

Model Theory

no. 1

vol. 4

2025

MA=

Pseudo T -closed fields

Samaria Montenegro and Silvain Rideau-Kikuchi



Pseudo T -closed fields

Samaria Montenegro and Silvain Rideau-Kikuchi

À Zoé, pour toutes les maths qu'elle nous a fait apercevoir.

Pseudo algebraically closed, pseudo real closed, and pseudo p -adically closed fields are examples of unstable fields that share many similarities, but have mostly been studied separately. We propose a unified framework for studying them: the class of pseudo T -closed fields, where T is an enriched theory of fields. These fields verify a “local-global” principle for the existence of points on varieties with respect to models of T . This approach also enables a good description of some fields equipped with multiple V -topologies, particularly pseudo algebraically closed fields with a finite number of valuations. One important result is a (model theoretic) classification result for bounded pseudo T -closed fields, in particular we show that under specific hypotheses on T , these fields are NTP_2 of finite burden.

1. Introduction

One of the main examples (and motivations) for the development of unstable theories, especially simple theories, was that of pseudo algebraically closed fields (PAC fields), especially those that have a small Galois group — those with finitely many extensions of any given degree d , usually called *bounded* fields. This class of fields was introduced in [Ax 1968] to describe the theory of finite fields. A PAC field is a field K such that any geometrically integral variety over K admits a K -rational point; in other words, they are existentially closed (in the language of rings) in any regular extension.

This notion has proven to be very fruitful, whether from the arithmetic point of view (see, for example, [Fried and Jarden 2008]) or the model theoretic point of view, where it continues to be the source of many examples of moderate structures (see, for example, [Chatzidakis 2002]). It was eventually reinterpreted as a special case of a wide range of local-global principles for the existence of rational points with the emergence of the pseudo real closed fields (PRC) of [Prestel 1981] (the existence of smooth points in any real closure implies the existence of rational points) and

Montenegro was partially supported by Proyecto Semilla 821-C0-464, Vicerrectoría de Investigación, UCR. Rideau-Kikuchi was partially supported by GeoMod AAPG2019 (ANR-DFG), Geometric and Combinatorial Configurations in Model Theory.

MSC2020: primary 03C45, 03C98, 12L12; secondary 12J10, 12J15.

Keywords: model theory, valued fields, ordered fields, NTP_2 , PAC, PRC and PpC fields.

pseudo p -adically closed fields (PpC) [Grob 1987; Haran and Jarden 1988] (the analogous notion for p -adic valuations) and, eventually, the pseudo classically closed fields of [Pop 2003] or the regularly T -closed fields of [Schmid 2000].

Despite the great activity in model theory surrounding PAC fields, for quite some time, the model theoretic study of these fields was mostly confined to questions of completeness and decidability. Montenegro [2017b] initiated the classification (in the sense of Shelah) of bounded PRC and PpC fields by showing, among other things, that bounded PRC and PpC fields are NTP_2 of finite burden. This was followed by further work on imaginaries [Montenegro 2017a; Montenegro and Rideau-Kikuchi 2021] and definable groups [Montenegro et al. 2020].

Notwithstanding an apparent similarity in techniques, PRC fields and PpC fields are studied separately in those works. The purpose of this text is to propose a common framework for the study of these fields. This has, of course, already been attempted before, and our approach is closely related to that of [Schmid 2000]: given a theory T of (large) fields, we say that a field K is pseudo T -closed if every geometrically integral variety over K with a smooth point in every model of T containing K has a point in K (see Definition 3.4). This class generalizes several of the notions that appear in the literature (see Example 3.6).

What sets this work apart, however, is that we aim for a more model theoretic flavor both in our definitions and in our goals. In particular, we also aim to describe pseudo algebraically closed fields endowed with a finite number of valuations, a family of structures that has to the best of our knowledge escaped scrutiny so far, despite its apparent similarities with the aforementioned examples.

One striking phenomenon that seems to arise in the study of all these “local-global” principles is the fact that an apparently arithmetic hypothesis, phrased uniquely in terms of existence of rational points in varieties, has strong topological implications on the density of points for a priori unrelated topologies. In other words, existential closure as a topological field comes for free from existential closure as a field. This was first observed for PRC fields by Prestel [1981], and was later generalized to other classes. However, in all of these examples, that behavior is not completely surprising since the topologies considered can be defined in the field structure alone. Somewhat more surprisingly, this is also true of algebraically closed fields by the historic theorem of Robinson [1977] which states that algebraically closed fields are existentially closed as valued fields despite the valuation very much not being definable. This was generalized much later to PAC fields in [Kollár 2007] and was also noticed by Johnson [2022], who showed that this happens as well when adding independent valuations to a Henselian valued field.

This stronger form of existential closure has been shown to have model theoretic consequences [Geyer and Jarden 2013; Hong 2023], and plays a central (albeit somewhat hidden) role in the first author’s work. It is therefore natural, given a set I

of completions of T — which we think of as “localizations of T ” coming with their topologies — to define the notion of I -pseudo T -closed fields (see [Definition 4.4](#)), which isolates this stronger notion of existential closure. Two obvious questions then arise:

- Are all pseudo T -closed fields I -pseudo T -closed?
- Are I -pseudo T -closed fields interesting from a model theoretic perspective?

The first half of this text gives a positive answer to the first question (under some finiteness hypothesis), showing that earlier instances of this phenomenon were not isolated incidents and that, indeed, I -pseudo T -closure is essentially just independence of the involved topologies on top of pseudo T -closure — see [Theorem 5.11](#) for a precise statement. For example, PAC fields remain existentially closed when endowed with independent valuations.

The second half of the text aims at giving a partial answer to the second question by showing that, at least in the bounded perfect case, I -pseudo T -closed fields display tame model theoretic behavior, in line with what we could expect from generalized PAC fields. The presence of topologies is an obvious obstruction to the simplicity of these fields, but we show that they are essentially the only obstructions and that they are essentially only as complicated as their “localizations”. For example, we show that their burden is the sum of the burden of the “localizations” ([Theorem 7.9](#)).

Paired with the positive answer to the first question, this provides a wealth of new examples of finite burden, strong and NTP₂ fields. One of the most natural finite burden examples is the aforementioned case of bounded perfect PAC fields with several independent valuations. In the [Appendix](#), we also explain how to get rid of the independence assumption.

The organization of the text is as follows. In [Section 2](#), we give some preliminaries on burden, forking, large fields and V -topologies. In [Section 3](#), we define the class of pseudo T -closed fields for a given theory T of (enriched) large fields and give several equivalent definitions in terms of “local-global” principles and existential closure. We then proceed to prove an approximation theorem for rational points of varieties ([Theorem 3.17](#)) under some finiteness hypothesis.

In [Section 4](#), we define, given a set I of completions of T , the class of I -pseudo T -closed fields, and we give several equivalent definitions in terms of approximation properties and existential closure. We also prove that this is an elementary class under some mild hypotheses that will always be verified when the theories in I are theories of Henselian valued (or real closed) fields. In [Section 6](#), we initiate the model theoretic study of perfect bounded I -pseudo T -closed fields. We show that the theory is model complete once the Galois group is fixed, and types are given by the isomorphism type of their algebraic closures ([Proposition 6.7](#)). We

then deduce that these fields enjoy a form of quantifier elimination “up to density” (see [Corollary 6.13](#)).

This last property can also be naturally rephrased as an “open core” property that is typical of generic topological structure: any open definable set is quantifier free definable ([Theorem 6.12](#)). Note that our methods are, in fact, more general and also allow proving such “open core” properties for topological fields with additional nontopological generic structure.

In [Section 7](#), generalizing work of the first author, we prove that although the presence of topologies is an obvious obstruction to the simplicity of these fields, it is essentially the only obstruction. For example, 3-amalgamation holds up to quantifier free obstructions (see [Theorem 7.1](#)). Along with the earlier density result, this allows us to compute burden in perfect bounded I -pseudo T -closed fields ([Theorem 7.9](#)) and characterize forking ([Corollary 8.5](#)).

We conclude with a short appendix describing how, although I -pseudo T -closure forces all the considered topologies to be independent, following an approach of Johnson [\[2019\]](#), we can also describe what happens for dependent valuations.

2. Preliminaries

In this section we recall some notions (and fix some notation) that are used throughout the paper.

Model theory. Most of the structures in this paper, if not all, are fields that we consider in (some expansion of) the ring language $\mathcal{L}_{\text{rg}} := \{+, -, \cdot, 0, 1\}$.

Notation 2.1. Let \mathcal{L} be a language and M an \mathcal{L} -structure. We write acl and dcl for the model theoretic algebraic and definable closure in M . If a and b are tuples in M and $A \subseteq M$, then we say that $a \equiv_A b$ if a and b have the same type over A , and $a \equiv_A^{\text{qf}} b$ if they have the same quantifier free type over A . We write $\mathcal{S}(A)$ for the space of \mathcal{L} -types with parameters in A .

Definition 2.2. If K is a field (considered as a structure in some expansion of \mathcal{L}_{rg}) and X is an $\mathcal{L}(K)$ -definable set, we write $\dim(X)$ for the algebraic dimension of X :

$$\dim(X) := \max\{\text{trdeg}(a/K) : a \in X(M) \text{ where } K \preceq M\}.$$

Definition 2.3. Let X be definable in some \mathcal{L} -structure M and κ a cardinal.

- (1) An *inp-pattern* of depth κ in X consists of definable subsets $\phi_l(M, a_{lj})$ of X , with $l < \kappa$ and $j < \omega$, such that
 - for every $l < \kappa$, $(\phi_l(x, a_{lj}))_{j < \omega}$ is k_l -inconsistent for some $k_l \geq 1$;
 - for every $f : \kappa \rightarrow \omega$, $(\phi_l(x, a_{lf(l)}))_{l < \kappa}$ is consistent.

- (2) The *burden* of X , denoted by $\text{bdn}(X)$, is greater than or equal to κ if there exists an inp-pattern of depth κ in X and

$$\text{bdn}(X) = \sup\{\kappa : \text{bdn}(X) \geq \kappa\}.$$

To keep track of finer estimates, burden can be computed in Card^* [Touchard 2023, Definition 1.27].

- (3) The burden of M , or equivalently the burden of $\text{Th}(M)$, is the burden of the home sort (provided M has a unique dominant sort).

Remark 2.4. (1) By Lemma 2.2 of [Chernikov 2014], if $\phi_l(M, a_{lj})$ is an inp-pattern, then you can always assume that the sequences $(a_{lj})_{j < \omega}$ are mutually indiscernible over any given $A \subseteq M$, meaning that $(a_{lj})_{j < \omega}$ is indiscernible over

$$A \cup \{a_{l'j} : l' \neq l, j < \omega\}.$$

- (2) By Lemma 7.1 of [Chernikov 2014], if $\phi_{l,0}(M, a_{l,0,j}) \vee \phi_{l,1}(M, a_{l,1,j})$ is an inp-pattern, then $\phi_{l,f(l)}(M, a_{l,f(l),j})$ is an inp-pattern for some $f : \lambda \rightarrow \{0, 1\}$.
- (3) If X, Y are definable, then $\text{bdn}(X) + \text{bdn}(Y) \leq \text{bdn}(X \times Y)$.
- (4) If X, Y are definable, then $\text{bdn}(X \cup Y) = \max(\text{bdn}(X), \text{bdn}(Y))$.
- (5) If $f : Y \rightarrow X$ is definable and surjective, then $\text{bdn}(Y) \geq \text{bdn}(X)$.
- (6) If $f : Y \rightarrow X$ is definable and finite-to-one, then $\text{bdn}(X) \geq \text{bdn}(Y)$.

Definition 2.5. A theory T is called *strong* if there are no inp-patterns of infinite depth in its models. It is NTP_2 if there is a (cardinal) bound on the depth of inp-pattern in its models.

