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SOME PROPERTIES OF ANALYTIC DIFFERENCE VALUED FIELDS

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Abstract We prove field quantifier elimination for valued fields endowed with both an analytic structure that is σ-Henselian and an automorphism that is σ-Henselian. From this result we can deduce various Ax–Kochen–Eršov type results with respect to completeness and the independence property. The main example we are interested in is the field of Witt vectors on the algebraic closure of $\mathbb{F}_p$ endowed with its natural analytic structure and the lifting of the Frobenius. It turns out we can give a (reasonable) axiomatization of its first-order theory and that this theory does not have the independence property.

Keywords: valued fields with an automorphism; analytic structure; separated power series; σ-Henselianity; Witt vectors; NIP

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Introduction

Since the work of Ax, Kochen and Eršov on valued fields (see, for example, [2]) and their proof that the theory of a Henselian valued field is essentially controlled (in equicharacteristic zero) by the theory of the residue field and the value group, model theory of Henselian valued fields has been very active and productive. Among later developments one may note Macintyre's result in [26] of elimination of quantifiers for p-adic fields and the proof by Pas of valued fields quantifier elimination for equicharacteristic zero Henselian fields with angular components in [28], which implies the Ax–Kochen–Eršov principle. Another notable result is the one by Basarab and Kuhlmann (see [5, 6, 22]) of valued field quantifier elimination for Henselian valued fields with amc-congruences (additive multiplicative congruences), a language that does not make the class of definable sets grow (whereas angular components might). Another result in the Ax–Kochen–Eršov spirit is the proof by Delon in [13] (extended by Bélair in [7]) that Henselian valued fields do not have the independence property if and only if their residue field does not have it (their value group never has the independence property by [18]).

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But model theorists have not limited themselves to giving an increasingly refined description of the model theory of Henselian valued fields. There have also been attempts at extending those results to valued fields with more structure. The two most notable enrichments that have been studied are, on the one hand, analytic structures as initiated by [14] and studied thereafter by a great number of people (among many others [9, 11, 15, 16, 24, 25]) and, on the other hand, $\mathcal{D}$-structures (a generalization of both difference and differential structures), first for derivations and certain isometries in [31] but also for greater classes of isometries in [4, 8, 32], and then for automorphisms that might not be isometries [3, 17, 19, 20, 27]. The model theory of valued differential fields is also quite central to the model-theoretic study of transseries (see, for example, [1]), but the techniques and results in this last field seem quite orthogonal to those in other references given above and to our work here.

The goal of the present paper is to study valued fields with *both* an analytic structure and an automorphism. The main result of this paper is Theorem A, which states that σ-Henselian (cf. Definition 4.10) valued fields with analytic structure and any automorphism σ eliminate field quantifiers resplendently in the leading term language (cf. Definition 1.1). We then deduce various Ax–Kochen–Eršov type results for analytic difference valued fields (with respect to both the theory and the independence property). We also try to give a systematic and comprehensive approach to quantifier elimination in (enriched) valued fields through some more abstract considerations (mainly found in the Appendix).

In [33], Scanlon already attempted to study analytic difference valued fields in the case of an isometry, but the definition of σ-Henselianity given there is too weak to actually work, although some incorrect computations hide this fact. The axiomatization and all the proofs had to be redone entirely but, as stated earlier, this paper does not only contain a corrected version of the results in [33], it also generalizes these results from the isometric case to the case of any valued field automorphism.

Some ideas from [33] could be salvaged though, among them the fact that Weierstrass preparation (see Definition 3.22) allows us to be close enough to the polynomial case to adapt the proofs from the purely valued difference setting. Nevertheless this adaptation is not as straightforward as one would hope, essentially because Weierstrass preparation only holds in one variable, but one variable in the difference world actually gives rise to many variables in the non-difference world. The main ingredient to overcome this obstacle is a careful study of differentiability of terms in many variables (see Definition 4.4) that allows us to give a new definition of σ-Henselianity in 4.10. These techniques can probably be used to prove results in greater generality: for example, valued fields with both analytic structure and $\mathcal{D}$-structure.

Our interest in the model theory of valued fields with both analytic structure and difference structure is not simply a wish to see Ax–Kochen–Eršov type results extended to more and more complicated structures and in particular to the combination of two structures where things are known to work well. It is also motivated by possible applications to diophantine and number-theoretic problems. In particular, it is the right model-theoretic setting in which to understand Buium's p-differential geometry. More precisely, any p-differential function over $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ is definable in $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ equipped

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with the lifting of the Frobenius and symbols for all p-adic analytic functions $\sum a_I x^I$, where $\mathrm{val}(a_I) \to \infty$ as $|I| \to \infty$. See [33, § 4] for an example of how a good model-theoretic understanding of this structure can help to show uniformity of certain diophantine results.

The organization of this text is as follows. Section 1 is a description of valued field languages, with either angular components or **RV**-structure. In § 2, we show that it is possible to transfer elimination of quantifier results from equicharacteristic zero to mixed characteristic (using the theoretical framework of Appendix B). Sections 3 and 4 describe the class of analytic difference valued fields we will be studying. Section 5 is concerned with purely analytical matters: it describes the link between analytic 1-types and the underlying algebraic 1-type. In § 6 we prove the main result of this paper, Theorem A, a field quantifier elimination result for σ-Henselian analytic difference valued fields. We also prove an Ax–Kochen–Eršov principle for these fields. Finally, § 7 shows how this quantifier elimination result also allows us to give conditions on the residue field and the value group for such fields to have (or not have) the independence property. The appendix contains an account of the more abstract model theory at work in the rest of the paper to help smooth out the arguments. Appendix B, in particular, sets up a general setting for transfer of elimination of quantifier results.

1. Languages of valued fields

We will be considering valued fields of *characteristic zero*. They will mainly be considered in two kinds of language: on the one hand, the language with leading terms, also known in the work of Basarab and Kuhlmann (cf. [5, 6, 22]) as amc-congruences and in later work as **RV**-sorts (see, for example, [21]), and on the other hand the language with angular components, also known as the Denef–Pas language.

Definition 1.1 ($\mathcal{L}^{\mathbf{RV}}$, the leading term language)**.** The language $\mathcal{L}^{\mathbf{RV}}$ has the following sorts: a sort $\mathbf{K}$ and a family of sorts $(\mathbf{RV}_n)_{n\in\mathbb{N}_{>0}}$. On the sort $\mathbf{K}$, the language consists of the ring language. The language also contains functions $\mathrm{rv}_n : \mathbf{K} \to \mathbf{RV}_n$ for all $n \in \mathbb{N}_{>0}$ and $\mathrm{rv}_{m,n} : \mathbf{RV}_n \to \mathbf{RV}_m$ for all $m|n$.

Any valued field can be considered as an $\mathcal{L}^{\mathbf{RV}}$-structure by interpreting $\mathbf{K}$ as the field and $\mathbf{RV}_n$ as $(\mathbf{K}^\star/1+n\mathfrak{M})\cup\{0\}$, where $\mathfrak{M}$ is the maximal ideal of the valuation ring $\mathcal{O}$. We will write $\mathbf{RV}_n^\star$ for $(\mathbf{K}^\star/1+n\mathfrak{M}) = \mathbf{RV}_n \smallsetminus \{0\}$. Then rv_n is interpreted as the canonical surjection $\mathbf{K}^\star \to \mathbf{RV}_n^\star$, and it sends 0 to 0; $\mathrm{rv}_{n,m}$ is interpreted likewise. Note that the sorts $\mathbf{RV}_n$ have a rich structure given by the following commutative diagram (where $\mathbf{R}_n := \mathcal{O}/n\mathfrak{M}$, $\mathbf{\Gamma}$ denotes the value group and all the lines are exact):

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{O}^\star \arrow[r] \arrow[dd, "\mathrm{res}_n"] \arrow[ddd, bend right, "\mathrm{res}_m"'] & \mathbf{K}^\star \arrow[dr, "\mathrm{val}"] \arrow[dd, "\mathrm{rv}_n"] \arrow[ddd, bend right, "\mathrm{rv}_m"'] & & \\
& & & \mathbf{\Gamma} \arrow[r] & 0 \\
1 \arrow[r] & \mathbf{R}_n^\star \arrow[r] \arrow[d, "\mathrm{res}_{m,n}"] & \mathbf{RV}_n^\star \arrow[ur, "\mathrm{val}_n"'] \arrow[d, "\mathrm{rv}_{m,n}"] & & \\
1 \arrow[r] & \mathbf{R}_m^\star \arrow[r] & \mathbf{RV}_m^\star \arrow[uur, bend right, "\mathrm{val}_m"'] & &
\end{tikzcd}$$

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We will denote the $\mathcal{L}^{\mathbf{RV}}$-theory of characteristic zero valued fields by T_{vf}. If we need to specify the residual characteristic, we will write $\mathrm{T}_{\mathrm{vf},0,0}$ or $\mathrm{T}_{\mathrm{vf},0,p}$. We will be denoting $\bigcup_n \mathbf{RV}_n$ by $\mathbf{RV}$. These sorts are closed in $\mathcal{L}^{\mathbf{RV}}$ (see Definition A.7). In order to eliminate $\mathbf{K}$-quantifiers, we will have to add some structure on the $\mathbf{RV}$ sorts.

Definition 1.2. The language $\mathcal{L}^{\mathbf{RV}^+}$ is the enrichment of $\mathcal{L}^{\mathbf{RV}}$ with, on each $\mathbf{RV}_n$, the language of (multiplicative) groups $\{1_n, \cdot_n\}$, a symbol 0_n and a binary predicate $|_n$, and functions $+_{m,n} : \mathbf{RV}_n^2 \to \mathbf{RV}_m$ for all $m|n$.

The multiplicative structure on $\mathbf{RV}_n$ is interpreted as its multiplicative (semi-)group structure, i.e., the group structure of $\mathbf{RV}_n^\star$, and $0_n \cdot_n x = x \cdot_n 0_n = 0_n$. The relation $x|_n y$ is interpreted as $\mathrm{val}_n(x) \leqslant \mathrm{val}_n(y)$. For all $x, y \in \mathbf{K}$ such that $\mathrm{val}(x+y) \leqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\} + \mathrm{val}(n) - \mathrm{val}(m)$, $\mathrm{rv}_n(x) +_{m,n} \mathrm{rv}_n(y)$ is interpreted as $\mathrm{rv}_m(x+y)$ and 0_n otherwise. This is well defined.

We will denote by $\mathrm{T}_{\mathrm{Hen}}$ the theory of characteristic zero Henselian valued fields in $\mathcal{L}^{\mathbf{RV}^+}$.

Remark 1.3.

1. If K has equicharacteristic zero, then, for all $m|n$, $\mathrm{rv}_{m,n}$ is an isomorphism. Hence, if we are working in equicharacteristic zero, we will only need to consider $\mathbf{RV}_1$. In that case we also have that $\mathbf{R}_1 = \mathbf{R}_1^\star \cup \{0\} \subseteq \mathbf{RV}_1^\star \cup \{0\} = \mathbf{RV}_1$. The additive structure is also simpler: we only need to consider the $+_{1,1}$ function on $\mathbf{RV}_1$. It extends the additive structure of $\mathbf{R}_1$ and makes every fiber of val_1 into an $\mathbf{R}_1$-vector space of dimension 1 (if we consider 0_1 to be the zero of every fiber).

2. If K has mixed characteristic p, then, whenever $m|n$ and $\mathrm{val}(n) = \mathrm{val}(m)$ (i.e., when p does not divide n/m), $\mathrm{rv}_{m,n}$ is an isomorphism. In particular, for all $n \in \mathbb{N}_{>0}$, $\mathrm{rv}_{n,p^{\mathrm{val}(n)}}$ is an isomorphism (where we identify $\mathrm{val}(p)$ and 1).

3. One could wonder then why consider all the $\mathbf{RV}_n$ when the only relevant ones are the $\mathbf{RV}_{p^n}$ in mixed characteristic p and $\mathbf{RV}_1$ in equicharacteristic zero. The main reason is that we want enough uniformity to be able to talk of T_{vf} without specifying the residual characteristic or adding a constant for the characteristic exponent (in particular if one wishes to consider ultraproducts of valued fields with growing residual characteristic, although we will not do so here).

The use of this language is mainly motivated by the following result that originates in [5, 6], although the phrasing in terms of resplendence first appears in [30]. By resplendent quantifier elimination relative to $\mathbf{RV}$, we mean that quantifiers on the sorts other than those in $\mathbf{RV}$ can be eliminated (namely the field quantifiers here) and that this result is true for any enrichment on the $\mathbf{RV}$-sorts (see Appendix A for precise definitions).

Theorem 1.4. *The theory* $\mathrm{T}_{\mathrm{Hen}}$ *eliminates* $\mathbf{K}$*-quantifiers resplendently relatively to* $\mathbf{RV}$.

Later, we will add analytic and difference structures; hence we will consider an enrichment of $\mathcal{L}^{\mathbf{RV}}$ by new terms on $\mathbf{K}$ and predicates and terms on $\mathbf{RV}$ (although none on both $\mathbf{K}$ and $\mathbf{RV}$; this is what we call in Appendix A an $\mathbf{RV}$-enrichment of a $\mathbf{K}$-term enrichment of $\mathcal{L}^{\mathbf{RV}}$). Let $\mathcal{L}$ be such a language, and let $\Sigma_{\mathbf{RV}}$ denote the new sorts coming from the **RV**-enrichment.

Remark 1.5. Any quantifier free $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$, where $\overline{x}$ are **K**-variables and $\overline{y}$ are **RV**-variables, is equivalent modulo T_{vf} to a formula of the form $\psi(\mathrm{rv}_{\overline{n}}(\overline{u}(\overline{x})), \overline{y})$, where ψ is a quantifier free $\mathcal{L}|_{\mathbf{RV}\cup\Sigma_{\mathbf{RV}}}$-formula and $\overline{u}$ are $\mathcal{L}|_{\mathbf{K}}$-terms. Indeed the only predicate involving **K** is the equality, and $t(\overline{x}) = s(\overline{x})$ is equivalent to $\mathrm{rv}_1(t(\overline{x}) - s(\overline{x})) = 0$. The statement follows immediately.

Here is an easy corollary that will be very helpful later on to uniformize certain results.

Corollary 1.6. *Let T be an $\mathcal{L}$-theory that eliminates* **K***-quantifiers, $M \models T$, $C \leqslant M$ (i.e., C is a substructure of M), and $\overline{x}$, $\overline{y} \in \mathbf{K}(M)$ be such that, for all $\mathcal{L}|_{\mathbf{K}}(C)$-terms $\overline{u}$, and all $n \in \mathbb{N}_{>0}$, $\mathrm{rv}_n(\overline{u}(\overline{x})) = \mathrm{rv}_n(\overline{u}(\overline{y}))$. Then $\overline{x}$ and $\overline{y}$ have the same $\mathcal{L}(C)$-type.*

Proof. Let $f : M \to M$ be the identity on $\mathbf{RV} \cup \Sigma_{\mathbf{RV}}(M)$ and send $u(\overline{x})$ to $u(\overline{y})$ for all $\mathcal{L}|_{\mathbf{K}}(C)$-terms u. By Remark 1.5, f is a partial $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism. But **K**-quantifiers elimination implies that f is in fact elementary. ■

The other kind of valued field language, the one with angular components, essentially boils down to giving oneself a section of the short sequences defining the $\mathbf{RV}_n$. That statement is made explicit in 1.8.

Definition 1.7 ($\mathcal{L}^{\mathrm{ac}}$, the angular component language)**.** The language $\mathcal{L}^{\mathrm{ac}}$ has the following sorts: **K**, $\mathbf{\Gamma}^{\infty}$, and $(\mathbf{R}_n)_{n\in\mathbb{N}_{>0}}$. The sorts **K** and $\mathbf{R}_n$ come with the ring language, and the sort $\mathbf{\Gamma}^{\infty}$ comes with the language of ordered (additive) groups and a constant ∞. The language also contains a function $\mathrm{val} : \mathbf{K} \to \mathbf{\Gamma}^{\infty}$, for all n, functions $\mathrm{ac}_n : \mathbf{K} \to \mathbf{R}_n$, $\mathrm{res}_n : \mathbf{K} \to \mathbf{R}_n$, $\mathrm{val}_{\mathbf{R},n} : \mathbf{R}_n \to \mathbf{\Gamma}^{\infty}$, $\mathrm{s}_{\mathbf{R},n} : \mathbf{\Gamma}^{\infty} \to \mathbf{R}_n$, and, for all $m|n$, functions $\mathrm{res}_{m,n} : \mathbf{R}_n \to \mathbf{R}_m$ and $\mathrm{t}_{\mathbf{R},m,n} : \mathbf{R}_n \to \mathbf{R}_m$.

As one might guess, the $\mathbf{R}_n$ are interpreted as the residue rings $\mathcal{O}/n\mathfrak{M}$. As with **RV**, we will write $\mathbf{R} := \bigcup_n \mathbf{R}_n$. The res_n and $\mathrm{res}_{m,n}$ denote the canonical surjections $\mathcal{O} \to \mathbf{R}_n$ and $\mathbf{R}_n \to \mathbf{R}_m$, extended by zero outside their domains. The function ac_n denotes an angular component, i.e., a multiplicative homomorphism $\mathbf{K}^{\star} \to \mathbf{R}_n^{\star}$ which extends the canonical surjection on $\mathcal{O}^{\star}$ and sends 0 to 0_n. Moreover, the system of the ac_n should be consistent; i.e., $\mathrm{res}_{m,n} \circ \mathrm{ac}_n = \mathrm{ac}_m$. The function $\mathrm{val}_{\mathbf{R},n}$ is interpreted as the function induced by val on $\mathbf{R}_n \smallsetminus \{0\}$ and sending 0_n to ∞. The function $\mathrm{s}_{\mathbf{R},n}$ is defined by $\mathrm{s}_{\mathbf{R},n}(\mathrm{val}(x)) = \mathrm{res}_n(x)\mathrm{ac}_n(x)^{-1}$ and $\mathrm{s}_{\mathbf{R},n}(\infty) = 0_n$. Finally, the function $\mathrm{t}_{\mathbf{R},m,n}$ is defined by $\mathrm{t}_{\mathbf{R},m,n}(\mathrm{res}_n(x)) = \mathrm{ac}_m(x)$ when $\mathrm{val}(x) \leqslant \mathrm{val}(n) - \mathrm{val}(m)$ and 0_m otherwise (this is well defined).

It should be noted that any valued field that is sufficiently saturated can be endowed with angular components (cf. [29, Corollary 1.6]).

The following pages, up to Definition 1.10, contain a (very technical) account of how to deduce quantifier elimination results in the angular component language from results in the leading term language. Although they are essential to prove Ax–Kochen–Eršov type results, angular components only appear again in §§ 6.3 and 7, and a reader mostly interested in the broader picture can safely skip those pages.

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Let $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$ be the enrichment of $\mathcal{L}^{\mathbf{RV}^{+}}$ obtained by adding a sort $\mathbf{\Gamma}^{\infty}$, equipped with the language of ordered (additive) groups, a family of sorts $\mathbf{R}_n$, equipped with the ring language, symbols $\mathrm{val}_n : \mathbf{RV}_n \to \mathbf{\Gamma}^{\infty}$ for the functions induced by the valuation, symbols $\mathrm{i}_n : \mathbf{R}_n \to \mathbf{RV}_n$ for the injection of $\mathbf{R}_n^{\star} \to \mathbf{RV}_n$ extended by 0 outside $\mathbf{R}_n^{\star}$, symbols $\mathrm{res}_{\mathbf{RV},n} : \mathbf{RV}_n \to \mathbf{R}_n$ for the map sending $a(1+n\mathfrak{M})$ to $a+n\mathfrak{M}$, $\mathrm{s}_n : \mathbf{\Gamma}^{\infty} \to \mathbf{RV}_n$ for a coherent system of sections of val_n compatible with the $\mathrm{rv}_{m,n}$, and symbols $\mathrm{t}_n : \mathbf{RV}_n \to \mathbf{R}_n$ interpreted as $\mathrm{t}_n(x) = \mathrm{i}_n^{-1}(x\,\mathrm{s}_n(\mathrm{val}_n(x))^{-1})$. Let $\mathrm{T}_{\mathrm{vf}}^{\mathrm{s}}$ be the $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$-theory of characteristic zero valued fields and $\mathrm{T}_{\mathrm{vf}}^{\mathrm{ac}}$ the $\mathcal{L}^{\mathrm{ac}}$-theory of characteristic zero valued fields.

Let $\mathcal{L}^{\mathrm{s},e}$ be an $\mathbf{RV}$-enrichment (with potentially new sorts $\Sigma_{\mathbf{RV}}$) of a $\mathbf{K}$-enrichment (with potentially new sorts $\Sigma_{\mathbf{K}}$) of $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$, and let T^e be an $\mathcal{L}^{\mathrm{s},e}$-theory extending $\mathcal{L}^{\mathbf{RV}^{\mathrm{s}}}$. We define $\mathcal{L}^{\mathrm{ac},e}$ to be the language containing the following:

(i) $\mathcal{L}^{\mathrm{ac}} \cup \mathcal{L}^{\mathrm{s},e}|_{\mathbf{K}\cup\Sigma_{\mathbf{K}}}$;

(ii) the new sorts $\Sigma_{\mathbf{RV}}$;

(iii) for each new function symbol $f : \prod S_i \to \mathbf{RV}_n$, two functions symbols $f_{\mathbf{R}} : \prod T_i \to \mathbf{R}_n$ and $f_{\mathbf{\Gamma}} : \prod T_i \to \mathbf{\Gamma}^{\infty}$, where $T_i = \mathbf{R}_m \times \mathbf{\Gamma}^{\infty}$ whenever $S_i = \mathbf{RV}_m$ and $T_i = S_i$ otherwise;

(iv) for each new function symbol $f : \prod S_i \to S$, where $S \neq \mathbf{RV}_n$, the same symbol f but with domain $\prod T_i$ as above;

(v) for each new predicate $R \subseteq \prod S_i$, the same symbol R but as a predicate in $\prod T_i$ for T_i as above.

We also define $\mathrm{T}^{\mathrm{ac},e}$ to be the theory containing the following:

(i) $\mathrm{T}_{\mathrm{vf}}^{\mathrm{ac}}$;

(ii) for all new function symbol f, whenever f or $f_{\mathbf{R}}$ and $f_{\mathbf{\Gamma}}$ (depending on the case) is applied to an argument (corresponding to an $\mathbf{RV}_n$-variable of f) outside of $\mathbf{R}_n^{\star} \times \mathbf{\Gamma} \cup \{0,\infty\}$, then f has the same value as if f were applied to $(0,\infty)$ instead;

(iii) for each new symbol f with image $\mathbf{RV}_n$, $\mathrm{Im}(f_{\mathbf{R}}, f_{\mathbf{\Gamma}}) \subseteq \mathbf{R}_n^{\star} \times \mathbf{\Gamma} \cup (0,\infty)$;

(iv) for each new predicate R, R applied to an argument outside of $\mathbf{R}_n^{\star} \times \mathbf{\Gamma} \cup \{0,\infty\}$ is equivalent to R applied to $(0,\infty)$ instead;

(v) the theory T^e translated in $\mathcal{L}^{\mathrm{ac},e}$ as explained in the following proposition.

In the following proposition, $\mathrm{Str}(T)$ denotes the category of substructures of models of T, i.e., models of $T_{\forall}$. See Appendix B for precise definitions.

Proposition 1.8. *There exist functors* $F : \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e}) \to \mathrm{Str}(T^e)$ *and* $G : \mathrm{Str}(T^e) \to \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ *that respect models, cardinality, and elementary submodels. These functors induce an equivalence of categories between* $\mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ *and* $\mathrm{Str}(T^e)$*. Moreover, G sends* $\mathbf{R} \cup \mathbf{\Gamma}^{\infty}$ *to* $\mathbf{RV} \cup \mathbf{R} \cup \mathbf{\Gamma}^{\infty}$.

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Proof. Let C be an $\mathcal{L}^{\mathrm{ac},e}$-substructure (inside some $M \models \mathrm{T}^{\mathrm{ac},e}$). We define $F(C)$ to have the same underlying sets for all sorts common to $\mathcal{L}^{\mathrm{ac},e}$ and $\mathcal{L}^{\mathrm{s},e}$ and $\mathbf{RV}_n(F(C)) = (\mathbf{R}_n^\star(C) \times (\mathbf{\Gamma}^\infty(C) \smallsetminus \{\infty\})) \cup \{(0_n, \infty)\}$. All the structure on the sorts common to $\mathcal{L}^{\mathrm{s},e}$ and $\mathcal{L}^{\mathrm{ac},e}$ is inherited from C. We define $\mathrm{rv}_n(x) = (\mathrm{ac}_n(x), \mathrm{val}(x))$ and $\mathrm{rv}_{m,n}(x, \gamma) = (\mathrm{res}_{m,n}(x), \gamma)$. The (semi-)group structure on $\mathbf{RV}_n$ is the product (semi-)group structure; 0_n is interpreted as $(0_n, \infty)$. We set $(x, \gamma)|_n(y, \delta)$ to hold if and only if $\gamma \leqslant \delta$, and we define $(x, \gamma) +_{m,n} (y, \delta)$ as $(\mathrm{res}_{m,n}(x), \gamma)$ if $\gamma < \delta$, $(\mathrm{res}_{m,n}(y), \delta)$ if $\delta < \gamma$, and $(\mathrm{t}_{\mathbf{R},m,n}(x + y), \gamma + \mathrm{val}_{\mathbf{R},n}(x + y))$ if $\delta = \gamma$. The functions val_n are interpreted as the right projections and the functions t_n as the left projections. Finally, define $\mathrm{i}_n(x) = (x, 0)$ on $\mathbf{R}_n^\star$ and $\mathrm{i}_n(x) = (0, \infty)$ otherwise, $\mathrm{res}_{\mathbf{RV},n}(x, \gamma) = x\, \mathrm{s}_{\mathbf{R},n}(\gamma)$, $\mathrm{s}_n(\gamma) = (1, \gamma)$ if $\gamma \neq \infty$, and $\mathrm{s}_n(\infty) = (0, \infty)$. For each function $f : \prod S_i \to \mathbf{RV}_n$ for some n, define $\overline{u} : \prod S_i \to \prod T_i$ to be such that $u_i(\overline{x}) = x_i$ if $S_i \neq \mathbf{RV}_m$ and $u_i(\overline{x}) = (\mathrm{t}_m(x_i), \mathrm{val}_m(x_i))$ if $S_i = \mathbf{RV}_m$. Then $f^{F(C)}(\overline{x}) = (f_{\mathbf{R}}^C(\overline{u}(\overline{x})), f_{\mathbf{\Gamma}}^C(\overline{u}(\overline{x})))$. If $f : \prod S_i \to S$, where $S \neq \mathbf{RV}_n$ for any n, then define $f^{F(C)}(\overline{x}) = f^C(\overline{u}(\overline{x}))$, and finally $F(C) \models R(\overline{x})$ if and only if $C \models R(\overline{u}(\overline{x}))$.

If $f : C_1 \to C_2$ is an $\mathcal{L}^{\mathrm{ac},e}$-isomorphism, we define $F(f)$ to be f on all sorts common to $\mathcal{L}^{\mathrm{ac},e}$ and $\mathcal{L}^{\mathrm{s},e}$ and $F(f)(x, \gamma) = (f(x), f(\gamma))$. It is easy to check that $F(f)$ is an $\mathcal{L}^{\mathrm{s},e}$-isomorphism.

Let D be an $\mathcal{L}^{\mathrm{s},e}$-structure (inside some $N \models T^e$), and define $G(D)$ to be the restriction of D to all $\mathcal{L}^{\mathrm{ac},e}$-sorts enriched with $\mathrm{val} = \mathrm{val}_n \circ \mathrm{rv}_1$, $\mathrm{res}_n = \mathrm{res}_{\mathbf{RV},n} \circ \mathrm{rv}_n$, $\mathrm{ac}_n = \mathrm{t}_n \circ \mathrm{rv}_n$. Moreover, for any function $f : \prod S_i \to \mathbf{RV}_n$ for some n, let $\overline{v} : \prod T_i \to \prod S_i$ be such that $v_i(\overline{x}) = x_i$ if $S_i \neq \mathbf{RV}_m$ for any m and $v_i(\overline{x}) = \mathrm{i}_m(y_i)\, \mathrm{s}_m(\gamma_i)$, where $x_i = (y_i, \gamma_i)$, if $S_i = \mathbf{RV}_m$. Then define $f_{\mathbf{R}}^{G(D)}(\overline{x}) = \mathrm{t}_n(f^D(\overline{v}(\overline{x})))$ and $f_{\mathbf{\Gamma}}^{G(D)}(\overline{x}) = \mathrm{val}_n(f^D(\overline{v}(\overline{x})))$. If $f : \prod S_i \to S$, where $S \neq \mathbf{RV}_n$ for any n, then $f^{G(D)}(\overline{x}) = f^D(\overline{v}(\overline{x}))$, and finally $G(D) \models R(\overline{x})$ if and only if $D \models R(\overline{v}(\overline{x}))$. If $f : D_1 \to D_2$ is an $\mathcal{L}^{\mathrm{s},e}$-isomorphism, it is easy to show that the restriction of f to the $\mathcal{L}^{\mathrm{ac},e}$-sorts is an $\mathcal{L}^{\mathrm{ac},e}$-isomorphism.

Now, one can check that, for any $\mathcal{L}^{\mathrm{s},e}$-formula $\varphi(\overline{x})$, there exists an $\mathcal{L}^{\mathrm{ac},e}$-formula $\varphi^{\mathrm{ac},e}(\overline{y})$ such that, for any $C \in \mathrm{Str}(\mathrm{T}^{\mathrm{ac},e})$ and $\overline{c} \in C$, $C \models \varphi(\overline{c})$ if and only if $F(C) \models \varphi^{\mathrm{ac},e}(\overline{u}(\overline{c}))$, where u is as above (for the sorts corresponding to $\overline{x}$). Similarly, to any $\mathcal{L}^{\mathrm{ac},e}$-formula $\psi(\overline{x})$ we can associate an $\mathcal{L}^{\mathrm{s},e}$-formula $\psi^{\mathrm{s},e}(\overline{x})$ such that, for any $D \in \mathrm{Str}(T)$ and $d \in D$, $D \models \psi(\overline{d})$ if and only if $G(D) \models \psi^{\mathrm{s},e}(\overline{d})$. One can also check that, for all $\mathcal{L}^{\mathrm{s},e}$-formula φ, $T \models (\varphi^{\mathrm{ac},e})^{\mathrm{s},e}(\overline{u}(\overline{x})) \iff \varphi(\overline{x})$ and, for all $\mathcal{L}^{\mathrm{ac},e}$-formula ψ, $\mathrm{T}^{\mathrm{ac},e} \models (\psi^{\mathrm{s},e})^{\mathrm{ac},e} \iff \psi$. The rest of the proposition follows. ∎

Remark 1.9.

