \downfree^{\mathrm{def}}_{Ca} b$, then $ad \downfree^{\mathrm{def}}_C b$.
2. If $C = \mathrm{acl}(C)$ and $a \downfree^{\mathrm{def}}_C b$, then $acl(Ca) \downfree^{\mathrm{def}}_C b$.

However existence is a serious issue outside of stability, limiting its usefulness :

1. The set of C-definable types is not, in general, closed and proposition 4.2.3 fails for definable types. In particular, it is possible for every E-definable set to be contained in an E-definable type, but for the only E-definable types to be realized — for example, in the infinite discrete 2-branching tree.
2. If, for every $E \subseteq M^{\mathrm{eq}}$ and every tuple $a \in M$, we have $a \downfree^{\mathrm{def}}_{\mathrm{acl}(E)} \mathrm{acl}(E)$, then M is stable. So, unless M is stable, condition (a) of proposition 4.1.8 will fail for some $E = \mathrm{acl}(E)$, making it impossible to prove elimination of imaginaries using proposition 4.1.8 with $\downfree^{\mathrm{def}}$.

In practice, we can use definable types (and their canonical bases) to prove the equivalent conditions of proposition 4.2.3. This leads to the following proof strategy for elimination of imaginaries first outlined by Hrushovski [Hru14]:

Proposition 4.2.7. *Let $F \subseteq E \subseteq M^{\mathrm{eq}}$. Assume the following:*

(a) any E-definable set is contained an E-definable type;
(b) any E-definable type is F-definable.

Then $E \subseteq \mathrm{dcl}(F)$.

Remark 4.2.8. If both hypotheses (a) and (b) hold, then any E-definable set is contained an F-definable type which is, in particular, F-invariant and hence, as seen in remark 4.2.4.1, we have $\mathrm{acl}(E) = \mathrm{dcl}(E) = \mathrm{acl}(F) = \mathrm{dcl}(F)$. So proposition 4.2.7 only ever applies if $E = \mathrm{acl}(E)$. This is why these methods are only relevant to proving weak elimination of imaginaries.

This criterion can be further improved by noting that left transitivity of $\downfree^{\mathrm{def}}$ — and fibrations — allows us to reduce hypothesis (a) to a question in arity one:

Proposition 4.2.9. *The following are equivalent:*

(i) for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable set is contained an E-definable type;
(ii) for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable set $X \subseteq M$ is contained an E-definable type.

Proof. Let $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ and X be an E-definable set. We proceed by induction on the arity of X. Let $\pi : X \to Y$ be any projection onto one coordinate. By hypothesis (i), there exists $a \in X$ with $a \downfree^{\mathrm{def}}_E E$. Let $X_a = \{x \in X : \pi(x) = a\}$. By induction, we have some $b \in X_a$ such that $b \downfree^{\mathrm{def}}_{acl(Ea)} \mathrm{acl}(Ea)$, in particular, $b \downfree^{\mathrm{def}}_{acl(Ea)} E$. It follows from remark 4.2.6.1 that $\mathrm{acl}(Ea) \downfree^{\mathrm{def}}_E E$ and hence, by remark 4.2.6.2, that $ab \downfree^{\mathrm{def}}_E E$. In particular, there exists an E-definable type containing $\mathrm{tp}(b/E)$, and hence C.

The converse is immediate. ☐

4.3 Independence in algebraically closed valued fields

When looking for a geometrically flavoured independence relation in valued fields, one natural place to start at is by looking at forking in the model completion : the theory ACVF of algebraically closed non trivially valued fields. As this theory is NIP (see [Sim15, definition 2.1 and theorem A.11]), forking is known to be strongly related to invariant independence:

Proposition 4.3.1 (*cf.* [Sim15, corollary 5.29]). *Let M be a sufficiently saturated and homogeneous* NIP *structure. Let $C \subseteq M^{\mathrm{eq}}$ and let p be a type over M. The following are equivalent:*

(i) the type p does not fork over C;

(ii) the type p is $\mathrm{bdd}(C)$-invariant.

Furthermore, work of Haskell, Hrushovski and Macpherson has led to a comprehensive understanding of imaginaries (and less visibly so, bounded hyperimaginaries) in algebraically closed non trivially valued fields. This work is the starting point of the general program of the "geometric model theory of valued fields" which encompasses much of the work presented here.

Recall that, for every positive integer n, we let $\mathrm{Gr}_n = \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O})$ denote the space on lattices in K^n — that is free rank n $\mathcal{O}$-submodules. We also consider the space $\mathrm{Lin}_n = \coprod_{s\in\mathrm{Gr}_n} s/\mathfrak{m}s$. Each of the fibres $s/\mathfrak{m}s$ is a k-vector space of dimension n. The geometric sorts $\mathcal{G}$ consists of K, Gr_n and Lin_n, for all $n \geqslant 1$.

Theorem 4.3.2 ([HHM06; Hru14]). *Let $M \models$ ACVF be sufficiently saturated and homogeneous and $E = \mathrm{dcl}(E) \subseteq M^{\mathrm{eq}}$.*

1. *Any E-definable set $X \subseteq \mathrm{K}$ is contained in an $\mathrm{acl}(E)$-definable type.*
2. *Any E-definable type is $\mathcal{G}(E)$-definable.*
3. *For any tuple $a \in M$, $a \downarrow^{\mathrm{inv}}_{\mathcal{G}(\mathrm{acl}(E))} E$.*

The third assertion follows from the first two and the discussion in section 4.2 (in particular, propositions 4.2.3 and 4.2.7).

The proof of the first assertion relies on two main ingredients.

- A canonical description of every definable $X \subseteq \mathrm{K}$, due to Holy [Hol95]: the set X is a finite union of balls in which finitely many balls have been removed. Moreover, if we take a minimal such decomposition, it is unique up to permutation of the "outer balls" on one hand and of the "holes" on the other.
- The existence of the definable generics η_b of any ball b.

Definition 4.3.3. Let B be a chain of balls in M. The generic η_B of B is the unique type such that for any ball b, we have $b \in \eta_B$ if and only if $\bigcap B \subseteq b$.

In general η_B is B-invariant. If B has a minimal element b, then it is $\ulcorner b \urcorner$-definable.

The original proof of assertion 2. is very involved and led to the definition of "stable domination" which turned out to be of independent interest (*eg.* [HHM08; HL16; HR19]). Johnson [Joh] later proposed a direct computation of the canonical basis of any definable type in ACVF, via definable valued vector spaces and the space of $\mathcal{O}$-modules of K^n, for all n. This alternative proof will play an important role in the work presented here (see proposition 4.4.5).

Corollary 4.3.4. *Let* $M \vDash \mathrm{ACVF}$ *and* $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. *Then*

1. *any* E*-definable set is contained in a* $\mathcal{G}(E)$*-definable type;*
2. *we have* $E \subseteq \mathrm{dcl}(\mathcal{G}(E))$.

In other words, ACVF has weak elimination of imaginaries down to $\mathcal{G}$. By [HHM06, proposition 3.4.1], finite sets in $\mathcal{G}$ are also coded in $\mathcal{G}$, yielding full elimination of imaginaries down to $\mathcal{G}$. The original arguments are of a very different nature than what we have considered so far, but, in [Joh, section 6.2], Johnson outlines a general proof strategy for coding finite sets that also relies on the density of (symmetric) definable types.