The burden of a real closed field (in the ring language, or equivalently the ordered ring language) is 1. In examples, we also consider valued fields. Given a valued field (K, v) , we denote by \mathcal{O} its valuation ring with maximal ideal \mathfrak{m} , by $\mathbf{k} = \mathcal{O}/\mathfrak{m}$ its residue field, by $\Gamma = v(K^\times)$ its valuation group and by $\text{RV} = K/(1 + \mathfrak{m})$ its leading terms. We refer the reader to [Touchard 2023] for a very thorough introduction to their model theory and the proof of the following burden computations:

Fact 2.6 [Touchard 2023, Corollary 3.13 and Theorem 3.21]. *Let (K, v) be a valued field.*

- (1) *If K eliminates quantifiers relative to RV_1 , is elementarily equivalent to a maximally complete valued field and the $\mathbf{k}^*/\mathbf{k}^{*n}$ are finite, then*

$$\text{bdn}(K) = \max\{\text{bdn}(\mathbf{k}), \text{bdn}(\Gamma)\}.$$

- (2) *If K eliminates quantifiers relative to $\bigcup_n \text{RV}_n$, is elementarily equivalent to a maximally complete valued field and K is finitely ramified—that is, for*

all $n \geq 1$, $(0, v(n))$ is finite — then

$$\text{bdn}(K) = \max\{\aleph_0 \cdot \text{bdn}(\mathbf{k}), \text{bdn}(\Gamma)\}.$$

These equalities hold resplendently for Γ -expansions of \mathbf{k} -expansions.

In [Touchard 2023], the results are stated under stronger assumptions, but as noted there, they hold at this level of generality — up to the fact that Touchard considers unramified mixed characteristic fields (and not finitely ramified ones), but the proof there also applies since the higher residue rings $R_n = \mathcal{O}/n\mathfrak{m}$ are also interpretable in \mathbf{k} in that case.

Definition 2.7. Let M be a (sufficiently saturated) \mathcal{L} -structure and $A \leq M$.

- (1) We say that the formula $\phi(x, a)$ *divides over* A if there exists an indiscernible sequence $(a_j)_{j \in \omega}$ over A such that $a_0 = a$ and $\{\phi(x, a_j) : j \in \omega\}$ is inconsistent.
- (2) We say that the formula $\phi(x, a)$ *forks over* A if there is an integer m and formulas $\psi_j(x, a_j)$ for $j < m$ such that $\phi(x, a) \vdash \bigvee_{j < m} \psi_j(x, a_j)$ and $\psi_j(x, a_j)$ divides over A for every $j < m$.
- (3) A partial type p *forks* (respectively, *divides*) *over* A if there is a formula in p which forks (respectively, divides) over A .
- (4) We say that A is an *extension base* if for all tuples a in M , $\text{tp}(a/A)$ does not fork over A .

By Theorem 1.2 of [Chernikov and Kaplan 2012], in an NTP_2 theory, forking equals dividing over extension bases.

Large fields. By a variety V over a field K , we mean a separated integral scheme of finite type over K . We denote by $V(K)$ the set of its K -rational points. We also denote by $V_{\text{sm}}(K)$ the set of smooth K -rational points.

Remark 2.8 [Stacks, Tag 01V7]. A point $x \in V(K)$ is smooth if there exists an affine neighborhood $U \subseteq V$ of x and an étale morphism $U \rightarrow \mathbb{A}^d$ — that is, U is isomorphic to $\text{Spec}(S)$, where $S = K[x_1, \dots, x_{c+d}]/(f_1, \dots, f_c)$ and the matrix $(df_i/dx_j)_{i \leq c, j \leq c}$ is invertible in S .

Definition 2.9. A field K is *large* if for every irreducible variety V , if $V_{\text{sm}}(K) \neq \emptyset$, then $V(K)$ is Zariski dense in V — equivalently, by compactness, $K(V)$ can be $\mathcal{L}_{\text{rg}}(K)$ -embedded in some $K^* \succ K$.

We know that algebraically closed fields, real closed fields, p -adically closed fields and, more generally, Henselian fields are large, as well as pseudo algebraically closed fields, pseudo real closed fields and pseudo p -adically closed fields, and further generalizations of these notions.

Finally, given two extensions M and K of a field F , we write $M \downarrow_F^\ell K$ if the field M is linearly disjoint from K over F and $L \downarrow_F^a K$ if M and K are algebraically independent over F . We also denote by K^a the algebraic closure of K and by K^s its separable closure. Recall that a variety V over K is geometrically integral (equivalently, absolutely irreducible defined over K) if the extension $K \leq K(V)$ is regular—that is, $K(V) \downarrow_K^\ell K^a$.

V-topological fields.

Definition 2.10. Let (K, τ) be a (nondiscrete) topological field.

- (1) A subset $S \subseteq K$ is said to be *bounded* if for every open neighborhood of the identity $U \subset K$, there exists $x \in K^\times$ such that $xS = \{x \cdot y : y \in S\} \subseteq U$ —equivalently, $\{x \in K : xS \subseteq U\}$ is a (nontrivial) neighborhood of 0.
- (2) A subset $S \subseteq K$ is said to be *bounded away from 0* if there exists a neighborhood U of 0 such that $S \cap U = \emptyset$.
- (3) The topology τ is said to be a *V-topology* if, whenever $S^{-1} = \{x^{-1} : x \in S\}$ is bounded away from 0, S is bounded.
- (4) The topology τ is said to be *Henselian* if for every integer $n \geq 1$ there is a neighborhood U of 0 such that every polynomial in $X^{n+1} + X^n + U[X]^{n-1}$ has a zero in K —where $U[X]^{n-1}$ denotes the set of polynomials of degree at most $n - 1$ with coefficients in U .

By a theorem of Dürbaum–Kowalsky, and independently Fleischer, a topology is a V -topology if and only if it is induced by either an order or a valuation. We refer the reader to [Prestel and Ziegler 1978] for further detail on V -topologies.

In this text, the most useful consequence of Henselianity (and an equivalent characterization) is the inverse function theorem:

Fact 2.11 [Halevi et al. 2020, Proposition 2.8]. *Let (K, τ) be a Henselian V -topological field and $f : V \rightarrow W$ be an étale morphism of varieties. Then any $x \in V(K)$ admits a neighborhood U such that $f|_U$ is a homeomorphism onto its open image.*

Definition 2.12. A V -topology on a field K is said to be *definable* if it admits a uniformly definable basis—equivalently, if there exists one definable bounded open set.

Remark 2.13. In fact, by (the proof of) [Engler and Prestel 2005, Lemma B.2], if τ is a definable V -topology on K , then there exists an open bounded definable neighborhood U of 0 such that $K = U \cup (U \setminus \{0\})^{-1}$.

For more details on definable V -topologies, we refer the reader to [Halevi et al. 2020, Section 3].

3. The class of pseudo T -closed fields

Let \mathcal{L} enrich the language of rings, T be an \mathcal{L} -theory of large fields and K be an \mathcal{L} -structure that is a field — we refer to those as \mathcal{L} -fields from now on. We write T_K for the theory of models of T containing K .

Definition 3.1. • We say that a field extension $K \leq F$ is *totally T* if, for every $M \models T_K$, F can be $\mathcal{L}_{\text{rg}}(K)$ -embedded in some $M^* \succcurlyeq M$.

• We say that a variety V over K is *totally T* if, for every $M \models T_K$, $V_{\text{sm}}(M) \neq \emptyset$.

Observe that if V is a geometrically irreducible variety over K , then V_{sm} is Zariski open in V [Stacks, Tag 056V]. In particular $V_{\text{sm}}(K(V)) \neq \emptyset$. It follows that, since models of T are large, V is totally T if and only if $K \leq K(V)$ is totally T .

Remark 3.2. Observe that if T is RCF in \mathcal{L}_{rg} , we have that $K \leq F$ is totally T if and only if it is a totally real extension, meaning that every order on K extends to some order on F . Similarly, if T is p CF in \mathcal{L}_{rg} , then $K \leq F$ is totally T if every p -adic valuation on K extends to some p -adic valuation on F .

Proposition 3.3. *The following are equivalent:*

- (i) *For every geometrically integral (affine) totally T variety V over K , we have $V(K) \neq \emptyset$.*
- (ii) *For every geometrically integral (affine) totally T variety V over K , $V(K)$ is Zariski dense in V .*
- (iii) *Any regular totally T extension $K \leq F$ is \mathcal{L}_{rg} -existentially closed.*

Proof. We obviously have that (ii) implies (i).

(i) \Rightarrow (iii) Let F be as in (iii) and let $\phi(x)$ be some quantifier free \mathcal{L}_{rg} -formula such that $F \models \exists x \phi(x)$. Adding existential quantifiers, we may assume that ϕ is of the form $\bigwedge_{i < n} P_i(x) = 0$, where $P_i \in K[x]$. Let $a \models \phi$ in F and V be the locus of a over K . Since $K \leq K(a) \leq F$ is regular, V is geometrically integral and, by [Stacks, Tag 056V], $V_{\text{sm}}(F) \neq \emptyset$. Moreover, by the hypothesis on F , for every $M \models T_K$, there exists $M^* \succcurlyeq M$ such that $F \leq M^*$. So $V_{\text{sm}}(M^*) \neq \emptyset$ and hence $V_{\text{sm}}(M) \neq \emptyset$. By (i), we find $c \in V(K)$. In particular, $K \models \bigwedge_i P_i(c) = 0$.

(iii) \Rightarrow (ii) Let V be as in (ii). Any $M \models T_K$ is large and hence, if $V_{\text{sm}}(M) \neq \emptyset$, then some $M^* \succcurlyeq M$ contains a K -generic point a — that is, a is not in any proper subvariety of V over K . In other terms, $F := K(V)$ is $\mathcal{L}_{\text{rg}}(K)$ -embeddable in M^* . By (iii), F is $\mathcal{L}_{\text{rg}}(K)$ -embeddable in $K^* \succcurlyeq K$, or equivalently, $V(K)$ is Zariski dense in V . \square

Definition 3.4. The \mathcal{L} -field K is said to be *pseudo T -closed* (PTC) if the statements of Proposition 3.3 hold.

Remark 3.5. (1) It follows from Proposition 3.3(ii) that PTC fields are large.

(2) The class of PTC fields is inductive in \mathcal{L} ; see Proposition 4.9.

Example 3.6. For different choices of T , the following are examples of PTC fields:

- If $T = \text{ACF}$ in \mathcal{L}_{rg} , then PTC fields are exactly PAC fields.
- If $T = \text{RCF}$ in \mathcal{L}_{rg} , then PTC fields are exactly PRC fields.
- If $T = \text{RCF}_{<}$ in $\mathcal{L}_{\text{rg}} \cup \{<\}$, then PTC fields are exactly 1-PRC fields, i.e., PRC fields whose only order is $<$. The equivalence of those two classes is not obvious from the definition and relies crucially on [Proposition 3.12](#) to show the only order on a PTC field is $<$.
- Let $\mathcal{L} = \mathcal{L}_{\text{rg}} \cup \{<_1, \dots, <_n\}$ and $T = \bigvee_i \text{RCF}_{<_i}$, whose models are the \mathcal{L} -structures M with $M|_{\mathcal{L}_i} \models \text{RCF}_{<_i}$, for some i . Then PTC fields where the $<_i$ define distinct orders are exactly n -PRC fields. This equivalence relies again on [Proposition 3.12](#).
- If $T = p\text{CF}$ in \mathcal{L}_{rg} , then PTC-fields are exactly PpC fields.
- For $i < n$, let \mathcal{L}_i be copies of Macintyre's language sharing the ring language and $p\text{CF}_{v_i}$ the \mathcal{L}_i -theory of p -adically closed fields. Then PTC fields whose \mathcal{L}_i -structure induces distinct p -adic valuations are exactly n -PpC fields. Once again the equivalence relies on [Proposition 3.12](#).
- Fehm's pseudo \mathcal{S} -closed fields [\[2013\]](#) also fit in this framework.