1. The functions $\mathrm{t}_{\mathbf{R},m,n}$ are actually not needed, if we Morleyize on $\mathbf{R} \cup \mathbf{\Gamma}^\infty$, as they are definable using only quantification in the $\mathbf{R}_n$.

2. As with the leading terms structure, in equicharacteristic zero, the angular component structure is a lot simpler. We only need val and ac_1 (and none of the val_n, $\mathrm{s}_{\mathbf{R},n}$, or $\mathrm{t}_{\mathbf{R},m,n}$).

3. In mixed characteristic with finite ramification (i.e., $\mathbf{\Gamma}$ has a smallest positive element 1 and $\mathrm{val}(p) = k \cdot 1$ for some $k \in \mathbb{N}_{>0}$) the structure is also simpler. The functions $\mathrm{val}_{\mathbf{R},n}$, $\mathrm{s}_{\mathbf{R},n}$, and $\mathrm{t}_{\mathbf{R},m,n}$ can be redefined (without $\mathbf{K}$-quantifiers)

knowing only $\mathrm{s}_{\mathbf{R},n}(1)$. Let $\mathcal{L}^{\mathrm{ac,fr}}$ be the language $(\mathcal{L}^{\mathrm{ac}} \setminus \{\mathrm{val}_{\mathbf{R},n}, \mathrm{s}_{\mathbf{R},n}, \mathrm{t}_{\mathbf{R},m,n} : m, n \in \mathbb{N}_{>0}\}) \cup \{c_n\}$, where c_n will be interpreted as $\mathrm{s}_{\mathbf{R},n}(1)$, i.e., as $\mathrm{res}_n(x)\mathrm{ac}_n(x)^{-1}$ for x with minimal positive valuation. This is the language in which finitely ramified mixed characteristic fields with angular components are usually considered, and in which they eliminate field quantifiers.

To finish this section let us define balls and Swiss cheeses.

Definition 1.10 (Balls and Swiss cheeses)**.** Let (K, v) be a valued field, $\gamma \in \mathrm{val}(K)$, and $a \in K$. Write $\mathring{\mathcal{B}}_\gamma(a) := \{x \in \mathbf{K}(M) : \mathrm{val}(x - a) > \gamma\}$ for the open ball of center a and radius γ, and $\overline{\mathcal{B}}_\gamma(a) := \{x \in \mathbf{K}(M) : \mathrm{val}(x - a) \geqslant \gamma\}$ for the closed ball of center a and radius γ.

A Swiss cheese is a set of the form $b \setminus (\bigcup_{i=1,\ldots,n} b_i)$, where b and the b_i are open or closed balls.

We allow closed balls to have radius ∞ (i.e., singletons are balls), and we allow open balls to have radius $-\infty$ (i.e., $\mathbf{K}$ itself is an open ball).

Definition 1.11 ($\mathcal{L}_{\mathrm{div}}$)**.** The language $\mathcal{L}_{\mathrm{div}}$ has a unique sort $\mathbf{K}$ equipped with the ring language and a binary predicate $|$.

In a valued field (K, val), the predicate $x|y$ will denote $\mathrm{val}(x) \leqslant \mathrm{val}(y)$. If $C \subseteq K$, we will denote by $\mathcal{SC}(C)$ the set of all quantifier free $\mathcal{L}_{\mathrm{div}}(C)$-definable sets in one variable. Note that all those sets are finite unions of Swiss cheeses.

Note that, later on, our valued fields may be endowed with more than one valuation. In that case, we will write $\mathring{\mathcal{B}}^{\mathcal{O}}_\gamma(a)$ or $\mathcal{SC}^{\mathcal{O}}(C)$ to specify that we are considering the valuation associated to $\mathcal{O}$. For a tuple $\overline{a} \in \mathbf{K}$, we will extend the notation for balls by writing $\mathring{\mathcal{B}}_\gamma(\overline{a}) := \{\overline{b} : \mathrm{val}(\overline{b} - \overline{a}) > \gamma\}$ and $\overline{\mathcal{B}}_\gamma(\overline{a}) := \{\overline{b} : \mathrm{val}(\overline{b} - \overline{a}) \geqslant \gamma\}$, where $\mathrm{val}(\overline{a}) := \min_i\{\mathrm{val}(a_i)\}$.

2. Coarsening

The goal of this section is to provide the necessary tools for the reduction to the equicharacteristic zero case. This is a classical method, which underlies most existing proofs of K-quantifier elimination for enriched mixed characteristic Henselian fields. We present it here on its own, as a general transfer principle which we will then be able to invoke directly, in order, hopefully, to make the proofs clearer.

Definition 2.1 (Coarsening valuations)**.** Let (K, val) be a valued field, $\Delta \subseteq \mathbf{\Gamma}(K)$ a convex subgroup, and $\pi : \mathbf{\Gamma}(K) \to \mathbf{\Gamma}(K)/\Delta$ the canonical projection. Let $\mathrm{val}^\Delta := \pi \circ \mathrm{val}$, extended to 0 by $\mathrm{val}^\Delta(0) = \infty$.

Remark 2.2. The valuation val^Δ is a valuation coarser than val. Its valuation ring is $\mathcal{O}^\Delta := \{x \in K : \exists \delta \in \Delta,\ \delta < \mathrm{val}(x)\} \supseteq \mathcal{O}(K)$, and its maximal ideal is $\mathfrak{M}^\Delta := \{x \in K : \mathrm{val}(x) > \Delta\} \subseteq \mathfrak{M}(K)$. Its residue field $\mathbf{R}_1^\Delta$ is in fact a valued field for the valuation $\widetilde{\mathrm{val}}^\Delta$ defined by $\widetilde{\mathrm{val}}^\Delta(x + \mathfrak{M}^\Delta) := \mathrm{val}(x)$ for all $x \in \mathcal{O}^\Delta \setminus \mathfrak{M}^\Delta$ and $\widetilde{\mathrm{val}}^\Delta(\mathfrak{M}^\Delta) = \infty$. Then $\widetilde{\mathrm{val}}^\Delta(\mathbf{R}_1^\Delta) = \Delta^\infty = \Delta \cup \{\infty\}$. The valuation ring of $\mathbf{R}_1^\Delta$ is $\widetilde{\mathcal{O}}^\Delta := \mathcal{O}/\mathfrak{M}^\Delta$, its maximal ideal is $\mathfrak{M}/\mathfrak{M}^\Delta$,

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and its residue field is $\mathbf{R}_1$. Moreover, if $\mathrm{rv}_n^\Delta : K \to K^\star/(1+n\mathfrak{M}^\Delta) \cup \{0\} =: \mathbf{RV}_n^\Delta$ is the canonical projection, rv_n factorizes through rv_n^Δ; i.e., there is a function $\pi_n : \mathbf{RV}_n^\Delta \to \mathbf{RV}_n$ such that $\mathrm{rv}_n = \pi_n \circ \mathrm{rv}_n^\Delta$.

$$\begin{tikzcd}
\mathcal{O}^\star \arrow[r, hook] \arrow[d] \arrow[dd, bend right=60, "\mathrm{res}_1"'] & (\mathcal{O}^\Delta)^\star \arrow[r, hook] \arrow[d, "\mathrm{res}_1^\Delta"] & \mathbf{K}^\star \arrow[d, "\mathrm{rv}_1^\Delta"] \\
(\widetilde{\mathcal{O}}^\Delta)^\star \arrow[r, hook] \arrow[d, "\widetilde{\mathrm{res}}_1^\Delta"] & (\mathbf{R}_1^\Delta)^\star \arrow[r, hook] \arrow[d, "\widetilde{\mathrm{val}}^\Delta"] & (\mathbf{RV}_1^\Delta)^\star \\
\mathbf{R}_1^\star & \Delta &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{K} \arrow[d, "\mathrm{rv}_n"] \arrow[dl, "\mathrm{rv}_n^\Delta"'] \arrow[r, "\mathrm{val}"] \arrow[dr, "\mathrm{val}^\Delta"] & \mathbf{\Gamma}^\infty \arrow[d, "\pi"] \\
\mathbf{RV}_n^\Delta \arrow[r, "\pi_n"] \arrow[drr, "\mathrm{val}_n{}^\Delta"'] & \mathbf{RV}_n \arrow[ur, "\mathrm{val}_n"'] & (\mathbf{\Gamma}/\Delta)^\infty
\end{tikzcd}$$

Before we go on, let us explain the link between open balls for the coarsened valuations and open balls for the original valuation.

Proposition 2.3. *Let* (K, val) *be a valued field and* Δ *a convex subgroup of its valuation group. Let* S *be an* $\mathcal{O}$*-Swiss cheese,* b *an* $\mathcal{O}^\Delta$*-ball, and* $c, d \in K$ *such that* $b = \mathring{\mathcal{B}}^{\mathcal{O}^\Delta}_{\mathrm{val}^\Delta(d)}(c)$*. If* $b \subseteq S$*, there exists* $d' \in K$ *such that* $\mathrm{val}^\Delta(d') = \mathrm{val}^\Delta(d)$ *and* $b \subseteq \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d')}(c) \subseteq S$*.*

Proof. Let (g_α) be a cofinal (ordinal indexed) sequence in Δ. We have $b = \bigcap_\alpha \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_\alpha)}(c)$. Indeed, $\mathrm{val}^\Delta(dg_\alpha) = \mathrm{val}^\Delta(d)$, and hence $b = \mathring{\mathcal{B}}^{\mathcal{O}^\Delta}_{\mathrm{val}^\Delta(dg_\alpha)}(c) \subseteq \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_\alpha)}(c)$. Conversely, if $x \in \bigcap_\alpha \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_\alpha)}(c)$, then $\mathrm{val}((x-c)/d) > \mathrm{val}(g_\alpha)$ for all α; hence $(x-c)/d \in \mathfrak{M}^\Delta$.

Let b' be any $\mathcal{O}$-ball. Then $b = \bigcap_\alpha \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_\alpha)}(c) \subseteq b'$ if and only if there exists α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \subseteq b'$, and $b \cap b' = \bigcap_\alpha \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_\alpha)}(c) \cap b' = \varnothing$ if and only if there exists α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \cap b' = \varnothing$. These statements still hold for Boolean combinations of balls; hence there is some α_0 such that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(dg_{\alpha_0})}(c) \subseteq S$. ∎

When (K, val) is a mixed characteristic valued field, we are interested in the coarsened valuation associated to Δ_p, the convex group generated by $\mathrm{val}(p)$. The main reason is that $(K, \mathrm{val}^{\Delta_p})$ has equicharacteristic zero. We will write $\mathrm{val}_\infty := \mathrm{val}^{\Delta_p}$, $\mathbf{R}_\infty := \mathbf{R}_1^{\Delta_p}$, $\mathcal{O}_\infty := \mathcal{O}^{\Delta_p} = \mathcal{O}_{p^{-1}}$, and $\mathfrak{M}_\infty := \mathfrak{M}^{\Delta_p} = \bigcap_{n\in\mathbb{N}} p^n\mathfrak{M}$. As the coarsened field has equicharacteristic zero, all $\mathbf{RV}_n^{\Delta_p}$ are the same, and we will write $\mathbf{RV}_\infty := K^\star/(1+\mathfrak{M}_\infty) \cup \{0\} = \mathbf{RV}_1^{\Delta_p}$.

Remark 2.4. We can (and we will) identify $\mathbf{RV}_\infty$ (canonically) with a subgroup of $\varprojlim \mathbf{RV}_n$, and the canonical projection $K \to \mathbf{RV}_\infty$ then coincides with $\varprojlim \mathrm{rv}_n : K \to \varprojlim \mathbf{RV}_n$; in particular, $\mathbf{RV}_\infty = (\varprojlim \mathrm{rv}_n)(K)$. Similarly, $\widetilde{\mathcal{O}}^{\Delta_p}$ can be identified with a subring of $\varprojlim \mathbf{R}_n$ and $\mathbf{R}_\infty = \mathrm{Frac}(\widetilde{\mathcal{O}}^{\Delta_p}) \subseteq \mathrm{Frac}(\varprojlim \mathbf{R}_n) = (\varprojlim \mathbf{R}_n)[\mathrm{rv}_\infty(p)^{-1}]$. The inclusions are equalities if K is $\aleph_1$-saturated. In particular, $\varprojlim \mathrm{rv}_n$ is surjective.

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$$\begin{tikzcd}
 & \mathbf{K} \arrow[r, "\mathrm{rv}_\infty"] \arrow[dl, "\mathrm{rv}_m"'] \arrow[d, "\mathrm{rv}_n"] \arrow[dr, "\varprojlim \mathrm{rv}_n"] & \mathbf{RV}_\infty \arrow[d, hook] \\
\mathbf{RV}_m \arrow[r, "\mathrm{rv}_{m,n}"'] & \mathbf{RV}_n & \varprojlim \mathbf{RV}_n \arrow[l, "\pi_n"] \arrow[ll, bend left, "\pi_m"]
\end{tikzcd}$$

Hence (K, val_∞) is prodefinable (i.e., a prolimit of definable sets) in (K, val) with its $\mathcal{L}^{\mathbf{RV}}$-structure.

Let $\mathcal{L}$ be an **RV**-enrichment of a **K**-enrichment of $\mathcal{L}^{\mathbf{RV}}$ with new sorts $\Sigma_{\mathbf{K}}$ and $\Sigma_{\mathbf{RV}}$ respectively. Somewhat abusing notation, when writing **K** we will mean $\mathbf{K} \cup \Sigma_{\mathbf{K}}$, and when writing **RV** we will mean $\bigcup_n \mathbf{RV}_n \cup \Sigma_{\mathbf{RV}}$ (and rely on the context for it to make sense). Let $T \supseteq \mathrm{T}_{\mathrm{vf},0,p}$ be an $\mathcal{L}$-theory. Let $\mathcal{L}^{\mathbf{RV}_\infty}$ be a copy of $\mathcal{L}^{\mathbf{RV}}$ (as $\mathcal{L}^{\mathbf{RV}_\infty}$ will only be used in equicharacteristic zero, we will only need its $\mathbf{RV}_1$, which we will denote $\mathbf{RV}_\infty$ to avoid confusion with the original $\mathbf{RV}_1$). Let $\mathcal{L}^\infty$ be $\mathcal{L}^{\mathbf{RV}_\infty} \cup \mathcal{L}|_{\mathbf{K} \cup \Sigma_K} \cup \mathcal{L}|_{\mathbf{RV} \cup \Sigma_{\mathbf{RV}}} \cup \{\pi_n : n \in \mathbb{N}_{>0}\}$, where π_n is a function symbol $\mathbf{RV}_\infty \to \mathbf{RV}_n$. Let T^∞ be the theory containing the following:

- $\mathrm{T}^\infty_{\mathrm{vf},0,0}$, i.e., the theory of equicharacteristic zero valued fields in $\mathcal{L}^{\mathbf{RV}_\infty}$;
- the translation of T into $\mathcal{L}^\infty$ by replacing rv_n by $\pi_n \circ \mathrm{rv}_\infty$.

Recall that $\mathrm{Str}(T)$ is the category of substructures of models of T, and whenever $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ is a functor and κ a cardinal, we denote by $\mathrm{Str}_{F,\kappa}(T_2)$ the full subcategory of $\mathrm{Str}(T_2)$ of structures that embed into some $F(M)$ for $M \models T_1$ κ-saturated. See Appendix B for precise definitions.

The main goal of the following proposition is to show that quantifier elimination results in equicharacteristic zero can be transferred to mixed characteristic using the results from Appendix B.

Proposition 2.5 (Reduction to equicharacteristic zero)**.** *We can define functors $\mathfrak{C}^\infty : \mathrm{Str}(T) \to \mathrm{Str}(T^\infty)$ and $\mathfrak{UC}^\infty : \mathrm{Str}(T^\infty) \to \mathrm{Str}(T)$ which respect cardinality up to $\aleph_0$ and induce an equivalence of categories between $\mathrm{Str}(T)$ and $\mathrm{Str}_{\mathfrak{C}^\infty,\aleph_1}(T^\infty)$. Moreover, $\mathfrak{C}^\infty$ respects $\aleph_1$-saturated models, and $\mathfrak{UC}^\infty$ respects models and elementary submodels and sends* **RV** *to* $\mathbf{RV} \cup \mathbf{RV}_\infty$ *(which are closed).*

Proof. Let $C \leqslant M \models T$ be $\mathcal{L}$-structures. Then $\mathfrak{C}^\infty(C)$ has underlying sets $\mathbf{K}(\mathfrak{C}^\infty(C)) = K(C)$, $\mathbf{RV}_\infty(\mathfrak{C}^\infty(C)) = \varprojlim \mathbf{RV}_n(C)$, and $\mathbf{RV}(\mathfrak{C}^\infty(C)) = \mathbf{RV}(C)$, keeping the same structure on **K** and **RV**, defining rv_∞ to be $\varprojlim \mathrm{rv}_n$ and π_n to be the canonical projection $\mathbf{RV}_\infty \to \mathbf{RV}_n$. Now, if $f : C_1 \to C_2$ is an $\mathcal{L}$-embedding, let us write $f_\infty := \varprojlim f|_{\mathbf{RV}_n}$. By definition, we have $\pi_n \circ f_\infty = f|_{\mathbf{RV}_n} \circ \pi_n$, and, by immediate diagrammatic considerations, $\mathrm{rv}_\infty \circ f|_{\mathbf{K}} = f_\infty \circ \mathrm{rv}_\infty$ and f_∞ is injective. Then, let $\mathfrak{C}^\infty(f)$ be $f|_{\mathbf{K}} \cup f_\infty \cup f|_{\mathbf{RV}}$. As f is an $\mathcal{L}$-embedding, $f|_{\mathbf{K}}$ respects the structure on **K**, $f|_{\mathbf{RV}}$ respects the structure on **RV**, and, as we have already seen, $\mathfrak{C}^\infty(f)$ respects rv_∞ and π_n. Hence $\mathfrak{C}^\infty(f)$ is an $\mathcal{L}^\infty$-embedding.

If $M \models T$ is $\aleph_1$-saturated, it follows from Remark 2.4 that $\mathfrak{C}^\infty(M) \models T^\infty$. Beware though that $\mathfrak{C}^\infty(M)$ is never $\aleph_0$-saturated because, if it were, we would find $x \neq y \in \mathbf{RV}_\infty(M_1)$ such that, for all $n \in \mathbb{N}_{>0}$, $\pi_n(x) = \pi_n(y)$, contradicting the fact that $\mathbf{RV}_\infty(M_1) = \varprojlim \mathbf{RV}_n(M_1)$. Let C be a substructure of M and let $f : C \to M$ be the

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injection. Then $\mathfrak{C}^\infty(f)$ is an embedding of $\mathfrak{C}^\infty(C)$ into $\mathfrak{C}^\infty(M)$, and $\mathfrak{C}^\infty$ is indeed a functor to $\mathrm{Str}(T)$.

The functor $\mathfrak{UC}^\infty$ is defined as the restriction functor that restricts $\mathcal{L}^\infty$-structures to the sorts $\mathbf{K}$ and $\mathbf{RV}$. It is clear that, if C is an $\mathcal{L}$-structure in some model of T, then $\mathfrak{UC}^\infty \circ \mathfrak{C}^\infty(C)$ is trivially isomorphic to C. Now if D is in $\mathrm{Str}(T^\infty)$ there will be three leading term structures (and hence valuations) on D: the one associated with the $\mathcal{L}^{\mathbf{RV}_\infty}$-structure of C (which is definable), whose valuation ring is $\mathcal{O}$, the one given by $\mathrm{rv}_n = \pi_n \circ \mathrm{rv}_\infty$ (which is definable), whose valuation ring is $\mathcal{O}_\infty$, and the one given by $\varprojlim \mathrm{rv}_n$ (which is only prodefinable), whose valuation ring is $\mathcal{O}_{p^{-1}}$. In general, we have $\mathcal{O} \subset \mathcal{O}_{p^{-1}} \subset \mathcal{O}_\infty$, but if $D = \mathfrak{C}^\infty(C)$ (or D embeds in some $\mathfrak{C}^\infty(C)$), $\mathcal{O}_{p^{-1}} = \mathcal{O}_\infty$ and $\varprojlim \mathrm{rv}_n(D) = \mathrm{rv}_\infty(D)$. Hence, if C embeds in some $\mathfrak{C}^\infty(M)$ then $\mathfrak{C}^\infty \circ \mathfrak{UC}^\infty(C)$ is (naturally) isomorphic to C.

Functoriality of all the previous constructions is a tedious but easy verification. ∎

3. Analytic structure

In [9], Cluckers and Lipshitz study valued fields with analytic structure. Let us recall some of their results. From now on, A will be a Noetherian ring separated and complete for its I-adic topology for some ideal I. Let $A\langle \overline{X}\rangle$ be the ring of power series with coefficients in A whose coefficients I-adically converge to 0. Let us also define $\mathcal{A}_{m,n} := A\langle \overline{X}\rangle[[\overline{Y}]]$, where $|\overline{X}| = m$, $|\overline{Y}| = n$, and $\mathcal{A} := \bigcup_{m,n} \mathcal{A}_{m,n}$. Note that $\mathcal{A}$ is a separated Weierstrass system over (A, I) as defined in [9, Example 4.4.(1)]. The main example to keep in mind here will be $\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}]\langle \overline{X}\rangle[[\overline{Y}]]$, which is a separated Weierstrass system over $(\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}], p\mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}])$.

Definition 3.1 ($\mathcal{Q}$)**.** We will extensively use a quotient symbol $\mathcal{Q} : \mathbf{K}^2 \to \mathbf{K}$ that is interpreted as $\mathcal{Q}(x, y) = x/y$, when $y \neq 0$ and $\mathcal{Q}(x, 0) = 0$.

Definition 3.2 ($\mathcal{R}$)**.** Let $\mathcal{R}$ be a valuation ring of K included in $\mathcal{O}$, and let $\mathfrak{N}$ be its maximal ideal and $\mathrm{val}^{\mathcal{R}}$ its valuation. We have $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathcal{R} \subseteq \mathcal{O}$. Also, note that $1 + n\mathfrak{M} \subseteq 1 + n\mathfrak{N} \subseteq \mathcal{R}^\star$, and hence the valuation $\mathrm{val}^{\mathcal{R}}$ corresponding to $\mathcal{R}$ factors through rv_n; i.e., there is some function f_n such that $\mathrm{val}^{\mathcal{R}} = f_n \circ \mathrm{rv}_n$. We will also be using a new predicate $x|_1^{\mathcal{R}} y$ on $\mathbf{RV}_1$ interpreted by $f_1(x) \leqslant f_1(y)$.

Note that $\mathcal{O}$ is the coarsening of $\mathcal{R}$ associated to the convex subgroup $\mathcal{O}^\star/\mathcal{R}^\star$ of $\mathbf{K}^\star/\mathcal{R}^\star$. Note also that $\mathcal{R}$ is then definable by the (quantifier free) formula, $\mathrm{rv}_1(1)|_1^{\mathcal{R}}\mathrm{rv}_1(x)$. In fact the whole leading term structure associated to $\mathcal{R}$ is interpretable in $\mathcal{L}^{\mathbf{RV}} \cup \{|_1^{\mathcal{R}}\}$.

Definition 3.3 (Fields with separated analytic $\mathcal{A}$-structure)**.** Let $\mathcal{L}_{\mathcal{A}}$ be the language $\mathcal{L}^{\mathbf{RV}^+}$ enriched with a symbol for each element in $\mathcal{A}$ (we will identify the elements in $\mathcal{A}$ and the corresponding symbols). For each $E \in \mathcal{A}^\star_{m,n}$ let also $E_k : \mathbf{RV}_k^{m+n} \to \mathbf{RV}_k$ be a new symbol, and $\mathcal{L}_{\mathcal{A},\mathcal{Q}} := \mathcal{L}_{\mathcal{A}} \cup \{|_1^{\mathcal{R}}, \mathcal{Q}\} \cup \{E_k : E \in \mathcal{A}^\star_{m,n}, m, n, k \in \mathbb{N}\}$. The theory $\mathrm{T}_{\mathcal{A}}$ of fields with separated analytic $\mathcal{A}$-structure consists of the following:

(i) T_{vf};

(ii) $\mathcal{Q}$ is interpreted as in Definition 3.1;

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(iii) $|_1^{\mathcal{R}}$ comes from a valuation subring $\mathcal{R} \subseteq \mathcal{O}$ with fraction field $\mathbf{K}$;

(iv) each symbol $f \in \mathcal{A}_{m,n}$ is interpreted as a function $\mathcal{R}^m \times \mathfrak{N}^n \to \mathcal{R}$ (the symbols will be interpreted as 0 outside $\mathcal{R}^m \times \mathfrak{N}^n$);

(v) the interpretations $i_{m,n} : \mathcal{A}_{m,n} \to \mathcal{R}^{\mathcal{R}^m \times \mathfrak{N}^n}$ are morphisms of the inductive system of rings $\bigcup_{m,n} \mathcal{A}_{m,n}$ to $\bigcup_{m,n} \mathcal{R}^{\mathcal{R}^m \times \mathfrak{N}^n}$, where the inclusions are the obvious ones;

(vi) $i_{0,0}(I) \subseteq \mathfrak{N}$;

(vii) $i_{m,n}(X_i)$ is the ith coordinate function and $i_{m,n}(Y_j)$ is the $(m+j)$th coordinate function;

(viii) for every $E \in \mathcal{A}^{\star}_{m,n}$, E_k is interpreted as the function induced by E on $\mathbf{RV}_k$ when it is well defined (we will see shortly, in Corollary 3.9, that it is, in fact, always well defined).

To specify the characteristic we will write $\mathrm{T}_{\mathcal{A},0,0}$ or $\mathrm{T}_{\mathcal{A},0,p}$.

Remark 3.4.

- These axioms imply a certain number of properties that it seems reasonable to require. First, (iv) implies that every constant in $A = \mathcal{A}_{0,0}$ is interpreted in $\mathcal{R}$. By (v) and (vii), polynomials in $\mathcal{A}$ are interpreted as polynomials. And (v) implies that any ring equality between functions in $\mathcal{A}_{m,n}$ for some m and n is also true in models of $\mathrm{T}_{\mathcal{A}}$. Using Weierstrass division (see Proposition 3.12) one can also show that compositional identities in $\mathcal{A}$ are also true in models of $\mathrm{T}_{\mathcal{A}}$.
- We allow the analytic structure to be over a smaller valuation ring in order to be able to coarsen the valuation while staying in our setting of analytic structures.

From now on, we will write $\langle C\rangle := \langle C\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q}}}$ and $C\langle\overline{c}\rangle := C\langle\overline{c}\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q}}}$ for the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structures generated by C and $C\overline{c}$ (cf. Definition A.12).

We could be working in a larger context here. What we really need in the proof is not that $\mathcal{A}$ is a separated Weierstrass system, as in [9], but the consequences of this fact, namely: Henselianity, (uniform) Weierstrass preparation, differentiability of the new function symbols, and extension of the analytic structure to algebraic extensions. One could give an axiomatic treatment along those lines, but, to simplify the exposition, we restrict to a more concrete case.

Also note that if $\mathcal{A}$ is not countable we may now be working in an uncountable language Let us now describe all the nice properties of models of $\mathrm{T}_{\mathcal{A}}$.

Proposition 3.5. *Let $M \models \mathrm{T}_{\mathcal{A}}$; then M is Henselian.*

Proof. If $\mathcal{O} = \mathcal{R}$, this is proved exactly as in [23, Lemma 3.3]. The case when $\mathcal{R} \neq \mathcal{O}$ follows, as coarsening preserves Henselianity. ■

Remark 3.6. As $\mathrm{T}_{\mathcal{A}}$ implies $\mathrm{T}_{\mathrm{Hen}}$, by resplendent elimination of quantifiers in $\mathrm{T}_{\mathrm{Hen}}$ (cf. Theorem 1.4), as $\mathcal{L}_{\mathcal{A},\mathcal{Q}} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$ is an $\mathbf{RV}$-enrichment of $\mathcal{L}^{\mathbf{RV}^+}$, any $\mathcal{L}_{\mathcal{A},\mathcal{Q}} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$-formula is equivalent modulo $\mathrm{T}_{\mathcal{A}}$ to a $\mathbf{K}$-quantifier free formula.

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Let us now show that functions from $\mathcal{A}$ have nice differential properties.

Definition 3.7. Let K be a valued field and $f : K^n \to K$. We say that f is differentiable at $\overline{a} \in K^n$ if there exist $\overline{d} \in K^n$ and ξ and $\gamma \in \mathrm{val}(\mathbf{K}^\star)$ such that, for all $\overline{\varepsilon} \in \mathring{\mathcal{B}}_\xi(\overline{a})$,

$$\mathrm{val}(f(\overline{a}+\overline{\varepsilon}) - f(\overline{a}) - \overline{d} \cdot \overline{\varepsilon}) \geqslant 2\mathrm{val}(\overline{\varepsilon}) + \gamma.$$

There is a unique such $\overline{d} = (d_i)$, and we will denote it $df_{\overline{a}}$. The d_i are usually called the derivatives of f at $\overline{a}$. We will denote them $\partial f/\partial x_i(\overline{a})$.