Another consequence of theorem 4.3.2 is a very natural description of forking in ACVF:

Corollary 4.3.5. *Let* $M \vDash \mathrm{ACVF}$ *be sufficiently saturated and homogeneous. Let* $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ *and let* p *be a type over* M. *The following are equivalent:*

(i) *the type* p *does not fork over* E*;*

(ii) *the type* p *is* $\mathcal{G}(\mathrm{acl}(E))$*-invariant.*

Proof. If p does not fork over E, it is $\mathrm{bdd}(E)$-invariant by proposition 4.3.1. But note that, by theorem 4.3.2, any E-definable set X is contained in a $\mathcal{G}(E)$-invariant type. So, by remark 4.2.4, the type p is in fact $\mathcal{G}(E)$-invariant. $\square$

Remark 4.3.6. This characterization is not entirely satisfactory as the existence of an invariant extension is not easily checked. If $E \leqslant B \leqslant M$ are subfields and $a \in M$ is a tuple such that the extension $E \leqslant E(a)$ is Abhyankar — that is, if $\mathrm{trdeg}(a/E) = \mathrm{trdeg}(\mathrm{k}(E(a))/\mathrm{k}(E)) + \dim_{\mathbb{Q}}(\Gamma(E(a))/\Gamma(E))$ — then,

$$a \mathop{\downarrow}^{\mathrm{f}}_{E} B \text{ if and only if } \begin{cases} \Gamma(E(a)) \mathop{\downarrow}^{\mathrm{f}}_{\Gamma(E)} \Gamma(B), \text{ and} \\ \mathrm{trdeg}(\mathrm{k}(B(a))/\mathrm{k}(B)) = \mathrm{trdeg}(\mathrm{k}(E(a))/\mathrm{k}(E)) \end{cases}$$

Note that, by work of Hossain [Hos21, theorem 1.14], forking in Γ is very easy to check: we have $\Gamma(E(a)) \mathop{\downarrow}^{\mathrm{f}}_{\Gamma(E)} \Gamma(B)$ if and only if, for every $c \in \mathrm{dcl}(\Gamma(E(a)))$ and any $b_1, b_2 \in \mathrm{dcl}(\Gamma(B))$, if $c \in [b_1, b_2]$, then $\Gamma(E) \cap [b_1, b_2] \neq \varnothing$.

However, if the extension is not Abhyankar and transcendental immediate extensions are involved, non forking seems hard to detect algebraically or geometrically.

4.4 Independence in valued fields

Given the characterization of forking in ACVF provided in corollary 4.3.5, it seems natural to consider the following independence relation in arbitrary valued fields. Let $\mathfrak{L}^{\mathrm{a}}$ denote the language of rings with an additional predicate $x|y$ interpreted in valued fields as $|x| \geqslant |y|$.

Definition 4.4.1. Let M be a sufficiently saturated and homogeneous valued field with potentially additional structure. let $E \subseteq B \subseteq M^{\mathrm{eq}}$ and $a \in \mathrm{K}(M)$ a tuple. Define $a \mathop{\downarrow}^{\mathrm{vf}}_{E} B$ to hold whenever $\mathrm{tp}(a/B)$ is consistent with an E-invariant quantifier free $\mathfrak{L}^{\mathrm{a}}$-type over M.

An important part of the work on imaginaries in (henselian) valued fields presented here has been to better understand this independence relation. Although it is far from obvious what are (if any) the good properties of this independence relation. For example, it is far from obvious if this relation enjoys left transitivity which is one of the most basic properties we could wish for and whose *a priori* absence complicates many proofs.

As outlined in proposition 4.1.8 and the discussion above, a question we want to settle is that of existence : if for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ and every tuple $a \in M$, do we have

$$a \downarrow^{\mathrm{vf}}_E E.$$

Remark 4.4.2. If we forget the valuation for now (equivalently, if we assume the valuation on M to be trivial), existence for $\downarrow^{\mathrm{vf}}$ is an immediate consequence of local stability: any type over E is consistent with the generic of its Zariski closure, which is $E \cap M$-definable in M^{a}.

Actually, in that case, $\downarrow^{\mathrm{vf}}_E$ is easily seen to coincide with forking in $M^{\mathrm{a}} \vDash \mathrm{ACF}$ over $E \cap M$.

If M is not trivially valued and the definable sets in M have a complicated geometry, existence for $\downarrow^{\mathrm{vf}}$ seems rather implausible (or, at least very hard to prove). We will therefore concentrate on valued fields with a "tame geometry". Recall the definition of a benign field (definition 1.1.1).

Conjecture 4.4.3. *Let M be a benign valued field.*

1. *Existence of $\downarrow^{\mathrm{vf}}$ holds over* acl*-closed sets: for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ and every tuple $a \in M$, we have*
$$a \downarrow^{\mathrm{vf}}_E E.$$
2. *Assuming that the residue field is infinite or that the value group is not discrete, quantifier free $\mathfrak{L}^{\mathrm{a}}$-types are dense among definable sets: for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable subset of some K^n is consistent with an E-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.*

One can easily formulate similar conjectures for finitely ramified mixed characteristic fields.

Remark 4.4.4.

1. It follows from the first statement that $\downarrow^{\mathrm{vf}}$ also enjoys the stronger property of 2-existence: if p is a quantifier free E-invariant $\mathfrak{L}^{\mathrm{a}}$-type consistent with $\mathrm{tp}(a/E)$, then for any $B \subseteq M^{\mathrm{eq}}$ and $a^\star \vDash p|_B$, we have $a^\star \equiv a$ and $a^\star \downarrow^{\mathrm{vf}}_E B$.
2. The second statement can only hold if the analog statement holds in the value group: any E-definable subset of some Γ^n should be consistent with an E-definable quantifier free $\mathfrak{L}_{\mathrm{oag}}$-type. I don't know of an ordered abelian group where this fails.
3. By compactness of the space of E-invariant quantifier free $\mathfrak{L}^{\mathrm{a}}$-types, the second statement implies the first (in the relevant case). But if there are many invariant types and few definable types, the second might fail while the first does hold.
4. This is the case if the residue field is finite and the value group discrete – for example in $\mathbb{Q}_p$. One can find a ball b whose only proper subballs definable over its code $\ulcorner b \urcorner$ are those such that b is covered by finitely many translates. Then, essentially since neither b not any of its $\ulcorner b \urcorner$-definable subballs have a generic type, b is not consistent with any $\ulcorner b \urcorner$-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type. However, b is consistent with the generic type of any maximal chain of $\ulcorner b \urcorner$-definable subballs.

We will later discuss a positive answer to both conjectures in residue characteristic zero, under some further assumption on the value group (theorem 4.7.2). Similar results should hold for sufficiently tame expansions of valued fields. We will also discuss a result in that direction (theorem 4.6.1).

In order to classify imaginaries using $\downfree^{\mathrm{vf}}$ (*cf.* proposition 4.1.8), we are actually interested in some further information: the "canonical basis" of a over E, that is the smallest $F \subseteq E$ such that $a \downfree^{\mathrm{vf}}_F E$, if it exists. Building on a result of Johnson [Joh, section 5.2], we can actually compute the canonical basis of any definable type.

Let M be an expansion of a valued field and $n > 0$ an integer. We write Mod_n for the (ind)-interpretable space of all definable $\mathcal{O}$-submodules of K^n. Let $\mathrm{Mod} = \coprod_n \mathrm{Mod}_n$.

The following result is closely related to [RV24, proposition 4.3], although it is not stated explicitly.

Proposition 4.4.5. *Assume that M is definably spherically complete. Let $E = \mathrm{dcl}(E) \subseteq M^{\mathrm{eq}}$. Any E-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type is $\mathrm{Mod}(E)$-definable.*

Given conjecture 4.4.3.2, it would follow that $\mathrm{Mod}(E)$ is also the canonical basis for $\downfree^{\mathrm{vf}}$. The following strengthening of conjecture 4.4.3.1 is therefore natural.