Lemma 3.7. *Let F, K be \mathcal{L} -fields, $F \leq M$ be regular and totally T , and $f : F \rightarrow K$ be an \mathcal{L} -embedding. Then $K \leq K \otimes_F M$ is regular and totally T .*

Proof. We may assume that M is finite type over F . Then $M = F(V)$, where V is a geometrically integral totally T variety over F . Let $E \models T_K$. Then $E \models T_F$ and hence $V_{\text{sm}}(E) \neq \emptyset$. Since E is large, we find an $\mathcal{L}_{\text{rg}}(E)$ -embedding $g : E \otimes_F M \simeq E(V) \rightarrow E^* \succcurlyeq E$ — in particular, we have an $\mathcal{L}_{\text{rg}}(K)$ -embedding $K(V) \simeq K \otimes_F M \rightarrow E^*$. \square

Convention 3.8. Replacing T by the theory whose models are either models of T or algebraically closed does not change the notion of pseudo T -closed fields: algebraically closed fields always have rational smooth points in any variety. So, from now on, we assume that any algebraically closed field is a model of T .

We now study the topological properties of PTC fields. We fix K a PTC field.

Notation 3.9. • For every $M \models T_K$, let $K_M := M \cap K^s$.

- For every V -topology τ on K , let C_τ be the set of those K_M that can be endowed with a Henselian V -topology that induces τ on K .
- We say that $F \in C_\tau$ is minimal if any $\mathcal{L}_{\text{rg}}(K)$ -embedding $E \rightarrow F$, with $E \in C_\tau$, is surjective.
- For every V -topology τ on K , let $K_\tau = \widehat{(K, \tau)} \cap K^s$, where $\widehat{(K, \tau)}$ is the completion of (K, τ) .

Note that, by convention, $K^a \models T$ and hence $K^s \in C_\tau$, for every τ . Also, by uniqueness of the Henselian V -topology on a nonseparably closed field [Prestel and Ziegler 1978, Theorem 7.9], if τ and τ' are distinct V -topologies on K , then $C_\tau \cap C_{\tau'} = \{K^s\}$.

Hypothesis 3.10. Until the end of Section 3, we assume:

(H) For every $M \models T_K$, K_M can be endowed with at least one Henselian V -topology.

Lemma 3.11. *For every τ , C_τ admits minimal elements.*

Proof. By Zorn's lemma, it suffices to show that any decreasing chain (of nonseparably closed fields) $(K_i)_{i \in I}$ in C_τ admits a lower bound. Let $M_i \models T_K$ such that $K_i = M_i \cap K^s$. The theory

$$T_K \cup \{\forall x P(x) \neq 0 : P \in K[x] \text{ such that, for some } i \in I, M_i \models \forall x P(x) \neq 0\}$$

is consistent since any finite subset is realized in an M_i . Let M be a model. For all $i \in I$ and $P \in K[x]$, if $\widehat{K_M} \models \exists x P(x) = 0$, then we also have $M_i \models \exists x P(x) = 0$, so K_M can be embedded in M_i over K , i.e., it is a lower bound. Moreover, by Hypothesis (H), K_M admits a Henselian V -topology which extends to the unique Henselian V -topology on every K_i . In particular, it induces τ on K . \square

As in [Schmid 2000, Proposition 4.5], we will see that minimal elements are unique (up to isomorphism).

Proposition 3.12. *Let τ be a V -topology on K and $F \in C_\tau$ be minimal. Then F is $\mathcal{L}_{\text{rg}}(K)$ -homeomorphic to K_τ . In particular, K is τ -dense in F .*

Proof. We start with two intermediary results:

Claim 3.12.1. *There exists a τ -continuous $\mathcal{L}_{\text{rg}}(K)$ -embedding $\sigma : K_\tau \rightarrow F$.*

Proof. There is a continuous $\mathcal{L}_{\text{rg}}(K)$ -embedding $\sigma : (\widehat{K}, \tau) \rightarrow (\widehat{F}, \tau)$, by universality of the completion. Since $\widehat{K} \subseteq K_\tau$ is separable and F , being topologically Henselian, is separably closed in (\widehat{F}, τ) [Prestel and Ziegler 1978, Corollary 7.6], we have $\sigma(K_\tau) \leq (\widehat{F}, \tau) \cap K^s = F$. \square

Claim 3.12.2. *There exists $M \models T_K$ and an $\mathcal{L}_{\text{rg}}(K)$ -embedding $\rho : K_M \rightarrow (\widehat{K}, \tau)$.*

Proof. If no K_M embeds in (\widehat{K}, τ) , then, by compactness, there exists a (nontrivial) separable $P \in K[x]$ with a zero in each K_M but no zero in (\widehat{K}, τ) . Then there exists a neighborhood $U \subseteq K$ of 0 such that $P((\widehat{K}, \tau)) \cap U = \emptyset$. In particular, $P(K) \cap U = \emptyset$. It follows that $P(K)^{-1}$ is bounded and hence so is $1 - P(0)P(K)^{-1}$. Let $c \in K^*$ be such that $c(1 - P(0)P(K)^{-1}) \subseteq U$. Let

$$Q(x, y) = P(x)(1 - c^{-1}P(y)) - P(0) \in K[x, y].$$

By [Heinemann and Prestel 1984, Proposition 1.1], the zero locus V of Q is geometrically integral. For every $M \models T$, one can check that $(0, a) \in V_{\text{sm}}(M)$, where $a \in M$ is a zero of P . So there exists $(b, a) \in V(K)$, that is, $P(b) = c(1 - P(0)P(a)^{-1}) \in U$, a contradiction. \square

Note that, since $K \leq K_M$ is separable, $\rho(K_M) \leq K_\tau$. If $K_M \in C_{\tau'}$, for some $\tau' \neq \tau$, then τ' extends to a Henselian V -topology on K_τ , distinct from τ . Moreover, K_τ is, by definition, separably closed in its τ -completion. It follows by [Prestel and Ziegler 1978, Improved Theorem 7.9], that $K_\tau = K^s$, and hence $F \mathcal{L}_{\text{rg}}(K)$ -embeds in K_τ . So we may assume that $K_M \in C_\tau$.

Now, the $\mathcal{L}_{\text{rg}}(K)$ -embedding $\sigma \circ \rho : K_M \rightarrow K_\tau \rightarrow F$ must be surjective by minimality of F and hence σ is a τ -continuous $\mathcal{L}_{\text{rg}}(K)$ -isomorphism. In particular, K is τ -dense in F . It follows that the identity on K extends to a (unique) τ -continuous $\mathcal{L}_{\text{rg}}(K)$ -morphism $F \rightarrow K_\tau$ which is the inverse of σ , and σ is indeed an $\mathcal{L}_{\text{rg}}(K)$ -homeomorphism. \square

Notation 3.13. Fix $(\tau_i)_{0 \leq i < n}$ distinct V -topologies on K and $K_i = K_{\tau_i} \in C_{\tau_i}$ minimal.

Since, by Stone's approximation theorem [Prestel and Ziegler 1978, Theorem 4.1], distinct V -topologies are independent, we deduce the following approximation result on the affine line:

Corollary 3.14. *For every nonempty τ_i -open $O_i \leq K_i$, $\bigcap_i O_i \cap K \neq \emptyset$.*

Our goal now is to extend this approximation result to arbitrary geometrically integral totally T varieties. We don't know, however, if we can do so without the following finiteness hypothesis:

Hypothesis 3.15. Until the end of Section 3, we assume:

(F₁) There are only finitely many nonseparably closed K_τ , among all possible τ .

Proposition 3.16. *Let C be a totally T smooth projective geometrically integral curve over K . For every nonempty τ_i -open set $O_i \subseteq C(K_i)$, $\bigcap_{i < n} O_i \cap C(K)$ is infinite.*

This statement (and its proof) is inspired by similar statements in two important and orthogonal subcases: Henselian fields enriched with further valuations [Johnson 2022, Theorem 4.1] (see also [Halevi et al. 2020, Proposition 4.2]) and PAC fields with a valuation [Kollár 2007, Theorem 2].

Proof. We proceed by induction on n . If $n = 0$, since $C(K)$ is Zariski dense in C , it is, in particular, infinite. Let us now assume $n > 0$. By induction, $\bigcap_{i > 0} O_i \cap C(K)$ contains a finite set P of size d larger than the genus of C . By Riemann–Roch, there exists a nonconstant rational function f on C over K , whose set of poles is contained in P and whose poles are all simple. This gives rise to a morphism

$f : C \rightarrow \mathbb{P}^1$ over K with $f^{-1}(1 : 0) \subseteq P \subseteq \bigcap_{0 < i < m} O_i \cap C(K)$ and which is étale above $(1 : 0)$.

Let τ be any V -topology on K distinct from τ_0 . If $\tau = \tau_i$, let $O_\tau = O_i$ and otherwise, let $O_\tau = V(K_\tau)$. By the inverse function theorem, [Fact 2.11](#), for every $p \in P \subseteq O_\tau$ there exists a τ -open neighborhood $W_{p,\tau} \subseteq O_\tau$ of p such that $f|_{W_{p,\tau}}$ is a homeomorphism unto its open image. We may assume that the $W_{p,\tau}$ do not intersect when p varies. Let $U_\tau := \bigcap_{p \in P} f(W_{p,\tau})$. It is open (and hence Zariski dense) in $\mathbb{P}^1(K_\tau)$. For every $y \in U_\tau$, $f^{-1}(y)$ consists of d distinct elements of O_τ .

Let $e \geq 1$ be maximal such that

$$\{t \in \mathbb{P}(K_0) : \exists^{\neq} y_0 \dots y_{e-1} \in C(K_0) \ f(y_j) = t \text{ and } y_0 \in O_0\}$$

is infinite. Let D be the normalization of the irreducible component of the e -fold product of C over \mathbb{P}^1 containing $y_{<e}$. Then $K \leq K(D) = K(y_{<e})$ is regular and D is geometrically integral. Let $g : D \rightarrow C$ be the projection on the first coordinate and $h = f \circ g$. Let $W_{\tau_0} := \{x \in O_0 : \exists y \in D(K_0) : g(y) = x \text{ and } g \text{ is smooth at } y\}$ and $U_{\tau_0} = f(W_{\tau_0})$. Recall that, τ_0 being Henselian, f and g are local homeomorphisms at smooth points, so both sets have nonempty τ_0 -interior. Shrinking them, we may assume that they are τ_0 -open and that U_{τ_0} is a ball.

Let $J := \{\tau : K_\tau \neq K_\tau^\circ\} \cup \{\tau_i : i < n\}$, which is finite by [\(F₁\)](#). By [Corollary 3.14](#), there exists $a \in \bigcap_{\tau \in J} U_\tau \cap K$. Composing f with a degree one morphism, we may assume that $a = (1 : 0)$. For every $s \in \mathbb{A}^1$, let $B_s \subseteq D \times D$ be defined by $(s^2 u_1 u_2 - v_1 v_2) \circ (h \times h) = 0$, where $([u_1 : v_1], [u_2, v_2])$ are coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. By [\[Kollár 2007, Lemma 15\]](#), for all but finitely many s , B_s is geometrically integral. Applying [Corollary 3.14](#) again, we find $s \in K^\times$ such that B_s is geometrically integral and $(1 : s) \in \bigcap_{\tau \in J} U_\tau$. Then, for every $\tau \in J$ and $y \in D(K_\tau)$ with $h(y) = (1 : s)$, (y, y) is a smooth point of $B_s(K_\tau)$ — with $(h \times h)(y, y) \in U_\tau \times U_\tau$. Note that for every $M \models T_K$, if K_M is not separably closed, it contains some nonseparably closed K_τ and hence there is a smooth M -point on B_s . If K_M is separably closed, B_s also has a smooth M -point, so B_s is totally T .