Proposition 3.8. *Let $M \models \mathrm{T}_{\mathcal{A}}$ and $f \in \mathcal{A}_{m,n}$ for some m and n. Then, for all $i < m+n$, there is $g_i \in \mathcal{A}_{m,n}$ such that, for all $\overline{a} \in K^{m+n}$, f is differentiable at $\overline{a}$ and $\partial f/\partial x_i(\overline{a}) = g_i(\overline{a})$.*

Proof. If $\overline{a} \notin \mathcal{R}^m \times \mathfrak{N}^n$ then f is equal to 0 on $\mathring{\mathcal{B}}_0(\overline{a})$, and the statement is trivial. If not, as $f \in A\langle X\rangle[[\overline{Y}]]$, it has a (formal) Taylor development:

$$f(X_0+X, Y_0+Y) = f(X_0, Y_0) + \sum_i g_i(X_0, Y_0)X_i + \sum_j g_{m+j}(X_0, Y_0)Y_j + h(X_0, Y_0, X, Y),$$

where h is a sum of terms each divisible by some quadratic monomial in $\overline{X}$ and $\overline{Y}$. As the interpretation morphisms are ring morphisms, this immediately implies that the interpretation of f is differentiable in $\mathbf{K}(M)$ at $\overline{a}$, and that the derivatives are given by the g_i. ∎

Corollary 3.9. *Let $M \models \mathrm{T}_{\mathcal{A}}$, $E(\overline{x}) \in \mathcal{A}_{m,n}$, and $S \subseteq \mathbf{K}(M)^{m+n}$. If, for all $\overline{x} \in S$, $\mathrm{val}(E(\overline{x})) = 0$, then, for all $\overline{x} \in S$, $\mathrm{rv}_n(E(\overline{x}))$ only depends on $\mathrm{res}_n(\overline{x})$.*

In particular, if $E \in \mathcal{A}^\star_{m,n}$, then, for all $\overline{x} \in \mathcal{R}^m \times \mathfrak{N}^n$, $\mathrm{val}^{\mathcal{R}}(E(\overline{x})) = 0$, and hence $\mathrm{val}(E(\overline{x})) = 0$, and thus $\mathrm{rv}_n(E(\overline{x}))$ is a function of $\mathrm{res}_n(\overline{x})$ which is a function of $\mathrm{rv}_n(\overline{x})$. Outside of $\mathcal{R}^m \times \mathfrak{N}^n$, $\mathrm{rv}_n(E(\overline{x}))$ is constant equal to 0, and hence it is also a function of $\mathrm{rv}_n(\overline{x})$. Hence, as announced earlier, E does induce a well-defined function on $\mathbf{RV}_k$ for any k.

Proof (Corollary 3.9). Any element with the same res_n residue as $\overline{x}$ is of the form $\overline{x} + n\overline{m}$ for some $\overline{m} \in \mathfrak{M}$. By Proposition 3.8, $E(\overline{x}+n\overline{m}) = E(\overline{x}) + \overline{G}(\overline{x}) \cdot (n\overline{m}) + H(\overline{x}, n\overline{m})$, where $\overline{G}(\overline{x}) \in \mathcal{R} \subseteq \mathcal{O}$ and $\mathrm{val}(H(\overline{x}, n\overline{m})) \geqslant 2\mathrm{val}(n\overline{m}) > \mathrm{val}(n)$; hence $\mathrm{res}_n(E(\overline{x}+n\overline{m})) = \mathrm{res}_n(E(\overline{x}))$. As, for all $\overline{z} \in S$, $\mathrm{val}(E(\overline{z})) = 0$, $\mathrm{rv}_n(E(\overline{z})) = \mathrm{res}_n(E(\overline{z}))$, and we have the expected result. ∎

Let us now (re)prove a well-known result from papers by Cluckers, Lipshitz, and Robinson. There are two main reasons for which to reprove this result. The first reason is that, although the proof given here is very close to the classical Denef–van den Dries proof as explained in [25, Theorem 4.2], the proof there only shows quantifier elimination for algebraically closed fields with analytic structures over $(\mathbb{Z}, 0)$. The second reason is to make sure that $\mathcal{O} \neq \mathcal{R}$ does not interfere.

Theorem 3.10.

$\mathrm{T}_{\mathcal{A}}$ *eliminates* $\mathbf{K}$*-quantifiers resplendently.*

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The proof of this theorem will need many definitions and properties that will only be used here and that will be introduced now.

For all $m, n \in \mathbb{N}$, we define $J_{m,n}$ to be the ideal $\{\sum_{\mu,\nu} a_{\mu,\nu}\overline{X}^{\mu}\overline{Y}^{\nu} \in \mathcal{A}_{m,n} : a_{\mu,\nu} \in I\}$ of $\mathcal{A}_{m,n}$. Most of the time we will only write J and rely on context for the indices. We will also write $\overline{X}_{\neq n}$ for the tuple $\overline{X}$ without its nth component.

Note that, in the definition below and in most of this proof, elements of $\mathcal{A}$ will be considered as formal series (and not as their interpretation in some $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure), and hence infinite sums as the one below do make sense.

Definition 3.11 (Regularity)**.** Let $f \in \mathcal{A}_{m_0,n_0}$, $m < m_0$, $n < n_0$. We say that

(i) $f = \sum_i a_i(\overline{X}_{\neq m}, \overline{Y})X_m^i$ is regular in X_m of degree d if f is congruent to a monic polynomial in X_m of degree d modulo $J + (\overline{Y})$;

(ii) $f = \sum_i a_i(\overline{X}, \overline{Y}_{\neq n})Y_n^i$ is regular in Y_n of degree d if f is congruent to Y_n^d modulo $J + (\overline{Y}_{\neq n}) + (Y_n^{d+1})$.

If we do not want to specify the degree, we will just say that f is regular in X_m (respectively, Y_n).

Proposition 3.12 (Weierstrass division and preparation)**.** *Let $f, g \in \mathcal{A}_{m_0,n_0}$, and suppose that f is regular in X_m (respectively, in Y_n) of degree d. Then there exist unique $q \in \mathcal{A}_{m,n}$ and $r \in A\langle\overline{X}_{\neq m}\rangle[[\overline{Y}]][X_m]$ (respectively, $r \in A\langle\overline{X}\rangle[[\overline{Y}_{\neq n}]][Y_n]$) of degree strictly lower than d such that $g = qf + r$.*

Moreover, there exist unique $P \in A\langle\overline{X}_{\neq m}\rangle[[\overline{Y}]][X_m]$ (respectively, $P \in A\langle\overline{X}\rangle[[\overline{Y}_{\neq n}]][Y_n]$) regular in X_m (respectively, in Y_n) of degree at most d and $u \in \mathcal{A}_{m,n}^{\star}$ such that $f = uP$.

Proof. See [25, Corollary 3.3]. ∎

We will be ordering multi-indices μ of the same length by lexicographic order, and we write $|\mu| = \sum_i \mu_i$.

Definition 3.13 (Preregularity)**.** Let $f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2, \overline{Y}_2)\overline{X}_1^{\mu}\overline{Y}_1^{\nu} \in \mathcal{A}_{m_1+m_2,n_1+n_2}$. We say that f is preregular in $(\overline{X}_1, \overline{Y}_1)$ of degree (μ_0, ν_0, d) when

(i) $f_{\mu_0,\nu_0} = 1$;

(ii) for all μ and ν such that $|\mu| + |\nu| \geqslant d$, $f_{\mu,\nu} \in J + (\overline{Y}_2)$;

(iii) for all $\nu < \nu_0$ and for all μ, $f_{\mu,\nu} \in J + (\overline{Y}_2)$;

(iv) for all $\mu > \mu_0$, $f_{\mu,\nu_0} \in J + (\overline{Y}_2)$.

Remark 3.14. Note that, if $f = \sum_\nu f_\nu(\overline{X})\overline{Y}^{\nu}$ is preregular in $(\overline{X}, \overline{Y})$ of degree (μ_0, ν_0, d), then f_{ν_0} is preregular in $\overline{X}$ of degree $(\mu_0, 0, d)$.

Let $T_d(\overline{X}) := (X_0 + X_{m-1}^{d^{m-1}}, \ldots, X_i + X_{m-1}^{d^{m-1-i}}, \ldots, X_{m-2} + X_{m-1}^{d}, X_{m-1})$, where $m = |\overline{X}|$. We call T_d a Weierstrass change of variables. Note that Weierstrass changes of variables are bijective.

Proposition 3.15. *Let* $f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2, \overline{Y}_2)\overline{X}_1^{\mu}\overline{Y}_1^{\nu} \in \mathcal{A}_{m_1+m_2,n_1+n_2}$. *Then,*

(i) *if* f *is preregular in* $(\overline{X}_1, \overline{Y}_1)$ *of degree* $(\mu_0, 0, d)$, *then* $f(T_d(\overline{X}_1), \overline{X}_2, \overline{Y})$ *is regular in* X_{1,m_1-1}*;*

(ii) *if* f *is preregular in* $(\overline{X}_1, \overline{Y}_1)$ *of degree* $(0, \nu_0, d)$, *then* $f(\overline{X}, T_d(\overline{Y}_1), \overline{Y}_2)$ *is regular in* Y_{1,n_1-1}.

Proof. Let $m = m_1 - 1$ and $n = n_1 - 1$. First assume that f is preregular in $(\overline{X}_1, \overline{Y}_1)$ of degree $(\mu_0, 0, d)$. Then

$$f \equiv \sum_{\mu<\mu_0,|\mu|<d} f_{\mu,0}\overline{X}_1^{\mu} \mod J + (\overline{Y_2}) + (\overline{Y_1}).$$

Furthermore, $T_d(\overline{X}_1)^{\mu} = (\prod_{i=0}^{m-1}(X_{1,i} + X_{1,m}^{d^{m-i}})^{\mu_i})X_{1,m}^{\mu_m}$ is a sum of monomials whose highest degree monomial only contains the variable $X_{1,m}$ and has degree $\sum_{i=0}^{m} d^{m-i}\mu_i$. It now suffices to show that this degree is maximal when $\mu = \mu_0$, but that is exactly what is shown in the following claim.

Claim 3.16. Let μ and ν be two multi-indices such that $\mu < \nu$ and $|\mu| < d$. Then

$$\sum_{i=0}^{m} d^{m-i}\mu_i < \sum_{i=0}^{m} d^{m-i}\nu_i.$$

Proof. Let i_0 be minimal such that $\mu_i < \nu_i$. Then, for all $j < i_0$, $\mu_j = \nu_j$. Moreover,

$$\begin{aligned}\sum_{i=i_0+1}^{m} d^{m-i}\mu_i &\leqslant \sum_{i=i_0+1}^{m} d^{m-i}(d-1)\\ &= d^{m-i_0} - 1\\ &< d^{m-i_0},\end{aligned}$$

and hence

$$\begin{aligned}\sum_{i=0}^{m} d^{m-i}\mu_i &< \sum_{i=0}^{i_0-1} d^{m-i}\mu_i + d^{m-i_0}\mu_{i_0} + d^{m-i_0}\\ &\leqslant \sum_{i=0}^{i_0-1} d^{m-i}\mu_i + d^{m-i_0}\nu_{i_0}\\ &\leqslant \sum_{i=0}^{m} d^{m-i}\nu_i,\end{aligned}$$

and we have proved our claim. ♦

Let us now suppose that f is preregular in $(\overline{X}_1, \overline{Y}_1)$ of degree $(0, \nu_0, d)$. Then

$$f \equiv \overline{Y}_1^{\nu_0} + \sum_{\nu>\nu_0,\mu} f_{\mu,\nu}\overline{X}_1^{\mu}\overline{Y}_1^{\nu} \mod J + (\overline{Y_2}).$$

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Now,

$$T_d(\overline{Y}_1)^\nu = \left(\prod_{i=0}^{n-1}(Y_{1,i} + Y_{1,n}^{d^{n-i}})^{\nu_i}\right) Y_{1,n}^{\nu_n} \equiv Y_{1,n}^{\sum_{i=0}^{n} d^{n-i}\nu_i} \mod J + (\overline{Y_2}) + (\overline{Y_1}_{\neq n}),$$

and we conclude again by Claim 3.16. ∎

Proposition 3.17 (Bound on the degree of preregularity)**.** *Let*

$$f = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{X}_2, \overline{Y}_2)\overline{X}_1^{\mu}\overline{Y}_1^{\nu} \in \mathcal{A}_{m_1+m_2,n_1+n_2}.$$

There exists d such that, for any (μ, ν) with $|\mu| + |\nu| < d$, there exist $g_{\mu,\nu} \in \mathcal{A}_{m_1+m_3,n_1+n_3}$ preregular in $(\overline{X}_1, \overline{Y}_1)$ of degree (μ, ν, d), and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-terms $\overline{u}_{\mu,\nu}$ and $\overline{s}_{\mu,\nu}$ such that, for all $M \models \mathrm{T}_{\mathcal{A}}$ and every $\overline{a} \in \mathcal{R}(M)$ and $\overline{b} \in \mathfrak{N}(M)$, if $f(\overline{X}_1, \overline{a}, \overline{Y}_1, \overline{b})$ is not the zero function, then there exists (μ_0, ν_0) with $|\mu_0| + |\nu_0| \leqslant d$ and

$$f(\overline{X}_1, \overline{a}, \overline{Y}_1, \overline{b}) = f_{\mu_0,\nu_0}(\overline{a}, \overline{b}) g_{\mu_0,\nu_0}(\overline{X}_1, \overline{u}_{\mu_0,\nu_0}(\overline{a}, \overline{b}), \overline{Y}_1, \overline{s}_{\mu_0,\nu_0}(\overline{a}, \overline{b})).$$

Proof. This follows from the strong Noetherian property [9, Theorem 4.2.15 and Remark 4.2.16] as in [25, Corollary 3.8]. ∎

As $\mathcal{R}$ and $\mathfrak{N}$ are not sorts in $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$, there is no way in this language to have a variable that ranges uniquely over one or the other. Hence we introduce the notion of a well-formed formula that essentially simulates that sorted behavior.

A **K**-quantifier free $\mathcal{L}_{\mathcal{A}}$-formula $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ will be said to be well formed if $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ are **K**-variables and $\overline{R}$ are **RV**-variables, symbols of functions from $\mathcal{A}$ are never applied to anything but variables, and $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ implies that $\bigwedge_i \mathrm{val}^{\mathcal{R}}(X_i) \geqslant 0$, $\bigwedge_i \mathrm{val}^{\mathcal{R}}(Z_i) \geqslant 0$, and $\bigwedge_i \mathrm{val}^{\mathcal{R}}(Y_i) > 0$. The $(\overline{X}, \overline{Y})$-rank of φ is the tuple $(|\overline{X}|, |\overline{Y}|)$. We order ranks lexicographically.

Lemma 3.18. *Let $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ be a well-formed* **K***-quantifier free $\mathcal{L}_{\mathcal{A}}$-formula. Then there exist a finite set of well-formed* **K***-quantifier free $\mathcal{L}_{\mathcal{A}}$-formulas $\varphi_i(\overline{X}_i, \overline{Y}_i, \overline{Z}_i, \overline{R})$ of $(\overline{X}_i, \overline{Y}_i)$-rank strictly smaller than the $(\overline{X}, \overline{Y})$-rank of φ, and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-terms $\overline{u}_i(\overline{Z})$ such that*

$$\mathrm{T}_{\mathcal{A}} \models \forall\overline{Z}\forall\overline{R}\left(\exists\overline{X}\exists\overline{Y}\varphi \iff \bigvee_i \exists\overline{X}_i\exists\overline{Y}_i\varphi_i(\overline{X}_i, \overline{Y}_i, \overline{u}_i(\overline{Z}), \overline{R})\right).$$

Proof. Let us define $m := |X|$ and $n := |Y|$. As polynomials with variables in $\mathcal{R}$ are in fact elements of $\mathcal{A}$, and $\mathcal{A}$ is closed under composition (for the $\mathcal{R}$-variables), we may assumes that any $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}$-term appearing in φ is an element of $\mathcal{A}$. Let $f_i(\overline{X}, \overline{Y}, \overline{Z})$ be the $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}$-terms appearing in φ. Splitting φ into different cases, we may assume that whenever a variable S appears as an $\mathfrak{N}$-variable of an f_i then φ implies that $\mathrm{val}^{\mathcal{R}}(S) > 0$ (in the part of the disjunction where $\mathrm{val}^{\mathcal{R}}(S) \leqslant 0$, we replace this f_i by zero).

If an X_i appears as an $\mathfrak{N}$-variable in an f_i, then φ implies that $\mathrm{val}^{\mathcal{R}}(X_i) > 0$, and hence we can safely rename this X_i as Y_n, and we obtain an equivalent formula of lower rank.

If Y_i appears as an $\mathcal{R}$-variable in an f_i, we can change this f_i so that Y_i appears as an $\mathfrak{N}$-variable. Thus we may assume that the X_i only appear as $\mathcal{R}$-variables and the Y_i as $\mathfrak{N}$-variables. Similarly adding new Z_j variables, we may assume that each Z_j appears only once (and in the end we can put the old variables back in), and that φ implies that $\mathrm{val}^{\mathcal{R}}(Z_j) > 0$ if it is an $\mathfrak{N}$-variable.

Applying Proposition 3.17 to each of the $f_i(\overline{X}, \overline{Y}, \overline{Z}) = \sum_{\mu,\nu} f_{\mu,\nu}(\overline{Z})\overline{X}^{\nu}\overline{Y}^{\mu}$, we find d, $g_{i,\mu,\nu}$, and $u_{i,\mu,\nu}(\overline{Z})$ such that $g_{i,\mu,\nu}$ is preregular in $(\overline{X}, \overline{Y})$ of degree (μ, ν, d) and, for every $M \models \mathrm{T}_{\mathcal{A}}$ and $\overline{a} \in M$, if $f_i(\overline{X}, \overline{Y}, \overline{a})$ is not the zero function, then there exists (μ, ν) such that $|\mu| + |\nu| < d$ and $f_i(\overline{X}, \overline{Y}, \overline{a}) = f_{i,\mu,\nu}(\overline{a})g_{i,\mu,\nu}(\overline{X}, \overline{Y}, \overline{u}_{i,\mu,\nu}(\overline{a}))$. Splitting the formula into the different cases, we may assume that, for each i, there are μ_i and ν_i such that $f_i(\overline{X}, \overline{Y}, \overline{a}) = f_{i,\mu_i,\nu_i}(\overline{a})g_{i,\mu_i,\nu_i}(\overline{X}, \overline{Y}, \overline{u}_i(\overline{a}))$ (in the case where no such μ_i and ν_i exist, we can replace f_i by 0). Let us now introduce a new variable T_j to replace any argument of a $g_{i,\mu,\nu}$ that is not in $\overline{X}$ or $\overline{Y}$; and, for each of these new T_j, we add to the formula $\mathrm{val}^{\mathcal{R}}(T_j) \geqslant 0$ if T_j is an $\mathcal{R}$-argument of $g_{i,\nu,\mu}$ or $\mathrm{val}^{\mathcal{R}}(T_j) > 0$ if it is an $\mathfrak{N}$-argument. Let us write $g_{i,\mu_i,\nu_i} = \sum_{\nu} g_{i,\nu}\overline{Y}^{\nu}$. Note that g_{i,ν_i} is preregular in $\overline{X}$ of degree $(\mu_i, 0, d)$. We can split the formula some more (and still call it φ) so that, for each i, one of the two conditions $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) > 0$ or $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) = 0$ holds.

If a condition $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) > 0$ occurs, let us add $\mathrm{val}^{\mathcal{R}}(Y_n) > 0 \wedge g_{i,\nu_i} - Y_n = 0$ to the formula. By Proposition 3.15, after a Weierstrass change of variable on the $\overline{X}$, we may assume that $g_{i,\nu_i} - Y_n$ is regular in X_{m-1}. By Weierstrass division, we can replace every f_j by a term polynomial in X_{m-1}, and by Weierstrass preparation we can replace the equality $g_{i,\nu_i} - Y_n = 0$ by the equality of a term polynomial in X_{m-1} to 0. In the resulting formula, no $f \in \mathcal{A}$ is ever applied to a term containing X_{m-1}, and we can apply Remark 3.6 to the formula where every $f \in \mathcal{A}$ is replaced by a new variable S_f to obtain a **K**-quantifier free formula $\psi(\overline{X}_{\neq m-1}, \overline{Y}, \overline{Z}, \overline{T}, \overline{S}, \overline{R})$ such that

$$\mathrm{T}_{\mathcal{A}} \models \exists X_{m-1}\varphi \iff \psi(\overline{X}_{\neq m-1}, \overline{Y}, \overline{Z}, \overline{u}(\overline{Z}), \overline{f}(\overline{X}_{\neq m-1}, \overline{Y}, \overline{Z}), \overline{R})$$

and $\psi(\overline{X}_{\neq m-1}, \overline{Y}, \overline{Z}, \overline{T}, \overline{f}(\overline{X}_{\neq m-1}, \overline{Y}, \overline{Z}), \overline{R})$ is well-formed of $(\overline{X}, \overline{Y})$-rank $(m-1, n+1)$.

If for all i we have $\mathrm{val}^{\mathcal{R}}(g_{i,\nu_i}) = 0$, we add $\mathrm{val}^{\mathcal{R}}(X_m) \geqslant 0 \wedge X_m \prod_i g_{i,\nu_i} - 1 = 0$ to the formula. As every g_{i,ν_i} is preregular in $\overline{X}$ of degree $(\mu_i, 0, d)$, $g = X_m \prod_i g_{i,\nu_i} - 1$ is preregular in $\overline{X}$ of degree $(\mu, 0, d')$ for some μ and d'. After a Weierstrass change of variables in $\overline{X}$, we may assume that g and each g_{i,ν_i} are in fact regular in X_m. Hence by Weierstrass preparation we may replace g in $g = 0$ by a term polynomial in X_m. Furthermore, by Remark 1.5 the f_i appear as $\mathrm{rv}_{n_i}(f_i)$ for some n_i in the formula. Replacing f_i by $f_{\mu_i,\nu_i}g_{i,\mu_i,\nu_i}$, we only have to show that $\mathrm{rv}_{n_i}(g_{i,\mu_i,\nu_i})$ can be replaced by a term polynomial in Y_{n-1} (and X_m). Let $h_i = X_M(\prod_{j\neq i} g_{j,\nu_j})g_{i,\nu_i,\mu_i} = \sum_{\nu} h_{i,\nu}Y^{\nu}$. Then $h_{i,\nu_i} = X_M \prod_i g_{i,\nu_i} = 1$, and, if $\nu < \nu_i$, $h_{i,\nu} = X_M(\prod_{j\neq i} g_{j,\nu_j})g_{i,\nu} \equiv 0 \mod J + (Z_j : Z_j$ is an $\mathfrak{N}$-argument). Hence h_i is preregular in $(\overline{X}, \overline{Y})$ of degree $(0, \nu_i, d)$. After a Weierstrass change of variables of the $\overline{Y}$, we may assume that h_i is in fact regular in Y_{n-1}.

Note that $\mathrm{rv}_{n_i}(g_{i,\nu_i,\mu_i}) = \mathrm{rv}_{n_i}(X_m)^{-1} \prod_{j\neq i} \mathrm{rv}_{n_i}(g_{i,\nu_i})^{-1}\mathrm{rv}_{n_i}(h_i)$. By Weierstrass preparation we can replace h_i by the product of a unit and p_i a polynomial in Y_{n-1}. As we have included the trace of units on the $\mathbf{RV}_n$ in our language, the unit is taken care of, and, by Weierstrass division by g, we can replace each coefficients in the p_i and each of the g_{i,ν_i} by a term polynomial in X_m. Note that, because we allow quantification on **RV**,

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although the language does not contain the inverse on **RV**, the inverses can be taken care of by quantifying over **RV**. Hence we obtain a formula where X_m and Y_{n-1} only occur polynomially, and we can proceed as in the previous case to eliminate them. ∎

Corollary 3.19. *Let $\varphi(\overline{X}, \overline{Y}, \overline{Z}, \overline{R})$ be a well-formed* **K***-quantifier free $\mathcal{L}_{\mathcal{A}}$-formula. Then there exists an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula $\psi(\overline{Z}, \overline{R})$ such that $\mathrm{T}_{\mathcal{A}} \models \exists\overline{X}\exists\overline{Y}\varphi \iff \psi$.*

Proof. This follows from Lemma 3.18 and an immediate induction. ∎

Proof (Theorem 3.10). Resplendence comes for free (see Proposition A.9). Hence, it suffices to show that, if $\varphi(X, \overline{Z})$ is a quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula, then there exists a quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula $\psi(\overline{Z})$ such that $\mathrm{T}_{\mathcal{A}} \models \exists X\varphi \iff \psi$. First, splitting the formula φ, we can assume that, for any of its variables S, φ implies either $\mathrm{val}^{\mathcal{R}}(S) \geqslant 0$ or $\mathrm{val}^{\mathcal{R}}(S) < 0$; in the second case, replacing S by S^{-1}, we also have $\mathrm{val}^{\mathcal{R}}(S) > 0$. We also add one variable X_i (respectively, Y_i) per $\mathcal{R}$-argument (respectively, $\mathfrak{N}$-argument) of any $f \in \mathcal{A}$ applied to some non-variable term u, and we add the corresponding equality $X_i = u$ (respectively, $Y_i = u$) and the corresponding inequalities $\mathrm{val}^{\mathcal{R}}(X_i) \geqslant 0$ (respectively, $\mathrm{val}^{\mathcal{R}}(Y_i) > 0$) and quantify existentially over this variable. Splitting the formula further (whether denominators in occurrences of $\mathcal{Q}$ are zero or not) we can transform φ such that it contains no $\mathcal{Q}$. Now $\exists X\varphi$ is equivalent to a disjunction of formulas $\exists\overline{X}\exists\overline{Y}\psi$, where ψ is well formed, and we conclude by applying Corollary 3.19.

This concludes the proof of Theorem 3.10. ∎

Recall that we denote by $\mathcal{SC}^{\mathcal{R}}(C)$ the set of all quantifier free $\mathcal{L}_{\mathrm{div}}(C)$-definable sets.

Definition 3.20 (Strong unit)**.** Let $M \models \mathrm{T}_{\mathcal{A}}$, $C = \mathbf{K}(\langle C\rangle)$, and $S \in \mathcal{SC}^{\mathcal{R}}(C)$. We say that an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term $E : \mathbf{K} \to \mathbf{K}$ is a strong unit on S if, for any open $\mathcal{O}$-ball $b := \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c) \subseteq S$, there exist $\overline{a}$, $e \in C\langle cd\rangle$, and $F(t, \overline{z}) \in \mathcal{A}$ such that $e \neq 0$, and, for all $x \in b$,

$$\mathrm{val}(F((x-c)/d, \overline{a})) = 0$$

and

$$E(x) = eF((x-c)/d, \overline{a}).$$

Note that if M is taken saturated saturated (i.e., at least $(|\mathcal{A}| + |C|)^{+}$-saturated) and if E is a strong unit on S, then, by compactness, there exist a tuple $\overline{a}(y, z)$ of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms, a finite number of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms $e_i(y, z)$, and $F_i[t, \overline{u}] \in \mathcal{A}$ such that, for all balls $b = \mathring{\mathcal{B}}_{\mathrm{val}(d)}(c) \subseteq S$, there is an i such that, for all $x \in b$,

$$E(x) = e_i(c, d)F_i((x-c)/d, \overline{a}(c, d))$$

and

$$F_i((x-c)/d, \overline{a}(c, d)) \in \mathcal{O}^{\star}.$$

Hence, if E is a strong unit on S, there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C)$-formula that witnesses it. If E and S are defined using some parameters $\overline{y}$ and, for all $\overline{y}$ in some definable set Y, $E = E_{\overline{y}}$ is a strong unit on $S = S_{\overline{y}}$, then we can choose this formula uniformly in $\overline{y}$.

We will say that E is an $\mathcal{R}$-strong unit on S if it verifies all the requirements of a strong unit, where all references to $\mathcal{O}$ are replaced by references to $\mathcal{R}$ (and references to $\mathcal{R}$ remain the same).

Proposition 3.21. *If E is an $\mathcal{R}$-strong unit on S, then it is also a strong unit on S.*

Proof. If $b \subseteq S$ is an $\mathcal{O}$-ball, then by Proposition 2.3 there exist d and c such that $b = \mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c) \subseteq \mathring{\mathcal{B}}^{\mathcal{R}}_{\mathrm{val}^{\mathcal{R}}(d)}(c) \subseteq S$. But E being a strong unit on S for $\mathcal{R}$, it has the expected form on $\mathring{\mathcal{B}}^{\mathcal{R}}_{\mathrm{val}^{\mathcal{R}}(d)}(c)$, and hence also on $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(d)}(c)$. ■

Definition 3.22 (Weierstrass preparation for terms)**.** Let M be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure, $C = \mathbf{K}(\langle C\rangle) \subseteq M$, $t : \mathbf{K} \to \mathbf{K}$ an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term, and $S \in \mathcal{SC}^{\mathcal{R}}(C)$. We say that t has a Weierstrass preparation on S if there exist an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term E that is a strong unit on S and a rational function $R \in C(X)$ with no poles in $S(\overline{\mathbf{K}(M)}^{\mathrm{alg}})$ such that, for all $x \in S$, $t(x) = E(x)R(x)$.

The structure M has a Weierstrass preparation if for any $C = \mathbf{K}(\langle C\rangle)$ and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms t and $u : \mathcal{R} \to K$ we have the following:

(i) there exists a finite number of $S_i \in \mathcal{SC}^{\mathcal{R}}(C)$ that cover $\mathcal{R}$ such that t has a Weierstrass preparation on each of the S_i;

(ii) if t and u have a Weierstrass preparation on some open ball b, and, for all $x \in b$, $\mathrm{val}(t(x)) \geqslant \mathrm{val}(u(x))$, then $t+u$ also has a Weierstrass preparation on b.

Remark 3.23.

1. An immediate consequence of Weierstrass preparation is that all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-terms in one variable have only finitely many isolated zeros. Indeed a zero of t is the zero of one of the R_i appearing in its Weierstrass preparation. That zero is isolated if R_i is non-zero or the corresponding S_i is discrete, i.e., is a finite set. In particular, let $\overline{m}$ be the parameters of t. Then any isolated zero of t is in the algebraic closure (in ACVF) of $\mathbf{K}(\langle\overline{m}\rangle)$. As the algebraic closure in ACVF coincides with the field-theoretic algebraic closure, any isolated zero of t is in fact also the zero of a polynomial (with coefficients in $\mathbf{K}(\langle\overline{m}\rangle)$).

2. As for strong units, for each choice of term $t_{\overline{y}}$ (with parameters $\overline{y}$), there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(\overline{y})$-formula that states that (i) holds for $t_{\overline{y}}$ in M, and we can choose this formula to be uniform in $\overline{y}$. For each choice of terms t, u, and formula defining S, there also is a (uniform) formula saying that (ii) holds for t, u, and b in M.

Proposition 3.24. *Any $M \models \mathrm{T}_{\mathcal{A}}$ has a Weierstrass preparation.*

Proof. If $\mathcal{R} = \mathcal{O}$, then the proposition is shown in [9, Theorem 5.5.3] and (ii) (called from now on invariance under addition) is clear from the proof given there. The one difference in the Weierstrass preparation is that, in [9], there is a finite set of points algebraic over the parameters where the behavior of the term is unknown. But this finite set can be replaced by discrete S_i, and as these exceptional points are common zeros of terms u and v such that $Q(u, v)$ is a subterm of t, it suffices to replace $Q(u, v)$ by 0 and apply the theorem to the new term to obtain the Weierstrass preparation also on the discrete S_i. The fact that the strong units in [9] have the proper form on open balls follows, for example, from the proof of [9, Lemma 6.3.12].