Conjecture 4.4.6. *Let M be a benign valued field, let $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ and let $a \in M$ be a tuple, we have*

$$a \downfree^{\mathrm{vf}}_{\mathrm{Mod}(E)} E.$$

Remark 4.4.7. Although density of definable sets is known to fail in finite extensions of $\mathbb{Q}_p$ (*cf.* remark 4.4.4.4), conjecture 4.4.6 does hold in finite extensions of $\mathbb{Q}_p$ [HMR18, remark 4.7].

4.5 Definable modules

As outlined by proposition 4.4.5, the space Mod of (codes for) definable $\mathcal{O}$-submodules of K^n will play an important role in this chapter. Let us describe some of its structure. First, in the valued fields we are interested in, definable modules have a very specific shape:

Proposition 4.5.1 ([RV24, proposition 3.3]). *Let M be a definably spherically complete valued field, potentially with additional structure. Any $R \in \mathrm{Mod}_n$ is of the form $\sum_{j<n} I_j \cdot a_j$, where $(a_j)_{j<n} \in \mathrm{B}_n(\mathrm{K})$ is a triangular basis of K^n and I_j is a $\ulcorner R \urcorner$-definable $\mathcal{O}$-submodule of K.*

Note that for any definable $\mathcal{O}$-submodule $I \subseteq \mathrm{K}$, $C = |I|$ is downward closed; we call such sets definable cuts of Γ. Moreover $I = I_C = \{x \in \mathrm{K} : |x| \in C\}$. So R is entirely determined by $a \in \mathrm{B}_n(\mathrm{K})$ and the tuple (C_j). Note that the choice of a is not canonical: it is only defined up to multiplication by $\mathrm{B}_n(\mathcal{O})$.

Let Cut denote the (ind)-interpretable space of all definable cuts in Γ. For every $c \in \mathrm{Cut}^n$, let Λ_c denote the $\mathcal{O}$-module $\sum_i I_{c_i} \cdot e_i$ where $e \in \mathrm{B}_n(\mathrm{K})$ is the identity. Let also

$$\mathrm{Mod}_c = \mathrm{B}_n(\mathrm{K})/\mathrm{Stab}(\Lambda_c).$$

Then $\mathrm{Mod}_n \simeq \bigcup_{c\in\mathrm{Cut}^n}\mathrm{Mod}_c$.

The existence of triangular bases is the cornerstone of our understanding of Mod. For example it implies that, for every $R \in \mathrm{Mod}_n$, the k vector space $V = R/\mathfrak{m}R$, of dimension at most n, has an $\ulcorner R \urcorner$-definable complete flag — that is, a sequence a subspaces $(V_i)_{i\leqslant \dim_{\mathrm{k}}(V)} \subseteq V$ with $\dim_{\mathrm{k}}(V_i) = i$. But the solvability of $\mathrm{B}_n(\mathrm{K})$ also gives rise to a unary analysis of Mod_c.

For every $c \in \mathrm{Cut}$, let $\Delta_c = \{\gamma \in \Gamma : \gamma + c = c\}$. It is a convex subgroup of Γ.

Proposition 4.5.2 ([RV24, proposition 3.9]). *Let $s \in \mathrm{Mod}_c$. There exists a tuple $b = (b_\ell)_\ell \in M^{\mathrm{eq}}$ and, for every ℓ, a $cb_{<\ell}$-interpretable sets X_ℓ containing b_ℓ such that:*

- $\mathrm{dcl}(s) = \mathrm{dcl}(cb)$*;*
- *if $\ell < |c|$, $X_\ell = \Gamma/\Delta_{c_\ell}$;*
- *if $\ell \geqslant |c|$, then X_l is a $cb_{<\ell}$-definable K/I_ℓ-torsor, where $I_\ell \subseteq \mathrm{K}$ is a cb_ℓ-definable $\mathcal{O}$-submodule.*

Note that the analysis can be made more precise but the statement gets quite technical.

A consequence of the unary analysis is that we can identify which subsets of Mod are k-internal, at least when the value group is dense. Recall that an interpretable set X is k-internal if there exists a definable surjection $f : \mathrm{k}^n \to X$ for some $n \geqslant 1$. Note that we only require f to be defined over a model and not over any parameter set over which X is defined.

Sets internal to the residue field play an important role in the the original proof for elimination of imaginaries in ACVF, as well as the subsequent developments of stable domination [HHM08; HR19] and of the stable completion [HL16]. They also play an important role in our understanding of leading term domination (section 4.8) and residue domination (section 4.11).

Definition 4.5.3. Let $E \subseteq M^{\mathrm{eq}}$. We define

$$\mathrm{Lin}_E = \coprod_{R\in\mathrm{Mod}(\mathrm{acl}(E))} R/\mathfrak{m}R.$$

This set (with its E-induced) structure is what Hrushovski calls a k-linear structure with flags in [Hru12, section 5].

Let M be a residue characteristic zero valued field, potentially with additional structure on RV.

Proposition 4.5.4. *Assume that $\Gamma(M)$ is dense and orthogonal to k. Let $X \subseteq \mathrm{Mod}^n$ be E-definable. The following are equivalent:*

1. *X is internal to k;*
2. *X is almost internal to k — that is, there is a definable map $f : X \to k^{\mathrm{eq}}$ with finite fibers;*
3. *X is orthogonal to Γ — that is, any definable function $f : X \to \Gamma^{\mathrm{eq}}$ has finite image;*
4. *X is in E-definable bijection with a subset of $\mathrm{acl}(E) \cup \mathrm{Lin}_E$.*

The only implication that requires actual work (and the density hypothesis) is that condition 3 implies condition 4. It is proved in [RV24, corollary 3.22]. Note that this implication fails if $\Gamma(M)$ is discrete with uniformizer π; indeed $\mathcal{O}/\pi^2\mathcal{O}$ verifies the first three condition but not the fourth.

4.6 Conjecture 4.4.6 and the elimination of $\exists^\infty$

Let us now focus on a first known case of conjecture 4.4.3.2 which (at least if he residue field is infinite or the value group non discrete) is the strongest of the three.

Theorem 4.6.1 ([HR21, Theorem 3.1.1]). *Let M be a expansion of a valued field such that:*

- *it is definably spherically complete;*
- *the value group is definably complete — any bounded definable subset of Γ has a supremum;*
- *the value group eliminates $\exists^\infty$ — in any definable family of subsets of Γ, there is a bound on the size of finite sets;*
- *the residue field eliminates $\exists^\infty$.*

Then, for any $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable subset of K^n is consistent with a $\mathcal{G}(E)$-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.

Remark 4.6.2.

1. If Γ is definably complete, then $\mathrm{Mod}(E) \subseteq \mathrm{dcl}(\mathcal{G}(E))$, explaining why statements like theorem 4.3.2 or theorem 4.6.1 can reduce to this apparently simpler set.
2. The assumption that Γ eliminates $\exists^\infty$ prevents discrete value groups.