By induction, we can find infinitely many points $(y_1, y_2) \in B_s(K)$ such that $h(y_k) \in \bigcap_{0 < i} U_{\tau_i} \setminus \{(0 : 1)\}$. Considering the affine coordinates v/u on \mathbb{P}^1 , we see that one of the $h(y_k)$ is τ_0 -closer to 0 than s and hence $h(y_k) \in U_{\tau_0}$. In other words, we have found infinitely many $t \in \bigcap_{i < n} U_{\tau_i}(K)$ and $(x_{t,i})_{i < e} \in C(K)$ with $f(x_{t,i}) = t$. By construction there exist $x_t \in O_0$ with $f(x_t) = t$. By maximality of e , x_t is distinct from the $x_{t,i}$ for at most finitely many t . So we may assume that $x_{t,0} = x_t \in O_0(K)$. Moreover, for every $i > 0$, since $f(x_t) = t \in U_{\tau_i}$, we have $x_t \in f^{-1}(U_{\tau_i}) \subseteq O_i$ and hence $x_t \in \bigcap_i O_i(K)$. \square

Theorem 3.17. *Let V be a geometrically integral K -variety. Let $O_i \subseteq V_{\text{sm}}(K_i)$ be nonempty τ_i -open sets, for every $i < n$. Then $\bigcap_{i < n} O_i \cap V(K) \neq \emptyset$.*

Proof. For every $i < n$, fix an $a_i \in O_i \subseteq V(K^s)$. By [Jarden and Razon 1998, Lemma 10.1] there exists a smooth geometrically integral affine K -curve $C \subseteq V$ containing the a_i . Let $U_i = O_i \cap C$, which is a τ_i -open nonempty subset of $C(K_i)$. Let \bar{C} be a smooth projective model of C . By Proposition 3.16, there exists $y \in \bigcap_i U_i \cap \bar{C}(K) \subseteq \bigcap_i O_i \cap V(K)$. \square

Remark 3.18. Schmid proved a similar result [2000, Theorem 4.9], without any finiteness hypothesis, but requiring that no minimal K_M is separably closed — a case that we certainly do not want to omit if we want to say anything about valued pseudo algebraically closed fields.

On the other hand, the proof we give here relies heavily on hypothesis (\mathbf{F}_1) . Without it, we would have to find points in infinitely many (uniformly defined) open sets for independent topologies. This is reminiscent of known approximation theorems (e.g., [Anscombe et al. 2020]), but it is not obvious that they apply here.

4. The class of I -pseudo T -closed fields

Let us fix the following notation for the rest of the text. As before, let \mathcal{L} enrich the language of rings and T be an \mathcal{L} -theory of large fields.

Let I be a (potentially infinite) set of theories which eliminate quantifiers in (disjoint) relational expansions of the language of rings. We often denote by T_i the element $i \in I$ and \mathcal{L}_i its language. From now on, all \mathcal{L}_i -formulas are assumed to be quantifier free. Let $\mathcal{L}_I = \bigcup_i \mathcal{L}_i$ and $T_I = \bigcup_i T_{i,\forall}$.

For every $K \models T_{i,\forall}$, we denote by $T_{i,K}$ the theory of models of T_i containing K . We also assume that any $M_i \models T_{i,K}$ can be expanded to a model of T_K . When I is finite, this can always be assumed by replacing T by $T \vee \bigvee_i T_i$, whose models are the $\mathcal{L} \cup \mathcal{L}_I$ -structures M such that $M|_{\mathcal{L}} \models T$ or $M|_{\mathcal{L}_i} \models T_i$ for some i .

For each $i \in I$, we denote by acl_i and dcl_i the algebraic and definable closure in models of T_i . We also write $S^i(K)$ for the space of (quantifier free) \mathcal{L}_i -types with parameters in K . We use the notation $\text{tp}_i(a/A)$ for the type of a over A in the language \mathcal{L}_i .

Let us fix some \mathcal{L} -field $K \models T_I$ and $M_i \models T_{i,K}$, for all $i \in I$.

Lemma 4.1. *Let V be an irreducible variety over K and let us assume that $\text{dcl}_i(K) \subseteq K^a$. Then, for any $\mathcal{L}_i(K)$ -definable $X \subseteq V$, $X(M_i)$ is Zariski dense in V if and only if it is K -Zariski dense in V (i.e., is not contained in any proper subvariety of V over K).*

Proof. Note first that this statement is clear for $K = M_i$ — i.e., $X(M_i)$ is Zariski dense in V if and only if it is M_i -Zariski dense in V . Let us now assume that $X(M_i)$ is K -Zariski dense in V . We may assume M_i is sufficiently saturated and homogeneous. Let $W \subseteq V$ be the Zariski closure of $X(M_i)$. Then W is

$\text{aut}(M_i/K)$ -invariant and hence it is defined over $\text{dcl}_i(K) \subseteq K^a$. Let $(W_j)_{j \leq n}$ be the K -conjugates of W . Then $X(M_i) \subseteq \bigcup_i W_i \subseteq V$. So $V = \bigcup_i W_i = W$. \square

Remark 4.2. Since finite sets in fields can be coded using symmetric polynomials, we have that $\text{acl}_i(K) \subseteq \text{dcl}_i(K)^a$. It follows that $\text{acl}_i(K) \subseteq K^a$ if and only if $\text{dcl}_i(K) \subseteq K^a$.

Proposition 4.3. *The following are equivalent:*

- (i) Any regular totally T extension $K \leq F \models T_I$ is \mathcal{L}_I -existentially closed.
- (ii) For every geometrically integral (affine) totally T variety V over K , and, for all $i \in I$, every $p_i \in \mathcal{S}^i(K)$ K -generic in V , $\bigcup_i p_i$ is realized in some $K^* \succ K$.
- (iii) For every geometrically integral (affine) totally T variety V over K , and, for all $i \in I_0 \subseteq I$ finite, every $\mathcal{L}_i(K)$ -definable K -Zariski dense $X_i(M_i) \subseteq V$, $\bigcap_i X_i(K) \neq \emptyset$. Equivalently, it is Zariski dense in V .
- (iv) For all $i \in I$, $\text{dcl}_i(K) \subseteq K^a$ and for every geometrically integral (affine) totally T variety V over K , and, for all $i \in I' \subseteq I$ finite, every $\mathcal{L}_i(K)$ -definable Zariski dense $X_i(M_i) \subseteq V$, $\bigcap_i X_i(K) \neq \emptyset$. Equivalently, it is Zariski dense in V .

Proof. (i) \Rightarrow (ii) Let p_i and V be as in (ii) and let $a_i \models p_i$ (in some $M_i^* \succ M_i$). Note that $K \leq K(a_i)$ is isomorphic to $K(V)$. These isomorphisms allow us to make $K(V)$ into a model of T_I , where all the a_i coincide. Since $K \leq K(V)$ is regular and totally T , we find an $\mathcal{L}_I(K)$ -embedding $f : K(V) \rightarrow K^* \succ K$. The common image of the a_i in K^* is a realization of $\bigcup_i p_i$.

(ii) \Rightarrow (iii) Let X_i and V be as in (iii). If $i \notin I_0$, let $X_i(M_i) = V(M_i)$. Note that since M_i can be made into a model of T_K and V is totally T , $V(M_i)$ is Zariski dense in V . For all $i \in I$, by compactness, we find $a_i \in X_i(M_i^*)$, where $M_i^* \succ M_i$, which is K -generic in V . Applying (ii) to $p_i = \text{tp}_i(a_i/K)$, we find $a \models \bigcup_i p_i$ in some $K^* \succ K$; in particular, $a \in \bigcap_{i \in I_0} X_i \setminus W$, where $W \subseteq V$ is any K -subvariety.

(iii) \Leftrightarrow (iv) Let us first assume (iii) and prove that $\text{dcl}_i(K) \subseteq K^a$. By contradiction, consider some $a \in \text{acl}_i(K) \setminus K^a$. Let X_i be $\mathcal{L}_i(K)$ -definable such that $X_i(M_i)$ is minimal finite containing a . Then $X_i(M_i)$ is K -Zariski dense in \mathbb{A}^1 . So $X_i(K) \neq \emptyset$, contradicting the minimality of X_i . The equivalence now follows from [Lemma 4.1](#).

(iii) \Rightarrow (i) Let F be as in (i) and X be a quantifier free $\mathcal{L}(K)$ -definable set with $X(F) \neq \emptyset$. We may assume that $X = \bigcap_{i \in I_0} X_i$, where $I_0 \subseteq I$ is finite and X_i is $\mathcal{L}_i(K)$ -definable. Fix some $a \in X(F)$. Let V be the locus of a over K . Since $K \leq F$ is regular and totally T , V is geometrically integral and totally T . By construction, $X_i(M_i)$ is K -Zariski dense in V . By (iv), $X(K) = \bigcap_i X_i(K) \neq \emptyset$. \square

Definition 4.4. The \mathcal{L} -field K is said to be I -pseudo T -closed (PTC_I) if the statements of [Proposition 4.3](#) hold.

Conditions (i) and (ii) have a similar flavor to the interpolative fusion of [Kruckman et al. 2021], although the main difference here is that K is not required to be a model of the T_i but only of $T_{i,\forall}$.

Example 4.5. The following are examples of PTC_I fields for different choices of T_i and T :

- By [Prestel 1981, Theorem 1.7], n -PRC fields are PTC_I , for $T_i = \text{RCF}_{<_i}$ and $T = \bigvee_i T_i$, as in Example 3.6.
- By [Montenegro and Rideau-Kikuchi 2021, Theorem 2.17], n -PpC fields are PTC_I , for $T_i = p\text{CF}_{v_i}$ and $T = \bigvee_i T_i$, as in Example 3.6.
- More generally, it follows from [Schmid 2000, Theorem 4.9] that if K is PRC and $(<_i)_{i \in I}$ are distinct orders on K , then it is PTC_I , for $T = \text{RCF}$ and $T_i = \text{RCF}_{<_i}$. The same holds for PpC fields with named p -adic valuations.
- It follows from [Kollár 2007, Theorem 2] that a PAC field with one valuation is PTC_I , for $T = \text{ACF}$ and $T_0 = \text{ACVF}$.
- In the present paper we generalize this result by showing that a PAC field with n distinct valuations is PTC_I , for $T = \text{ACF}$ and $(T_i)_{i < n}$ copies of ACVF . This is a consequence Theorem 5.11.
- By [Johnson 2022, Theorem 4.1], if K is real closed (resp. p -adically closed, algebraically closed), and the $(v_i)_{0 < i \leq n}$ are distinct valuations, then it is PTC_I , for $T_0 = \text{RCF}_{<_0}$ (resp. $p\text{CF}_{v_0}$, ACVF), $T_i = \text{ACVF}$, for $0 < i < n$ and $T = \bigvee_i T_i$.
- As a consequence of Theorem 5.11, we also generalize this result by showing that if $I = \{\text{Th}(K_i) : i < n\}$, where K_i is either real closed or a characteristic zero Henselian valued field—in an adequate language to eliminate quantifiers—and $T = \bigvee_i T_i$, then every PTC field where the T_i induce distinct topologies is PTC_I . Note that if all T_i but T_0 are equal to ACVF , then any model of T_0 is PTC.

Lemma 4.6. Fix an $i \in I$ and let $K, M \models T_{i,\forall}$, $F \leq M$ be regular with $\text{acl}_i(F) \cap M \subseteq F$ and $f : F \rightarrow K$ be an \mathcal{L}_i -embedding. Then $K \otimes_F M$ can be made into a model of $T_{i,\forall}$, extending the \mathcal{L}_i -structure on both K and M .