If $\mathcal{R} \neq \mathcal{O}$, the proposition follows from the $\mathcal{O} = \mathcal{R}$ case and Proposition 3.21. ■

Remark 3.25.

1. Let $t_{\overline{y}}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-term with parameters $\overline{y}$. As shown in Remark 3.23.2, there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula θ that states that Weierstrass preparation holds for $t_{\overline{y}}$ in models of T. More explicitly, there are finitely many choices of S_i^k, E_i^k, and R_i^k (with parameters $\overline{u}(\overline{y})$, where $\overline{u}$ are $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-terms) such that for each $\overline{y}$ there is a k such that the S_i^k, E_i^k, and R_i^k work for $t_{\overline{y}}$. As $\mathrm{T}_{\mathcal{A}}$ eliminates $\mathbf{K}$-quantifiers, for each k there is a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ formula $\theta_k(\overline{y})$ that is true when the kth choice works for t (and not the ones before). Hence, taking $S_{i,k}$ to be $S_i^k \wedge \theta_k$, we could suppose that Weierstrass preparation for terms is uniform, but we will not be using that fact.
2. Conversely, the proof of Proposition 5.3 can be adapted to show that uniform Weierstrass preparation for terms implies $\mathbf{K}$-quantifier elimination. This is exactly the proof of quantifier elimination given in [9], although its authors did not see at the time that they were relying on a more uniform version of Weierstrass preparation for terms than what they had actually shown. Hence it would be interesting to know if one could prove uniform Weierstrass preparation for terms without using $\mathbf{K}$-quantifier elimination to recover their proof (see [10] for more on this subject).

Proposition 3.26. *Let $M \models \mathrm{T}_{\mathcal{A}}$. Then the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-structure of M can be extended (uniquely) to any algebraic extension of $\mathbf{K}(M)$, so that it remains a model of $\mathrm{T}_{\mathcal{A}}$. Moreover, if $C \leqslant M$ and $a \in \mathbf{K}(M)$ is algebraic over $\mathbf{K}(C)$, then $\mathbf{K}(C\langle a\rangle) = \mathbf{K}(C)[a]$.*

Proof. The case when $\mathcal{R} = \mathcal{O}$ is proved in [11, Theorem 2.18]. The same proof applies when $\mathcal{R} \neq \mathcal{O}$. ■

To conclude this section, let us show that under certain circumstances analytic terms have a linear behavior.

Proposition 3.27. *Let $M \models \mathrm{T}_{\mathcal{A}}$, and suppose that $\mathbf{K}(M)$ is algebraically closed. Let $t : \mathbf{K} \to \mathbf{K}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(M)$-term and b be an open ball in M with radius $\xi \neq \infty$. Suppose that t has a Weierstrass preparation on b (and hence t is differentiable at any $a \in b$) and $\mathrm{rv}(dt_x)$ is constant on b. Also assume that $\mathrm{val}(t(x))$ is constant on b or $t(x)$ is polynomial. Then, for all a, $e \in b$, $\mathrm{rv}(t(a) - t(e)) = \mathrm{rv}(dt_a) \cdot \mathrm{rv}(a - e)$.*

Moreover, if $v(t(x))$ is constant on b, then $\mathrm{val}(t(a)) \leqslant \mathrm{val}(dt_a) + \xi$.

Proof. If $\mathrm{val}(t(x))$ is constant on b, then $\mathrm{val}(t(x)) - \mathrm{val}(t(a)) \geqslant 0$, and, by invariance under addition, $t(x) - t(a)$ has a Weierstrass preparation on b. If $t(x)$ is polynomial this is also clear. Hence there are $F_a \in \mathcal{A}$ (with other parameters in $\mathbf{K}(M)$), P_a, $Q_a \in \mathbf{K}(M)[X]$, such that, for all $x \in b$,

$$t(x) - t(a) = F_a\left(\frac{x-a}{g}\right)\frac{P_a(x)}{Q_a(x)},$$

where $\mathrm{val}(F_a(y)) = 0$ for all $y \in \mathfrak{M}$ and $\mathrm{val}(g) = \xi$. If t is constant on b, i.e., $P_a = 0$, then the proposition follows easily. If not, P_a has only finitely many zeros. Let a_i be the zeros of P_a in $\mathbf{K}(M)$ (recall that M is assumed algebraically closed), and let m_i be the multiplicity of a_i. Let c_j be the zeros of Q_a and n_j be their multiplicities. Note that every zero of $Q_a(x)$ is outside b; hence, for all j, $\mathrm{val}(c_j - a) \leqslant \xi$. For all $e \in b$, note that $t(x) - t(a)$ is also differentiable at e with differential dt_e, and hence, if e is distinct from all a_i, then

$$\begin{aligned} \mathrm{rv}\left(\frac{dt_a}{t(e)-t(a)}\right) &= \mathrm{rv}\left(\frac{dt_e}{t(e)-t(a)}\right) \\ &= \mathrm{rv}\left(\frac{\partial\left(F_a\left(\frac{x-a}{g}\right)\right)/\partial x_x(e)}{F_a\left(\frac{e-a}{g}\right)} + \frac{d(P_a)_e}{P_a(e)} + \frac{d(Q_a)_e}{Q_a(e)}\right) \\ &= \mathrm{rv}\left(\frac{d(F_a)_{\frac{e-a}{g}}}{gF_a\left(\frac{e-a}{g}\right)} + \sum_i \frac{m_i}{e-a_i} + \sum_j \frac{n_j}{e-c_j}\right). \end{aligned}$$

For any $y \in \mathfrak{M}$, $\mathrm{val}(d(F_a)_y) \geqslant 0 = \mathrm{val}(F_a(y))$; hence $\mathrm{val}(d(F_a)_y/(gF_a(y))) \geqslant -\mathrm{val}(g) > -\mathrm{val}(e-a)$. We also have that, for all j, $\mathrm{val}(1/(e-c_j)) = -\mathrm{val}(e-c_j) > -\mathrm{val}(e-a)$. Finally, suppose that there is a unique a_{i_0} such that $\mathrm{val}(e-a_{i_0})$ is maximal. Then, for all $i \neq i_0$, $\mathrm{val}(1/(e-a_i)) > \mathrm{val}(1/(e-a_{i_0}))$, and hence $\mathrm{rv}(m_{i_0})\mathrm{rv}(e-a_{i_0})^{-1} = \mathrm{rv}(dt_a)\mathrm{rv}(t(e)-t(a))^{-1}$; i.e., $\mathrm{rv}(t(e)-t(a)) = \mathrm{rv}(dt_a m_{i_0}^{-1}(e-a_{i_1}))$.

As $t(e) \neq t(a)$, this immediately implies that $dt_a \neq 0$. Let us now show that if $a_i \in b$ it cannot be a multiple zero. If it were,

$$dt_{a_i} = d(F_a((x-a)/c)/Q_a(x))_{a_i} P_a(a_i) + P_a'(a_i)F_a((a_i-a)/c)/Q_a(a_i) = 0,$$

which is absurd. Hence, for all $a_i \in b$, $m_i = 1$, and if we could show that there is a unique $a_i \in b$ (namely a itself) we would be done.

Suppose there is more that one a_i in b, and let $\gamma := \min\{\mathrm{val}(a_i - a_j) : a_i, a_j \in b \wedge i \neq j\}$. We may assume that $\mathrm{val}(a_0 - a_1) = \gamma$. Let us also assume that the a_i have been numbered so that there exists i_0 such that, for all $i \leqslant i_0$, $\mathrm{val}(a_i - a_0) = \gamma$, and, for all $i > i_0$, $\mathrm{val}(a_i - a_0) < \gamma$. In particular, for all $i \neq j \leqslant i_0$, $\mathrm{val}(a_i - a_j) = \gamma$. For each $i \leqslant i_0$, let e_i be such that $\mathrm{val}(e_i - a_i) > \gamma$. Then we can apply the previous computation to e_i, and we get that $\mathrm{rv}(t(e_i) - t(a)) = \mathrm{rv}(dt_a)\mathrm{rv}(e_i - a_i)$. But

$$\mathrm{rv}(t(e_i)-t(a)) = \mathrm{rv}\left(F_a\left(\frac{e_i - x_{\alpha_0+1}}{g}\right)\right)\mathrm{rv}(p)\prod_k(\mathrm{rv}(e_i-a_k))^{m_k}\mathrm{rv}(q)^{-1}\prod_j(\mathrm{rv}(e_i-c_j))^{-n_j},$$

where p and q are the dominant coefficients of, respectively, P_a and Q_a, and hence

$$\mathrm{rv}(dt_a) = \mathrm{rv}\left(F_a\left(\frac{e_i - x_{\alpha_0+1}}{g}\right)\right)\mathrm{rv}(p)\prod_{k\neq i}(\mathrm{rv}(e_i-a_k))^{m_k}\mathrm{rv}(q)^{-1}\prod_j(\mathrm{rv}(e_i-c_j))^{-n_j}.$$

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As $\mathrm{rv}(F_a((e_i - x_{\alpha_0+1})/g))$, $\mathrm{rv}(e_i - a_k)$ for all $k > i_0$ and $\mathrm{rv}(e_i - c_j)$ do not depend on i, and, for all $k \leqslant i_0$, $k \neq i$, $\mathrm{rv}(e_i - a_k) = \mathrm{rv}(a_i - a_k)$, we obtain that, for all $i, j \leqslant i_0$,

$$\prod_{i \neq k \leqslant i_0} \mathrm{rv}(a_i - a_k) = \prod_{j \neq k \leqslant i_0} \mathrm{rv}(a_j - a_k).$$

Replacing a_i by $(a_i - a_0)/g$, where $\mathrm{val}(g) = \gamma$, we obtain the same equalities, but we may assume that, for all $i \leqslant i_0$, $a_i \in \mathcal{O}$, and for all $i \neq j$, $a_i - a_j \in \mathcal{O}^\star$. The equations can now be rewritten as $\prod_{i \neq k} \mathrm{res}(a_i - a_k) = \prod_{i \neq k} (\mathrm{res}(a_i) - \mathrm{res}(a_k)) = c$ for some $c \in \mathbf{R}(M)$. Let $P = \prod_k (X - \mathrm{res}(a_k))$. Then our equations state that $P'(\mathrm{res}(a_i)) - c = 0$ for all $i \leqslant i_0$. But $P' - c$ is a degree i_0 polynomial, and it cannot have $i_0 + 1$ roots.

Finally, if $\mathrm{val}(t(x))$ is constant on b, then, for all a and $e \in b$, $\mathrm{val}(t(a)) \leqslant \mathrm{val}(t(a) - t(e)) = \mathrm{val}(dt_a) + \mathrm{val}(a - e)$. As this holds for any e, we must have $\mathrm{val}(t(a)) \leqslant \mathrm{val}(dt_a) + \zeta$. ∎

Remark 3.28. The conclusion of Proposition 3.27 seems very close to the Jacobian property (see, for example, [9, Definition 6.3.5]). In fact, this lemma is very similar (both in its hypothesis and its conclusion) to [9, Lemma 6.3.9].

4. σ-Henselian fields

Definition 4.1 (Analytic field with an automorphism)**.** Let us suppose that each $\mathcal{A}_{m,n}$ is given with an automorphism of the inductive system $t \mapsto t^\sigma : \mathcal{A}_{m,n} \to \mathcal{A}_{m,n}$. An analytic field M with an automorphism is a model of $\mathrm{T}_\mathcal{A}$ with a distinguished $\mathcal{L}^{\mathbf{RV}} \cup \{|^{\mathcal{R}}_1\}$-automorphism σ such that, for symbols $t \in \mathcal{A}_{m,n}$ and $\overline{x} \in \mathbf{K}(M)^{m+n}$, $\sigma(t(\overline{x})) = t^\sigma(\sigma(\overline{x}))$.

Let $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma} := \mathcal{L}_{\mathcal{A},\mathcal{Q}} \cup \{\sigma\} \cup \{\sigma_n : n \in \mathbb{N}\}$. An analytic field M with an automorphism τ can be made into an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-structure by interpreting σ as $\tau|_\mathbf{K}$ and σ_n as $\tau|_{\mathbf{RV}_n}$. Note that σ also induces a ring automorphism on every $\mathbf{R}_n$ and an ordered group automorphism $\sigma_\mathbf{\Gamma}$ on $\mathbf{\Gamma}$. We will write $\mathrm{T}_{\mathcal{A},\sigma}$ for the $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of analytic fields with an automorphism. We will most often write σ instead of σ_n as there should not be any confusion.

If K is a field with an automorphism σ (also referred to as a difference field in this text), we will write $\mathrm{Fix}(K) := \{x \in K : \sigma(x) = x\}$ for its fixed field. For all $x \in K$, we will write $\overline{\sigma}(x)$ for the tuple $x, \sigma(x), \dots, \sigma^n(x)$, where the n should be explicit from the context.

Remark 4.2. In fact σ induces an action on all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}\big|_\mathbf{K}$-terms, and we have $\mathrm{T}_{\mathcal{A},\sigma} \models \sigma(t(\overline{x})) = t^\sigma(\sigma(\overline{x}))$. It follows immediately that for any $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}\big|_\mathbf{K}$-term t there is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}\big|_\mathbf{K}$-term u such that $\mathrm{T}_{\mathcal{A},\sigma} \models t(\overline{x}) = u(\overline{\sigma}(\overline{x}))$.

Definition 4.3 (Linearly closed difference field)**.** A difference field (K, σ) is linearly closed if every equation of the form $\sum_{i=0}^n a_i \sigma^i(x) = b$, where $a_n \neq 0$, has a solution.

Definition 4.4 (Linear approximation)**.** Let K be a valued field with an automorphism σ, $f : K^n \to K^n$ a (partial) function, and $\overline{d} \in K^n$.

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(i) Let $\overline{b}$ be a tuple of open balls in M. We say that $\overline{d}$ linearly approximates f on $\overline{b}$ if for all $\overline{a}$ and $\overline{c} \in \overline{b}$ we have

$$\mathrm{val}(f(\overline{c}) - f(\overline{a}) - \overline{d} \cdot (\overline{c} - \overline{a})) > \min_i \{\mathrm{val}(d_i) + \mathrm{val}(c_i - a_i)\}.$$

(ii) Let b be an open ball of M. We say that $\overline{d}$ linearly approximates f at prolongations on b if for all $a, c \in b$ we have

$$\mathrm{val}(f(\overline{\sigma}(c)) - f(\overline{\sigma}(a)) - \overline{d} \cdot \overline{\sigma}(c - a)) > \min_i \{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(c - a))\}.$$

Remark 4.5.

1. Let $M \models \mathrm{T}_{\mathcal{A},\sigma}$. Suppose that σ is an isometry; i.e., $\sigma_{\mathbf{\Gamma}} = \mathrm{id}$. Let t be an $\mathcal{L}_{\mathcal{A}}|_{\mathbf{K}}(\mathcal{O}(M))$-term and $a \in \mathcal{O}(M)$. We can show that $dt_{\overline{a}}$ linearly approximates t on $\mathring{\mathcal{B}}_{\gamma}(a)$, where $\gamma = \min_i\{\mathrm{val}(\partial f / \partial x_i(\overline{a}))\}$.
2. We allow a slight abuse of notation by saying that terms constant on a ball are linearly approximated (at prolongations) by the zero tuple, even though the required inequality does not hold as $\infty \not> \infty$.

Let us first prove that it suffices to show linear approximation variable by variable to obtain linear approximation for the whole function. We will write $(\overline{a}_{\neq i}, x_i)$ for the tuple $\overline{a}$ where the ith component is replaced by x_i (with a slight abuse of notation as the x_i does not appear in the right place) and $\overline{a}^{\leqslant i}$ for the tuple $\overline{a}$ where the jth components for $j > i$ are replaced by zeros.

Proposition 4.6. *Let (K, val) be a valued field, $f : K^n \to K$, $\overline{d} \in K^n$, and $\overline{b}$ a tuple of balls. If, for all $\overline{a} \in \overline{b}$ and $j < n$, d_j linearly approximates $f(\overline{a}_{\neq j}, x_j)$ on b_j, then $\overline{d}$ linearly approximates f on $\overline{b}$.*

Proof. Let $\overline{a}$ and $\overline{e} \in \overline{b}$, and $\overline{\varepsilon} = \overline{e} - \overline{a}$. Then, we have

$$\begin{aligned}\mathrm{val}(f(\overline{a} + \overline{\varepsilon}) - f(\overline{a}) - \overline{d} \cdot \overline{\varepsilon}) &= \mathrm{val}\left(\sum_j f(\overline{a} + \overline{\varepsilon}^{\leqslant j}) - f(\overline{a} + \overline{\varepsilon}^{\leqslant j-1}) - d_j \varepsilon_j\right) \\ &\geqslant \min_j \{f(\overline{a} + \overline{\varepsilon}^{\leqslant j}) - f(\overline{a} + \overline{\varepsilon}^{\leqslant j-1}) - d_j \varepsilon_j\} \\ &> \min_j \{\mathrm{val}(d_j) + \mathrm{val}(\varepsilon_j)\}.\end{aligned}$$

And that concludes the proof. ■

Although linear approximation (at prolongations) looks like differentiability, one must be aware that linear approximations are not uniquely determined, because, among other reasons, we are only looking at tuples that are prolongations but also because the error term is only linear. But when σ is an isometry, we can recover some uniqueness, and give an alternative definition (perhaps of a more geometric flavor) of linear approximation at prolongations.

Definition 4.7 $(\mathbf{R}_{1,\gamma})$**.** Let (K, val) be a valued field, and let $\gamma \in \mathrm{val}(\mathbf{K}^{\star})$. We define $\mathbf{R}_{1,\gamma} := \overline{\mathcal{B}}_{\gamma}(0) / \mathring{\mathcal{B}}_{\gamma}(0)$ and let $\mathrm{res}_{1,\gamma}$ denote the canonical projection $\overline{\mathcal{B}}_{\gamma}(0) \to \mathbf{R}_{1,\gamma}$. Note that $\mathbf{R}_{1,\gamma}$ can be identified (canonically) with $\mathrm{val_1}^{-1}(\gamma) \cup \{0\} \subseteq \mathbf{RV}_1$.

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Proposition 4.8. *Let (K, val) be a valued field with an isometry σ and a linearly closed residue field. Let $f : K^n \to K$, $\overline{d}$ be a linear approximation of f at prolongations on some open ball b with radius ξ, $\overline{e} \in K^n$, $\delta := \mathrm{val}(\overline{d})$, and $\eta := \mathrm{val}(\overline{e})$. The following are equivalent.*

(i) *$\overline{e}$ is a linear approximation of f at prolongations on b;*

(ii) $\mathrm{val}(\overline{d} - \overline{e}) > \min\{\delta, \eta\}$*;*

(iii) *$\eta = \delta$ and $\mathrm{res}_{1,\delta}(\overline{d}) = \mathrm{res}_{1,\delta}(\overline{e})$.*

Proof. (i)⇒(ii) Suppose that $\overline{d} \neq \overline{e}$. Let ε be such that $\mathrm{val}(\varepsilon) > \xi$, and let $g \in K$ be such that $\mathrm{val}(g) = \mathrm{val}(\overline{d} - \overline{e})$. Then $P(\overline{\sigma}(x)) := \sum_i (d_i - e_i)\sigma^i(\varepsilon)g^{-1}\varepsilon^{-1}\sigma^i(x)$ is a linear difference polynomial with a non-zero residue. As K is residually linearly closed, the residue of P cannot always be zero, and hence there exists $c \in \mathcal{O}^\star$ such that $\mathrm{val}(P(\overline{\sigma}(c))) = 0$; i.e., $\mathrm{val}((\overline{d} - \overline{e}) \cdot \overline{\sigma}(\varepsilon c)) = \mathrm{val}(g) + \mathrm{val}(\varepsilon)$. But then, for all $a \in b$,

$$\begin{aligned} \mathrm{val}(g) + \mathrm{val}(\varepsilon) &= \mathrm{val}((\overline{d} - \overline{e}) \cdot \overline{\sigma}(\varepsilon c)) \\ &= \mathrm{val}(f(\overline{\sigma}(a + \varepsilon c)) - f(\overline{\sigma}(a)) - \overline{e} \cdot \overline{\sigma}(\varepsilon b)) \\ &\quad - f(\overline{\sigma}(a + \varepsilon c)) + f(\overline{\sigma}(a)) + \overline{d} \cdot \overline{\sigma}(\varepsilon b) \\ &> \mathrm{val}(\varepsilon) + \min\{\delta, \eta\}; \end{aligned}$$

i.e., $\mathrm{val}(\overline{d} - \overline{e}) > \min\{\delta, \eta\}$.

(ii)⇒(iii) First, suppose that $\delta < \eta$. Then, if $\mathrm{val}(d_i)$ is minimal, $\mathrm{val}(d_i) = \delta < \eta \leqslant \mathrm{val}(e_i)$, and hence $\mathrm{val}(d_i - e_i) = \mathrm{val}(d_i) = \delta = \min\{\delta, \eta\}$, contradicting our previous inequality. Hence we must have, by symmetry, $\delta = \eta$. Now inequality (ii) can be rewritten $\mathrm{val}(\overline{d} - \overline{e}) > \delta$, which exactly means that $\mathrm{res}_{1,\delta}(\overline{d}) = \mathrm{res}_{1,\delta}(\overline{e})$.

(iii)⇒(i) For all ε such that $\mathrm{val}(\varepsilon) > \xi$, as $\mathrm{val}(\overline{d} - \overline{e}) > \delta$, we have

$$\begin{aligned} &\mathrm{val}(f(\overline{\sigma}(a + \varepsilon)) - f(\overline{\sigma}(a)) - \overline{e} \cdot \overline{\sigma}(\varepsilon)) \\ &\quad = \mathrm{val}(f(\overline{\sigma}(a + \varepsilon)) - f(\overline{\sigma}(a)) - \overline{d} \cdot \overline{\sigma}(\varepsilon) + (\overline{d} - \overline{e}) \cdot \overline{\sigma}(\varepsilon)) \\ &\quad > \delta + \mathrm{val}(\varepsilon) \\ &\quad = \eta + \mathrm{val}(\varepsilon). \end{aligned}$$

This concludes the proof. ■

Remark 4.9.

1. In the isometry case, linear approximations describe the trace of a given function on $\mathbf{RV}_1$. More precisely, a function f is linearly approximated at prolongations on some open ball b with radius ξ if and only if there exist $\delta \in \mathrm{val}(K)$ and $\overline{d} \in \mathbf{R}_{1,\delta}(K)$ such that, for all $\gamma > \xi$ and $a \in b$, the function $\mathrm{res}_{1,\gamma}(\varepsilon) \mapsto \mathrm{res}_{1,\gamma+\delta}(f(\overline{\sigma}(a + \varepsilon)) - f(\overline{\sigma}(a))) : \mathbf{R}_{1,\gamma} \to \mathbf{R}_{1,\gamma+\delta}$ is well defined and coincides with the function $x \mapsto \overline{d} \cdot \overline{\sigma}(x)$ (where the sum is given by $+_{1,1}$).

2. If we are working in a valued field with a linearly closed residue field, it follows from Proposition 4.8 that δ and $\overline{d}$ from 4.9.1 are actually uniquely defined.

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Definition 4.10 (σ-Henselianity)**.** Let $M \models \mathrm{T}_{\mathcal{A},\sigma}$, t be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term, $\overline{d} \in \mathbf{K}(M)$, $a \in \mathbf{K}(M)$, and $\xi \in \mathbf{\Gamma}(M)$. We say that $(t, a, \overline{d}, \xi)$ is in σ-Hensel configuration if $\overline{d}$ linearly approximates t at prolongations on $\mathring{\mathcal{B}}_{\xi}(a)$ and

$$\mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \sigma^i(\xi)\}.$$

We say that M is σ-Henselian if, for all $(t, a, \overline{d}, \xi)$ in σ-Hensel configuration, there exists $c \in \mathbf{K}(M)$ such that $t(\overline{\sigma}(c)) = 0$ and $\mathrm{val}(c - a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$.

Remark 4.11. By Remark 4.5.1, when σ is an isometry, this form of the σ-Hensel lemma is equivalent to classical forms for difference polynomials (i.e., without any analytic structure) as stated in [4, 31, 32], for example. In particular, it implies Hensel's lemma (for polynomials).

Definition 4.12 (Pseudo-convergence)**.** Let $M \models \mathrm{T}_{\mathcal{A},\sigma}$.

(i) A sequence $(x_\alpha)_{\alpha\in\beta}$ of (distinct) points in $\mathbf{K}(M)$ indexed by an ordinal is said to be pseudo-convergent if, for all $\alpha, \gamma, \delta \in \beta$ such that $\alpha < \gamma < \delta$, we have $\mathrm{val}(x_\alpha - x_\delta) < \mathrm{val}(x_\gamma - x_\delta)$.

(ii) We say that $a \in \mathbf{K}(M)$ is a pseudo-limit of the pseudo-convergent sequence (x_α) (and we write $x_\alpha \rightsquigarrow a$) if, for all $\alpha < \gamma < \beta$, $\mathrm{val}(x_\alpha - a) < \mathrm{val}(x_\gamma - a)$.

(iii) A pseudo-convergent sequence of elements of $C \subseteq \mathbf{K}(M)$ is said to be maximal (over C) if it has no pseudo-limit in C.

(iv) We say that a sequence $(\overline{x}_\alpha)$ of tuples pseudo-solves an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term t if $t = 0$ or for $\alpha \gg 0$ (i.e., for α in a final segment) $t(\overline{x}_\alpha) \rightsquigarrow 0$.

(v) We say that a sequence (x_α) σ-pseudo-solves an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term t if $(\overline{\sigma}(x_\alpha))$ pseudo-solves t.

(vi) We say that M is maximally complete if any pseudo-convergent sequence in M (indexed by a limit ordinal) has a pseudo-limit in M.

(vii) We say M is σ-algebraically maximally complete if any pseudo-sequence (x_α) from M (indexed by a limit ordinal) σ-pseudo-solving an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term $t \neq 0$ has a pseudo-limit in M.

Remark 4.13.

1. Let (x_α) be a pseudo-convergent sequence. Then, for all $\alpha_0 < \alpha_1$, $\mathrm{val}(x_{\alpha_0} - x_{\alpha_1}) = \mathrm{val}(x_{\alpha_0} - x_{\alpha_0+1}) =: \gamma_{\alpha_0}$. The γ_α form a strictly increasing sequence. If $x_\alpha \rightsquigarrow a$, then $\mathrm{val}(a - x_{\alpha_0}) = \gamma_{\alpha_0}$, and if b is such that, for all α, $\mathrm{val}(b - a) > \gamma_\alpha$, then we also have $x_\alpha \rightsquigarrow b$.

2. As, in any valued field, balls with a non-infinite radius always have more than one point, if (x_α) is a maximal pseudo-convergent sequence over K then either γ_α is cofinal in $\mathrm{val}(\mathbf{K}^\star)$ and (x_α) is indexed by the successor of a limit ordinal, or (x_α) is indexed by a limit ordinal.

Proposition 4.14. *If M is σ-algebraically maximally complete and $\mathbf{R}_1(M)$ is linearly closed then M is σ-Henselian.*

Proof. First we present an easy claim.

Claim 4.15. Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration. Then

$$\max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \xi.$$

Proof. As $(t, a, \overline{d}, \xi)$ is in σ-Hensel configuration, there exists i_0 such that $\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}(d_{i_0}) + \sigma^{i_0}(\xi)$, and hence $\mathrm{val}(\sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1})) = \sigma^{-i_0}(\mathrm{val}(t(\overline{\sigma}(a))) - \mathrm{val}(d_{i_0})) > \xi$. ♦

And now, we present two lemmas about finding better approximations to zeros of terms.

Lemma 4.16. *Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration such that $t(\overline{\sigma}(a)) \neq 0$. Then there exists c such that $\mathrm{val}(c - a) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$, $\mathrm{val}(t(\overline{\sigma}(c))) > \mathrm{val}(t(\overline{\sigma}(a)))$, and $(t, c, \overline{d}, \xi)$ is also in σ-Hensel configuration.*

Proof. Choose any $\varepsilon \in \mathbf{K}(M)$ with $\mathrm{val}(\varepsilon) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$. By Claim 4.15, $\mathrm{val}(\varepsilon) > \xi$. For all $x \in \mathcal{O}$, let $R(a, \varepsilon, x) := t(\overline{\sigma}(a) + \overline{\sigma}(\varepsilon x)) - t(\overline{\sigma}(a)) - \overline{d} \cdot \overline{\sigma}(\varepsilon x)$ and

$$u(x) := \frac{t(\overline{\sigma}(a) + \overline{\sigma}(\varepsilon x))}{t(\overline{\sigma}(a))} = 1 + \sum_i \frac{d_i \sigma^i(\varepsilon)}{t(\overline{\sigma}(a))}\sigma^i(x) + \frac{R(a, \varepsilon, x)}{t(\overline{\sigma}(a))}.$$

For all i,

$$\mathrm{val}\left(\frac{d_i \sigma^i(\varepsilon)}{t(\overline{\sigma}(a))}\right) \geqslant \mathrm{val}(d_i) + \mathrm{val}(t(a)) - \mathrm{val}(d_i) - \mathrm{val}(t(a)) = 0,$$

and for any i_0 such that $\mathrm{val}(\varepsilon) = \mathrm{val}(\sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1}))$ it is an equality. As $\overline{d}$ linearly approximates t at prolongations on $\mathring{\mathcal{B}}_\xi(a)$, we also have

$$\mathrm{val}(R(a, \varepsilon, x)) > \min_i\{\mathrm{val}(\sigma^i(\varepsilon)) + \mathrm{val}(d_i)\} \geqslant \mathrm{val}(t(\overline{\sigma}(a))),$$

and $\mathrm{res}_1(u(x)) = 0$ is a non-trivial linear equation in the residue field. As $\mathbf{R}_1(M)$ is linearly closed, this equation has a solution $\mathrm{res}_1(e)$. Note that we must have $\mathrm{res}_1(e) \neq 0$.