Conjecture 4.4.6 follows in that case:

Corollary 4.6.3. *Let M and E be as above. Then*

$$a \downarrow^{\mathrm{vf}}_{\mathcal{G}(E)} E.$$

Given proposition 4.4.5, to prove theorem 4.6.1, it suffices to find an E-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type consistent with X. If $X \subseteq \mathrm{K}$ this can be done in the following way:

Lemma 4.6.4. *Let $X \subseteq \mathrm{K}$ be definable (and the hypotheses of theorem 4.6.1 hold). Then there exists an $\mathrm{acl}(\ulcorner X \urcorner)$-definable quantifier free type p — that is a type with a finite orbit under the action of $\{\sigma \in \mathrm{aut}(M) : \sigma(X) = X\}$ — which is consistent with X.*

Proof. For any two balls b_1, b_2, write $b_1 \trianglelefteq b_2$ if $b_1 \cap X \subseteq b_2 \cap X$. This is an $\ulcorner X \urcorner$-definable pre-order whose associated ($\ulcorner X \urcorner$-definable) equivalence relation we denote $\equiv$. For any ball b, $b/\equiv$, the $\ulcorner b/\equiv \urcorner$ definable type $\eta_{b/\equiv}$ is consistent with X if and only $b \cap X$ is not covered by finitely many proper subballs of b. Starting from the maximal element $\mathrm{K}/\equiv$, we see that either the lemma holds or, there is an infinite discrete finitely branching tree below the maximal element.

Looking at closed balls of radius γ in that tree, as γ varies, we get a family of finite sets of balls of unbounded cardinality. This contradicts eliminations of $\exists^\infty$ either in Γ or k. ☐

The proof of theorem 4.6.1 revolves around applying inductively the above construction to images and fibers of coordinate wise projections of X in K. Given enough left transitivity, this would be a rather straightforward proof along the lines of proposition 4.2.9. However, since we only construct quantifier free types, this proofs fails when we need to apply the analog of remark 4.2.6.2. Instead we run the unary construction above realizations of previously built quantifier free types. There are three main obstacles:

- We rely on a counting argument to avoid an infinite discrete finitely branching tree. Working over a type, we need to count germs of functions to the space of balls instead. This is achieved by showing that these counting questions can also be reduced to counting in k and Γ.
- We need to ensure a sufficiently finitary situation to have a hope of finding definable types and not invariant types — this is crucial to later compute canonical bases. In particular, we need to bound the degree of the polynomials involved.
  This is achieved by working with Δ-types for well chosen finite sets of quantifier free formulas $\Delta(x, y)$. These finite sets of formulas are strongly related to cell decomposition in $M^{\mathrm{a}} \vDash \mathrm{ACVF}$.
- We are looking for definable Δ-types but to run the inductive construction we need to know (definably) when certain $\mathfrak{L}(M)$-definable sets are consistent with (or implied by) a previously constructed type p. So we need the filter of $\mathfrak{L}(M)$-definable sets generated by p to be definable — such partial types are called quantifiable in [HR21]. We prove that, if k eliminates $\exists^\infty$, this is always the case.

Those complications in the proof do mean that we prove a strictly stronger result:

Corollary 4.6.5. *Let M as in theorem 4.6.1 and $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. Then, any partial E-quantifiable $\mathfrak{L}(M)$-type is consistent with a $\mathcal{G}(E)$-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.*

4.7 Conjecture 4.4.6 and benign fields

Theorem 4.6.1, via lemma 4.6.4 relies on strong combinatorial properties of the residue field and value group (namely elimination of $\exists^\infty$) but it does not leverage at all the geometry of X beyond what these combinatorial properties imply. However, by definition, in benign valued fields, definable subsets of K have a very constrained geometry.

Here we will only need the following variation of lemma 4.6.4:

Lemma 4.7.1. *Let $X \subseteq \mathrm{K}$ be a definable set which is prepared by some finite set. Then there exists an $\mathrm{acl}(\ulcorner X \urcorner)$-definable quantifier free type p which is consistent with X.*

Proof. With the notations introduced in the proof of lemma 4.6.4, if there is a discrete finitely branching tree below the maximal element $\mathrm{K}/{\equiv}$, all but finitely many branches of the tree are eventually disjoint from the finite set preparing C. Choose one such branch and let b be sufficiently far along that branch. Then, by preparation, we must have $b \subseteq X$, contradicting that b has only finitely many successors in the tree.

So the initial discrete finitely branching tree is finite and the lemma holds. □

In work with Vicaría, we leverage this version of the construction of a generic type in the unary case to prove conjecture 4.4.6 in a somewhat orthogonal context to the previously considered case.

Theorem 4.7.2 ([RV24, Theorem 4.1]). *Let M be a henselian valued field of residue characteristic zero, potentially with additional structure on* RV *and with a stably embedded value group such that:*

(D) *For any finite set $\Delta(x, y)$ of formulas (where x is a variable on Γ) containing $x < y_0$ and any $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable Δ-type $p(x)$ is contained in an E-definable complete type $q(x)$.*

Then, for any $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, any E-definable subset of K^n is consistent with a $\mathrm{Mod}(E)$-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.

Note that here we consider very mild expansions of a residue characteristic zero valued field, contrary to theorem 4.6.1 where the characteristic is arbitrary as is the expansion. But this allows us to relax the conditions of the residue field (there are none) and the value group.

Remark 4.7.3. This result essentially generalizes earlier work of Hils and myself [HR21, theorem 3.1.3]. However, note that this earlier result does not rely on property **(D)** as long as the value group is definably complete (and the residue field eliminates $\exists^\infty$); thus, it applies in expansions of valued fields where theorem 4.7.2 does not.

Note that [HR21, theorem 3.1.3] also applies in finitely ramified mixed characteristic. But it seems probable that the proof of theorem 4.7.2 can be adapted to finitely ramified mixed characteristic.

Again, the proof of theorem 4.7.2 relies on the fibration argument outlined in the proof of proposition 4.2.9. However, in that case the *a priori* failure of the analog of remark 4.2.6.2 is dealt with by actually proving it in that setting; to be precise, we prove it for "quantifier free definable types" in a definable expansion of the valued field structure.

An important obstacle to proving an analog to remark 4.2.6.2 for quantifier free definable types is that there might be definable functions to Γ (or even worse Γ^{eq}!) that don't make any sense in M^{a} and hence interact in a complicated way with quantifier free types. The solution to that particular issue is to work instead in the maximal unramified algebraic extension M^{ur} of M, which has the same value group. This field is known to eliminate quantifiers [Vic23, corollary 2.33] in the language $\mathfrak{L}^{\mathrm{ur}}$ obtained by adding all (pullbacks of) sets definable in Γ to $\mathfrak{L}^{\mathrm{a}}$.

We can then prove the following analog of remark 4.2.6.2. Let M be as in theorem 4.7.2 and let $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. Let $\mathcal{G}^{\mathrm{ur}} = \mathrm{Mod} \cup \Gamma^{\mathrm{eq}}$.

Proposition 4.7.4. *Let $X \subseteq \mathrm{K}^n$ be E-definable and let p be an E-definable quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type consistent with X. Then there exists $a \models p$ such that $a \in X$ and the quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type of $\mathcal{G}^{\mathrm{ur}}(\mathrm{acl}(Ea))$ over M is E-definable.*

The proof of this result relies heavily on the unary analysis of Mod (proposition 4.5.2).

Proposition 4.4.5 can also be extended to quantifier free $\mathfrak{L}^{\mathrm{ur}}$-types:

Proposition 4.7.5 ([RV24, proposition 4.3]). *The unique extension to M^{ur} of any E-definable quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type is $\mathfrak{L}^{\mathrm{ur}}(\mathcal{G}^{\mathrm{ur}}(E))$-definable.*

Note that in that case, we even find a definable type in M^{ur}. So we can rely on remark 4.2.6.1 in M^{ur} for any of our left transitivity needs.