Proof. By quantifier elimination, we can extend f to some $g : M \rightarrow N \models T_i$. Since $\text{acl}_i(F) \cap M \subseteq F$, any $\mathcal{L}_i(F)$ -definable set containing a tuple in $M \setminus F$ is infinite and thus, if N is a sufficiently saturated extension of M , it contains a realization in $N \setminus K$. By compactness, we may assume that $K \downarrow_F^a g(M)$. Since $F \leq M$ is regular, we have $K \downarrow_F^\ell M$. Then $K \otimes_F M \simeq Kg(M) \models T_{i,\forall}$. \square

Corollary 4.7. Let $K, M \models T_I$, $F \leq M$ be regular and totally T , with $\text{acl}_i(F) \cap M \subseteq F$, for all $i \in I$, and $f : F \rightarrow K$ be an \mathcal{L}_I -embedding. Then $K \leq K \otimes_F M$ is a regular totally T extension that can be made into a model of T_I extending the \mathcal{L}_I -structure on K and M .

Proof. This follows from Lemmas 3.7 and 4.6. \square

We also need the following orthogonal case of free amalgamation:

Lemma 4.8. *Fix an $i \in I$ and let $K, M \models T_{i,\forall}$, $F \leq M$ be algebraic and $f : F \rightarrow K$ be an \mathcal{L}_i -embedding with $M \otimes_F K$ integral. Then $K \otimes_F M$ can be made into a model of $T_{i,\forall}$, extending the \mathcal{L}_i -structure on both K and M .*

Proof. By quantifier elimination we can extend f to some $g : M \rightarrow N \models T_i$. Since $F \leq M$ is algebraic and $K \otimes_F M$ is integral, we have $K \otimes_F M \simeq Kg(M) \models T_{i,\forall}$. \square

Let us now prove that, in the cases which we will later consider, PTC_I is elementary:

Proposition 4.9. *Assume that for all $i \in I$,*

(A_i) *For every $F \models T_{i,\forall}$, $\text{acl}_i(F) \subseteq F^a$.*

(Z_i) *Maximal Zariski dimension is definable in family: for every \mathcal{L}_i definable sets $X \subseteq \mathbb{A}^{n+m}$, the set $\{s \in \mathbb{A}^n : \dim(X_s) = m\}$ is \mathcal{L}_i -definable, where X_s is the fiber of X in s .*

Then the class of PTC_I fields is elementary — in fact, inductive — in $\mathcal{L} \cup \mathcal{L}_I$.

Remark 4.10. Instead of assuming Hypothesis (A_i), we can allow the following generalization. For all $i \in I$, let $T'_i \supseteq T_{i,\forall}$ be such that for every $F \models T'_i$, $\text{dcl}_i(F) \subseteq F^a$. Then the class of PTC_I models of $T'_i := \bigcup_i T'_i$ is elementary.

For example, one could take T'_i to be the class of dcl_i -closed models of $T_{i,\forall}$.

Proof of Proposition 4.9. Let us first show that if $F \leq K \models T_I$ is \mathcal{L}_I -existentially closed and K is PTC_I , then so is F . Let $F \leq M \models T_I$ be regular and totally T . By Hypothesis (A_i), for all i , $\text{acl}_i(F) \cap M \subseteq F^a \cap M = F$ and hence, by Corollary 4.7, $K \leq K \otimes_F M \models T_I$ is regular and totally T and the \mathcal{L}_I -structure on $K \otimes_F M$ extends both that of M and K . Since K is PTC_I , it follows that this extension is \mathcal{L}_I -existentially closed and hence so is $F \leq M$.

Now, we let J be a set of indices and let K_j be PTC_I , for all $j \in J$, and \mathfrak{U} be an ultrafilter on J . We wish to show that $K := \prod_{j \rightarrow \mathfrak{U}} K_j$ is PTC_I . Let V be a smooth geometrically integral totally T variety over K such that for all $M \models T_K$, $V(M) \neq \emptyset$. By definability of irreducibility in ACF, we can find smooth geometrically irreducible varieties V_j over K_j such that $V = \prod_{j \rightarrow \mathfrak{U}} V_j$. Let $Y := \{j : \text{for all } M_j \models T_{K_j}, V_j(M_j) \neq \emptyset\}$. If $Y \notin \mathfrak{U}$, then for every $j \notin Y$, let $M_j \models T_{K_j}$ such that $V_j(M_j) = \emptyset$. Then $M := \prod_{j \rightarrow \mathfrak{U}} M_j \models T_K$, but $V(M) = \emptyset$, a contradiction. So, we may assume that $Y = J$. For $i \in I_0 \subseteq I$ finite, let now $M_i \models T_{i,K}$ and $X_i(M_i) \subseteq V$ be $\mathcal{L}_i(K)$ -definable and Zariski dense — in other words, $\dim(X_i) = \dim(V)$ or equivalently, any coordinate projection which is dominant from V is dominant from X_i . By Hypothesis (Z_i), we find $M_{ji} \models T_{i,K_j}$ and

$X_{ji}(M_{ji}) \subseteq V_j$ $\mathcal{L}_i(K)$ -definable and Zariski dense with $\prod_{j \rightarrow \mathfrak{U}} X_{ji} = X_i$. Since K_j is PTC_I , $\bigcap_i X_{ji}(K_j) \neq \emptyset$. Hence, $\bigcap_i X_i(K) \neq \emptyset$.

It follows that the class PTC_I is elementary in $\mathcal{L} \cup \mathcal{L}_I$. Note that, since it is in fact closed under existentially closed substructures, it is, in fact, inductive this can be seen, e.g., by working in the enrichment of $\mathcal{L} \cup \mathcal{L}_I$ by all existential formulas which yields a class closed under all substructures, i.e., a universal class. \square

5. V -topological theories

We now fix an $i \in I$ and $M_i \models T_i$. We wish to consider that \mathcal{L}_i -definable sets are essentially open. This is closely related to the notion of “ t -theory” in [van den Dries 1978, Chapter III].

Hypothesis 5.1. In this section, we assume that:

(H_{*i*}) T_i admits a definable (nondiscrete) Henselian V -topology τ_i and, for every $M_i \models T_i$, any $\mathcal{L}_i(M_i)$ -definable set X_i has nonempty τ_i -interior in (the M_i -points of) its Zariski closure.

Example 5.2. (1) Hypothesis **(H_{*i*})** holds if T_i is the theory of real closed fields. Indeed, the order topology is a definable Henselian V -topology and any definable set is a disjoint union of sets of the form $V \cap U$, where V is Zariski closed and U is open.

(2) Similarly, Hypothesis **(H_{*i*})** also holds if T_i is a theory of $(\text{RV}_1$ -enriched) Henselian valued fields which eliminates quantifiers relatively to RV_1 — or, more generally, relatively to $\bigcup_n \text{RV}_n$ in characteristic zero.

Remark 5.3. (1) Whenever **(H_{*i*})** holds, Hypothesis **(Z_{*i*})** of Proposition 4.9 also holds since a definable set is Zariski dense in \mathbb{A}^n if and only if it has nonempty τ_i -interior.

(2) As we will see in Lemma 5.4, Hypothesis **(A_{*i*})** of Proposition 4.9 also holds provided $\text{dcl}_i(\emptyset) \subseteq F^a$, where F is the field generated by the constants — constants that we may add for this exact purpose.

As it turns out, we can almost control the parameters in Hypothesis **(H_{*i*})**:

Lemma 5.4. *Let $F \leq M_i$. The following are equivalent:*

- (i) $\text{dcl}_i(\emptyset) \subseteq F^a$.
- (ii) $\text{acl}_i(F) \subseteq F^a$.
- (iii) Any $\mathcal{L}_i(F)$ -definable set X has nonempty τ_i -interior in its F -Zariski closure.

Proof. It is clear that (ii) implies (i).

(i) \Rightarrow (ii) Let $E = \text{dcl}_i(\emptyset)$. We prove, by induction on $\dim(V)$, that for any variety V over E and $X \subseteq V \times \mathbb{A}^1$ \mathcal{L}_i -definable with $X_a := \{y \in \mathbb{A}^1 : (a, y) \in X\}$ finite for every $a \in X(M_i)$, we have $X_a(M_i) \subseteq E(a)^a$. We may assume that V

is irreducible. By finiteness of X_a , X is not τ_i -open in $V \times \mathbb{A}^1$ and hence is not Zariski dense. So there exists $P = \sum_j p_j(x)y^j \in E[x, y]$ with $P(V \times \mathbb{A}^1) \neq 0$ but $P(X) = 0$. Let V_j be the zero locus of p_j in V and $J := \{j : V_j \subset V\} \neq \emptyset$. By induction, for any $j \in J$ and $a \in V_j$, $X_a(M_i) \subseteq E(a)^a$ and by construction, for any $a \in V \setminus \bigcup_j V_j$, $P(a, y) \neq 0$ and $P(a, X_a) = 0$.

(ii) \Rightarrow (iii) By [Lemma 4.1](#), $X(M_i)$ is Zariski dense in its F -Zariski closure V . It now follows from [\(H_i\)](#) that X has nonempty interior in V .

(iii) \Rightarrow (ii) Let $X \subseteq \mathbb{A}^1$ be a minimal finite $\mathcal{L}_i(F)$ -definable set and V its F -Zariski closure. Then either $\dim(V) = 0$, in which case $X \subseteq F^a$ as required, or $\dim(V) = 1$ and $V = \mathbb{A}^1$. By (iii), the finite set X is τ_i -open in \mathbb{A}^1 , contradicting its nondiscreteness. \square

Lemma 5.5. *Let $F \leq M_i$ be such that any nonempty open $\mathcal{L}_i(F)$ -definable subset of $\mathbb{A}^1(M_i)$ has an F -point. Then the topology generated by the $X(F)$, where $X \subseteq M_i$ is open and $\mathcal{L}_i(F)$ -definable, is a V -topology. If, moreover, $F^s \cap M_i \subseteq F$, then it is Henselian.*

We also denote this topology on F by τ_i even if it might not be the induced topology, per se.

Proof. Let S be $\mathcal{L}_i(F)$ -definable and bounded (in M_i) and let U be an $\mathcal{L}_i(F)$ -definable neighborhood of zero. Then $\{x \in M_i : xS \subseteq U\}$ is an $\mathcal{L}_i(F)$ -definable neighborhood of 0. So, by hypothesis, its τ_i interior contains a point distinct from 0; in other terms, $S(F)$ is bounded in F . So, if $U \subseteq M_i$ is an $\mathcal{L}_i(F)$ -definable neighborhood of 0, $(M_i \setminus U)^{-1}$ is bounded in M_i and hence in F , proving that the topology is a V -topology.

The second statement follows from the fact that Henselianity states the existence of roots to certain separable polynomials. \square

Lemma 5.6. *Let $F \leq M_i \models T_i$ be such that $F^s \cap M_i \subseteq F$ and any nonempty τ_i -open $\mathcal{L}_i(F)$ -definable subset of $\mathbb{A}^1(M_i)$ has an F -point. Let V be a geometrically irreducible variety over F and $X(M_i) \subseteq V(M_i)$ be τ_i -open and $\mathcal{L}_i(F)$ -definable. The following are equivalent:*

- (i) $X(M_i)$ is Zariski dense in V .
- (ii) $X(M_i) \cap V_{\text{sm}} \neq \emptyset$.
- (iii) $X(F) \cap V_{\text{sm}} \neq \emptyset$.

Proof. Note that (i) obviously implies (ii), and so does (iii). Also, shrinking V , we may assume [\(Remark 2.8\)](#) that it is affine smooth and that there exists an étale map $f : V \rightarrow \mathbb{A}^n$ over F .

(ii) \Rightarrow (iii) By [Fact 2.11](#), $f(X) \subseteq \mathbb{A}^n$ has a nonempty interior, so there exists $y \in f(X) \cap \mathbb{A}^n(F)$ and hence $x \in X(M_i)$ with $f(x) = y$. Then $x \in F(f(x))^s \cap M_i = F$.