Let $c = a + \varepsilon e$. It is clear that $\mathrm{val}(c - a) = \mathrm{val}(\varepsilon) = \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \xi$ and that $\mathrm{val}(t(\overline{\sigma}(c))) = \mathrm{val}(t(\overline{\sigma}(a))u(e)) > \mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \sigma_i(\xi)\}$. ♦

Lemma 4.17. *Let (x_α) be a pseudo-convergent sequence (indexed by a limit ordinal). Assume that, for all α, $(t, x_\alpha, \overline{d}, \xi)$ is in σ-Hensel configuration, $\mathrm{val}(x_{\alpha+1} - x_\alpha) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(x_\alpha))d_i^{-1}))\}$, and $t(\overline{\sigma}(x_\alpha)) \rightsquigarrow 0$. If c is such that $x_\alpha \rightsquigarrow c$, then $(t, c, \overline{d}, \xi)$ is in σ-Hensel configuration, and, for all α, $\mathrm{val}(t(\overline{\sigma}(c))) > \mathrm{val}(t(\overline{\sigma}(x_\alpha)))$.*

Proof. First of all, as $(t, x_0, \overline{d}, \xi)$ is in σ-Hensel configuration, $\overline{d}$ continuously linearly approximates t at prolongations on $\mathring{\mathcal{B}}_\xi(x_0)$. By Claim 4.15, $\mathrm{val}(c - x_0) = \mathrm{val}(x_1 - x_0) > \xi$. Moreover, let $R(x, c) := t(\overline{\sigma}(c)) - t(\overline{\sigma}(x)) - \overline{d} \cdot \overline{\sigma}(c - x)$. Then, for all α,

$$\begin{aligned}\mathrm{val}(t(\overline{\sigma}(c))) &= \mathrm{val}(t(\overline{\sigma}(x_\alpha)) + \overline{d}(\overline{\sigma}(x_\alpha)) \cdot \overline{\sigma}(c - x_\alpha) + R(x_\alpha, c)) \\ &\geqslant \min_i\{\mathrm{val}(t(\overline{\sigma}(x_\alpha))), \mathrm{val}(d_i) + \mathrm{val}(\sigma^i(c - x_\alpha))\} \\ &\geqslant \mathrm{val}(t(\overline{\sigma}(x_\alpha))).\end{aligned}$$

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Finally, as $\mathrm{val}(t(\overline{\sigma}(c))) \geqslant \mathrm{val}(t(\overline{\sigma}(x_0))) > \min_i\{\mathrm{val}(d_i) + \sigma^i(\xi)\}$, $(t, c, \overline{d}, \xi)$ is also in σ-Hensel configuration. ♦

Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration. If $t = 0$, we are done; if not, let $(x_\alpha)_{\alpha\in\beta}$ be a maximal sequence (with respect to the length) such that $x_0 = a$ and, for all α, $(t, x_\alpha, \overline{d}, \xi)$ is in σ-Hensel configuration, $\mathrm{val}(x_{\alpha+1} - x_\alpha) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(x_\alpha))d_i^{-1}))\}$, and $t(\overline{\sigma}(x_\alpha)) \rightsquigarrow 0$. If α is a limit ordinal, as M is σ-algebraically maximally complete, and $t \neq 0$, (x_α) has a pseudo-limit x_β. By Lemma 4.17, the sequence $(x_\alpha)_{\alpha\in\beta+1}$ still meets the same requirements, contradicting the maximality of $(x_\alpha)_{\alpha\in\beta}$. It follows that $\beta = \gamma + 1$. If $t(\overline{\sigma}(x_\gamma)) \neq 0$, then applying Lemma 4.16 to (t, x_γ), we obtain an element x_β such that $(x_\alpha)_{\alpha\in\beta+1}$ still meets the same requirements, a contradiction of the maximality of $(x_\alpha)_{\alpha\in\beta}$ once again. Hence we must have that $t(\overline{\sigma}(x_\gamma)) = 0$ and $c = x_\gamma$ is a solution to the σ-Hensel configuration $(t, a, \overline{d}, \xi)$. ■

Definition 4.18 ($\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$)**.** Let $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be the $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of analytic fields with an automorphism that are σ-Henselian and have a non-trivial valuation group. To specify the characteristic we will write $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ or $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$.

Proposition 4.19. *Let $\mathcal{A} := \bigcup_{\overline{X},\overline{Y}} \mathrm{W}[\overline{\mathbb{F}_p}^{\mathrm{alg}}]\langle\overline{X}\rangle[[\overline{Y}]]$, and let W_p be the $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-structure with base set $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$, the obvious valued field structure and analytic structure, and interpreting σ as the lifting to $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ of the Frobenius automorphism on $\overline{\mathbb{F}_p}^{\mathrm{alg}}$. Then $\mathrm{W}_p \models \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$.*

Proof. It is clear that $\mathrm{W}_p \models \mathrm{T}_{\mathcal{A}}$. As $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$ is complete with a discrete valuation, it is maximally complete. It follows from Proposition 4.14 that it is σ-Henselian. ■

In the definition of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, we have not required the residue field to be linearly closed, since it comes for free.

Proposition 4.20. *Let $M \models \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, then $\mathbf{K}(M)$ is linearly closed.*

Proof. Let $c \in \mathbf{K}(M)$, and let $P(x) = \sum_i a_i x_i$ be a non-zero linear polynomial. Let $\varepsilon \in \mathbf{K}(M)$ be such that $\mathrm{val}(\varepsilon) < \mathrm{val}(c) - \mathrm{val}(a_0)$. Then $P(\overline{\sigma}(\varepsilon x)) = \sum a_i\sigma^i(\varepsilon)\sigma^i(x)$ and $\min_i\{\mathrm{val}(a_i\sigma^i(\varepsilon))\} \leqslant \mathrm{val}(a_0) + \mathrm{val}(\varepsilon) < \mathrm{val}(c)$. Thus, we may assume that $\min_i\{\mathrm{val}(a_i)\} < \mathrm{val}(c) = \mathrm{val}(P(\overline{\sigma}(0)) - c)$. But, as P is linear, $\overline{a}$ linearly approximates P at prolongations on $\mathfrak{M}$, and $(P - c, 0, \overline{a}, 0)$ is in σ-Hensel configuration. As M is σ-Henselian, there exists $e \in \mathbf{K}(M)$ such that $P(\overline{\sigma}(e)) = c$. ■

To conclude this section, let us show that $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ behaves well with respect to coarsening. The following results will mainly be used in § 6.3 to transfer quantifier results from equicharacteristic zero to mixed characteristic.

Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$, and let T be an $\mathcal{L}$-theory containing $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$ Morleyized on $\mathbf{RV}$ (cf. Definition A.3). By § 2 we can find an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty$ of $\mathcal{L}^{\mathbf{RV}_\infty}$ (the ∞ in $\mathcal{L}^{\mathbf{RV}_\infty}$ is there to recall that the leading term structure is given by $\mathbf{RV}_\infty$ and not the $\mathbf{RV}_n$, although, to add to the general confusion, the $\mathbf{RV}_n$ are indeed present in

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the enrichment) an $\mathcal{L}^{\infty}$-theory $T_1^{\infty} \supseteq \mathrm{T}^{\infty}_{\mathrm{vf},0,0}$ and two functors $\mathfrak{C}_1^{\infty} : \mathrm{Str}(T) \to \mathrm{Str}(T_1^{\infty})$ and $\mathfrak{UC}_1^{\infty} : \mathrm{Str}(T_1^{\infty}) \to \mathrm{Str}(T)$. For any C in $\mathrm{Str}(T)$ we enrich $\mathfrak{C}_1^{\infty}(C)$ by defining

- $\cdot_{\infty}$ and 1_{∞} to be the multiplicative group structure of $\mathbf{RV}_{\infty}$;
- 0_{∞} to be $(0_n)_{n\in\mathbb{N}_{>0}}$;
- $x|_{\infty}y$ to hold if, for some n, $\pi_1(x)|_1\mathrm{rv}_1(p^{-n})\pi_1(y)$ holds;
- $x +_{\infty,\infty} y$ to be $(\pi_{mn}(x) +_{mn,m} \pi_{mn}(y))_{m\in\mathbb{N}_{>0}}$ if there exists $n \in \mathbb{N}_{>0}$ such that $\pi_n(x) +_{n,1} \pi_n(y) \neq 0_1$ and 0_{∞} otherwise;
- $x|^{\mathcal{R}}_{\infty}y$ to hold if $\pi_1(x)|^{\mathcal{R}}_1\pi_1(y)$ holds;
- $E_{\infty}(x)$ to be $(E_k(x))_{k\in\mathbb{N}_{>0}}$ for all E in some $\mathcal{A}^{\star}_{m,n}$;
- σ_{∞} to be $(\sigma_n(x))_{n\in\mathbb{N}_{>0}}$;

and we obtain a new functor $\mathfrak{C}_2^{\infty} : \mathrm{Str}(T) \to \mathrm{Str}(T_2^{\infty})$, where $T_2^{\infty} := T_1^{\infty} \cup \mathrm{T}^{\infty}_{\mathcal{A},\sigma,0,0}$. One can check that we still have an equivalence of categories induced by $\mathfrak{C}_2^{\infty}$ and $\mathfrak{UC}_1^{\infty}$, and that $\mathfrak{C}_2^{\infty}$ also respect cardinality up to $\aleph_0$ and respects $\aleph_1$-saturated models. Finally, by Corollary B.4, as T is Morleyized on $\mathbf{RV}$, we obtain functors $\mathfrak{C}_3^{\infty} : \mathrm{Str}(T) \to \mathrm{Str}(T_2^{(\mathbf{RV}_{\infty}\cup\mathbf{RV})-\mathrm{Mor}})$ and $\mathfrak{UC}_3^{\infty} : \mathrm{Str}(T_2^{(\mathbf{RV}_{\infty}\cup\mathbf{RV})-\mathrm{Mor}}) \to \mathrm{Str}(T)$. Note that, in this case, because we only enrich by predicates, the full subcategory $\mathfrak{F}$ of $\mathrm{Str}(T)$ is not actually needed.

Let us now show that, for all $M \models T$, $\mathfrak{C}_3^{\infty}(M) \models \mathrm{T}^{\infty}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$.

Proposition 4.21. *Let $M \models T$ and $t : \mathbf{K}^n \to \mathbf{K}$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(M)$-term, $\overline{d} \in \mathbf{K}(M)$, and b an open $\mathcal{O}_{\infty}$-ball. If, in $\mathfrak{C}_3^{\infty}(M)$, $\overline{d}$ linearly approximates t at prolongations on b, then, for any open $\mathcal{O}$-ball $b' \subseteq b$, $\overline{d}$ also linearly approximates t at prolongations on b' in M.*

Proof. For all a and $e \in b' \subseteq b$, we have

$$\mathrm{val}_{\infty}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(e)) - \overline{d}\cdot\overline{\sigma}(a-e)) > \min_i\{\mathrm{val}_{\infty}(d_i) + \mathrm{val}_{\infty}(\sigma^i(a-e))\}.$$

Let i_0 be such that $\mathrm{val}_{\infty}(d_{i_0}) + \mathrm{val}_{\infty}(\sigma^{i_0}(a-e))$ is minimal. Then we have $\mathrm{val}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(e)) - \overline{d}\cdot\overline{\sigma}(a-e)) > \mathrm{val}(d_{i_0}) + \mathrm{val}(\sigma^{i_0}(a-e)) \geqslant \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a-e))\}$. ■

Proposition 4.22. *Let $M \models T$. Then $\mathfrak{C}_3^{\infty}(M)$ is σ-Henselian (for the valuation val_{∞}).*

Proof. Let $(t, a, \overline{d}, \xi)$ be in σ-Hensel configuration in $\mathfrak{C}_3^{\infty}(M)$. Let i_0 be such that $\mathrm{val}_{\infty}(d_{i_0}) + \sigma^{i_0}(\xi)$ is minimal. As $(t, a, \overline{d}, \xi)$ is in σ-Hensel configuration, $\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}_{\infty}(d_{i_0}) + \sigma^{i_0}(\xi)$. Let $r = \sigma^{-i_0}(t(\overline{\sigma}(a))d_{i_0}^{-1}p^{-1})$. Then

$$\mathrm{val}(t(\overline{\sigma}(a))) > \mathrm{val}(d_{i_0}) + \mathrm{val}(\sigma^{i_0}(r)) \geqslant \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(r))\}.$$

Moreover,

$$\mathrm{val}_{\infty}(\sigma^{i_0}(r)) = \mathrm{val}_{\infty}(t(a)) - \mathrm{val}_{\infty}(d_{i_0}) > \sigma^{i_0}(\xi);$$

i.e., $\mathrm{val}_{\infty}(r) > \xi$. It follows that $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(r)}(a) \subseteq \mathring{\mathcal{B}}^{\mathcal{O}_{\infty}}_{\xi}(a)$, and hence, by Proposition 4.21, $\overline{d}$ linearly approximates t at prolongations on $\mathring{\mathcal{B}}^{\mathcal{O}}_{\mathrm{val}(r)}(a)$, and $(t, a, \overline{d}, \mathrm{val}(r))$ is in σ-Hensel configuration.

By σ-Henselianity of M, we can find $c \in \mathbf{K}(M)$ such that $t(\overline{\sigma}(c)) = 0$ and $\mathrm{val}(c-a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\}$. But then we have that, for all i, $\mathrm{val}_\infty(c-a) \geqslant \mathrm{val}_\infty(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))$. ∎

It follows from those two propositions that we can further enrich $T_2^{(\mathbf{RV}_\infty \cup \mathbf{RV})-\mathrm{Mor}}$ so that it is an $\mathbf{RV}$-enrichment of $\mathrm{T}^\infty_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$. Hence we have proved the following.

Proposition 4.23. *Let $\mathcal{L}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$ and T be an $\mathcal{L}$-theory containing $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$ Morleyized on $\mathbf{RV}$. There exist an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty$ of $\mathcal{L}^\infty_{\mathcal{A},\mathcal{Q},\sigma}$ (with new sorts $\mathbf{RV} = \bigcup_n \mathbf{RV}_n$) and an $\mathcal{L}^\infty$-theory $T^\infty \supseteq \mathrm{T}^\infty_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty : \mathrm{Str}(T) \to \mathrm{Str}(T^\infty)$ and $\mathfrak{U}\mathfrak{C}^\infty : \mathrm{Str}(T^\infty) \to \mathrm{Str}(T)$ that respect cardinality up to $\aleph_0$ and induce an equivalence of categories between $\mathrm{Str}(T)$ and $\mathrm{Str}_{\mathfrak{C}^\infty,(|\mathcal{L}|^{\aleph_1})^+}(T^\infty)$ and such that $\mathfrak{U}\mathfrak{C}^\infty$ respects models and elementary submodels and sends $\mathbf{RV}_\infty \cup \mathbf{RV}$ to $\mathbf{RV}$ and $\mathfrak{C}^\infty$ respects $(|\mathcal{A}|^{\aleph_1})^+$-saturated models.*

Similarly, we can prove the existence of these functors in the analytic setting and in the algebraic setting, and these functors are actually induced by those in the analytic difference case.

Proposition 4.24. *Let $\mathcal{L}_{\mathrm{an}}$ be any $\mathbf{RV}$-extension of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ contained in $\mathcal{L}$, and let $\mathcal{L}_{\mathrm{alg}}$ be any $\mathbf{RV}$-extension of $\mathcal{L}^{\mathbf{RV}^+}$ contained in $\mathcal{L}_{\mathrm{an}}$. Let $T_{\mathrm{an}} := T|_{\mathcal{L}_{\mathrm{an}}}$ and $T_{\mathrm{alg}} := T|_{\mathcal{L}_{\mathrm{alg}}}$. Assume that both T_{an} and T_{alg} are Morleyized on $\mathbf{RV}$.*

(i) *There exist an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty_{\mathrm{an}}$ of $\mathcal{L}^\infty_{\mathcal{A},\mathcal{Q}}$ and an $\mathcal{L}^\infty_{\mathrm{an}}$-theory $T^\infty_{\mathrm{an}} \supseteq \mathrm{T}^\infty_{\mathcal{A},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty_{\mathrm{an}} : \mathrm{Str}(T_{\mathrm{an}}) \to \mathrm{Str}(T^\infty_{\mathrm{an}})$ and $\mathfrak{U}\mathfrak{C}^\infty_{\mathrm{an}} : \mathrm{Str}(T_{\mathrm{an}}) \to \mathrm{Str}(T_{\mathrm{an}})$ with the same properties as in Proposition 4.23.*

   Moreover, $\mathfrak{C}^\infty_{\mathrm{an}}(\cdot\ |_{\mathcal{L}_{\mathrm{an}}}) = \mathfrak{C}^\infty(\cdot)\big|_{\mathcal{L}^\infty_{\mathrm{an}}}$, and similarly for $\mathfrak{U}\mathfrak{C}^\infty_{\mathrm{an}}$.

(ii) *There exist an $\mathbf{RV}_\infty$-enrichment $\mathcal{L}^\infty_{\mathrm{alg}}$ of $\mathcal{L}^{\mathbf{RV}_\infty{}^+}$ and an $\mathcal{L}^\infty_{\mathrm{alg}}$-theory $T^\infty_{\mathrm{alg}} \supseteq \mathrm{T}^\infty_{\mathrm{Hen},0,0}$ Morleyized on $\mathbf{RV}_\infty \cup \mathbf{RV}$, and functors $\mathfrak{C}^\infty_{\mathrm{alg}} : \mathrm{Str}(T_{\mathrm{alg}}) \to \mathrm{Str}(T^\infty_{\mathrm{alg}})$ and $\mathfrak{U}\mathfrak{C}^\infty_{\mathrm{alg}} : \mathrm{Str}(T^\infty_{\mathrm{alg}}) \to \mathrm{Str}(T_{\mathrm{alg}})$ with the same properties as in Proposition 4.23.*

   Moreover, $\mathfrak{C}^\infty_{\mathrm{alg}}(\cdot\ |_{\mathcal{L}_{\mathrm{alg}}}) = \mathfrak{C}^\infty_{\mathrm{an}}(\cdot\ |_{\mathcal{L}_{\mathrm{an}}})\big|_{\mathcal{L}^\infty_{\mathrm{alg}}} = \mathfrak{C}^\infty(\cdot)\big|_{\mathcal{L}^\infty_{\mathrm{alg}}}$, and similarly for $\mathfrak{U}\mathfrak{C}^\infty_{\mathrm{alg}}$.

5. Reduction to the algebraic case

In this section, let $\mathcal{L}_{\mathrm{an}}$ be an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$, and let T_{an} be an $\mathcal{L}_{\mathrm{an}}$-theory containing $\mathrm{T}_{\mathcal{A}}$, Morleyized on $\mathbf{RV}$. We define $\mathcal{L}_{\mathrm{alg}} := \mathcal{L}_{\mathrm{an}} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$ (it is an $\mathbf{RV}$-enrichment of $\mathcal{L}^{\mathbf{RV}^+}$) and $T_{\mathrm{alg}} = T_{\mathrm{an}}|_{\mathcal{L}_{\mathrm{alg}}}$. As previously, if there are new sorts $\Sigma_{\mathbf{RV}}$, we write $\mathbf{RV}$ for $\mathbf{RV} \cup \Sigma_{\mathbf{RV}}$.

Remark 5.1. Let M_1 and $M_2 \models T_{\mathrm{an}}$, $C_i \subseteq M_i$, and let $f : C_1 \to C_2$ be an $\mathcal{L}_{\mathrm{an}}$-isomorphism. Obviously, f extends uniquely to $\langle C_1 \rangle$. As $\mathcal{L}_{\mathrm{an}}$ contains $\mathcal{Q}$, $\mathbf{K}(\langle C_1 \rangle)$ is a field. Hence any partial $\mathcal{L}_{\mathrm{an}}$-isomorphism with domain C has a unique extension to $\mathrm{Frac}(\mathbf{K}(C))$.

Although it is well known, the algebraic case (i.e., in $\mathcal{L}_{\mathrm{alg}}$) is a bit more complicated because we do not have $\mathcal{Q}$ in $\mathcal{L}_{\mathrm{alg}}$.

Proposition 5.2. *Let M_1 and $M_2 \models T_{\mathrm{alg}}$ be two $\mathcal{L}_{\mathrm{alg}}$-structures, $C_i \subseteq M_i$, and let $f : C_1 \to C_2$ be an $\mathcal{L}^{\mathbf{RV}^+}$-isomorphism. Assume that* $\mathrm{rv}(\mathrm{Frac}(\mathbf{K}(C_1))) \subseteq \mathbf{RV}(C_1)$. *Then f has a unique extension to* $\mathrm{Frac}(\mathbf{K}(C_1))$.

Proof. Let $f'|_{\mathbf{K}}$ be the unique extension of $f|_{\mathbf{K}}$ to $\mathrm{Frac}(\mathbf{K}(C_1))$. It is a ring morphism. By Lemma A.13, it suffices to show that $f'|_{\mathbf{K}} \cup f|_{\mathbf{RV}}$ respects the rv_n. As $\mathrm{rv}(\mathrm{Frac}(\mathbf{K}(C_1))) \subseteq \mathbf{RV}(C_1)$, $f|_{\mathbf{RV}}$ commutes with the inverse on any rv_n, and hence

$$\mathrm{rv}_n(f'(a/b)) = \mathrm{rv}_n(f(a)f(b)^{-1}) = f(\mathrm{rv}_n(a))f(\mathrm{rv}_n(b)^{-1}) = f(\mathrm{rv}_n(a/b)).$$

This concludes the proof. ∎

In the following proposition we will be working in equicharacteristic zero; hence, to avoid needlessly cluttered notation, we will write $\mathbf{R}$, res, $\mathbf{RV}$, and rv for $\mathbf{R}_1$, res_1, $\mathbf{RV}_1$, and rv_1.

Proposition 5.3 (Reduction to the algebraic case)**.** *Suppose that $T_{\mathrm{an}} \supseteq \mathrm{T}_{\mathcal{A},0,0}$. Let M_1 and $M_2 \models T_{\mathrm{an}}$, $f : M_1 \to M_2$ be a partial $\mathcal{L}_{\mathrm{an}}$-isomorphism with domain $C_1 \leqslant M_1$, and $a_1 \in M_1$. If f can be extended to an $\mathcal{L}_{\mathrm{alg}}$-isomorphism f' whose domain contains a_1, then f can be extended to an $\mathcal{L}_{\mathrm{an}}$-isomorphism sending a_1 to $f'(a_1)$.*

Proof. First, because $T_{\mathrm{alg}}|_{\mathbf{RV}} = T_{\mathrm{an}}|_{\mathbf{RV}}$, T_{alg} is also Morleyized on $\mathbf{RV}$. By Lemma A.11, we can extend f' on $\mathbf{RV}$, and we may assume that $\mathbf{RV}(C_1\langle a_1\rangle) \subseteq \mathbf{RV}(C_1)$. Moreover, as f' respects $|_1^{\mathcal{R}}$, f' respects $\mathcal{R}$, and by Remark 5.1 and Proposition 5.2, replacing, if need be, a_1 by its inverse, we can assume that $a_1 \in \mathcal{R}$.

Let $a_2 = f'(a_1)$, and let us define f'' on $\mathbf{K}(\langle C_1\rangle a_1)$ by $f''(t(a_1)) = t^f(a_2)$, where t^f is the term obtained by applying f to the parameters of t. This morphism f'' clearly coinciding with f' on $\mathbf{K}(C_1)[a_1]$, and it is well defined. Indeed, it suffices to check that, if $t(a_1) = 0$, then $t^f(a_2) = 0$. But, by Weierstrass preparation, there exist $S \in \mathcal{SC}^{\mathcal{R}}(C_1)$, an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C_1)$ term E (a strong unit on S), and P, $Q \in \mathbf{K}(C_1)[X]$ such that Q does not have any zero in $S(\overline{\mathbf{K}(C_1)}^{\mathrm{alg}})$, $a_1 \in S$, and, for all $x \in S$, $t(x) = E(x)P(x)/Q(x)$. As $t(a_1) = 0$ and $E(x) \neq 0$, we must have $P(a_1) = 0$. As f' is a partial $\mathcal{L}_{\mathrm{alg}}$-isomorphism, we have $a_2 \in S^f$ and $P^f(a_2) = 0$. As f is an $\mathcal{L}_{\mathrm{an}}$-isomorphism, by Theorem 3.10 it is in fact an elementary partial $\mathcal{L}_{\mathrm{an}}$-isomorphism, and we also have that, for all $x \in S^f$, $t^f(x) = E^f(x)P^f(x)/Q^f(x)$, and E^f is a strong unit on S^f. Hence, $t^f(a_2) = E^f(a_2)P^f(a_2)/Q^f(a_2) = 0$.

Let us show that $f'' \cup f|_{\mathbf{RV}}$ is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-isomorphism. By Lemma A.13, it suffices to show that, for all $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C_1)$-terms t, $\mathrm{rv}(t^f(a_2)) = f(\mathrm{rv}(t(a_1)))$. By Remark 1.5, S is defined by a formula of the form $\theta(\mathrm{rv}(\overline{R}(x)))$, where θ is an $\mathcal{L}_{\mathrm{alg}}|_{\mathbf{RV}}$-formula and the R_i are polynomials in $\mathbf{K}(C_1)[X]$. By [12, proof of Theorem 7.5], there exists an $\mathcal{L}_{\mathrm{alg}}(C_1)$-definable function $g : K \to \prod_i \mathbf{RV}_{n_i}$ such that every fiber is an open $\mathcal{O}$-ball and, for any polynomial T equal to P, Q, or one of the R_i, $\mathrm{rv}(T(x))$ is constant on any fiber of g. It follows immediately that every fiber of g is either in S or in its complement. Let $\overline{\alpha} = g(a_1)$ and $\beta = \mathrm{rv}(t(a_1))$. As E is a strong unit, on $g^{-1}(\overline{\alpha}) = \mathring{\mathcal{B}}_{\mathrm{val}(d)}(c)$ it is of the form $eF((x-c)/d)$ with $\mathrm{val}(F((x-c)/d)) = 0$. As $\mathrm{res}((x-c)/d) = 0$ on all of $g^{-1}(\overline{\alpha})$, by Corollary 3.9, $\mathrm{rv}(E(x))$ is constant on $g^{-1}(\overline{\alpha})$, and hence $\mathrm{rv}(t(x))$ is

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constant on $g^{-1}(\overline{\alpha})$. As f is a partial elementary $\mathcal{L}_{\mathrm{an}}$-isomorphism and $\overline{\alpha}$ and $\beta \in \mathbf{RV}(C_1)$, the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-formula $\forall x, g(x) = \overline{\alpha} \Rightarrow \mathrm{rv}(t(x)) = \beta$ is preserved by f. And as f' is a partial elementary $\mathcal{L}_{\mathrm{alg}}$-isomorphism (by Theorem 1.4) and g is $\mathcal{L}_{\mathrm{alg}}(C_1)$-definable, $g^f(a_2) = f(\overline{\alpha})$, and we have that $\mathrm{rv}(t^f(a_2)) = f(\beta) = f(\mathrm{rv}(t(a_1)))$. ■

Corollary 5.4. *The previous proposition holds without any assumption on residue characteristic.*

Proof. Recall Proposition 4.24 and assume that M_1 and M_2 have mixed characteristic, and that f and f' are as in Proposition 5.3.

Then $\mathfrak{C}^{\infty}_{\mathrm{alg}}(f')$ is an extension of $\mathfrak{C}^{\infty}_{\mathrm{an}}(f)$ whose domain contains a_1. Applying Proposition 5.3, we obtain f'', an $\mathcal{L}^{\infty}_{\mathrm{an}}$-isomorphism extending $\mathfrak{C}^{\infty}_{\mathrm{an}}(f)$, whose domain contains a_1, and we conclude by applying $\mathfrak{U}\mathfrak{C}^{\infty}_{\mathrm{an}}$. ■

Corollary 5.5. *Let $\varphi(x, \overline{y}, \overline{r})$ be any $\mathcal{L}_{\mathrm{an}}$-formula where x and $\overline{y}$ are $\mathbf{K}$-variables and $\overline{r}$ are $\mathbf{RV} \cup \Sigma_{\mathbf{RV}}$-variables. Then there exist a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathrm{alg}}$-formula $\psi(x, \overline{z}, \overline{r})$ and $\mathcal{L}_{\mathrm{an}}|_{\mathbf{K}}$-terms $\overline{u}(\overline{y})$ such that $T_{\mathrm{an}} \models \varphi(x, \overline{y}, \overline{r}) \iff \psi(x, \overline{u}(\overline{y}), \overline{r})$.*

Proof. This follows from the previous corollary by a (classic) compactness argument. For the sake of completeness (and also because the uniformization part of that argument may be less usual), let us state it. Consider the set of formulas

$$T_{\mathrm{an}} \cup \{\varphi(x_1, \overline{y}, \overline{r}), \neg\varphi(x_2, \overline{y}, \overline{r})\} \cup$$
$$\{\psi(x_1, \overline{u}(\overline{y}), \overline{r}) \iff \psi(x_2, \overline{u}(\overline{y}), \overline{r}) : \psi \text{ is an } \mathcal{L}_{\mathrm{alg}}\text{-formula and } \overline{u} \text{ are } \mathcal{L}_{\mathcal{A},\mathcal{Q}}\big|_{\mathbf{K}}\text{-terms}\}.$$

By Corollary 5.4, this set of formulas cannot be consistent. Hence there is a finite set of $\mathcal{L}_{\mathrm{alg}}$-formulas $(\psi_i)_{0 \leqslant i < n}$ (which we can take $\mathbf{K}$-quantifier free by Theorem 1.4) and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}\big|_{\mathbf{K}}$-terms $\overline{u}_i$ such that

$$T_{\mathrm{an}} \models \forall\, \overline{y} x_1 x_2 \left(\bigwedge_i \psi_i(x_1, \overline{u}_i(\overline{y}), \overline{r}) \iff \psi_i(x_2, \overline{u}_i(\overline{y}), \overline{r}) \right) \Rightarrow (\varphi(x_1, \overline{y}, \overline{r}) \iff \varphi(x_2, \overline{y}, \overline{r})).$$

For all $\varepsilon \in 2^n$, let $\theta_\varepsilon := \bigwedge \psi_i(x, \overline{u}_i(\overline{y}), \overline{r})^{\varepsilon(i)}$, where $\psi^1 = \psi$ and $\psi^0 = \neg\psi$. For fixed $\overline{y}$ and $\overline{r}$, the $\theta_\varepsilon(x, \overline{y}, \overline{r})$ form a partition of $\mathbf{K}$ compatible with $\varphi(x, \overline{y}, \overline{r})$. For all $\eta \in 2^{2^n}$, let $\chi_\eta(\overline{y}, \overline{r})$ be a $\mathbf{K}$-quantifier free $\mathcal{L}_{\mathrm{an}}$-formula equivalent to $\bigwedge_\varepsilon (\exists x\, \theta_\varepsilon(x, \overline{y}, \overline{r}) \wedge \varphi(x, \overline{y}, \overline{r}))^{\eta(\varepsilon)}$. Note that for any choice of $\overline{y}$ and $\overline{r}$ there is exactly one η such that $\chi_\eta(\overline{y}, \overline{r})$ holds. It is now quite easy to show that $\varphi(x, \overline{y}, \overline{r}) \iff \bigvee_\eta (\chi_\eta(\overline{y}, \overline{r}) \wedge \bigvee_{\varepsilon \in \eta} \theta_\varepsilon(x, \overline{y}, \overline{r}))$. ■

Remark 5.6.