The final ingredient of theorem 4.7.2 is that property **(D)** exactly fills the gap between finding a definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type and a definable quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type:

Lemma 4.7.6. *Assume property* **(D)**. *Then any E-definable quantifier $\mathfrak{L}^{\mathrm{a}}$-type containing* K *is contained in an E-definable quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type.*

Running the standard fibration-and-transitivity proof (see proposition 4.2.9), we actually get a strengthening of theorem 4.7.2:

Corollary 4.7.7. *Any E-definable subset of* K^n *is consistent with a $\mathcal{G}^{\mathrm{ur}}(E)$-definable quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type.*

In other words, writing $a \downarrow^{\mathrm{ur}}_{E} b$ if $\mathrm{tp}(a/b)$ is consistent with an E-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type, we have the following strengthening of conjecture 4.4.6. For any tuple $a \in M$, we have

$$a \downarrow^{\mathrm{ur}}_{\mathcal{G}^{\mathrm{ur}}(E)} E.$$

This strengthening might well hold in any benign valued field.

Let us conclude this section by showing that, although it is an artefact of the proof and might well not be necessary, property **(D)** is a non-trivial hypothesis:

Proposition 4.7.8. *Density of definable types (and hence* **(D)***) fails in $G = \mathbb{Z}^\omega$ with the lexicographic order.*

We refer the reader to [HLT23] for all necessary results on this structure. Fix some prime $p > 0$, then the valuation $v_n : G/nG \to \omega \cup \{\infty\}$ defined by $v_n(a) = \min\{i \in \omega : a_i \notin n\mathbb{Z}\}$ is definable. For every $\alpha \in \omega$, write $G_{n,\alpha} = \{x \in G : v_n(x) \geqslant \alpha\}$, then $G_{n,\alpha}/G_{n,\alpha+1} \simeq \mathbb{Z}/n\mathbb{Z}$ is finite.

So we can proceed as in $\mathbb{Q}_p$ (*cf.* remark 4.4.4.4). Let $b \subseteq G/nG$ (in some elementary extension) be a v_n ball which is covered by finitely many translates of any $\mathrm{acl}(\ulcorner b \urcorner)$-definable sub$v_n$-ball. Then no $\mathrm{acl}(\ulcorner b \urcorner)$-invariant type containing b is definable.

4.8 Leading term domination

In the previous sections, we have concentrated our efforts on proving condition (a) of proposition 4.1.8 for $\downarrow^{\mathrm{vf}}$, but, as we know from said proposition, if we want to use this relation to classify imaginaries we also need some form of 3-amalgamation. In this section, all the types we will consider will be concentrating on some cartesian power of K.

An important ingredient for that is to understand how close quantifier free $\mathfrak{L}^{\mathrm{a}}$-types are from being complete. Let M be a residue characteristic zero henselian valued field[2], with potentially additional structure on RV.

Theorem 4.8.1 ([Del82, théorème 6.1], [Bas91, theorem A]). *Let $A \subseteq \mathrm{K}(M)$. Then $\mathrm{tp}(A)$ is entirely determined by its restriction to quantifier free $\mathfrak{L}^{\mathrm{a}}$-formulas and by $\mathrm{tp}(\mathrm{rv}(A))$.*

[2] Similar results hold in the algebraically maximal roughly Kaplansky fields of remark 1.1.3 and finitely ramified mixed characteristic henselian fields.

So, if p is a quantifier free $\mathfrak{L}^{\mathrm{a}}$-type over M, $a \vDash p$,

$$p \cup \mathrm{tp}(\mathrm{rv}(M(a))/\mathrm{rv}(M)) \vdash \mathrm{tp}(a/M).$$

In that case, $\mathrm{rv}(M(a))$ is countably generated[3] over $\mathrm{rv}(M)$, we can find a countable tuple ρ of $\mathfrak{L}^{\mathrm{a}}(M)$-definable functions $p \to \mathrm{RV}$ such that

$$p \cup \mathrm{tp}(\rho(a)/\mathrm{rv}(M)) \vdash \mathrm{tp}(a/M),$$

Now, if p is E-invariant, for some $E \subseteq M^{\mathrm{eq}}$, we might expect (or wish that) ρ were $\mathfrak{L}(E)$-definable. This is indeed the case if E is a spherically complete submodel:

Proposition 4.8.2 ([HHM08, proposition 12.11]). *If $E \preccurlyeq \mathrm{K}(M)$ is spherically complete and $a \vDash p$, then $\mathrm{rv}(M(a))$ is generated by a countable tuple $\rho(a) \in \mathrm{rv}(E(a))$ over $\mathrm{rv}(M)$. Moreover,*

$$p|_A \cup \mathrm{tp}(\rho(a)/\mathrm{rv}(M)) \vdash \mathrm{tp}(a/M).$$

This is the prototypical domination result in valued fields and it is a cornerstone of most further developments in geometric model theory of valued fields. Note that the result of Haskell, Hrushovski and Macpherson is more precise as it isolates explicitly the residual and valuative conditions in p that are required for the result to hold.

However, this results fails over arbitrary bases. The main obstruction are the balls or chains of balls definable over E that do not have a point definable over E. This obstruction can take two distinct flavors.

Let B be a chain of E-definable balls and let $p = \eta_B$ be its generic over M (see definition 4.3.3).

Lemma 4.8.3. *For any $a \vDash p$ and any proper subball $b \subset \bigcap B$, $\mathrm{rv}(M(a))$ is generated by $\mathrm{rv}(a - b)$ over $\mathrm{rv}(M)$.*

If there exists such a $b \in \mathrm{dcl}(E)$, then $\mathrm{rv}(a - b) \in \mathrm{dcl}(Ea)$, as required. Otherwise, we might even have $\mathrm{RV}(\mathrm{dcl}(Ea)) = \mathrm{RV}(\mathrm{dcl}(E))$. There are two ways to go around this issue, depending on the nature of B.

Let us first recall the notion of germ of a definable function. If p is a type and f, g are definable functions defined at $p(x)$, we say that they have the same germ at p if $p(x) \vdash f(x) = g(x)$. We write $[f]_p$ for the class of definable functions with the same germ at p as f.

- If B is a proper chain or an open ball[4] then for any two $c_1, c_2 \in \bigcap B(M)$, we have $\mathrm{rv}(a - c_1) = \mathrm{rv}(a - c_2)$. So the function $\rho : a \vDash p \mapsto \mathrm{rv}(a - c_1)$ is not E-definable, but its germ $[\rho]_p$ is E-invariant.
- If B consists of a single closed ball b of radius γ, then we can consider $\mathrm{k}_b = \{$maximal open subballs of $b\} = b/\gamma\mathfrak{m}$. This is a torsor for $\gamma\mathcal{O}/\gamma\mathfrak{m} \simeq \mathrm{RV}_\gamma = \{\xi \in \mathrm{RV} : |xi| = \gamma\} \cup \{0\}$ which is naturally a dimension one k-vector space. In particular, if we name one point in k_b, it is naturally identified with RV_γ and if we name two points, it is identified with k. Let $\mathrm{res}_b : b \to \mathrm{k}_b$ denote the canonical projection.

[3] As a hyperfield, or equivalently as a multiplicative group with the addition on k.
[4] That is, it does not have a minimal element which is a closed ball.

For any $c \in b(M)$, we have $\mathrm{res}_b(a) = \mathrm{res}_b(c) + \mathrm{rv}(a - c)$ and hence $\mathrm{rv}(a - c)$ is interdefinable (in M^{a}) with $\mathrm{k}_b(a)$ over $E \cup \mathrm{k}_b(M)$. So $\mathrm{rv}(M(a))$ is generated[5] by $\mathrm{res}_b(a)$ over $\mathrm{rv}(M) \cup \mathrm{k}_b(M)$.

For types of higher arity, we need higher dimensional equivalents of k_b. The structure Lin_E (definition 4.5.3) will fill in that role.