(ii) \Rightarrow (i) By [Fact 2.11](#) we can find a nonempty $\mathcal{L}_i(M_i)$ -definable τ_i -open $U \subseteq X(M_i)$ such that $f|_U$ is a homeomorphism onto its open image. Since open subsets of \mathbb{A}^n are Zariski dense, (i) follows. \square

Lemma 5.7. *Let $F \leq M_i$, with $\text{dcl}_i(\emptyset) \subseteq F^a$. Let V be a geometrically irreducible variety over F and $X \subseteq V$ be $\mathcal{L}_i(F)$ -definable. The following are equivalent:*

- (i) $X(M_i)$ is Zariski dense in V .
- (ii) $X(M_i)$ is F -Zariski dense in V .
- (iii) The τ_i -interior of $X(M_i)$ in V contains a smooth point.

Proof. It is clear that (i) implies (ii) and, by [Lemma 5.6](#), (iii) implies (i). So let us prove that (ii) implies (iii). Assume that $X(M_i)$ is F -Zariski dense in V and let U be its τ_i -interior in V . Since $X \setminus U$ has empty τ_i -interior in V , by [Lemma 5.4](#), it is not F -Zariski dense in V . It follows that U is F -Zariski dense in V . In particular, $U \cap V_{\text{sm}} \neq \emptyset$. \square

Remark 5.8. Note that Hypothesis (\mathbf{Z}_i) of [Proposition 4.9](#) follows from (\mathbf{H}_i) (see [Hypothesis 5.1](#)). Indeed, by [Lemma 5.7](#), a definable set $X \subseteq \mathbb{A}^n$ is Zariski dense if and only if it has nonempty τ_i -interior.

Also, if $F \models T_{i,\forall}$ is such that $\text{dcl}_i(\emptyset) \subseteq F^a$, then, by [Lemma 5.4](#), $T_{i,F,\forall}$ satisfies the conditions of [Remark 4.10](#). Therefore, for topological theories, the conditions for PTC_I to be elementary given in [Proposition 4.9](#) and [Remark 4.10](#) are easy to ensure.

We now show the remarkable fact that, under some finiteness hypothesis, \mathcal{L}_I -existential closure comes essentially from independence of the τ_i topologies.

Hypothesis 5.9. We consider the following hypothesis:

- (\mathbf{F}_2) The set I is finite and for every $M \models T_K$, there exists $i \in I$ and an $\mathcal{L}_{\text{rg}}(K)$ -embedding $M_i \rightarrow M^* \succcurlyeq M$.

In particular, the minimal fields of the form $M \cap K^s$, where $M \models T_K$, are among the $K_i = M_i \cap K^s$, where $M_i \models T_{i,K}$. Note that it also follows that every extension $K \leq M \models T_I$ is totally T .

It follows that a field is then I -pseudo T -closed if and only if it is I -pseudo $\bigvee_{i \in I} T_i$ -closed, so the mention of T is now redundant.

Definition 5.10. If I is finite, we say that K is pseudo I -closed (PIC) if it is I -pseudo $\bigvee_{i \in I} T_i$ -closed.

Theorem 5.11. *Assume (\mathbf{H}_i) for all $i \in I$ (see [Hypothesis 5.1](#)) and (\mathbf{F}_2) . The following are equivalent:*

- (i) $K \models \text{PIC}$.
- (ii) The following hold:

- (a) $K \models \text{PTC}$.
- (b) For all $i \in I$, $\text{dcl}_i(\emptyset) \subseteq K^a$.
- (c) The τ_i are distinct topologies on K as $i \in I$ varies.
- (d) For all $i \in I$, any nonempty open $\mathcal{L}_i(K)$ -definable $X \subseteq \mathbb{A}^1$ contains a K -point.

Proof. Assuming (i), clearly condition (a) follows. Condition (b) follows from [Proposition 4.3\(iv\)](#) and conditions (c) and (d) follow from [Proposition 4.3\(iii\)](#) applied to \mathbb{A}^1 . Conversely, let us assume (ii). By [Proposition 4.3](#) and [Lemmas 5.7](#) and [5.6](#), it suffices to prove that, given V a geometrically integral totally T variety over K and $X_i \subseteq V$ $\mathcal{L}_i(K)$ -definable τ_i -open with $X_i(K_i) \cap V_{\text{sm}} \neq \emptyset$, we have $\bigcap_i X_i(K) \neq \emptyset$.

Note that, by Hypothesis (\mathbf{F}_2) , any $K_M = M \cap K^s$, where $M \models T_K$, is an algebraic extension of some K_i , which by [Lemma 5.5](#) admits a Henselian V -topology extending τ_i . So (\mathbf{H}) holds and if K_M is minimal in some C_τ and not separably closed, then $\tau = \tau_i$ and $K_M \simeq K_i$ — in particular, (\mathbf{F}_1) holds. Since the τ_i are all distinct, it also follows that every K_i is minimal in C_{τ_i} . We can, therefore, conclude with [Theorem 3.17](#). \square

Remark 5.12. If (\mathbf{F}_2) does not hold, but instead (\mathbf{H}) holds and no $M \models T_K$ contains K^s , then by [\[Schmid 2000, Theorem 4.9\]](#), [Theorem 5.11](#) also holds.

6. Bounded PIC fields

From now on, we assume that I is finite and that $T = \bigvee_{i \in I} T_i$. In particular, any extension $K \leq M \models T_I$ is totally T , and we therefore prefer the terminology pseudo I -closed (see [Definition 5.10](#)).

We fix some $I_\tau \subseteq I$, and we assume that, for every $i \in I_\tau$, (\mathbf{H}_i) holds (see [Hypothesis 5.1](#)). By [Remark 5.3](#), hypotheses (\mathbf{Z}_i) and (\mathbf{A}_i) of [Proposition 4.9](#) hold for every $i \in \tau$ (provided we add constants). We also assume that (\mathbf{Z}_i) holds for every $i \notin I_\tau$, and we fix T'_i as in [Remark 4.10](#).

Definition 6.1. A field K is called *bounded* if for any integer n , K has finitely many extensions of degree n .

Notation 6.2. Fix $\mathfrak{d} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$. Let $\mathcal{L}_{\mathfrak{d}} := \mathcal{L}_I \cup \{c_j : j > 0 \text{ and } |c_j| = \mathfrak{d}(j)\}$. We denote by $T_{\mathfrak{d}}$ the $\mathcal{L}_{\mathfrak{d}}$ -theory of fields such that $P_i := X^{\mathfrak{d}(i)} + \sum_{j < \mathfrak{d}(i)} c_{i,j} X^j$ is irreducible and every separable polynomial of degree i is split modulo P_i . Let $\text{PIC}_{\mathfrak{d}}$ be the (elementary) class of PIC models of $T_{\mathfrak{d}} \cup \bigcup_{i \notin I_\tau} T'_i$.

From now on, we always work in the language $\mathcal{L}_{\mathfrak{d}}$, so if K is $\text{PIC}_{\mathfrak{d}}$ and $A \leq K$, then $\text{acl}(A)$, $\text{dcl}(A)$, $\text{acl}_i(A)$, $\text{dcl}_i(A)$ always contain the constants.

Remark 6.3. Any model of $T_{\mathfrak{d}}$ is bounded. Conversely, any bounded model of T can be made into a model of $T_{\mathfrak{d}}$ for some well-chosen \mathfrak{d} .

Lemma 6.4. *Let $K \models T$ and $F \leq K$. The following are equivalent:*

- (1) $K \models T_\delta$ and $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ is surjective.
- (2) $F \models T_\delta$ and $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ is a homeomorphism.
- (3) $K \models T_\delta$ and $F \models T_\delta$.

This is rather standard, but we could not find a reference, so we sketch the proof.

Proof. Let us first assume (1) and prove (3). The polynomial P_n is clearly irreducible over F . Let $Q \in F[X]$ be an irreducible separable polynomial of degree n and a one of its roots. Since Q is split in $K[X]/P_n$, P_n admits a root $e \in F^s$ of P_n such that $a \in K[e]$. Then $\text{Gal}(F[e]) = \text{res}(\text{Gal}(K[e])) \leq \text{res}(\text{Gal}(K[a])) = \text{Gal}(F[a])$, where the equalities follow from surjectivity, and hence $F[a] \leq F[e]$.

If (3) holds, then every separable extension of K is included by one generated by an element of F^s and hence $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ is injective. Also, any separable extension of F is contained in one generated by a root of a P_n which is also irreducible over K . It follows that F^s is linearly disjoint from K over F and hence $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ is surjective, so (2) holds. Conversely, if (2) holds, then by Galois correspondence, the P_n are irreducible over K and $K^s = KF^s$ so $K \models T_\delta$ and (1) holds. \square

Lemma 6.5. *Let $K, M \models \text{PIC}_\delta$ with K perfect, and $F \leq K$ with $\text{acl}_i(F) \cap K \subseteq F$ for all $i \in I$. Then any \mathcal{L}_δ -embedding $f : F \rightarrow M$ can be extended to an \mathcal{L}_δ -embedding $g : K \rightarrow M^* \succcurlyeq M$ with $\text{acl}_i(g(K)) \cap L^* \subseteq g(K)$ for all $i \in I$.*

Proof. Note that F is perfect and hence $F \leq K$ is regular. So g exists by [Corollary 4.7](#) and the fact that $M \models \text{PIC}$. It remains to show that $\text{acl}_i(g(K)) \cap M^* \subseteq g(K)$. Since $\text{acl}_i(K) \subseteq K^a$, it suffices to show that $g(K)^a \cap M^* \subseteq g(K)$. By [Lemma 6.4](#), since $K \models T_\delta$ and $F \leq K$ is regular, $F \models T_\delta$ and hence, by [Lemma 6.4](#) again, $\text{Gal}(M^*) \rightarrow \text{Gal}(g(F))$ is a homeomorphism. It factorizes through the homeomorphism $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ and hence $\text{res} : \text{Gal}(M^*) \rightarrow \text{Gal}(g(K))$ is surjective; i.e., $g(K)^a \cap M^* = g(K)$. \square

Definition 6.6. For every $K \models \text{PIC}_\delta$ and $A \subseteq K$, let $\text{acl}_I(A) \subseteq K$ be the smallest subset $C \subseteq K$ containing A such that $\text{acl}_i(C) \cap K \subseteq C$ for all $i \in I$.

Proposition 6.7. *Let $K, M \models \text{PIC}_\delta$ be perfect, $F \leq K$ and $f : F \rightarrow M$ be an \mathcal{L}_δ -embedding. The following are equivalent:*

- (1) f extends to an \mathcal{L}_δ -embedding $g : \text{acl}_I(F) \rightarrow M$.
- (2) f is \mathcal{L}_δ -elementary (from K to M).

Proof. (1) \Rightarrow (2) Let $g : \text{acl}_I(F) \rightarrow M$ extend f . By a standard back and forth argument using [Lemma 6.5](#), we build $K \preceq K^*, M \preceq M^*$ and an \mathcal{L}_δ -isomorphism $h : K^* \rightarrow M^*$ extending g . Elementarity of f follows immediately.

(2) \Rightarrow (1) This is immediate. \square

Corollary 6.8. *Any $K \leq M$, with both fields PIC_δ perfect, is \mathcal{L}_δ -elementary.*

Proof. By Lemma 6.4, the morphism $\text{res} : \text{Gal}(K) \rightarrow \text{Gal}(F)$ is surjective. Since $\text{acl}_I(K) \subseteq K^a$, it follows that $\text{acl}_I(K) \cap M \subseteq K$ and hence, by Proposition 6.7 the inclusion is elementary. \square

For the rest of the section we fix a perfect $K \models \text{PIC}_\delta$ and $M_i \models T_{i,K}$.