1. This corollary is a stronger version of [16, Theorem B]. Not only is it resplendent but it also has better control of the parameters (essentially due to a better control of the parameters in Weierstrass preparation in [9]). In particular, it is uniform.

2. Let $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ be $\mathcal{L}^{\mathrm{ac}}$ enriched with symbols for all the functions from $\mathcal{A}$, a symbol $\mathcal{Q} : \mathbf{K}^2 \to K$, for all units $E \in \mathcal{A}$ a symbol $E_k : \mathbf{R}_k \to \mathbf{R}_k$, and a symbol $|^{\mathcal{R}} \subseteq (\mathbf{\Gamma}^\infty)^2$. Then, any $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$-formula (or even formulas in an $\mathbf{R} \cup \mathbf{\Gamma}$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$) can

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be translated into an **RV**-enrichment of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$ (see Proposition 1.8), and hence Corollary 5.5 also holds (resplendently) for the $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$-theory $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathrm{Hen}}$ of Henselian valued fields with separated $\mathcal{A}$-structure and angular components. Note that some of the symbols we should have added have disappeared, like the trace of E_k on $\mathbf{\Gamma}^\infty$ which is constant equal to 0. Similarly the E_k and $|^{\mathcal{R}}_1$ are missing one of their arguments (the $\mathbf{\Gamma}^\infty$-argument in the case of E_k and the $\mathbf{R}_n$-argument for $|^{\mathcal{R}}$) but they depend trivially on it.

6. K-quantifier elimination in $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$

Up to § 6.3, we will be working solely in equicharacteristic zero; hence, we will once again write **RV** and rv for $\mathbf{RV}_1$ and rv_1. We will also be considering that variables are indexed by $\mathbb{N}$, and we will sometimes identify a variable and its index. But hopefully no confusion should arise.

Let $M \models \mathrm{T}_{\mathcal{A}}$ and $C \leqslant M$.

Definition 6.1 (Order-degree)**.** We say that an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term $t = \sum_{i=0}^{d} t_i(\overline{x}_{\neq m})x_m^i$ is polynomial of order (at most) d in x_m. If t is not of this form, we take the convention that t has infinite degree in x_m. Let $\mathcal{T}(C)$ be the set of tuples (t, I, m, d), where I is a finite set of variables, $m \in I$, $d \in \mathbb{N} \cup \{\infty\}$, and $t \neq 0$ is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-term whose variables are contained in I and which is either polynomial in x_m of degree at most d or not polynomial in x_m (and $d = \infty$). Let $\mathcal{T}_0(C) = \mathcal{T}(C) \cup \{0\}$.

We (partially) order $\mathcal{T}(C)$ by saying that (u, J, n, e) has (strictly) lower order-degree than (t, I, m, d) if one of the following holds:

(i) $\max(J) < \max(I)$;

(ii) $\max(J) = \max(I)$ and $J \subset I$; i.e., J is *strictly* included in I;

(iii) $J = I$ and $n > m$;

(iv) $J = I$ and $n = m$ and $e < d$.

We extend this order to $\mathcal{T}_0(C)$ by making the zero term greater than any element of $\mathcal{T}(C)$.

Remark 6.2.

1. This is a well-founded (partial) order.
2. In condition (iii), the order is the inverse of what one would expect, but that is because we want minimal terms to be polynomial in the last variable.
3. We will also write $J < I$ to mean that condition (i) or condition (ii) holds.

When $\overline{a}$ is indexed by some set $I \subseteq \mathbb{N}$ and $n \in I$, we will denote by $\overline{a}_{\neq n}$ the tuple $\overline{a}$ missing its nth component and use $(\overline{a}_{\neq n}, x_n)$ for the tuple $\overline{a}$ where the nth component is replaced by x_n. We define $\overline{\sigma}_{\neq n}(a)$ and $(\overline{\sigma}_{\neq n}(a), x_n)$ similarly, and let $\overline{\sigma}_{\leqslant n}(a) := (a, \sigma(a), \ldots, \sigma^n(a))$. Finally, we will write $\langle C\rangle_\sigma := \langle C\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}}$ and $C\langle\overline{c}\rangle_\sigma := C\langle\overline{c}\rangle_{\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}}$ (cf. Definition A.12).

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6.1. Residual and ramified extensions

Definition 6.3 (Regularity)**.** Let $t(\overline{x}) = \sum_i t_i(\overline{x}_{\neq m})x_m^i$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(M)$ term, and let $\overline{a} \in \mathbf{K}(M)$. We say that t is regular at $\overline{a}$ in x_m if

$$\mathrm{val}(t(\overline{a})) = \min_i\{\mathrm{val}(t_i(\overline{a}_{\neq m})) + i\,\mathrm{val}(a_m)\}.$$

By convention the zero term is never regular.

First, we state a proposition which has nothing to do with automorphisms.

Proposition 6.4. *Let $\overline{\alpha} \in \mathrm{rv}(\mathcal{R}(M))$, $\overline{a} \in \mathrm{rv}^{-1}(\overline{\alpha})$, and $(t, I, m, d) \in \mathcal{T}_0(C)$ be of minimal order-degree such that $t(\overline{x})$ is polynomial in x_m and t is not regular at $\overline{a}$. Then, for all $(u, J, n, e) < (t, I, m, d)$, $\mathrm{rv}(u(\overline{x}))$ is constant on $\mathrm{rv}^{-1}(\overline{\alpha})$. Moreover, for all $\overline{a}_{\neq n} \in \mathrm{rv}^{-1}(\overline{\alpha}_{\neq n})$, $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on $\mathrm{rv}^{-1}(\alpha_n)$.*

Proof. First, we may assume that $\overline{\mathbf{K}(M)}^{\mathrm{alg}} = \mathbf{K}(M)$ (see Proposition 3.26). We work by induction on J for the order defined in Remark 6.2.3. The proposition is trivial for constant terms. Now, assume that the proposition is true for any (v, K, p, f) with $K < J$.

Let us first assume that u is polynomial in x_n. Then, $u = \sum_i u_i(\overline{x}_{\neq n})x_n^i$ must be regular at $\overline{a}$, and hence $\mathrm{val}(u(\overline{a})) = \min_i\{\mathrm{val}(u_i(\overline{a}_{\neq n})) + i\,\mathrm{val}(a_n)\}$, and hence $\mathrm{rv}(u(\overline{a})) = \sum_i \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i \neq 0$. For any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\alpha})$ and i, $\mathrm{rv}(u_i(\overline{e}_{\neq n}))\mathrm{rv}(e_n)^i = \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i$. Moreover, if $\sum_i \mathrm{rv}(c_i) \neq 0$, then $\mathrm{rv}(\sum_i c_i) = \sum_i \mathrm{rv}(c_i)$; hence we must also have $\mathrm{rv}(u(\overline{e})) = \sum_i \mathrm{rv}(u_i(\overline{a}_{\neq n}))\alpha_n^i \neq 0$. As u is polynomial in x_n, it has a Weierstrass preparation. Hence, for polynomial u, the proposition is proved.

Suppose now that u is of infinite degree in x_n, and hence that all terms (v, J, n, e) with $e \neq \infty$ have been taken care of in the previous paragraph. By Weierstrass preparation, there exists $S \in \mathcal{SC}^{\mathcal{R}}(C\langle\overline{a}_{\neq n}\rangle)$ such that $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on S and $a_n \in S$. But then either $\mathrm{rv}^{-1}(\alpha_n) \subseteq S$ or $\mathrm{rv}^{-1}(\alpha_n)$ contains a $\overline{\mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$-ball and hence a point $c \in \overline{\mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$. Let $P = \sum p_i(\overline{a}_{\neq n})X^i \in \mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)[X]$ be its minimal polynomial. Then, for all $e \in \mathrm{rv}^{-1}(\alpha_n)$, $\mathrm{rv}(P(e)) = 0$ (i.e., $P(e) = 0$), but that is absurd. Hence t has a Weierstrass preparation on $\mathrm{rv}^{-1}(\alpha_n)$ and there exist $F(x, \overline{z}) \in \mathcal{A}$, $\overline{c} \in \mathbf{K}(C\langle\overline{a}\rangle)$, and P and $Q \in \mathbf{K}(C\langle\overline{a}_{\neq n}\rangle)[X]$ such that, for all $x_n \in \mathrm{rv}^{-1}(\alpha_n)$,

$$u(\overline{a}_{\neq n}, x_n) = F\left(\frac{x_n - a_n}{a_n}, \overline{c}\right)\frac{P(x_n)}{Q(x_n)}$$

and $\mathrm{val}(F((x_n - a_n)/a_n, \overline{c})) = 0$. But $\mathrm{rv}(P(x_n))$ and $\mathrm{rv}(Q(x_n))$ do not depend on x_n, and $\mathrm{rv}(F((x_n - a_n)/a_n, \overline{c}))$ only depends on $\mathrm{res}((x_n - a_n)/a_n) = 0$ (see Corollary 3.9). Hence $\beta := \mathrm{rv}(u(\overline{a}_{\neq n}, x_n))$ does not depend on $x_n \in \mathrm{rv}^{-1}(\alpha_n)$. The $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-formula

$$\forall x_n\, \mathrm{rv}(x_n) = \alpha_n \Rightarrow \mathrm{rv}(u(\overline{a}_{\neq n}, x_n)) = \beta$$

is in the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{a}_{\neq n}$ over $C\alpha_n\beta$. By induction (and Corollary 1.6), all tuples $\overline{a}_{\neq n} \in \mathrm{rv}^{-1}(\overline{\alpha}_{\neq n})$ have the same $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C\alpha\beta)$-type, and $\mathrm{rv}(u(\overline{a})) = \beta$ for all $\overline{a} \in \mathrm{rv}^{-1}(\overline{\alpha})$. ■

Let us now prove the first embedding theorem we will need to eliminate quantifiers. Let M_1 and M_2 be models of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$, $C_i \leqslant M_i$, and let $f : C_1 \to C_2$ be an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism.

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Proposition 6.5. *Let $\alpha \in \mathrm{rv}(\mathcal{R}(M_1)) \cap \mathbf{RV}(C_1)$, $a \in \mathrm{rv}^{-1}(\alpha)$ and $(t, I, m, d) \in \mathcal{T}_0(C)$ be polynomial in x_m for some $m \in \mathbb{N}$. Assume that (t, I, m, d) is of minimal order-degree such that t is not regular at $\overline{\sigma}(a)$. Then*

(i) *there exist $a_1 \in \mathcal{R}(M_1)$ and $a_2 \in \mathcal{R}(M_2)$ such that $t(\overline{\sigma}(a_1)) = 0 = t^f(\overline{\sigma}(a_2))$, $\mathrm{rv}(a_1) = \alpha$, and $\mathrm{rv}(a_2) = f(\alpha)$;*

(ii) *for any such a_i, f can be extended to an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism sending a_1 to a_2.*

Proof. Let $t = \sum_{i=0}^{d} t_i(\overline{x}_{\neq m})x_m^i$. By minimality of t, we cannot have $t_d(\overline{\sigma}_{\neq m}(a)) = 0$. Dividing by t_d, we may assume that $t_d = 1$.

Claim 6.6. There exists $\overline{c} \in \mathbf{K}(M)$ that linearly approximates t on $\mathrm{rv}^{-1}(\alpha)$ at prolongations and such that

$$\min_j\{\mathrm{val}(c_j) + \mathrm{val}(\sigma^j(a))\} = \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\,\mathrm{val}(\sigma^m(a))\}.$$

Proof. Let $N_i = \overline{M_i}^{\mathrm{alg}}$ (see Proposition 3.26). Let $s_e := \sum_{i<e} t_i x_m^i$ and $s := s_d$. For all i and $j \neq m$, by Proposition 6.4 applied in N_1, $t_i(\overline{\sigma}_{\neq j}(a), x_j)$ has a Weierstrass preparation on the ball $b_j := \mathrm{rv}^{-1}(\alpha_j)$ and constant valuation. By Proposition 6.4, for all $e \leqslant d$, s_e also has constant valuation on $\mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. By invariance under addition (and an induction on e) we can show that $s(\overline{\sigma}_{\neq j}(a), x_j)$ also has a Weierstrass preparation on b_j. Moreover, $\partial s/\partial x_j(\overline{x})$ is also given by an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-term of degree $d-1$ in x_m; hence $\mathrm{rv}(\partial s/\partial x_j(\overline{\sigma}_{\neq j}(a), x_j))$ is constant on b_j (equal to some $\mathrm{rv}(c_j)$, where $c_j \in \mathbf{K}(M_1)$). By Proposition 3.27, for all y_j and $z_j \in b_j$,

$$\mathrm{rv}(t(\overline{\sigma}_{\neq j}(a), y_j) - t(\overline{\sigma}_{\neq j}(a), z_j)) = \mathrm{rv}(s(\overline{\sigma}_{\neq j}(a), y_j) - s(\overline{\sigma}_{\neq j}(a), z_j)) = \mathrm{rv}(c_j)\mathrm{rv}(y - z).$$

This last statement is in the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{\sigma}_{\neq j}(a)$ over $C_1\mathrm{rv}(c_j)$. By Proposition 6.4 and Corollary 1.6, any $\overline{e}_{\neq j} \in \mathrm{rv}^{-1}(\overline{\sigma}_{\neq j}(\alpha))$ has the same $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1\mathrm{rv}(c_j))$-type, and hence the same c_j works for any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$.

By minimality of t, s is regular at $\overline{\sigma}(a)$ and $\mathrm{val}(s(\overline{\sigma}(a))) = \min_{i<d}\{\mathrm{val}(t_i(\overline{\sigma}_{\neq j}(a))) + i\,\mathrm{val}(\sigma^m(a))\}$. Because $\mathrm{val}(s(\overline{\sigma}_{\neq j}(a), x_j))$ is constant on b_j, by the last statement of Proposition 3.27 and because $\mathrm{rad}(b_j) = \mathrm{rv}(\sigma^j(a))$, we obtain that

$$\mathrm{val}(c_j) + \mathrm{val}(\sigma^j(a)) \geqslant \mathrm{val}(s(\overline{\sigma}(a))) \geqslant \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\,\mathrm{val}(\sigma^m(a))\}.$$

When $j = m$, as t is polynomial in x_m and $\partial t/\partial x_m(\overline{x})$ is of degree $d-1$ in x_m, by Proposition 3.27, we can also find $c_m \in \mathbf{K}(M_1)$ that linearly approximates $t(\overline{e}_{\neq m}, x_m)$ on $\mathrm{rv}^{-1}(\sigma^m(\alpha))$ for any $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. And

$$\mathrm{val}(c_m) + \mathrm{val}(\sigma^m(a)) = \mathrm{val}(\partial t/\partial x_m(\overline{x})) + \mathrm{val}(\sigma^m(a)) = \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\,\mathrm{val}(\sigma^m(a))\}.$$

It now follows from Proposition 4.6 that $\overline{c}$ linearly approximates t on $\mathrm{rv}^{-1}(\alpha)$ at prolongations. ♦

If $t \neq 0$, as t is not regular at $\overline{\sigma}(a)$, $\mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\,\mathrm{val}(\sigma^m(a))\} = \mathrm{val}(c_m) + \mathrm{val}(\sigma^m(a)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(\sigma^j(a))\}$, and $(t, a, \overline{c}, \mathrm{val}(\alpha))$ is in σ-Hensel

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configuration. Hence there exists $a_1 \in M_1$ such that $t(a_1) = 0$ and $\mathrm{val}(a_1 - a) \geqslant \max_i\{\sigma^{-i}(t(\overline{\sigma}(a)c_i^{-1}))\}$. In particular,

$$\begin{aligned}
\mathrm{val}(\sigma^m(a_1 - a)) &\geqslant \mathrm{val}(t(\overline{\sigma}(a))) - \mathrm{val}(c_m) \\
&> \min_i\{\mathrm{val}(t_i(\overline{\sigma}(a))) + i\,\mathrm{val}(\sigma^m(a))\} - \mathrm{val}(c_m) \\
&= \mathrm{val}(\sigma^m(a)),
\end{aligned}$$

and thus $\mathrm{rv}(a_1) = \mathrm{rv}(a)$.

If x_m is not the highest variable appearing in t (which we call x_n) then, applying Proposition 6.4 to $(t, I, n, \infty) < (t, I, m, d)$, we get that $\mathrm{rv}(t(\overline{x}))$ is constant equal to 0 on all of $\mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. As $t(\overline{\sigma}_{\neq m}(a), x_m)$ is polynomial and has infinitely many zeros, we must have $t_i(\overline{\sigma}(a)) = 0$ for all i, but that contradicts the non-regularity of t in x_m at $\overline{\sigma}(a)$. Thus x_m must be the highest variable appearing in t. For the same reasons, we cannot have $\mathrm{rv}(c_m) = 0$.

Note that we have also proved that, for all $\overline{e} \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$, t is minimal such that it is not regular in $\overline{e}$; hence the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-type of $\overline{\sigma}(\alpha)$ says so, and hence, as $\mathrm{T}_{\mathcal{A}}$ eliminates field quantifiers, t^f has the same minimality property (relatively to $f(\alpha)$), and we find a_2 in the exact same way. If $t = 0$ then any a_1 and $a_2 \in \mathrm{rv}^{-1}(\alpha)$ will work.

Let us now show that f can be extended to send a_1 to a_2. For all $n < m$, any term $u(\overline{x}^{\leqslant n}) \in \mathcal{T}_0(C)$ has order-degree strictly smaller than (t, I, m, d), and hence $\beta := \mathrm{rv}(u(\overline{e}))$ does not depend on the choice of $e \in \mathrm{rv}^{-1}(\overline{\sigma}(\alpha))$. The formula '$\forall \overline{x}\, \mathrm{rv}(\overline{x}) = \overline{\sigma}(\alpha) \Rightarrow \mathrm{rv}(u(\overline{x})) = \beta$' is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}(C_1)$-formula respected by f, and thus $f(\beta) = f(\mathrm{rv}(u(\overline{\sigma}(a_1)))) = u(\overline{\sigma}(a_2))$. It follows immediately that the function f_n sending $u(\overline{\sigma}(a_1))$ to $u(\overline{\sigma}(a_2))$ is well defined. It is obviously an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$-morphism, and, as it respects rv, by Lemma A.13, it is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-morphism.

Let us now assume that $t \neq 0$.

Claim 6.7. Let $P := t(\overline{\sigma}_{\neq m}(a_1), x_m) \in \mathbf{K}(C_{1,m-1})[X]$. For all $n \geqslant m$, $\sigma^n(a_1)$ is the only zero of $P^{\sigma^{n-m}}$ whose leading term is $\sigma^n(\alpha)$.

Proof. Because σ is an automorphism of valued fields, it suffices to prove the case when $n = m$. Let $e \in \mathrm{rv}^{-1}(\sigma^m(\alpha)) \smallsetminus \{\sigma^m(a_1)\}$. Then $\mathrm{rv}(P(e)) = \mathrm{rv}(P(e) - P(a)) = \mathrm{rv}(c_m)\mathrm{rv}(e - \sigma^m(a_1)) \neq 0$. ♦

The same claim is true of $\sigma^m(a_2)$ with respect to $f(P^{\sigma^{n-m}})$ and $f(\sigma^n(\alpha))$. Therefore, it suffices to extend f_{m-1} to the $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-definable closure of $C_{1,m-1}$, which we can certainly do as f_{m-1} is an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-elementary isomorphism (by resplendent field quantifier elimination in $\mathrm{T}_{\mathcal{A}}$).

Thus we have obtained an $\mathcal{L}_{\mathcal{A},\mathcal{Q}}$-isomorphism between $C_1\langle\overline{\sigma}(a_1)\rangle$ and $C_2\langle\overline{\sigma}(a_2)\rangle$. By Remark 4.2, it is an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism. This morphism is also an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism by Lemma A.13. ■

Corollary 6.8. *Let $\alpha \in \mathbf{RV}(C_1)$. Then there exists $a_1 \in M_1$ such that $\mathrm{rv}(a_1) = \alpha$ and f extends to an isomorphism on $C_1\langle a_1\rangle_\sigma$.*

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Proof. If $\alpha \in \mathrm{res}(\mathcal{R}(M_1))$, then Proposition 6.5 applies. If not, apply Proposition 6.5 to α^{-1}, and conclude by extending the isomorphism to the analytic field generated by its domain by Remark 5.1. ■

6.2. Immediate extensions

Let $M \models \mathrm{T}_{\mathcal{A}}$ be sufficiently saturated, and let $C \leqslant M$.

Definition 6.9 (pseudo-convergent $\star$-sequences)**.** Let $(\overline{x}_\alpha)$ be a sequence of tuples of the same length. We say that it is a pseudo-convergent sequence if, for all i, $(\overline{x}_{i,\alpha})$ is pseudo-convergent. Moreover, we will say that $\overline{a}$ is a pseudo-limit of $(\overline{x}_\alpha)$ if, for all i, $\overline{x}_{i,\alpha} \rightsquigarrow a_i$.

Definition 6.10 (Equivalent pseudo-convergent sequences)**.** We will say that two pseudo-convergent sequences are equivalent if they have the same pseudo-limits.

Lemma 6.11. *Let $\overline{x}_\alpha$ be a pseudo-convergent sequence, $\overline{a}$ a pseudo-limit of this sequence, and $\overline{y}_\alpha$ a sequence such that, for all i, $\mathrm{val}(a_i - y_{i,\alpha}) = \mathrm{val}(a_i - x_{i,\alpha})$. Then (y_α) is also a pseudo-convergent sequence, equivalent to (x_α).*

Proof. We may assume that $|x_\alpha| = 1$. Note that, for all $\beta > \alpha$, $\mathrm{val}(y_\beta - y_\alpha) = \mathrm{val}(y_\beta - a + a - y_\alpha) = \mathrm{val}(a - x_\alpha) = \mathrm{val}(x_\beta - x_\alpha)$, as $\mathrm{val}(a - x_\beta) > \mathrm{val}(a - x_\alpha)$. Hence (y_α) is also pseudo-convergent. Moreover, if b is any pseudo-limit of (x_α), then $\mathrm{val}(b - y_\alpha) = \mathrm{val}(b - x_{\alpha+1} + x_{\alpha+1} - a + a - y_\alpha) = \mathrm{val}(a - y_\alpha) = \mathrm{val}(a - x_\alpha) = \mathrm{val}(b - x_\alpha)$ and $y_\alpha \rightsquigarrow b$. The symmetric argument shows that if $y_\alpha \rightsquigarrow b$ then $x_\alpha \rightsquigarrow b$. ■

Definition 6.12 (Rich enough families)**.** We say that a family $\mathcal{F}$ of equivalent pseudo-convergent sequences of C is rich enough if, for any linear polynomial $P(\overline{X}) = \sum_i \pi_i X_i \in \mathrm{rv}(\mathbf{K}(C))[\overline{X}]$, there exists $(\overline{x}_\alpha) \in \mathcal{F}$ such that, for all pseudo-limit $\overline{a}$ and all α, $P(\mathrm{rv}(\overline{a} - \overline{x}_\alpha)) \neq 0$; i.e., if $\mathrm{rv}(p_i) = \pi_i$ then $\mathrm{val}(\sum_i p_i(a_i - x_{i,\alpha})) = \min_i\{\mathrm{val}(p_i) + \mathrm{val}(a_i - x_{i,\alpha})\}$.

We will say that a term $u = \sum_{i=0}^{d} u_i(\overline{x}_{\neq m})\sigma^m(x)^i$ is monic if $u_d = 1$. As in § 6.1, let us begin with a proposition that does not seem to have anything to do with automorphisms.

Proposition 6.13. *Let $\mathcal{F}$ be a rich enough family of equivalent pseudo-convergent sequences of C that are eventually in $\mathcal{R}$ and let $(t, I, m, d) \in \mathcal{T}_0(C)$ does seem a better idea. Suppose that (t, I, m, d) has minimal order-degree such that t is a monic polynomial in x_m, and that there exists a pseudo-convergent sequence $(\overline{x}_\alpha) \in \mathcal{F}$ that pseudo-solves t. Then, for all $(u, J, n, e) < (t, I, m, d)$, there exists α_0 such that $\mathrm{rv}(u(\overline{x}))$ is constant on $\overline{b}_0 := \mathring{\mathcal{B}}_{\overline{\gamma}_0}(\overline{x}_{\alpha_0+1})$, where $\overline{\gamma}_0 := \mathrm{val}(\overline{x}_{\alpha_0+1} - \overline{x}_{\alpha_0})$ (it follows immediately that $\mathrm{rv}(u(\overline{x})) \in \mathrm{rv}(\mathbf{K}(C))$), and, for any $\overline{a} \in \overline{b}_0$, $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on $b_{n,0}$.*

Proof. We may assume that $\overline{\mathbf{K}(M)}^{\mathrm{alg}} = \mathbf{K}(M)$. The proof proceeds by induction on J. Suppose that Proposition 6.13 holds for any term (v, K, p, f) such that $K < J$. Let us prove a few claims to take care of certain induction steps.

Claim 6.14. Fix e and $n \in \mathbb{N}$. Suppose that the lemma holds for all (u, J, n, e). Then it holds for any $(u, J, n, e+1)$, where u is a monic polynomial in x_n.

Note that the case when $e = 0$ does not require any hypothesis (other than the induction hypothesis on J).

Proof. Let $u = x_n^{e+1} + \sum_{i \leqslant e} u_i(\overline{x}_{\neq n}) x_n^i$ and, for all $f \leqslant e$, $s_f := \sum_{i \leqslant f} u_i(\overline{x}_{\neq n}) x_n^i$. Let $\overline{a}$ be a pseudo-limit of $(\overline{x}_\alpha)$. Then we can find α_0 such that, for all $j \neq n$, $\mathrm{val}(s_f(\overline{a}_{\neq j}, x_j))$ and $\mathrm{val}(u_{f+1}(\overline{a}_{\neq j}, x_j))$ are constant on $b_{j,0}$, and $u_{f+1}(\overline{a}_{\neq j}, x_j)$ has a Weierstrass preparation. By induction on f and invariance under addition, $s_e(\overline{a}_{\neq j}, x_j)$ has a Weierstrass preparation on $b_{j,0}$. Let $s := s_e$. Making α_0 bigger we can also assume that $\mathrm{val}(\partial s / \partial x_j(\overline{a}_{\neq j}, x_j))$ is constant on $b_{j,0}$. By Proposition 3.27, we find $c_j \in \mathbf{K}(C)$ such that, for all y_j and $z_j \in b_{j,0}$, $\mathrm{rv}(u(\overline{a}_{\neq j}, y_j) - u(\overline{a}_{\neq j}, z_j)) = \mathrm{rv}(s(\overline{a}_{\neq j}, y_j) - s(\overline{a}_{\neq j}, z_j)) = \mathrm{rv}(c_j)\mathrm{rv}(y_j - z_j)$. By field quantifier elimination, this statement only depends on the value of $\mathrm{rv}(v(\overline{a}_{\neq j}))$ for a finite number of $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}(C)$-terms v, and hence, by induction, making α_0 bigger, we may assume that this statement is true of all $\overline{a} \in \overline{b}_0$. When $j = n$, the same arguments yield some $c_n \in \mathbf{K}(C)$ as u is already polynomial in x_n and $\partial u / \partial x_n(\overline{x})$ is polynomial in x_n of degree at most e. It now follows from Proposition 4.6 that $\overline{c}$ linearly approximates u on $\overline{b}_0$.

Let $(\overline{y}_\alpha) \in \mathcal{F}$ be such that, for any pseudo-limit $\overline{a}$, $\mathrm{val}(\overline{c} \cdot (\overline{a} - \overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$. Then $\mathrm{val}(u(\overline{a}) - u(\overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$. If, for all α, $\mathrm{val}(u(\overline{a})) > \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$, then $\mathrm{val}(u(\overline{y}_\alpha)) = \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$ and $u(\overline{y}_\alpha) \rightsquigarrow 0$, contradicting the minimality of t. Hence $\mathrm{val}(u(\overline{a})) < \min_j\{\mathrm{val}(c_j) + \mathrm{val}(a_j - y_{j,\alpha})\}$ for $\alpha \gg 0$, and $\mathrm{rv}(u(\overline{a})) = \mathrm{rv}(u(\overline{y_\alpha})) \in \mathrm{rv}(\mathbf{K}(C))$. By compactness, making α_0 bigger, this is true for any $\overline{a} \in \overline{b}_0$. ♦

Claim 6.15. Fix e and $n \in \mathbb{N}$. Suppose that the lemma holds for (u, J, n, e) monic polynomial in x_n. Then it holds for any (u, J, n, e).

Proof. Dividing by the dominant coefficient u_e (which has constant rv on $\overline{b}_0$ by induction), we obtain a term v monic polynomial of degree at most e in x_n and which must also have constant rv on $\overline{b}_0$ if we take α_0 big enough. ♦

Claim 6.16. Fix $n \in \mathbb{N}$. Suppose that, for all $e \in \mathbb{N}$, the lemma holds for all (u, J, n, e). Then it also holds for all (u, J, n, ∞).

Proof. Let $\overline{a}$ be a pseudo-limit of (x_α). Any $S \in \mathcal{SC}^{\mathcal{R}}(C\langle \overline{a}_{\neq n}\rangle)$ that contains a_n must contain $b_{n,0}$ for α_0 big enough. If not, there exists $c \in \overline{\mathbf{K}(C\langle \overline{a}_{\neq n}\rangle)}^{\mathrm{alg}}$ such that $x_{n,\alpha} \rightsquigarrow c$. Let $P(\overline{a}_{\neq n}, x_n) = \sum_i p_i(\overline{a}_{\neq n}) x_n^i$ be its minimal polynomial. Then, by hypothesis, for all $\overline{e} \in \overline{b}_0$ (for α_0 big enough), $\mathrm{rv}(P(\overline{e})) = 0$, and we must have $p_i(\overline{a}_{\neq n}) = 0$ for all i, but that is absurd.