Remark 4.8.4. Note that, by "projectivization", a translate $a+R$ of a definable $\mathcal{O}$-submodule $R \subseteq \mathrm{K}^n$ is coded by a definable module in K^{n+1}, namely the $\mathcal{O}$-module $S = \mathcal{O}\cdot(a,1) \oplus R \times \{0\}$ generated by $(a + R) \times \{1\}$. Indeed, we have $S \cap \mathrm{K}^n \times \{1\} = (a + R) \times \{1\}$.

In particular, any element of $a + \gamma\mathfrak{m} \in \mathrm{k}_b$ can be identified with the module generated by $\mathcal{O} \cdot (a, 1) + \gamma\mathfrak{m} \times \{0\}$ which is itself coded by a definable subspace of $R/\mathfrak{m}R$, where $R = \mathcal{O} \cdot (a, 1) + \gamma\mathcal{O} \times \{0\}$ is E-definable. Finally this subspace is coded in Lin_E.

So we can identify k_b with a subset of Lin_E.

Given the constraint outlined above, the optimal domination result that we can hope for is the following. Note that, the elements of $\mathrm{Lin}_E(M)$ cannot be naturally identified with imaginaries in M^{a} (unless Γ is definably complete). So we will need to work in M^{ur}.

Let M be a (sufficiently saturated and homogeneous) residue characteristic zero henselian field and let $E \subseteq M^{\mathrm{eq}}$. We define $\mathcal{R}_E = \mathrm{RV} \cup \mathrm{Lin}_E$.

Conjecture 4.8.5. *Let p be an E-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type over M. Then there exists a countable tuple ρ of $\mathfrak{L}^{\mathrm{ur}}(M)$-definable maps $p \to \mathcal{R}_E$ such that:*

- *the germ $[\rho]_p$ is $E \cup \mathcal{R}_E(M)$-invariant;*
- $\mathrm{rv}(M(a))$ *is generated[6] by $\rho(a)$ over $\mathcal{R}_E(M)$, for any $a \vDash p$.*

In work with Vicaria [RV24], building on work with Hils [HR21], we prove that the conjecture holds for "geometric" E, up to adding auxiliary variables and under some restrictions on the value group. Assume either:

- the value group is dense with property **(D)**;
- the value group is a discrete with no further structure and we add a constant for a uniformizer π — an element with minimal positive valuation.

Theorem 4.8.6. *Let $E \subseteq \mathrm{Mod}(M)$ and let p be an E-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type over M. There exists an E-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type $q(xy)$ over M (where y might be an infinite tuple) and an (infinite) tuple ρ of $\mathfrak{L}^{\mathrm{ur}}(M)$-definable maps $q \to \mathcal{R}_E$ such that:*

- *the germ $[\rho]_q$ is $E \cup \mathcal{R}_E(M)$-invariant;*
- $\mathrm{rv}(M(ae))$ *is generated by $\rho(ae)$ over $\mathcal{R}_E(M)$, for any $ae \vDash q$.*

The proof of that result is not explicit in [RV24, section 5] but it is a variation on the proof presented there. There are two main ingredients. First, a variant of proposition 4.8.2 over "bases with sufficiently many points". Let p be an E-invariant quantifier free $\mathfrak{L}^{\mathrm{a}}$-type over M.

[5] In the sense of definable closure in M^{a}.

[6] In the sense of definable closure in M^{ur}.

Proposition 4.8.7 ([HR21, corollary 4.3.16]). *Let $\widehat{E} \subseteq \mathrm{K}(M)$ contain a realization of every type over E and $a \vDash p$, then $\mathrm{rv}(M(a))$ is generated by a countable tuple $\rho(a) \in \mathrm{rv}(\widehat{E}(a))$ over $\mathrm{rv}(M)$.*

Remark 4.8.8. Although it is not stated there, the proof of proposition 4.8.7 yields the following stronger statement, which is better suited to transitivity arguments:

Let $q(y)$ be an (infinitary) $E \cup \mathcal{R}_E(M)$-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type over M such that some $c \vDash q$ contains a realization of every type over E. If $a \vDash p|_{Mc}$, then $\mathrm{rv}(M(ca))$ is generated by some countable tuple $\rho(ca) \in \mathrm{rv}(\mathbb{Q}(ca))$ over $\mathrm{rv}(M(c))$.

The second ingredient consists in finding such a q and a $c \vDash q$ that enumerates a model:

Proposition 4.8.9. *Assume $\Gamma(M)$ is as in theorem 4.8.6 and $E \subseteq \mathrm{Mod}(M)$. Then there exists $C \equiv_E M$ and an (infinite) tuple of $\mathfrak{L}^{\mathrm{ur}}(M)$-definable maps $q \to \mathcal{R}_E$ such that:*

- *C contains a realization of every type over E;*
- *the quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type of C over M is $E \cup \mathcal{R}_E(M)$-invariant;*
- *the germ $[\rho]_q$ is $E \cup \mathcal{R}_E(M)$-invariant;*
- *$\mathrm{rv}(MC)$ is generated by $\rho(c)$ over $\mathcal{R}_E(M)$.*

The hardest part of this result is finding a lift $C \subseteq \mathrm{K}(M)$ of E with the last three properties (see [RV24, proposition 5.17]). This relies on the unary analysis of Mod (proposition 4.5.2) and on showing that germs of some definable functions into RV lie in Lin_E. The classification of k-internal subsets of Mod in M^{ur} (proposition 4.5.4) plays a crucial role in this computation. But it is only known when the value group is dense. When the value group is discrete (with uniformizer π), we need to work in $M^{\mathrm{ur}}[\pi^{1/\infty}]$, requiring further "descent" results to get the various codes involved from $\mathrm{Mod}(M^{\mathrm{ur}}[\pi^{1/\infty}])$ down into $\mathrm{Mod}(M)$ (see [RV24, lemma 5.12]).

Once we have a lift $C \subseteq \mathrm{K}$ of E, the proof of proposition 4.8.9 is a matter of growing C sufficiently by lifting points of RV until it is a model.

The existence of a c as in remark 4.8.8 follows on abstract grounds, noting that the set of types such that RV is generated by the image of a function with invariant germ is closed and hence compact. Theorem 4.8.6 follows by transitivity.

4.9 Elimination of imaginaries

Let us now come back to our original question of classifying imaginaries in a characteristic zero henselian valued field M. Theorem 4.8.1 shows that there is no hope that $\downarrow^{\mathrm{vf}}$ has 3-amalgamation (or even the weaker version of proposition 4.1.8.(b)) unless it holds in RV; but if the residue field is ordered, for example, this cannot be the case.

The standard solution is to work over RV. In fact, conjecture 4.8.5 can be reinterpreted as a form of "relative 2-uniqueness", albeit for $\downarrow^{\mathrm{ur}}$. But it is somewhat cumbersome to use, due in part to the fact that we only know conjecture 4.8.5 up to adding variables.

In [HR21; RV24], we chose a different presentation which consists in coming back to $\downarrow^{\mathrm{inv}}$ but now over the whole of $\mathcal{R}_E(M)$. In that setting we can prove condition (a) of proposition 4.1.8:

Corollary 4.9.1. *Let M be as in theorem 4.8.6 and $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. For every tuple $a \in M$, there exists $a^\star \equiv_E a$ such that*

$$a^\star \downarrow^{\mathrm{inv}}_{\mathrm{Mod}(E)\cup\mathcal{R}_E(M)} M.$$

Proof. By corollary 4.7.7 we find an $\mathcal{G}^{\mathrm{ur}}(E)$-invariant quantifier free $\mathfrak{L}^{\mathrm{ur}}$-type $p(x)$ over M, which is consistent with $\mathrm{tp}(a/E)$. Let $q(xy)$ and ρ be as in theorem 4.8.6. Let $a^\star c^\star \vDash q$. By theorem 4.8.1,

$$q \cup \mathrm{tp}(\rho(a^\star c^\star)/\mathrm{rv}(M)) \vdash \mathrm{tp}(a^\star c^\star/M).$$

Since q and $[\rho]_q$ are $E \cup \mathcal{R}_E(M)$-invariant, it follows that $\mathrm{tp}(a^\star c^\star/M)$ also is; and hence so is $\mathrm{tp}(a^\star/M)$. □

Remark 4.9.2. As in remark 3.2.2, similar results hold in mixed characteristic provided we consider all $\mathrm{RV}_n = K/1 + n\mathfrak{m}$ and all $R/n\mathfrak{m}R$, for $n \geqslant 1$ and $R \in \mathrm{Mod}(\mathrm{acl}(E))$.