Proposition 6.9. *Let $F \leq K$. Then $\text{acl}(F) = \text{acl}_I(F)$.*

Proof. Let $E = \text{acl}_I(F) \subseteq \text{acl}(F)$. By Corollary 4.7, we find an $\mathcal{L}_\delta(E)$ -embedding $f : K \rightarrow K^* \succ K$ with $f(K) \downarrow_E^\ell K$. By Corollary 6.8, f is elementary. It follows that $\text{acl}(F) \subseteq K \cap f(K) = E$. \square

Notation 6.10. For every $p \in S(F)$, where $F \leq K$, let p_i denote the underlying (quantifier free) \mathcal{L}_i -type.

Proposition 6.11. *Let $F = \text{acl}(F) \leq K$, V an irreducible variety over F , $p \in S(F)$ F -generic in V and, for $i \in I_\tau$, let $O_i \subseteq V(M_i)$ be $\mathcal{L}_i(K)$ -definable τ_i -open sets consistent with p_i (modulo $T_{i,K}$). Then p is consistent with $\bigcap_{i \in I_\tau} O_i(K)$.*

Proof. Let $a \models p$ (in some $K^* \succ K$). Let ac enumerate $\text{acl}(Fa)$. By Proposition 6.7, it suffices to find $de \equiv_F^{\text{qf}} ac$ with $d \in \bigcap_{i \in I_\tau} O_i$. By compactness, we may assume c is finite. Let $q_i = \text{tp}_i(ac/F)$ for every $i \in I$, and W be the geometrically integral algebraic locus of ac . For every $i \in I_\tau$, we also denote by O_i the corresponding open subset of W .

Let us fix some $i \in I_\tau$. For every $\mathcal{L}_i(F)$ -definable X , by (\mathbf{H}_i) , $q_i \vdash X$ if and only if $q_i \vdash \dot{X} \cap W_{\text{sm}}$, where \dot{X} denotes the τ_i -interior of $X \cap W$ in W . In particular, $\dot{X} \cap W_{\text{sm}} \cap O_i \neq \emptyset$ and, by Lemma 5.6, $X \cap O_i$ is consistent with the K -generic of W . So, by compactness, we find $a_i c_i \models q_i$ K -generic in W with $a_i \in O_i$. If $i \notin I_\tau$, by Lemma 4.6 we can also find $a_i c_i \models q_i$ K -generic in W . Let $r_i := \text{tp}_i(a_i c_i/K) \supseteq q_i$. Note that since $\text{tp}(ac/F)$ is finitely satisfiable in K , $W_{\text{sm}}(K) \neq \emptyset$. By Proposition 4.3, there exists $de \models \bigcup_i r_i$ in some $K^* \succ K$. By construction, for every $i \in I_\tau$, we have $d \in O_i$ and, for every $i \in I$, we have $de \models p_i$, concluding the proof. \square

Let τ be the topology generated by the τ_i for $i \in I_\tau$.

Theorem 6.12. *Let $F = \text{acl}(F) \leq K$, and X be $\mathcal{L}_\delta(F)$ -definable and τ -closed (or τ -open) in its Zariski closure V . Then X is quantifier free $\bigcup_{i \in I_\tau} \mathcal{L}_i(F)$ -definable.*

Proof. Assume that X is τ -closed. Let $x \in X(K)$ and $y \in V(K)$ be such that $\bigcup_{i \in I_\tau} \text{tp}_i(x/F) = \bigcup_{i \in I_\tau} \text{tp}_i(y/F)$. By compactness, it suffices to show that $y \in X$. For every $i \in I_\tau$, $O_i \subseteq V(M_i)$ τ_i -open containing y . Let W be the locus of x (and hence y) over F . By Proposition 6.11, $\text{tp}(x/F)$ is consistent with $W \cap \bigcap_i O_i$. In particular, there exists $z \in X \cap \bigcap_i O_i$. So y is in the τ -closure of X , which is X itself. \square

Considering the τ -closure of a definable set, we have generalized the “density theorem” of [Montenegro 2017b, Theorem 3.17]:

Corollary 6.13. *Let $F = \text{acl}(F) \leq K$, and X be $\mathcal{L}_\delta(F)$ -definable with Zariski closure V . Then there exists an integer m , $\mathcal{L}_i(F)$ -definable τ_i -open sets $Y_{ij} \subseteq V$ for $i \in I_\tau$ and $j < m$, and a quantifier free $\mathcal{L}_\delta(F)$ -definable set Y with $\dim(Y) < \dim(X)$ such that*

$$X \subseteq \bigcup_j \bigcap_i Y_{ij} \cup Y \text{ is } \tau\text{-dense.}$$

Proof. Applying Theorem 6.12 to the τ -closure of X , we find $\mathcal{L}_i(F)$ -definable sets Y_{ij} such that $X \subseteq \bigcup_j \bigcap_i Y_{ij}$ is τ -dense. Further dividing, we may assume that Y_i is either τ_i -open or has empty τ_i -interior (and hence has lower dimension by (\mathbf{H}_i)). \square

Remark 6.14. The previous result generalizes the common pattern, that, when adding a “generic” topology to some structure, the “open core” consists only of sets definable in the topology on its own. In fact, this result provides a broad setting in which to consider these questions, which does not depend on the initial structure and allows for more than one topology.

Generic derivations. To illustrate that point, let us consider the following example. Let $T_1 = \text{DCF}_0$ be the (Morleyized) theory of characteristic zero differentially closed fields with a derivation, $T_2 = \text{ACVF}$ and $I := \{T_1, T_2\}$. Then existentially closed valued fields with a derivation are PIC . Hence, definable closed sets in such fields are definable using only the valued field structure, recovering a special case of [Cubides Kovacsics and Point 2023, Theorem 3.1.11].

The proof given here would extend immediately to bounded perfect PAC fields with independent valuations and generic commutative derivations (which are PIC for $T_1 = \text{DCF}_n$ and $T_2, \dots, T_m = \text{ACVF}$), if we knew that this class is indeed closed under elementary extensions. Proposition 4.9 does not apply in that context, but a geometric axiomatization seems plausible, along the lines of [Tressl 2005].

There is another issue if we wish to consider fields that are not PAC, namely our reliance on relational languages. If $T_2 = \text{RCF}_<$ and $T_1 = \text{DCF}_0$ (Morleyized), then existentially closed ordered differential fields might not be PIC . Indeed, the order on a subfield of a differential field might not extend to the whole field. However, the setup presented here can be adapted to allow for nonrelational \mathcal{L}_i ; the main issue being, again, the elementarity of the class.

7. Amalgamation and burden in PIC_δ

Recall that we now assume that I is finite and that $T = \bigvee_{i \in I} T_i$. In particular, any extension $K \leq M \models T_I$ is totally T . We now work with $I_\tau = I$ and therefore assume that (\mathbf{H}_i) holds for every $i \in I$ (see Hypothesis 5.1), and τ denotes the topology

generated by the τ_i for all $i \in I$. Finally, we fix $K \models \text{PIC}_\partial$ perfect, sufficiently saturated and homogeneous (see [Notation 6.2](#)) and $M_i \models T_{i,K}$.

We start by proving that obstructions to 3-amalgamation only arise from the quantifier free structure.

Theorem 7.1. *Let $F \leq K$ and $a_1, a_2, c_1, c_2, c \in K$. Suppose that $\text{dcl}_i(\emptyset) \subseteq F^a$ for all i , $F = \text{acl}(F)$, a_j enumerates $A_j := \text{acl}(Fa_j)$ and c_j enumerates $C_j := \text{acl}(Fc_j)$ for $j = 1, 2$. Assume also that $A_1 \cap A_2 = F$, $c \downarrow_F^a a_1 a_2$, $c_1 \equiv_F c_2$ and $c \equiv_{A_j}^{\text{qf}} c_j$ for $j = 1, 2$. Then*

$$\text{tp}(c_1/A_1) \cup \text{tp}(c_2/A_2) \cup \text{qftp}(c/A_1 A_2)$$

is consistent.

This is a minor variation on [\[Montenegro 2017b, Theorem 3.21\]](#).

Proof. First note that, since $A_1 A_2$ is a regular extension of A_1 and of A_2 , by [\[Chatzidakis 2002, Lemma 2.1\]](#) we have that $A_1^a \cap A_2^a = (A_1 \cap A_2)^a = F^a$. Note also that for any $F \leq E \leq K$, by [Lemma 5.4](#) $\text{acl}(E) = E^a \cap K$.

Let $C := F(c)$ and $A := \text{acl}(A_1 A_2)$. Since $c \equiv_{a_j}^{\text{qf}} c_j$, we have an $\mathcal{L}_I(A_j)$ -isomorphism $f_j : A_j C_j \rightarrow A_j C$ sending c_j to c . This map extends to an $\mathcal{L}_{\text{rg}}(A_j)$ -embedding $B_j := \text{acl}(A_j C_j) \rightarrow (A_j C)^a$. We endow $D_j := f_j(B_j)$ with the \mathcal{L}_I -structure making f_j into an $\mathcal{L}_I(A_j)$ -isomorphism. This structure extends the \mathcal{L}_I -structure on $A_j C$.

Note that both $A_2 \leq A$ and $A_2 \leq D_2 \simeq B_2$ are regular and that, since $a_1 \downarrow_{a_2}^a c$, we have $A \downarrow_{A_2}^a D_2$. It follows that $A \leq D_2 A$, $A_2 \leq D_2 A$ and $F \leq D_2 A$ are regular. So $D_2 A \downarrow_F^\ell F^a$ and hence $D_2 A \downarrow_{CA_1}^\ell F^a C A_1$. Moreover, since $C^a \downarrow_{F^a}^a A_1^a A_2^a$, by [\[Chatzidakis 2002, Lemma 2.5\(2\)\]](#) we have

$$(CA_1)^a \cap (CA_2)^a (A_1 A_2)^a = (C^a (A_1^a \cap A_2^a))^a A_1^a = C^a A_1^a.$$

By [Lemma 6.4](#), $\text{Gal}(C) \simeq \text{Gal}(F)$, and hence $C^a = F^a C$. Similarly, $A_1^a = F^a A_1$ and thus $C^a A_1^a = F^a C A_1$. So, $(CA_1)^a \cap D_2 A = F^a C A_1 \cap D_2 A = C A_1$; i.e., $CA_1 \leq D_2 A$ is regular. By symmetry, $CA_2 \leq D_1 A$ is also regular.

Since $D_j \downarrow_{CA_j}^\ell CA \leq D_k A$ for $k \neq j$, by [Lemma 4.6](#), $D_j A$ can be made into a model of T_I whose \mathcal{L}_I -structure extends that of D_j and CA . Since $D_1 \downarrow_{CA_1}^\ell D_2 A$, we also have $D_1 A \downarrow_{CA}^\ell D_2 A$ and, by [Lemma 4.8](#), $D_1 D_2 A$ can be made into a model of T_I whose \mathcal{L}_I -structure extends that of CA , D_1 and D_2 . Also, since $CA_1 \leq D_2 A$ is regular and $D_1 \leq (CA_1)^a$, it follows that $D_1 \leq D_1 D_2 A$, and thus $F \leq D_1 D_2 A$, are regular. So $D_1 D_2 A \downarrow_A^\ell F^a A = A^a$ and $A \leq D_1 D_2 A$ is regular.

By [Corollary 4.7](#), there exists an $\mathcal{L}_I(A)$ -embedding $g : D_1 D_2 A \rightarrow K^* \succ K$. Let $c^* = g(c)$. We have $c^* \equiv_{\mathcal{L}_I(a_1 a_2)}^{\text{qf}} c$. Moreover, by [Proposition 6.7](#), $g \circ f_j$ is a $\mathcal{L}_I(A_j)$ -elementary isomorphism sending c_j to c^* . \square

Burden. We now study the burden of perfect bounded PIC fields in terms of the burden of each theory T_i .