It follows that we can find α_0 such that $u(\overline{a}_{\neq n}, x_n)$ has a Weierstrass preparation on $b_{n,0}$; i.e., there exist $F \in \mathcal{A}$, $\overline{c} \in \mathbf{K}(C\langle \overline{a}_{\neq n}\rangle)$ and $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}(C)}$-terms P and Q polynomial in x_n such that, for all $x_n \in b_{j,0}$,

$$u(\overline{a}_{\neq n}, x_n) = F\left(\frac{x_n - x_{n,\alpha_0+1}}{x_{n,\alpha_0+1} - x_{n,\alpha_0}}, \overline{c}\right) \frac{P(\overline{a}_{\neq n}, x_n)}{Q(\overline{a}_{\neq n}, x_n)}$$

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and $\mathrm{val}(F((x_n - x_{n,\alpha_0+1})(x_{n,\alpha_0+1} - x_{n,\alpha_0}), \overline{c})) = 0$. In turn, this implies that $\mathrm{rv}(u(\overline{a}_{\neq n}, x_n))$ does not depend on $x_n \in b_{n,0}$, and, by the usual uniformization argument (making α_0 bigger), we can ensure that $\mathrm{rv}(u(\overline{e}))$ does not depend on $\overline{e} \in \overline{b}_0$. ♦

Proposition 6.13 follows by induction. ■

Remark 6.17. Note that the proof of Claim 6.14 also shows that there exists $\overline{d} \in \mathbf{K}(C)$ that linearly approximates t on $\overline{b}_0$ for α_0 big enough.

Let $M \models \mathrm{T}_{\mathcal{A},\sigma}$ be sufficiently saturated and $C \leqslant M$ such that $\mathrm{res}(\mathbf{K}(C))$ is linearly closed.

Proposition 6.18. *Let x_α be a pseudo-convergent sequence of C. Then the family $\{\overline{\sigma}(y_\alpha) : y_\alpha$ is a pseudo-convergent sequence of C equivalent to $x_\alpha\}$ is a rich enough family of equivalent pseudo-convergent sequences of C.*

Proof. Let $P(\overline{X}) = \sum_i p_i X_i \in \mathbf{K}(C)[\overline{X}]$. If, for all i, $p_i = 0$, we are done. Otherwise, let $\varepsilon_\alpha = x_{\alpha+1} - x_\alpha$, let i_0 such that $\mathrm{val}(p_{i_0}) + \mathrm{val}(\sigma^{i_0}(\varepsilon_\alpha))$ is minimal, and let

$$Q_\alpha(\overline{\sigma}(X)) = p_{i_0}^{-1}\sigma^{i_0}(\varepsilon_\alpha)^{-1}P(\overline{\sigma}(\varepsilon_\alpha X)) = \sum_i p_i p_{i_0}^{-1}\sigma^i(\varepsilon_\alpha)\sigma^{i_0}(\varepsilon_\alpha^{-1})\sigma^i(X).$$

As $\mathrm{res}(Q_\alpha)$ is linear with coefficients in $\mathrm{res}(\mathbf{K}(C))$, which is linearly closed, we can find $d_\alpha \in \mathbf{K}(C)$ such that $\mathrm{res}(Q_\alpha(\overline{\sigma}(d_\alpha))) \neq \mathrm{res}(Q_\alpha(\overline{\sigma}(1)))$. In particular, $\mathrm{res}(d_\alpha) \neq \mathrm{res}(1)$ and $\mathrm{val}(d_\alpha - 1) = 0$. Let $y_\alpha = x_\alpha + \varepsilon_\alpha d_\alpha$.

Let $\overline{a}$ be such that $\overline{\sigma}(x_\alpha) \rightsquigarrow \overline{a}$. Then

$$\begin{aligned} \mathrm{rv}(a_i - \sigma^i(y_\alpha)) &= \mathrm{rv}(a_i - \sigma^i(x_{\alpha+1}) + \sigma^i(x_{\alpha+1}) - \sigma^i(x_\alpha) + \sigma^i(x_\alpha) - \sigma^i(y_\alpha)) \\ &= \mathrm{rv}(\sigma^i(\varepsilon_\alpha))\mathrm{rv}(1 - \sigma^i(d_\alpha)). \end{aligned}$$

It follows that $\mathrm{val}(a_0 - y_\alpha) = \mathrm{val}(\varepsilon_\alpha) = \mathrm{val}(a_0 - x_\alpha)$. By Lemma 6.11, (y_α) is equivalent to (x_α). Let $c_i = (a_i - \sigma^i(y_\alpha))/\sigma^i(\varepsilon_\alpha)$. Then

$$\begin{aligned} \mathrm{res}(P(\overline{a} - \overline{\sigma}(y_\alpha))p_{i_0}^{-1}\varepsilon_\alpha^{-1}) &= \mathrm{res}(Q)(\mathrm{res}(\overline{c})) \\ &= \mathrm{res}(Q)(\mathrm{res}(\overline{\sigma}(1) - \overline{\sigma}(d_\alpha))) \\ &= \mathrm{res}(Q(\overline{\sigma}(1))) - \mathrm{res}(Q(\overline{\sigma}(d_\alpha))) \\ &\neq 0. \end{aligned}$$

Hence, we have $\mathrm{val}(P(\overline{a} - \overline{\sigma}(y_\alpha))) = \mathrm{val}(p_{i_0}) + \mathrm{val}(\sigma^{i_0}(\varepsilon_\alpha)) = \min_i\{\mathrm{val}(p_i) + \mathrm{val}(a_i - \sigma^i(y_\alpha))\}$. ■

And now let us prove another embedding theorem for immediate extensions. Let M_1 and $M_2 \models \mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be sufficiently saturated, $N_i \leqslant M_i$ have no immediate extension in M_i and be σ-Henselian (as we will see in Remark 6.21 this second hypothesis follows from the first one), and let $C_i \leqslant N_i$ be such that $\mathrm{res}(\mathbf{K}(C_1))$ is linearly closed and $f : C_1 \to C_2$ an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}^{\mathbf{RV}-\mathrm{Mor}}$-isomorphism.

Definition 6.19 (Minimal term of a pseudo-convergent sequence)**.** Let (x_α) be a pseudo-convergent sequence of C_1. We say that $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is its minimal term

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if it is minimal such that it is monic polynomial in x_n and it is σ-pseudo-solved by a pseudo-convergent sequence equivalent to (x_α).

Note that any pseudo-convergent sequence has a minimal term, as any pseudo-convergent sequence σ-pseudo-solves 0.

Proposition 6.20. *Let (x_α) be a pseudo-convergent sequence of $\mathbf{K}(C_1)$ (indexed by a limit ordinal) which is eventually in $\mathcal{R}$. Let (t, I, m, d) be its minimal term. Then*

(i) *there exist $a_1 \in N_1$ and $a_2 \in N_2$ such that $x_\alpha \rightsquigarrow a_1$, $f(x_\alpha) \rightsquigarrow a_2$ and $t(\overline{\sigma}(a_1)) = 0 = t^f(\overline{\sigma}(a_2))$;*

(ii) *for any such a_1, $C_1\langle a_1\rangle_\sigma$ is an immediate extension of C_1;*

(iii) *for any such a_i, f can be extended to an $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism sending a_1 to a_2.*

Proof. If t is zero, it suffices to choose any a_1 and a_2 such that $x_\alpha \rightsquigarrow a_1$ and $f(x_\alpha) \rightsquigarrow a_2$. These exist in M_i, and we will see in the end why they exist in N_i. Let us now assume that t is not zero. By Remark 6.17 (and Propositions 6.13 and 6.18), we find α_0 and $\overline{d} \in \mathbf{K}(C_1)$ that linearly approximate t at prolongations on $b_0 := \mathring{\mathcal{B}}_{\mathrm{val}(x_{\alpha_0+1}-x_{\alpha_0})}(x_{\alpha_0+1})$. By Proposition 6.18, we can find a pseudo-convergent sequence (z_α) of C_1 equivalent to (x_α) such that, for all pseudo-limit a of (x_α), $\mathrm{val}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(z_\alpha))) = \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\}$. If, for all such a, $\mathrm{val}(t(\overline{\sigma}(a))) < \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\}$ for α big enough, then $\mathrm{val}(t(\overline{\sigma}(a))) = \mathrm{val}(t(\overline{\sigma}(y_\alpha)))$. By compactness, $\mathrm{val}(t(\overline{\sigma}(x)))$ is constant on some b_0. But this contradicts the fact that we can find y_α equivalent to x_α that σ-pseudo-solves t.

Hence there exists a pseudo-limit a such that $\mathrm{val}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(a - z_\alpha))\} \geqslant \min_i\{\mathrm{val}(d_i) + \mathrm{val}(\sigma^i(\xi_0))\}$, where ξ_0 is the radius of b_0 and $(t, a, \overline{d}, \xi_0)$ is in σ-Hensel configuration. As N_1 is σ-Henselian, we can find $a_1 \in \mathbf{K}(N_1)$ such that $t(a_1) = 0$ and $\mathrm{val}(a_1 - a) \geqslant \max_i\{\mathrm{val}(\sigma^{-i}(t(\overline{\sigma}(a))d_i^{-1}))\} > \mathrm{val}(x_{\alpha+1} - x_\alpha)$; i.e., $x_\alpha \rightsquigarrow a_1$. As f is an $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-isomorphism, (t^f, I, m, d) is the minimal term of $(f(x_\alpha))$, and the same argument shows that there is $a_2 \in \mathbf{K}(N_2)$ such that $t^f(a_2) = 0$ and $f(x_\alpha) \rightsquigarrow a_2$.

If $t \neq 0$, let us now show that x_m must be the last variable appearing in t. If it is not, let x_n be that last variable. By Proposition 6.13, we can find $\mathcal{L}_{\mathcal{A},\mathcal{Q}}\big|_{\mathbf{K}}(A)$-terms E, P, and Q such that E is a strong unit in x_n, P and Q are polynomial in x_n, and, for all $\overline{a} \in \overline{b}_0$, $t(\overline{a}) = E(\overline{a})P(\overline{a})/Q(\overline{a})$. As $t(\overline{\sigma}(a_1)) = 0$, we also have $P(\overline{\sigma}(a_1)) = 0$, and because $(P, I, n, \infty) < (t, I, m, d)$, we have $\mathrm{rv}(P(\overline{a})) = 0$ (and hence $\mathrm{rv}(t(\overline{a})) = 0$) for all $\overline{a} \in \overline{b}_0$, contradicting the fact that we can find y_α equivalent to x_α that σ-pseudo-solves t.

We can now conclude as in Proposition 6.5 by extending f to $C_{1,n} := C_1\langle\overline{\sigma}_{\leqslant n}(a_1)\rangle$ progressively, by sending $\sigma^n(a_1)$ to $\sigma^n(a_2)$. For $n < m$, it is exactly the same, and for $n \geqslant m$, use the fact that $\sigma^n(a_1)$ is the only zero of $P^{\sigma^{n-m}}(X)$ in $(b_{n,0})$ for $\alpha_0 \gg 0$, where $P(X_m) = t(\overline{\sigma}_{\neq m}(a_1), X_m)$.

If $n < m$, we have proved in Proposition 6.13 that the extension is immediate. If $n \geqslant m$, we have just seen that $\sigma^n(a_1)$ is ACVF-definable over $\mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)$. It follows that $\sigma^n(a_1) \in \mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)^h$, which is an immediate extension of $\mathbf{K}(C_1\langle\overline{\sigma}_{\leqslant n-1}(a_1)\rangle)$, and we conclude by Proposition 3.26.

In the case where t is zero, we have yet to show that we can take $a_i \in N_i$. Let (u, J, n, e) be minimal over N_1 that is σ-pseudo-solved by a pseudo-converging sequence equivalent to x_α. We can find a_1 in M_1 such that $u(\overline{\sigma}(a_1)) = 0$ and $x_\alpha \rightsquigarrow a_1$. But then $\mathbf{K}(N_1\langle a_1\rangle_\sigma)$ is an immediate extension of N_1, and we must have $a_1 \in N_1$. ■

Remark 6.21. Note that we have just shown that, if we only assume that N_1 has no immediate extension in M_1 (and not that N_1 is σ-Henselian), then N_1 is maximally complete, and hence, by Proposition 4.14, it is σ-Henselian.

Definition 6.22 (Minimal term of a point)**.** Let $a \in M_1$. We say that $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is the minimal term of a over C_1 if it is minimal such that it is monic polynomial in x_m and $t(\overline{\sigma}(a)) = 0$.

Note that, because of Weierstrass preparation, minimal terms will always be polynomial in their last variable.

Definition 6.23 ((t, I, m, d)-fullness)**.** Let $(t, I, m, d) \in \mathcal{T}_0(C_1)$. We will say that C_1 is (t, I, m, d)-full if, for all pseudo-convergent sequences (x_α) (indexed by a limit ordinal) of elements in C_1 that are eventually in $\mathcal{R}$ with minimal term $(u, J, n, e) < (t, I, m, d)$, (x_α) has a pseudo-limit in C_1.

Corollary 6.24. *Let (x_α) be a maximal pseudo-convergent sequence in C_1 (indexed by a limit ordinal) pseudo-converging to some $a_1 \in \mathcal{R}(M_1)$. If $(t, I, m, d) \in \mathcal{T}_0(C_1)$ is its minimal term over C_1 and C_1 is (t, I, m, d)-full, then $\mathbf{K}(C_1\langle a_1\rangle_\sigma)$ is an immediate extension of $\mathbf{K}(C_1)$, and f extends to a morphism from $C_1\langle a_1\rangle_\sigma$ into N_2.*

Proof. Since C_1 is (t, I, m, d)-full, (x_α) (or any equivalent pseudo-convergent sequence) cannot pseudo-solve a term of order-degree strictly less than (t, I, m, d) (this would contradict either (t, I, m, d)-fullness of C_1 or maximality of (x_α)). By Propositions 6.13 and 6.18, there is a tuple $\overline{d}$ and there is a sequence (y_α) equivalent to (x_α) such that

$$\operatorname{val}(t(\overline{\sigma}(y_\alpha))) = \operatorname{val}(t(\overline{\sigma}(a)) - t(\overline{\sigma}(y_\alpha))) = \min_i\{\operatorname{val}(d_i) + \operatorname{val}(\sigma^i(a - y_\alpha))\};$$

i.e., $t(\overline{\sigma}(y_\alpha)) \rightsquigarrow 0$. We have just shown that t is the minimal term of the pseudo-convergent sequence (x_α), and thus we can now apply Proposition 6.20. ■

From now on, suppose that $\mathbf{K}(N_i)$ is an immediate extension of $\mathbf{K}(C_i)$; hence it is a maximal immediate extension of $\mathbf{K}(C_i)$ in M_i.

Corollary 6.25. *Suppose that all $a \in \mathcal{R}(N_1)$ with a minimal term of order-degree strictly smaller than (t, I, m, d) are already in C_1. Then C_1 is (t, I, m, d)-full.*

Proof. Let (x_α) be a pseudo-convergent sequence of C_1 (indexed by a limit ordinal) that is eventually in $\mathcal{R}$ and $(u, J, n, e) < (t, I, m, d)$ that is σ-pseudo-solved by (x_α). We may assume that (u, J, n, e) is its minimal term. By Proposition 6.20, there is $a_1 \in N_1$ such that $x_\alpha \rightsquigarrow a_1$ and $u(a_1) = 0$. Hence a_1 has a minimal polynomial of order-degree strictly lower than (t, I, m, d), so $a_1 \in C_1$, and C_1 is indeed (t, I, m, d)-full. ■

Corollary 6.26. *The isomorphism f extends to an isomorphism between N_1 and N_2; i.e., maximum immediate extensions (in some saturated model) — and hence maximally complete extensions — are unique up to isomorphism.*

We could prove this corollary without using the notion of fullness and without doing the extensions in the right order: just pick any maximal pseudo-convergent sequence indexed by a limit ordinal, find its minimal term, and apply Proposition 6.20 to extend f some more and iterate. But the following proof provides a better description of the information needed to describe the type of a given point in an immediate extension.

Proof. Let us consider the extensions $C_1 \leqslant L_\alpha \leqslant N_1$ defined by taking $L_{\alpha+1} = L_\alpha\langle c_\alpha\rangle_\sigma$, where $c_\alpha \in \mathcal{R}(N_1) \smallsetminus L_\alpha$ has a minimal term of minimal order-degree over $L_{\alpha+1}$ and $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for λ limit. Then we can show by induction that we can extend f to L_α in a coherent way.

Let us suppose that we have extended f to f_α on L_α. Let $a = c_\alpha$. Let $x_\beta \rightsquigarrow a$ be a maximal pseudo-converging sequence of L_α. If (t, I, m, d) is a minimal term of a, then by Corollary 6.25, L_α is (t, I, m, d)-full. Applying Corollary 6.24, we obtain that f_α can be extended to $L_\alpha\langle a\rangle_\sigma = L_{\alpha+1}$. The limit case is trivial.

As N_1 is the field generated by $\bigcup_\alpha L_\alpha$, by Remark 5.1 we can extend f to a morphism from N_1 into N_2. Now, if f is not onto, pick $a \in \mathbf{K}(N_2) \smallsetminus \mathbf{K}(f(N_1))$, (x_α) maximal pseudo-converging to a and (t, I, m, d) its minimal term. Then, applying Proposition 6.20 the other way round, we would find an immediate extension of N_1 in M_1, but that is absurd. ■

6.3. Relative quantifier elimination

Theorem A:

The theory $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ eliminates quantifiers resplendently relatively to **RV**.

Proof. By Proposition A.9, it suffices to show that $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ eliminates quantifiers relatively to **RV**. Note that if two models of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ contain isomorphic substructures they have the same characteristic and residual characteristic; hence it also suffices to prove the result for $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$ and $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen},0,p}$. Let us first consider the equicharacteristic zero case.

It suffices to show that, if M_1 and M_2 are sufficiently saturated models of $\mathrm{T}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$, f a partial $\mathcal{L}^{\mathbf{RV}-\mathrm{Mor}}_{\mathcal{A},\mathcal{Q}}$-isomorphism with (small) domain C_1, and $a_1 \in K(M_1)$, f can be extended to $C_1\langle a_1\rangle_\sigma$. Let $N_1 \leqslant M_1$ with no immediate extension in M_1 and containing both C_1 and a_1. By Morleyization on **RV** and Lemma A.11, we can extend f to $D_1 \leqslant N_1$ such that $\mathbf{RV}(D_1) = \mathbf{RV}(N_1)$. Then applying Corollary 6.8 repetitively we can extend f to $E_1 \leqslant N_1$ such that $\mathrm{rv}(\mathbf{K}(E_1)) = \mathbf{RV}(E_1)$. Now $\mathbf{K}(N_1)$ is a maximal immediate extension of $\mathbf{K}(E_1)$, and we can extend f to N_1 by Proposition 6.20.

Now that we know the equicharacteristic zero case, the mixed characteristic case follows from Propositions B.5 and 4.23. ■

We also obtain the corresponding results when there are angular components. Let $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ be $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ enriched with a symbol $\sigma : \mathbf{K} \to \mathbf{K}$, symbols $\sigma^n : \mathbf{R}^n \to \mathbf{R}^n$, and a

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symbol $\sigma_{\mathbf{\Gamma}} : \mathbf{\Gamma}^{\infty} \to \mathbf{\Gamma}^{\infty}$. Let $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be the $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of σ-Henselian analytic difference valued fields with a linearly closed residue field and angular components that are compatible with σ; i.e., $\mathrm{ac}_n \circ \sigma = \sigma_n \circ \mathrm{ac}_n$. Let $\mathcal{L}^{\mathrm{ac,fr}}_{\mathcal{A},\mathcal{Q},\sigma}$ be the enrichment of $\mathcal{L}^{\mathrm{ac,fr}}$ with the same symbols and $\mathrm{T}^{\mathrm{ac},e-\mathrm{fr}}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ be the theory of finitely ramified characteristic $(0, p)$ valued fields as above with ramification index at most e; i.e., $e \cdot 1 \geqslant \mathrm{val}(p)$.

Corollary 6.27. *$\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ and $\mathrm{T}^{\mathrm{ac},e-\mathrm{fr}}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ for all p and e eliminate $\mathbf{K}$-quantifiers resplendently.*

Proof. By Proposition A.9, resplendence comes for free once we have $\mathbf{K}$-quantifier elimination. Moreover, by Propositions 1.8 and B.5, we can transfer quantifier elimination in an $\mathbf{RV}$-enrichment of $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ (cf. Theorem A) to quantifier elimination in a definable $\mathbf{R} \cup \mathbf{\Gamma}$-enrichment of $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$, and hence $\mathbf{K}$-quantifier elimination in $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$.

The proof for $\mathrm{T}^{\mathrm{ac},e-\mathrm{fr}}_{\mathcal{A},\sigma-\mathrm{Hen},p}$ now follows by Remark 1.9.3. ■

Remark 6.28.

1. In a valued field with isometry and $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$, angular components that are compatible with σ are determined by their restriction to the fixed field. Indeed, if $\mathrm{val}(x) = \mathrm{val}(\varepsilon)$, where $\varepsilon \in \mathrm{Fix}(\mathbf{K})$, then $\mathrm{ac}_n(x) = \mathbf{R}_n(x\varepsilon^{-1})\mathrm{ac}_n(\varepsilon)$. In fact, any angular components on the fixed field can be extended using this formula to angular components on the whole field that are compatible with σ, and hence any valued field with an isometry and $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$ can be elementarily embedded into a valued field with an isometry and compatible angular components.

2. In fact, the existence of angular components in a σ-Henselian valued field with an isometry implies that $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$.

Until the end of this section, we will add constants to $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ and $\mathcal{L}^{\mathrm{ac,fr}}_{\mathcal{A},\mathcal{Q},\sigma}$ for $\mathrm{ac}_n(t)$ and $\mathrm{val}(t)$ for every $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}\big|_{\mathbf{K}}$-term t without any free variables. The reason for which we need to add these constants is that, although these are $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$-terms, we may have no trace of them in $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}\Big|_{\mathbf{R}}$ and $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}\Big|_{\mathbf{\Gamma}}$. Ax–Kochen–Eršov type results now follow by the usual arguments.

Corollary 6.29 (Ax–Kochen–Eršov principle for analytic difference valued fields)**.**

(i) *Let $\mathcal{L}$ be an $\mathbf{R}$-extension of a $\mathbf{\Gamma}$-extension of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$, T an $\mathcal{L}$-theory containing $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen},0,0}$, and M and $N \models T$. Then the following hold.*

(a) *$M \equiv N$ if and only if $\mathbf{R}_1(M) \equiv \mathbf{R}_1(N)$ as $\mathcal{L}|_{\mathbf{R}_1}$-structures and $\mathbf{\Gamma}^{\infty}(M) \equiv \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures.*

(b) *Suppose that $M \leqslant N$. Then $M \preccurlyeq N$ if and only if $\mathbf{R}_1(M) \preccurlyeq \mathbf{R}_1(M)$ as $\mathcal{L}|_{\mathbf{R}_1}$-structures and $\mathbf{\Gamma}^{\infty}(M) \preccurlyeq \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures.*

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(ii) *Let $\mathcal{L}$ be an $\mathbf{R}$-extension of a $\mathbf{\Gamma}$-extension of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$, T an $\mathcal{L}$-theory containing $\mathrm{T}^{\mathrm{ac},e-\mathrm{fr}}_{\mathcal{A},\sigma-\mathrm{Hen},p}$, and M and $N \models T$. Then the following hold.*

(a) *$M \equiv N$ if and only if $\mathbf{R}(M) \equiv \mathbf{R}(N)$ as $\mathcal{L}|_{\mathbf{R}}$-structures and $\mathbf{\Gamma}^{\infty}(M) \equiv \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures.*

(b) *Suppose that $M \leqslant N$. Then $M \preccurlyeq N$ if and only if $\mathbf{R}(M) \preccurlyeq \mathbf{R}(N)$ as $\mathcal{L}|_{\mathbf{R}}$-structures and $\mathbf{\Gamma}^{\infty}(M) \preccurlyeq \mathbf{\Gamma}^{\infty}(N)$ as $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structures.*

Remark 6.30.

1. In mixed characteristic with finite ramification, if $\mathcal{R} = \mathcal{O}$, we have better results. Indeed, the trace of any unit E on any $\mathbf{RV}_k$ is given by the trace of a polynomial (which depends only on E and not on its interpretation), and the E_k are in fact useless. Hence the $\mathbf{R}_n$ are pure rings with an automorphism. If there is no ramification (i.e., $e = 1$), the $\mathbf{R}_n$ are ring schemes over $\mathbf{R}_1$ (the Witt vectors of length n) (the ring scheme structure does not depend on the actual model we are looking at, contrary to the general finite ramification case), and the automorphism on $\mathbf{R}_n$ can be defined using the automorphism on $\mathbf{R}_1$, and hence $\mathbf{R}$ is definable in $\mathbf{R}_1$. Finally, if σ is a lifting of the Frobenius, σ_0 is definable in the ring structure of $\mathbf{R}_1$. It follows that we obtain Ax–Kochen–Eršov results looking only at $\mathbf{R}_1$ as a ring and $\mathbf{\Gamma}^{\infty}$ as an ordered abelian group (after adding some constants).

2. The fact that the E_k are useless is also true in equicharacteristic zero when $\mathcal{R} = \mathcal{O}$.

3. It also follows that, in equicharacteristic zero or mixed characteristic with finite ramification (with angular component), $\mathbf{R}$ and $\mathbf{\Gamma}^{\infty}$ are stably embedded and have pure $\mathcal{L}|_{\mathbf{R}}$-structure (respectively, $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structure), where $\mathcal{L}$ is either $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ or $\mathcal{L}^{\mathrm{ac,fr}}_{\mathcal{A},\mathcal{Q},\sigma}$. In particular, it will make sense to speak of the theory induced on $\mathbf{R}$ or $\mathbf{\Gamma}^{\infty}$.

Proposition 6.31. *Let $\mathcal{L}$ be the language $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$ enriched with predicates P_n on $\mathbf{RV}_1$ interpreted as $n|\mathrm{val}_1(x)$. The $\mathcal{L}$-theory of W_p is axiomatized by $\mathrm{T}_{\mathcal{A},\sigma-\mathrm{Hen}}$ and σ_1 is the Frobenius, the induced theory on $\mathbf{R}_1$ is ACF_p, p has minimal positive valuation, and $\mathbf{\Gamma}$ is a $\mathbb{Z}$-group. Moreover, $\mathbf{R}_1$ is a pure algebraically closed valued field and $\mathbf{\Gamma}$ is a pure $\mathbb{Z}$-group, and they are stably embedded.*

Proof. Any model of that theory has definable angular components compatible with σ. And these angular components extend the usual ones on the field of constants $\mathrm{W}(\overline{\mathbb{F}_p}^{\mathrm{alg}})$. Hence the only constants we add are for elements of $\overline{\mathbb{F}_p}^{\mathrm{alg}} \subseteq \mathbf{R}_1$ and $\mathbb{Z} \subseteq \mathbf{\Gamma}$. The proposition now follows from the discussion above (and the fact that ACF and $\mathbb{Z}$-groups are model complete). ■

7. The NIP property in analytic difference valued fields

Let us first recall what is shown by Delon and Bélair in the algebraic case [7, 13]. Let $\mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ be the $\mathcal{L}^{\mathrm{ac}}$-theory of Henselian valued fields with angular component maps.

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Theorem 7.1.

Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^{\infty}$-enrichment of $\mathcal{L}^{\mathrm{ac}}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathrm{Hen}}$ be an $\mathcal{L}$-theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^{\infty}$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structure) are* NIP *(not the independence property).*

Proof. See [7, Théorème 7.4]. The resplendence of the theorem is not stated there, but the proof is exactly the same after enriching on $\mathbf{R}$ and $\mathbf{\Gamma}^{\infty}$. ∎

This result can be extended first to analytic fields and then to analytic fields with an automorphism.

Corollary 7.2. *Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^{\infty}$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q}}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathrm{Hen}}$ be an $\mathcal{L}$ theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is* NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^{\infty}$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structure) are* NIP.

Proof. Suppose that T is not NIP. Then there is a formula $\varphi(x,\overline{y})$ which has the independence property and where $|x| = 1$. Note that, since for any sort there is an $\varnothing$-definable function from $\mathbf{K}$ onto that sort, we may assume that x and $\overline{y}$ are $\mathbf{K}$-variables. By Remark 5.6.2, there is an $\mathcal{L} \smallsetminus (\mathcal{A} \cup \{\mathcal{Q}\})$-formula $\psi(x,\overline{z})$ and there are $\mathcal{L}_{\mathcal{A},\mathcal{Q}}|_{\mathbf{K}}$ terms $\overline{u}(\overline{y})$ such that $\varphi(x,\overline{y})$ is equivalent to a $\psi(x,\overline{u}(\overline{y}))$. But then ψ would have the independence property too, contradicting Theorem 7.1. ∎

Corollary 7.3. *Let $\mathcal{L}$ be an $\mathbf{R}$-enrichment of a $\mathbf{\Gamma}^{\infty}$-enrichment of $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma}$ and $T \supseteq \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathrm{Hen}}$ be an $\mathcal{L}$ theory implying either equicharacteristic zero or finite ramification in mixed characteristic. Then T is* NIP *if and only if $\mathbf{R}$ (with its $\mathcal{L}|_{\mathbf{R}}$-structure) and $\mathbf{\Gamma}^{\infty}$ (with its $\mathcal{L}|_{\mathbf{\Gamma}^{\infty}}$-structure) are* NIP.

Proof. Suppose that T is not NIP. Then there is a formula $\varphi(x,\overline{y})$ which has the independence property (where x and the $\overline{y}$ are $\mathbf{K}$-variables). By Corollary 6.27, we may assume that φ is without $\mathbf{K}$-quantifiers; i.e., there is a $\mathbf{K}$-quantifier free $\mathcal{L}^{\mathrm{ac}}_{\mathcal{A},\mathcal{Q},\sigma} \smallsetminus \{\sigma\}$-formula $\psi(\overline{x},\overline{z})$ such that $\varphi(x,\overline{y})$ is equivalent to $\psi(\overline{\sigma}(x),\overline{\sigma}(\overline{y}))$. But then ψ would have the independence property too, contradicting Corollary 7.2. ∎

Remark 7.4. In the isometry case with $\mathrm{val}(\mathrm{Fix}(\mathbf{K})) = \mathrm{val}(\mathbf{K})$, this last result also holds without angular components because any such valued field can be elementarily embedded into a valued field with angular components compatible with σ.

Corollary 7.5. *The $\mathcal{L}_{\mathcal{A},\mathcal{Q},\sigma}$-theory of W_p is* NIP.