As noted in remark 4.2.2.2, condition (b) of proposition 4.1.8 always holds for $\downarrow^{\mathrm{inv}}$. We can thus deduce relative elimination of imaginaries:

Corollary 4.9.3. *Let M be as in theorem 4.8.6 and $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$. Then*

$$E \subseteq \mathrm{dcl}(\mathrm{Mod}(E), \mathcal{R}_E(M)).$$

One might complain that we are working over a large parameter set (namely $E \cup \mathcal{R}_E(M)$) as if it were small, but this is not an actual problem — at most a technical hurdle — since $\mathcal{R}_E$ is stably embedded.

Given that imaginaries in short exact sequences like RV are well-understood (see [RV24, proposition 6.2]), we can further refine this result:

Theorem 4.9.4. *Let M be as in theorem 3.2.1. For every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, we have*

$$E \subseteq \mathrm{dcl}(\mathrm{Mod}(E), \Gamma(M), \mathrm{Lin}_E(M)).$$

Theorem 3.2.1 is a reformulation of that result.

4.10 Valued fields with operators

The methods presented up to now extend naturally to valued fields with operators (be it derivations, automorphisms, λ-functions, ...) via the use of "prolongations".

Definition 4.10.1. Let M be a henselian valued field, potentially with additional structure, in some language $\mathfrak{L}$. Let $\mathfrak{L}_{\mathrm{h}} \subseteq \mathfrak{L}$ be an RV-expansion of $\mathfrak{L}^{\mathrm{a}}$ and let f be an (infinite) tuple of functions $\mathrm{K} \to \mathrm{K}$. We say that M is a valued fields with operators if:

- For every $A \subseteq \mathrm{K}(M)$, we have

$$\mathrm{tp}_{\mathrm{h}}(f(A)) \vdash \mathrm{tp}(A),$$

where tp_{h} is the (field) quantifier free $\mathfrak{L}_{\mathrm{h}}$-type.

The main examples are the following:

- σ-henselian valued fields with an automorphism σ [DO15] with
$$f(x) = (\sigma^n(x))_{n\geqslant 0};$$
- ∂-henselian valued fields with a monotone derivation ∂ and enough constants [Sca00] with
$$f(x) = (\partial^n(x))_{n\geqslant 0};$$
- algebraically maximal roughly Kaplansky fields with a generic derivation ∂ [FT23] with
$$f(x) = (\partial^n(x))_{n\geqslant 0};$$
- separably closed valued fields of finite imperfection degree [Hon16] with
$$f(x) = (\lambda^n_I(x))_{n,I},$$
or with iterated Hasse derivations.

We can then apply theorem 4.6.1 to get:

Theorem 4.10.2. *Further assume that M is as in theorem 4.6.1. If $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$, then any E-definable subset of K^n is consistent with a $\mathcal{G}(E)$-definable quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.*

In particular, for every and tuple $a \in \mathrm{K}(M)$, we have
$$f(a) \downarrow^{\mathrm{vf}}_{\mathcal{G}(E)} E.$$

Note that we must use the version of conjecture 4.4.3 that relies on stronger assumptions on the residue field and value group as we are dealing with expansions of M that are not benign.

If the $\mathfrak{L}^{\mathrm{a}}$-type of $f(a)$ generates a complete type, we immediately deduce weak elimination of imaginaries — and hence full elimination since finite sets are coded [HHM06, proposition 3.4.1].

Corollary 4.10.3. *Further assume that:*

- *for every tuple $a \in \mathrm{K}$, we have*
$$\mathrm{tp}_{\mathrm{a}}(f(a)/M) \vdash \mathrm{tp}(a/M),$$
where tp_{a} is the quantifier free $\mathfrak{L}^{\mathrm{a}}$-type.

Then M eliminates imaginaries down to $\mathcal{G}$.

In particular, this corollary applies in existentially closed valued fields with a monotone derivation [Rid19], separably closed fields of finite imperfection degree [HKR18] and algebraically closed fields with a generic derivation. But when the structure on RV is more complex, section 4.8 comes into play.

Theorem 4.10.4 ([HR21, Proposition 4.4.8]). *Let M be a (sufficiently saturated and homogeneous) residue characteristic zero henselian field with operators as in theorem 4.6.1, and with stably embedded* k *and* RV. *Then for every $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$ and for every tuple $a \in M$, there exists $a^\star \equiv_E a$ such that*

$$a^\star \downarrow^{\mathrm{inv}}_{\mathcal{G}(E)\cup\mathcal{R}_E(M)} M.$$

In particular,

$$E \subseteq \mathrm{dcl}(\mathcal{G}(E), \mathcal{R}_E(M)).$$

The proof follows that of corollary 4.9.1, replacing a by $f(a)$ and relying on theorem 4.10.2 to find a $\mathcal{G}(E)$-invariant quantifier $\mathfrak{L}^{\mathrm{a}}$-type consistent with $\mathrm{tp}(f(a)/E)$.

In some cases, the structure on $\mathcal{R}_E$ is sufficiently well understood and we can further eliminate imaginaries. For example in valued fields with a generic derivation, the derivation induces no new structure on RV — and hence on Lin_E.

Corollary 4.10.5. *Let M be a residue characteristic zero henselian field with a generic derivation as in theorem 4.6.1 (and potentially angular components). Then M eliminates imaginaries down to* $\mathrm{K} \cup \Gamma^{\mathrm{eq}} \cup \mathrm{k}^{\mathrm{leq}}$.

On the other hand, in existentially closed multiplicative difference valued fields, both RV and Lin_E have new structure induced by the automorphism, they eliminate imaginaries and we deduce theorem 3.3.1.

4.11 On residue domination

The domination results of section 4.8 can also help to understand another notion of domination in valued fields. Let M be a valued field and $E = \mathrm{acl}(E) \subseteq M^{\mathrm{eq}}$.

Definition 4.11.1.
1. Let $a \in \mathrm{K}$ be a tuple. The type $\mathrm{tp}(a/E)$ is said to be residually dominated if for every $B = \mathrm{acl}(EB) \subseteq M^{\mathrm{eq}}$, if $\mathrm{Lin}_E(\mathrm{dcl}(Ea)) \downarrow^{\mathrm{ur}}_{\mathrm{Mod}(E)} \mathrm{Lin}_E(B)$ then

$$\mathrm{tp}(B/E\mathrm{Lin}_E(\mathrm{dcl}(Ea))) \vdash \mathrm{tp}(B/Ea),$$

   equivalently $\mathrm{tp}(a/E) \cup \mathrm{tp}(\mathrm{Lin}_E(\mathrm{dcl}(Ea))/B) \vdash \mathrm{tp}(a/B)$.
2. Let p be an E-definable type. It is orthogonal to Γ if for every $a \vDash p$, $\Gamma(\mathrm{dcl}(Ma)) = \Gamma(M)$.