Proof. This is an immediate corollary of Remark 7.4, Corollary 7.3, and the fact that $\mathbf{R}$ is definable in $\mathbf{R}_1$ which is a pure algebraically closed field (where the Frobenius automorphism is definable) and that $\mathbf{\Gamma}$ is a pure $\mathbb{Z}$-group (see Proposition 6.31). ∎

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Appendices

A. Resplendent relative quantifier elimination

This section, although it may appear fastidious and nitpicking, is actually an attempt at clarifying some notions and properties that are often assumed to be clear when studying model theory of valued fields, but may actually need precise and careful presentation. In all this section, $\mathcal{L}$ will denote a language and Σ, Π a partition of its sorts.

Definition A.1 (Restriction)**.** If $\mathcal{L}' \subseteq \mathcal{L}$ is another language and T an $\mathcal{L}$-theory, we will denote by $T|_{\mathcal{L}'}$ the $\mathcal{L}'$-theory $\{\varphi$ an $\mathcal{L}'$-formula : $T \models \varphi\}$, and, if C is an $\mathcal{L}$-structure, $C|_{\mathcal{L}'}$ will have underlying set $\bigcup_{S\in\mathcal{L}'} S(C)$ with the obvious $\mathcal{L}'$-structure. In particular, when Σ is a set of $\mathcal{L}$ sorts, let $\mathcal{L}|_\Sigma$ be the restriction of $\mathcal{L}$ to the predicate and function symbols that only concern the sorts in Σ. Then we will write $T|_\Sigma := T|_{\mathcal{L}|_\Sigma}$ and $C|_\Sigma := C|_{\mathcal{L}|_\Sigma}$.

Note that the restriction is a functor from $\mathbf{Str}(T)$ to $\mathbf{Str}(T|_{\mathcal{L}'})$ respecting models, cardinality, and elementary submodels (see § B for the definitions).

Definition A.2 (Enrichment)**.** Let $\mathcal{L}_e \supseteq \mathcal{L}$ be another language and Σ_e the set of new $\mathcal{L}_e$-sorts, i.e., the $\mathcal{L}_e$-sorts that are not $\mathcal{L}$-sorts. The language $\mathcal{L}_e$ is said to be a Σ-enrichment of $\mathcal{L}$ if $\mathcal{L}_e \smallsetminus \mathcal{L}_e|_{\Sigma\cup\Sigma_e} \subseteq \mathcal{L}$; i.e., the enrichment is limited to the new sorts and the sorts in Σ. If, moreover, $\Sigma_e = \varnothing$ and $\mathcal{L}_e \smallsetminus \mathcal{L}$ consists only of function symbols, we will say that $\mathcal{L}_e$ is a Σ-term enrichment of $\mathcal{L}$.

Let T be an $\mathcal{L}$-theory. An $\mathcal{L}_e$-theory $T_e \supseteq T$ is said to be a definable enrichment of T if there are no new sorts and, for every predicate $P(\overline{x})$ (respectively, function $f(\overline{x})$) symbol in $\mathcal{L}_e \smallsetminus \mathcal{L}$, there is an $\mathcal{L}$-formula $\varphi_P(\overline{x})$ (respectively, $\varphi_f(\overline{x}, y)$ such that $T \models \forall\, \overline{x}\, \exists^{=1} y,\ \varphi_f(\overline{x}, y)$) and that $T_e = T \cup \{P(\overline{x}) \leftrightarrow \varphi_P(\overline{x})\} \cup \{\varphi_f(\overline{x}, f(\overline{x}))\}$.

Definition A.3 (Morleyization)**.** The Morleyization of $\mathcal{L}$ on Σ is the language $\mathcal{L}^{\Sigma-\mathrm{Mor}} := \mathcal{L} \cup \{P_\varphi(\overline{x}) : \varphi(\overline{x})$ an $\mathcal{L}|_\Sigma$-formula$\}$. If T is an $\mathcal{L}$-theory, the Morleyization of T on Σ is the following $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-theory, $T^{\Sigma-\mathrm{Mor}} := T \cup \{P_\varphi(\overline{x}) \leftrightarrow \varphi(\overline{x})\}$, and, if M is an $\mathcal{L}$-structure, $M^{\Sigma-\mathrm{Mor}}$ is the $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-structure with the same $\mathcal{L}$-structure as M and where P_φ is interpreted by $\varphi(M)$.

On the other hand, we will say that an $\mathcal{L}$-theory T is Morleyized on Σ if every $\mathcal{L}|_\Sigma$-formula is equivalent, modulo T, to a quantifier free $\mathcal{L}|_\Sigma$-formula.

Note that $T^{\Sigma-\mathrm{Mor}}$ is a definable Σ-enrichment of T, and if $M \models T$ then $M^{\Sigma-\mathrm{Mor}} \models T^{\Sigma-\mathrm{Mor}}$.

Definition A.4 (Elementary on Σ)**.** Let M_1 and M_2 be two $\mathcal{L}$-structures. A partial isomorphism $M_1 \to M_2$ is said to be Σ-elementary if it is a partial $\mathcal{L}^{\Sigma-\mathrm{Mor}}$-isomorphism.

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Definition A.5 (Resplendent relative elimination of quantifiers)**.** Let T be an $\mathcal{L}$-theory. We say that T eliminates quantifiers relatively to Σ if $T^{\Sigma-\mathrm{Mor}}$ eliminates quantifiers.

We say that T eliminates quantifiers resplendently relatively to Σ if, for any Σ-enrichment $\mathcal{L}_e$ of $\mathcal{L}$ (with possibly new sorts Σ_e) and any $\mathcal{L}_e$-theory $T_e \supseteq T$, T_e eliminates quantifiers relatively to $\Sigma \cup \Sigma_e$.

Definition A.6 (Resplendent elimination of quantifiers from a sort)**.** We will say that an $\mathcal{L}$-theory T eliminates Π-quantifiers if every $\mathcal{L}$-formula is equivalent modulo T to a formula where quantification only occurs on variables from the sorts in Σ.

We will say that T eliminates Π-quantifiers resplendently if, for any Σ-enrichment $\mathcal{L}_e$ of $\mathcal{L}$ and any $\mathcal{L}_e$-theory $T_e \supseteq T$, T_e eliminates Π-quantifiers.

Definition A.7 (Closed sorts)**.** We will say that Σ is closed if $\mathcal{L} \smallsetminus (\mathcal{L}|_\Pi \cup \mathcal{L}|_\Sigma)$ only consists of function symbols $f : \prod_i P_i \to S$, where $P_i \in \Pi$ and $S \in \Sigma$. Equivalently, any predicate involving a sort in Σ and any function with a domain involving a sort in Σ only involves sorts in Σ.

Remark A.8.

1. Note that if the sorts Σ are closed then, in any Σ-enrichment (with possibly new sorts Σ^e) of a Π-enrichment of $\mathcal{L}$ (or vice versa), the sorts $\Sigma \cup \Sigma^e$ are still closed.

2. Elimination of quantifiers relative to Σ implies elimination of Π-quantifiers. But the converse is in general not true. Indeed, if $\mathcal{L}$ is a language with two sorts S_1 and S_2 and a predicate on $S_1 \times S_2$, then the formula $\exists x R(x, y)$ is an S_2-quantifier free formula, but there is no reason for it to be equivalent to any quantifier free $\mathcal{L}^{S_1-\mathrm{Mor}}$-formula.

3. However, if the sorts Σ are closed, then it follows from Remark A.10.1 that T eliminates Π-quantifiers if and only if T eliminates quantifiers relatively to Σ. If $\mathcal{L}_e$ is a Σ-enrichment of $\mathcal{L}$ with new sorts Σ_e, then $\Sigma \cup \Sigma_e$ is still closed, and thus the equivalence is also true resplendently.

We will now suppose that Σ is *closed*, and we will denote by $\mathcal{F}$ the set of functions $f : \prod_i P_i \to S$, where $P_i \in \Pi$ and $S \in \Sigma$.

Proposition A.9. *Let T be an $\mathcal{L}$-theory. If T eliminates quantifiers relatively to Σ then T eliminates quantifiers resplendently relatively to Σ.*

Let us begin with some remarks and lemmas that will have a more general interest.

Remark A.10.

1. Any atomic $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$, where $\overline{x}$ are Π-variables and $\overline{y}$ are Σ-variables, is either of the form $\psi(\overline{x})$, where ψ is an atomic $\mathcal{L}|_\Pi$-formula, or of the form $\psi(\overline{f}(\overline{u}(\overline{x})), \overline{y})$, where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms, and $\overline{f}$ are functions from $\mathcal{F}$.

2. If T eliminates quantifiers relatively to Σ, it follows from Remark A.10.1 above that, for any $M \models T$, any $\mathcal{L}(M)$-definable set in a product of sorts from Σ is defined by

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a formula of the form $\varphi(\overline{x}, \overline{f}(\overline{a}), \overline{b})$, where φ is a $\mathcal{L}|_\Sigma$-formula. Hence Σ is stably embedded in T; i.e., any $\mathcal{L}(M)$-definable subset of Σ is in fact $\mathcal{L}(\Sigma(M))$-definable. Moreover, these sets are in fact $\mathcal{L}|_\Sigma(\Sigma(M))$-definable. In that case, we say that Σ is a pure $\mathcal{L}|_\Sigma$-structure.

Lemma A.11. *Suppose that T is an $\mathcal{L}$-theory Morleyized on Σ. Then, for any sufficiently saturated M_1, $M_2 \models T$, any partial $\mathcal{L}$-isomorphism $f : M_1 \to M_2$ with small domain C_1, and any $c_1 \in \Sigma(M_1)$, f can be extended to a partial $\mathcal{L}$-isomorphism whose domain contains c_1.*

Proof. First we may assume that $C_1 \leqslant M_1$, and in particular, for all $g \in \mathcal{F}$, $g(C_1) \subseteq \Sigma(C_1)$. Because f is a partial $\mathcal{L}$-isomorphism and T is Morleyized on Σ, $f|_\Sigma$ is a partial elementary $\mathcal{L}|_\Sigma$-isomorphism. By saturation of M_2 we can extend $f|_\Sigma$ to $f'\big|_\Sigma : M_1|_\Sigma \to M_2|_\Sigma$, a partial elementary $\mathcal{L}|_\Sigma$-isomorphism whose domain contains c_1. Let $f' = f|_\Pi \cup f'\big|_\Sigma$.

As $f|_\Pi$ is a partial $\mathcal{L}|_\Pi$-isomorphism, f' respects formulas $\varphi(\overline{x})$, where φ is an atomic $\mathcal{L}|_\Pi$-formula ($f|_\Pi$ also respects $\mathcal{L}|_\Pi$-terms). Moreover, as for all $g \in \mathcal{F}$, $f'\big|_{g(C_1)} = f|_{g(C_1)}$, f' still respects g. As $f'\big|_\Sigma$ is a partial $\mathcal{L}|_\Sigma$-isomorphism, it respects all atomic $\mathcal{L}|_\Sigma$-formulas. It follows that f' also respects formulas of the form $\psi(\overline{g}(\overline{u}(\overline{x})), \overline{y})$, where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms, and $\overline{g} \in \mathcal{F}$. By Remark A.10.1, f' respects all atomic $\mathcal{L}$-formulas, and hence is a partial $\mathcal{L}$-isomorphism. ■

Definition A.12 (Generated structure)**.** Let $\mathcal{L}$ be a language, M an $\mathcal{L}$-structure, and $C \subseteq M$. The $\mathcal{L}$-structure generated by C will be denoted $\langle C \rangle_\mathcal{L}$. If C is an $\mathcal{L}$-structure and $\overline{c} \in M$, the $\mathcal{L}$-structure generated by C and $\overline{c}$ will be denoted $C\langle \overline{c} \rangle_\mathcal{L}$.

Lemma A.13. *Let M_1, $M_2 \models T$, $f : M_1 \to M_2$ a partial $\mathcal{L}$-isomorphism with domain $C_1 \leqslant M_1$, and let $c_1 \in \Pi(M_1)$ be such that $\Sigma(C_1\langle c_1 \rangle_\mathcal{L}) \subseteq \Sigma(C_1)$. Suppose that f' is a partial $\mathcal{L}|_\Pi \cup \mathcal{F}$-isomorphism extending f whose domain is $C_1\langle c_1 \rangle_\mathcal{L}$. Then f' is also a partial $\mathcal{L}$-isomorphism.*

Proof. First, by hypothesis, f' respects atomic $\mathcal{L}|_\Pi$-formulas. Moreover, as $\Sigma(C_1\langle c_1 \rangle_\mathcal{L}) \subseteq \Sigma(C_1)$, $f'\big|_\Sigma = f|_\Sigma$, and it is a partial $\mathcal{L}|_\Sigma$-isomorphism. As, by hypothesis, f' respects $g \in \mathcal{F}$, it respects all formulas of the form $\psi(\overline{g}(\overline{u}(\overline{x})), \overline{y})$, where ψ is an atomic $\mathcal{L}|_\Sigma$-formula, $\overline{u}$ are $\mathcal{L}|_\Pi$-terms, and $g \in \mathcal{F}$. Hence, by Remark A.10.1, f' is a partial $\mathcal{L}$-isomorphism. ■

Proof (Proposition A.9). We want to show that, if $\mathcal{L}_e$ is a Σ-enrichment of $\mathcal{L}$ (with new sorts Σ_e) and $T_e \supseteq T$ an $\mathcal{L}_e$-theory, then $T_e^{\Sigma \cup \Sigma_e - \mathrm{Mor}}$ eliminates quantifiers. It suffices to show that, for all M_1 and $M_2 \models T_e$ that are $|\mathcal{L}_e|^+$-saturated, for all partial $\mathcal{L}_e^{\Sigma \cup \Sigma_e - \mathrm{Mor}}$-isomorphism $f : M_1 \to M_2$ of domain C_1 with $|C_1| \leqslant |\mathcal{L}_e|$, and for all $c_1 \in M_1$, f can be extended to a partial $\mathcal{L}_e^{\Sigma \cup \Sigma_e - \mathrm{Mor}}$-isomorphism whose domain contains c_1.

Note first that $\Sigma \cup \Sigma_e$ is closed. If $c_1 \in \Sigma \cup \Sigma_e(M_1)$, then we can conclude by Lemma A.11 (where $\mathcal{L}$ is now $\mathcal{L}_e^{\Sigma \cup \Sigma_e - \mathrm{Mor}}$). If $c_1 \in \Pi(M_1)$, by repetitively applying Lemma A.11, we can extend f to f' whose domain contains all of $\Sigma \cup \Sigma_e(C_1\langle c_1 \rangle_{\mathcal{L}_e})$. Then f' is in particular an $\mathcal{L}^{\Sigma - \mathrm{Mor}}$-isomorphism and, as T eliminates quantifiers relatively

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to Σ, f' is in fact a partial elementary $\mathcal{L}$-isomorphism that can be extended to a partial $\mathcal{L}$-isomorphism f'' whose domain contain c_1. But, by Lemma A.13, $f''|_{C_1\langle c_1\rangle_{\mathcal{L}_e}}$ is also a partial $\mathcal{L}_e^{\Sigma\cup\Sigma_e-\mathrm{Mor}}$-isomorphism. ■

B. Categories of structures

Recall that structures are always non-empty.

Definition B.1 (Str(T))**.** Let $\mathcal{L}$ be a language, and let T be an $\mathcal{L}$-theory. We will denote by $\mathrm{Str}(T)$ the category whose objects are the $\mathcal{L}$-structures that can be embedded in a model of T (i.e., models of $T_\forall$) and whose morphisms are the $\mathcal{L}$-embeddings between those structures.

Moreover, let T_i be an $\mathcal{L}_i$-theory for $i = 1, 2$, $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ be a functor, and κ be a cardinal. We will denote by $\mathrm{Str}_{F,\kappa}(T_2)$ the full subcategory of $\mathrm{Str}(T_2)$ of structures that embed into some $F(M)$ for $M \models T_1$ κ-saturated.

A functor $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ is said to respect

- models if, for all $M \models T_1$, $F(M) \models T_2$;
- κ-saturated models if, for all κ-saturated $M \models T_1$, $F(M) \models T_2$;
- cardinality if, for all $C \models T_{1,\forall}$, $|F(C)| \leqslant |C|$;
- cardinality up to κ if, for all $C \models T_{1,\forall}$, $|F(C)| \leqslant |C|^\kappa$;
- elementary submodels if, for all $M_1 \preccurlyeq M_2 \models T_1$, $F(M_1) \preccurlyeq F(M_2)$.

Let Σ_i be a closed set of $\mathcal{L}_i$-sorts for $i = 1, 2$. We say that $f : C_1 \to C_2$ in $\mathrm{Str}(T_1)$ is a Σ_1-extension if $C_2 \smallsetminus f(C_1) \subseteq \Sigma_1(C_2)$. We say that the functor F sends Σ_1 to Σ_2 if, for all Σ_1-extensions $C_1 \to C_2$, $F(C_1) \to F(C_2)$ is a Σ_2-extension.

Let me recall some basic notions of category theory. A natural transformation α between functors $F, G : \mathcal{C}_1 \to \mathcal{C}_2$ associates a morphism $\alpha_c \in \mathrm{Hom}_{\mathcal{C}_2}(F(c), G(c))$ to every object $c \in \mathcal{C}_1$ such that, for all morphism $f \in \mathrm{Hom}_{\mathcal{C}_1}(c, d)$, we have $G(f) \circ \alpha_c = \alpha_d \circ F(f)$. A natural transformation is said to be a natural isomorphism if, for all $c \in \mathcal{C}_1$, α_c is an isomorphism in $\mathcal{C}_2$. It is easy to check that, when α is a natural isomorphism, its inverse (namely the transformation that associates α_c^{-1} to any $c \in \mathcal{C}_1$) is also natural.

A pair of functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_2 \to \mathcal{C}_1$ are said to be an equivalence of categories between $\mathcal{C}_1$ and $\mathcal{C}_2$ if GF and FG are naturally isomorphic to the identity functor of, respectively, $\mathcal{C}_1$ and $\mathcal{C}_2$. We can always choose the natural isomorphisms $\alpha : FG \to \mathrm{Id}$ and $\beta : GF \to \mathrm{Id}$ such that $\alpha_F = F(\beta)$ and $\beta_G = G(\alpha)$, where $\alpha_F : c \mapsto \alpha_{F(c)}$ and $F(\alpha) : c \mapsto F(\alpha_c)$.

Until the end of this section, let κ be a cardinal, T_i be an $\mathcal{L}_i$-theory, Σ_i be a set of closed $\mathcal{L}_i$-sorts for $i = 1, 2$, and $\mathfrak{F}$ be a full subcategory of $\mathrm{Str}(T_1)$ containing the κ^+-saturated models such that, for any $C \to M_1 \models T_1$ where M_1 is κ^+-saturated and $|C| \leqslant \kappa$, there is some D in $\mathfrak{F}$ such that $C \to D \to M_1$ and $C \to D$ is a Σ_1-extension. Let $F : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2)$ and $G : \mathrm{Str}(T_2) \to \mathrm{Str}(T_1)$ be functors that respect cardinality up to κ and induce an equivalence of categories between $\mathfrak{F}$ and $\mathrm{Str}_{F,\kappa^+}(T_2)$. We will also suppose that G respects models and elementary submodels and sends Σ_2 to Σ_1, and that F respects κ^+-saturated models.

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The goal of this section is to show that these (somewhat technical) requirements are a way to transfer elimination of quantifiers results from one theory to another and to give a meaning to (and in fact extend) the impression that if theories are quantifier free bi-definable (whatever that means) then elimination of quantifiers in one theory should imply elimination in the other. Proposition B.5 will be used, for example, to deduce valued field quantifiers elimination with angular components from valued field quantifiers elimination with sectioned leading terms. It will also be used to reduce the mixed characteristic case to the equicharacteristic zero case.

Proposition B.2 is only used to prove Corollary B.4, which in turn will be very useful to show that the functors between mixed characteristic and equicharacteristic zero can be modified to take in account Morleyization on **RV** while remaining in the right setting to transfer elimination of quantifiers.

Proposition B.2. *Suppose that T_1 is Morleyized on Σ_1, and let M_1 and $M_2 \models T_1$ be $(|\mathcal{L}_2|^\kappa)^+$-saturated. Then any partial $\mathcal{L}_2$-isomorphism $f : F(M_1) \to F(M_2)$ is Σ_2-elementary.*

Proof. To show that f is Σ_2-elementary, it suffices to show that the restriction of f to any finitely generated structure is Σ_2-elementary. To do so it suffices to show that the restriction of f can be extended (on both its domain and its image) to any finitely generated Σ_2-extension. By symmetry, it suffices to prove the following property: if D_1, $D_2 \leqslant F(M_1)$ are such that $D_1 \to D_2$ is a Σ_2-extension, $|D_2| \leqslant |\mathcal{L}_2|$ and $f : D_1 \to F(M_2)$ is an $\mathcal{L}_2$-embedding, then f can be extended to some $g : D_2 \to F(M_2)$.

Applying G to the initial data, we obtain the following diagram:

$$\begin{tikzcd}
GF(M_1) & M_2 & GF(M_2) \arrow[l, "\beta_{M_2}"'] \\
G(D_2) \arrow[u] \arrow[ur, dotted, "g"] & & \\
G(D_1) \arrow[u] \arrow[uurr, "G(f)"'] & &
\end{tikzcd}$$

where g comes from the fact that, as T is Morleyized on Σ_1, $\beta_{M_2} \circ G(f)\big|_{\Sigma_1}$ is in fact elementary and, as $|G(D_2)| \leqslant |\mathcal{L}_2|^\kappa$, M_2 is $(|\mathcal{L}_2|^\kappa)^+$-saturated, and $G(D_1) \to G(D_2)$ is a Σ_1-extension, by Lemma A.11, $\beta_{M_2} \circ G(f)$ can be extended to $g : G(D_2) \to M_2$. Applying F, we now obtain

$$\begin{tikzcd}
D_2 \arrow[r, "\alpha_{D_2}^{-1}"] & FG(D_2) \arrow[r, "F(g)"] & F(M_2) \\
D_1 \arrow[u] \arrow[r] \arrow[urr, bend right=60, "f"'] & FG(D_1) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

and $F(g) \circ \alpha_{D_2}^{-1}$ is the extension we were looking for. ∎

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Remark B.3.

1. One could hope the proposition to be true without the saturation hypothesis. But, without some saturation, it is not even true that $M_1 \preccurlyeq M_2$ implies $F(M_1) \preccurlyeq F(M_2)$. Take for example the coarsening functor $\mathfrak{C}^\infty$ of §2 and $\mathbb{Q}_p \preccurlyeq M$, where M is $\aleph_0$-saturated. Then $\mathfrak{C}^\infty(\mathbb{Q}_p)$ is trivially valued but $\mathfrak{C}^\infty(M)$ is not.

2. One should beware that, as $F(M_1)$ and $F(M_2)$ are not saturated, we have not proved that T_2 eliminates quantifiers.

3. We have proved nonetheless that, if Σ_i is the set of all $\mathcal{L}_i$ sorts (in that case we ask that T_2 eliminates all quantifiers) then, for all M_1 and $M_2 \models T_1$ sufficiently saturated, $M_1 \equiv M_2$ implies that $F(M_1) \equiv F(M_2)$.

Corollary B.4. *Let T_2^e be a definable Σ_2-enrichment of T_2 (in the language $\mathcal{L}_2^e$). Then F induces a functor $F^e : \mathrm{Str}(T_1) \to \mathrm{Str}(T_2^e)$ and G induces a functor $G^e : \mathrm{Str}(T_2^e) \to \mathrm{Str}(T_1)$. We can also find a full subcategory $\mathfrak{F}^e$ of $\mathfrak{F}$ such that F^e and G^e induce an equivalence of categories between $\mathfrak{F}^e$ and $\mathrm{Str}_{F^e,(|\mathcal{L}_2|^\kappa)^+}(T_2^e)$. The functor G^e still respects cardinality up to κ, models, and elementary submodels, and sends Σ_2 to Σ_1, and F^e respects cardinality up to $\kappa + |\mathcal{L}_2|$ and $(|\mathcal{L}_2|^\kappa)^+$-saturated models. Finally, $\mathfrak{F}^e$ contains all $(|\mathcal{L}_2|^\kappa)^+$-saturated models, and any C in $\mathrm{Str}(T_1)$ has a Σ_1-extension D in $\mathfrak{F}^e$. Moreover, if $C \leqslant M_1 \models T_1$ and M_1 is $(|\mathcal{L}_2|^\kappa)^+$-saturated, then we can find such a $D \leqslant M_1$.*

Proof. Let $C \leqslant M \models T_1$. We can suppose that M is $(|\mathcal{L}_2|^\kappa)^+$-saturated. As $F(M) \models T_2$, we can enrich $F(M)$ to make it into an $\mathcal{L}_2^e$-structure $F(M)^e \models T_2^e$, and we take $F^e(C) = \langle C \rangle_{\mathcal{L}_2^e}$. Note that, if M_1 and M_2 are two $(|\mathcal{L}_2|^\kappa)^+$-saturated models containing C, then Proposition B.2 implies that $\mathrm{id}_{F(C)}$ is a partial isomorphism $F(M_1) \to F(M_2)$ Σ_2-elementary, and hence the generated $\mathcal{L}_2^e$-structures are $\mathcal{L}_2^e$-isomorphic. As $F^e(C)$ does not depend (up to $\mathcal{L}_2^e$-isomorphism) on the choice of $(|\mathcal{L}_2|^\kappa)^+$-saturated model containing C, F^e is well defined on objects. If $f : C_1 \to C_2$ is a morphism in $\mathrm{Str}(T_1)$, by the same Proposition B.2, $F(f)$ is Σ_2-elementary and can be extended to a $\mathcal{L}_2^e$-isomorphism on the $\mathcal{L}_2^e$-structure generated by its domain. Note that, if we denote by i_C the embedding $F(C) \to F^e(C)$, we have also defined a natural transformation from F to F^e (a meticulous reader might want to add the forgetful functor $\mathrm{Str}(T_2^e) \to \mathrm{Str}(T_2)$ for it all to make sense).

We define G^e to be G (precomposed by the same forgetful functor). All the statements about G^e follow immediately from those about G. As $\langle F(C) \rangle_{\mathcal{L}_2^e}$ has cardinality at most $|C|^\kappa |\mathcal{L}_2| \leqslant |C|^{\kappa + |\mathcal{L}_2|}$, F respects cardinality up to $\kappa + \mathcal{L}_2$, and if $M \models T_1$ is $(|\mathcal{L}_2|^\kappa)^+$-saturated then, seeing it as a substructure of itself, we obtain that $F^e(M) \models T_2^e$.

We define $\mathfrak{F}^e$ to be the full subcategory of $\mathfrak{F}$ containing the C such that i_C is an isomorphism. In particular, it contains $(|\mathcal{L}_2|^\kappa)^+$-saturated models. Let D be an $\mathcal{L}_2^e$-substructure of $F^e(M)$ for some $(|\mathcal{L}_2|^\kappa)^+$-saturated $M \models T_1$. Then $F^e G^e(D) = \langle FG(D) \rangle_{\mathcal{L}_2^e}$, where the generated structure is taken in $F(M)$. By Proposition B.2, the (natural) isomorphism $D \to FG(D)$ is Σ_2-elementary and can be extended (uniquely) into an $\mathcal{L}_2^e$-isomorphism between $D = \langle D \rangle_{\mathcal{L}_2^e}$ and $F^e G^e(D)$. This new isomorphism is also natural. It follows that $FG(D) = F^e G^e(D)$ and that $i_{G(D)}$ is in fact an isomorphism; hence $G(D) \in \mathfrak{F}^e$.

If $C \in \mathfrak{F}^e$, $\beta_C \circ G(i_C^{-1}) : G^e F^e(C) \to C$ is a natural isomorphism. Finally, it remains to show that any $C \to M \models T_1$, where M is $(|\mathcal{L}_2|^\kappa)^+$-saturated, can be embedded in some $E \in \mathfrak{F}^e$ such that $C \to E$ is a Σ_1-extension and $E \to M$. We already know that there exists $D \in \mathfrak{F}$ such that $C \to D \to M$ and $C \to D$ is a Σ_1-extension. Now $F(D) \to F^e(D)$ is a Σ_2-extension; hence $D \cong GF(D) \to GF^e(D)$ is a Σ_1-extension. Moreover, $GF^e(D) \to GF^e(M) \cong M$ and, as $F^e(D)$ is an $\mathcal{L}_2^e$-structure of $F^e(M)$, $GF^e(D) \in \mathfrak{F}^e$. Thus we can take $E = GF^e(D)$. ∎

Let us now prove a second result in the spirit of Proposition B.2, but the other way round.

Proposition B.5. *If T_1 is Morleyized on Σ_1 and T_2 eliminates quantifiers, then T_1 eliminates quantifiers.*

Proof. To show that T_1 eliminates quantifiers it suffices to show that, for all κ^+-saturated $M_i \models T_1$, $i = 1, 2$, $C_1 \leqslant C_2 \subseteq M_1$, and $f : C_1 \to M_2$ an $\mathcal{L}_1$-embedding, then f can be extended to an embedding from C_2 into some elementary extension of M_2. Let $D_1 \in \mathfrak{F}$ be such that $C_1 \to D_1 \to M_1$ and $C_1 \to D_1$ is a Σ_1-extension. As T_1 is Morleyized on Σ_1, by Lemma A.11, we can extend f to an embedding from D_1 into an elementary extension of M_2. Replacing C_1 by D_1, C_2 by $\langle D_1 C_2 \rangle_{\mathcal{L}_1}$, and M_2 by its elementary extension, we can consider that $C_1 \in \mathfrak{F}$. Applying F, we obtain the following diagram:

$$\begin{tikzcd}
F(M_1) & M_2^\star \\
F(C_2) \arrow[u] \arrow[ur, dotted, "g"] & F(M_2) \arrow[u, "\preccurlyeq"'] \\
F(C_1) \arrow[u] \arrow[ur, "F(f)"'] &
\end{tikzcd}$$

where $M_2^\star$ is a $(|C_1|^{\aleph_0})^+$-saturated extension of $F(M_2)$ and g comes from quantifier elimination in T_2 and saturation of $M_2^\star$. Applying G, we obtain

$$\begin{tikzcd}
C_2 \arrow[r] & GF(C_2) \arrow[r, "G(g)"'] & G(M_2^\star) & \\
C_1 \arrow[u] \arrow[r] \arrow[rrr, bend right, "f"'] & GF(C_1) \arrow[u] \arrow[r, "GF(f)"'] & GF(M_2) \arrow[u, "\preccurlyeq"'] \arrow[r, leftrightarrow] & M_2 \arrow[ul, "\preccurlyeq"']
\end{tikzcd}$$

and we have the required extension. ∎

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List of Notations

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