In ACVF, these two notions are known to coincide by work of Haskell, Hrushovski and Macpherson [HHM08] that stemmed from their work on imaginaries.

In [CRV24], we generalize this result to a large class of residue characteristic zero henselian valued fields.

First, we show that, if a type is orthogonal to Γ (which can be detected by the underlying $\mathfrak{L}^{\mathrm{a}}$-type), a stronger version of conjecture 4.8.5 holds. Let M be as in theorem 4.8.6.

Proposition 4.11.2 ([CRV24, proposition 3.19]). *Let p be an E-definable $\mathfrak{L}^{\mathrm{a}}$-type over M and $a \vDash p$. If $|M(a)| = |M|$ then $\mathrm{rv}(M(a))$ is generated by $\mathrm{Mod}(E)$ and $\mathrm{Lin}_E(\mathrm{dcl}(Ea))$ over $\mathcal{R}_E(M)$.*

From this we can deduce that for any tuple $a \in K(M)$, if the type of a is residually dominated in M^{ur} over $\mathrm{Mod}(E)$, then it is also residually dominated in M — as previously, in the discrete case, we have to consider $M^{\mathrm{ur}}[\pi^{1/\infty}]$ instead.

Using work of Simon and Vicaría [SV24], we can then deduce that both notions are equivalent and that they are equivalent to their counterpart in M^{a} (and hence in any valued field):

Theorem 4.11.3 ([CRV24, theorem 3.27]). *Let p be an E-definable type over M and $a \models p$. The following are equivalent:*

1. *$tp(a/E)$ is residually dominated;*
2. *p is orthogonal to Γ;*
3. *the $\mathfrak{L}^{\mathrm{a}}$-type of a over $\mathcal{G}(E)$ is residually dominated in M^{a}.*

4.12 Imaginaries in pseudo p-adically closed fields

To conclude this text, let us consider a result that relates the current discussion on imaginaries and independence to chapter 2. In [MR21], we classify imaginaries in bounded pseudo p-adically closed fields.

A field is pseudo p-adically closed if it is pseudo T-closed, where T is the theory of $\mathbb{Q}_p$ as a ring. A bounded pseudo p-adically closed field K admits only finitely many p-adic valuations — that is valuations with $|p|$ maximal below 1 and residue field $\mathbb{F}_p$. As in section 2.2, we now assume that we have added constants for a subfield of K with the same Galois group as K.

Let I be the set of p-adic valuations on K and, for every $|\cdot|_i \in I$, K has an algebraic extension $K_i \equiv \mathbb{Q}_p$ — the p-adic closure of $(K, |\cdot|_i)$. For every $i \in I$, K_i eliminates imaginaries down to K_i and $\mathrm{Gr}_{n,i} = \mathrm{GL}_n(\mathrm{K}_i)/\mathrm{GL}_n(\mathcal{O}_i)$ by [HMR18, theorem 2.6]. Let $\mathcal{G}_i$ denote the union of those sets.

We wish to classify imaginaries in K. To do so we wish to apply proposition 4.1.8 to a well chosen independence relation:

Definition 4.12.1. For every $E = \mathrm{acl}(E) \subseteq B = \mathrm{dcl}(B) \subseteq K^{\mathrm{eq}}$ and every tuple $a \in K$, write $a \downarrow^{\mathrm{qf}}_E B$ if for every $i \in I$, the type of a over $\mathcal{G}_i(B)$ in K_i is contained in a $\mathcal{G}_i(E)$-invariant type over K_i.

Remark 4.12.2. If $E \subseteq \mathrm{dcl}(\mathrm{K}(E))$, by theorem 2.2.9, this is exactly forking independence over E. This probably remains true when E is imaginary.

This independence relation satisfies existence:

Proposition 4.12.3 ([MR21, proposition 2.41]). *For every $E = \mathrm{acl}(E) \subseteq K^{\mathrm{eq}}$ and every tuple $a \in \mathrm{K}$, there exists $a^\star \equiv_E a$ such that*

$$a^\star \downarrow^{\mathrm{qf}}_E K.$$

Note that, by definition, $a^\star \downarrow^{\mathrm{qf}}_E K$ if and only if $a^\star \downarrow^{\mathrm{qf}}_{\bigcup_i \mathcal{G}_i(E)} K$.

The proof of this proposition follows closely that of theorem 4.7.2:

- we first prove the proposition in arity one;

- we then compute germs of functions over invariant quantifier free types in the K_i to deduce a version of proposition 4.7.4;
- we conclude by left transitivity.

The main difference is that there are very few definable types in a p-adically closed field and hence we have to work with invariant types instead, which does complicate the computation of germs of functions.

This independence relation also satisfies 3-amalgamation, provided that there are no obstructions in any of the K_i:

Proposition 4.12.4 ([MR21, theorem 2.54]). *Let* $E = \mathrm{acl}(E) \subseteq K^{\mathrm{eq}}$. *Let* $a_1, a_2, c_1, c_2, c \in \mathrm{K}$ *be tuples such that* $a_1 \downarrow_E^{\mathrm{qf}} a_2$, $c \downarrow_E^{\mathrm{qf}} a_1 a_2$, $c_1 \equiv_E c_2$, ca_1 *and* $c_1 a_1$ *(resp.* ca_2 *and* $c_2 a_2$*) have the same type over* $\mathcal{G}_i(E)$ *in* K_i*, for all* i*. Then there exists* $c^\star$ *such that* $c^\star a_1 \equiv_E c_1 a_1$, $c^\star a_2 \equiv_E c_2 a_2$, *and* $c^\star a_1 a_2$ *has the same type as* $ca_1 a_2$ *in every* K_i.

This implies condition b of proposition 4.1.8 so we deduce weak elimination. As finite sets are coded [MR21, corollary 2.57], we get that:

Theorem 4.12.5 ([MR21, theorem 2.58]). *The bounded pseudo p-adically closed field K eliminates imaginaries down to* K *and* $\mathrm{Gr}_{n,i}$ *for all* $n \geqslant 1$ *and* $i \in I$.

Note that if $\mathcal{O} = \bigcap_i \mathcal{O}_i$, we have $\mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O}) \simeq \prod_i \mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O}_i)$. So K also eliminates imaginaries down to K and $\mathrm{GL}_n(\mathrm{K})/\mathrm{GL}_n(\mathcal{O})$, for all $n \geqslant 1$.

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RÉSUMÉ

Ce mémoire traite de la géométrie des ensembles définissables dans les corps valués henséliens ainsi que dans les structures apparentées. Dans un premier temps, nous proposons un cadre axiomatique pour une géométrie modérée non-archimédienne, la h-minimalité, inspiré de l'o-minimalité dans le cas réel. Dans un second temps, nous étudions l'interaction, arithmétique et modèle théorique, entre les topologies dans des corps qui vérifient un principe local-global pour l'existence de points : les corps pseudo T-clos. Enfin, dans un troisième temps, nous nous intéressons à la description des espaces de modules pour les familles d'ensembles définissables (de manière équivalente, les quotients d'ensembles définissables par des relations d'équivalence définissable) dans les corps henséliens, potentiellement munis d'opérateurs.

ABSTRACT

This document explores the geometry of definable sets in henselian valued fields and related structures. First, we propose an axiomatic framework for a moderate non-archimedean geometry, h-minimality, inspired by o-minimality in the real case. Secondly, we study the interaction, arithmetic and model theoretic, between topologies in fields that verify a local-global principle for the existence of points: pseudo T-closed fields. Finally, we are interested in the description of moduli spaces for families of definable sets (equivalently, quotients of sets definable by definable equivalence relations) in henselian valued fields, potentially with operators.