Notation 7.2. We denote by bdn_i the burden computed in models of $T_{i,K}$.

Lemma 7.3. *Let X be an $\mathcal{L}_i(M_i)$ -definable set of dimension d . Then*

$$\text{bdn}_i(X) = \text{bdn}_i(\mathbb{A}^d).$$

Proof. By Remark 2.13, there exists an $\mathcal{L}_i(M_i)$ -definable bounded τ_i -open neighborhood O_i of 0 such that $M_i \subseteq O_i \cup (O_i \setminus \{0\})^{-1}$. Then

$$\mathbb{A}^d(M_i) = \bigcup_{\varepsilon \in 2^d} \prod_{j < d} O_i^{\varepsilon_j},$$

where $O_i^1 = O_i$ and $O_i^0 = (O_i \setminus \{0\})^{-1}$. Since $\text{bdn}_i(\prod_{j < d} O_i) = \prod_{j < d} O_i^{\varepsilon_j}$ for every ε , it follows that $\text{bdn}_i(\prod_{j < d} O_i) = \text{bdn}_i(\mathbb{A}^d)$. If $U \subseteq \mathbb{A}^d$ is an $\mathcal{L}_i(M_i)$ -definable open set, it contains a definable subset of the form $\prod_{j < d} U_j$, where $U_j \subseteq M_i$ is in (affine) $\mathcal{L}_i(M_i)$ -definable bijection with O_i . It follows that

$$\text{bdn}_i(\mathbb{A}^d) = \text{bdn}_i\left(\prod_{j < d} O_i\right) = \text{bdn}_i\left(\prod_{j < d} U_j\right) \leq \text{bdn}_i(U) \leq \text{bdn}_i(\mathbb{A}^d).$$

Now, since $\dim(X) = d$, there exists an $\mathcal{L}_i(M_i)$ -definable finite-to-one map $f : X \rightarrow \mathbb{A}^d$ whose image is Zariski dense and hence has nonempty interior. So, we have $\text{bdn}_i(X) = \text{bdn}_i(f(X)) = \text{bdn}_i(\mathbb{A}^d)$. \square

Proposition 7.4. *Let $A = \text{acl}(A) \subseteq K$, X be quantifier free $\mathcal{L}_I(A)$ -definable and nonempty, $\phi(x, y)$ be an \mathcal{L}_0 -formula and $(a_j)_{j \in \omega} \in K^\omega$ be an indiscernible sequence over A . Suppose that $\phi(K, a_0)$ is τ -dense in X . Then $\bigwedge_j \phi(x, a_j)$ is consistent.*

Proof. By compactness, extend $(a_j)_{j \in \omega}$ to an A -indiscernible sequence indexed by some sufficiently large cardinal κ . For every finite $J \subseteq \kappa$, let a_J enumerate $\text{acl}(Aa_j : j \in J)$ in a way compatible with inclusions: if $f : J_1 \rightarrow J_2$ is an increasing bijection, then for any $J \subseteq J_1$, a_J and $a_{f(J)}$ appear in the same place in a_{J_1} and a_{J_2} , respectively.

Claim 7.4.1. *There exists $c \in X$ such that $c \downarrow_A^a \bigcup_{j < \omega} a_j$ and, for all finite $J \subseteq \omega$, $\text{tp}(a_J / Ac)$ only depends on $|J|$.*

Proof. Note that, by invariance, the Zariski closure of X is over A and hence we can find $c \in X$ with $c \downarrow_A^a \bigcup_j a_j$. By iterated applications of Erdős–Rado and at the cost of reducing κ (exactly as when extracting an indiscernible sequence), we find $J_0 \subseteq \kappa$, with order type ω , such that for all finite $J \subseteq J_0$, $\text{tp}(a_J / Ac)$ only depends on $|J|$. By homogeneity (and changing c), we may assume that $J_0 = \omega$. \square

Note that if $J_1, J_2 \subseteq \omega$ are disjoint of the same cardinality, if $e \in a_{J_1} \cap a_{J_2}$, then there is some n, m with $a_{J_1, n} = e = a_{J_2, m}$. Spreading J_1 and J_2 apart, we can find J_3 such that $J_1 \cup J_2, J_1 \cup J_3$ and $J_2 \cup J_3$ are ordered in the same way. It follows that $a_{J_1, n} = a_{J_3, m} = a_{J_2, n}$. So e appears in the same place in all a_J , with J of the given cardinality, and $\text{tp}(a_J / \text{Ace})$ only depends on $|J|$. So we may assume that for disjoint $J_1, J_2 \subseteq \omega$, $a_{J_1} \cap a_{J_2} = A$.

Claim 7.4.2. *There exists $c \in \phi(K, a_0)$ such that $c \downarrow_A^a \bigcup_{j < \omega} a_j$ and, for all finite $J \subseteq \omega$, $\text{qftp}(a_J / \text{Ac})$ only depends on $|J|$.*

Proof. Let $\Sigma_i(x)$ be the (closure under consequence of the) partial type expressing that $\text{tp}_i(a_J / Ax)$ only depends on $|J|$. By compactness, it suffices, given $\psi_i(x) \in \Sigma_i(x)$ for every i and $U \subseteq V$ Zariski open over $A(a_j)_{j < \omega}$, to find $e \in U$ realizing $\phi(x, a_0)$ and the ψ_i .

Let c be such as in Claim 7.4.1 and let V be its algebraic locus over A , which is also its algebraic locus over $Aa_{j < \omega}$. Then c is in the τ_i -interior of $\psi_i(K)$ in V and, since $c \in \psi_i(K) \cap X$, $\psi_i(K)$ has nonempty τ_i -interior in X . The existence of e as above now follows from the τ -density of $\phi(K, a_0)$ in X and the fact U is τ -open (in V and hence in X). \square

Note that if E is an $\mathcal{L}_i(\text{Ac})$ -definable finite equivalence relation, for some finite $J_1, J_2 \subseteq \omega$ we have $a_{J_1} E a_{J_2}$ and hence this holds for all J_1, J_2 . In other words, $\text{tp}_i(a_J / \text{acl}_i(\text{Ac}))$ only depends on $|J|$ and hence, since $\text{acl}(\text{Ac}) \subseteq A(c)^a \subseteq \text{acl}_i(\text{Ac})$ for all i , $\text{qftp}(a_J / \text{acl}(\text{Ac}))$ only depends on $|J|$. Let d enumerate $\text{acl}(\text{Ac})$.

Claim 7.4.3. *For all $n > 0$, there is $d_n \equiv_{a_{\{0, \dots, 2^n-1\}}}^{\text{qf}} d$ such that $K \models \bigwedge_{j < 2^n} \phi(d_n, a_j)$.*

Proof. We proceed by induction on n and let us assume we have found d_n — note that $d_0 = d$ works. Let $a^1 = a_{\{0, \dots, 2^n-1\}}$ and $a^2 = a_{\{2^n, \dots, 2^{n+1}-1\}}$, $d^1 = d_n$ and d^2 be such that $d^2 a^2 \equiv_A d^1 a^1$; in particular, $d^2 \models \bigwedge_{2^n \leq j < 2^{n+1}} \phi(x, a_j)$. Then

$$\begin{aligned} a^1 \cap a^2 &= A, & d \downarrow_A^a a^1 a^2, & & d^1 \equiv_A d^2, \\ d^1 a^1 &\equiv_A^{\text{qf}} d a^1 & \text{and} & & d^2 a^2 \equiv_A d^1 a^1 \equiv_A^{\text{qf}} d a^1 \equiv_A^{\text{qf}} d a^2. \end{aligned}$$

By Theorem 7.1, we can find a d_{n+1} such that $d_{n+1} a^j \equiv_A d^j a^j$ — and hence $d_{n+1} \in \bigcap_{j < 2^{n+1}} \phi(K, a_j)$ — and $d_{n+1} \equiv_{a^1 a^2}^{\text{qf}} d$. \square

So $\bigwedge_j \phi(x, a_j)$ is indeed consistent. \square

Proposition 7.5. *Let X be $\mathcal{L}_\delta(K)$ -definable of dimension d . If for every $i \in I$, $\text{bdn}_i(\mathbb{A}^d) \leq \kappa_i$ for some cardinal κ_i , then*

$$\text{bdn}(X) \leq \sum_{i \in I} \kappa_i.$$

Moreover, if $\text{bdn}_i(\mathbb{A}^d) < \kappa$, where κ is an infinite cardinal, then

$$\text{bdn}(X) < \sum_{i \in I} \kappa.$$

Proof. We proceed by induction on d and assume K is sufficiently saturated. Let $\phi_l(x, a_{lj})_{l < \kappa, j < \omega}$ be an inp-pattern of depth κ in X — where $\kappa > \sum_i \text{bdn}_i(\mathbb{A}^d)$ in the first case. Let $A = \text{acl}(A) \leq K$ be such that the ϕ_l and X are all over A . We may assume that the $(a_{lj})_{j < \omega}$ are mutually indiscernible over A . Let V be the Zariski closure of X .

By [Chernikov 2014, Lemma 7.1] (see Remark 2.4(2)), Theorem 6.12 and indiscernibility, we find (uniformly) $\mathcal{L}_i(a_{lj})$ -definable $Y_{lji} \subseteq V$ such that $\phi_l(K, a_{lj})$ is τ -dense in $Y_{lj} := \bigcap_i Y_{lji}$. If any $Y_{l_0 j_0 i_0}$ has dimension smaller than d , then, by induction, $\text{bdn}(Y_{l_0 j_0 i_0}) \leq \sum_{i \in I} \lambda_i$, where $\lambda_i + 1 = \kappa_i$. However, the $\phi_l(x, a_{lj})_{l \neq l_0, j < \omega}$ is an inp-pattern of depth λ in $Y_{l_0 j_0 i_0}$, where $\lambda + 1 = \kappa$. So

$$\kappa = \lambda + 1 \leq \text{bdn}(Y_{l_0 j_0 i_0}) + 1 \leq \sum_{i \in I} \lambda_i + 1 \leq \sum_{i \in I} \kappa_i < \kappa,$$

a contradiction. In the second case, by induction, we would have $\kappa \leq \text{bdn}(Y_{l_0 j_0 i_0}) < \kappa$, which is also a contradiction.

So we may assume that all the Y_{lji} have dimension d . In particular, they have nonempty τ_i -interior in V . Since the border of Y_{lji} in V has lower dimension, by [Chernikov 2014, Lemma 7.1] and induction, we may further assume that the Y_{lji} are τ_i -open in V .

Since $\phi_l(x, a_{lj})$ is an inp-pattern, for all $f : \kappa \rightarrow \omega$ we have that $\bigwedge_l Y_{l, f(l), i}$ is consistent. It follows that there are at most κ_i many l (or strictly less than κ in the second case) such that $\bigwedge_l Y_{lji}$ is inconsistent. In both cases, it follows that we can find an l such that for all i , $\bigwedge_j Y_{lji}$ is consistent.

Since the Y_{lji} are τ_i -open (in V), by compactness, $\bigwedge_j Y_{lji}$ contains some $\mathcal{L}_i(K)$ -definable τ_i -open $U_i \subseteq V$. Let $U = \bigcap_i U_i$. Then for every j , $\phi(K, a_{lj})$ is τ -dense in U . Let $B \supseteq A$ be such that U is quantifier free $\mathcal{L}_I(B)$ -definable. We may assume that a_{lj} is B -indiscernible. By Proposition 7.4, $\bigwedge_j \phi(K, a_{lj})$ is consistent — contradicting that $\phi_l(x, a_{lj})_{l < \kappa, j < \omega}$ is an inp-pattern. \square

We also have the following qualitative corollary:

Corollary 7.6. *Suppose that for all $i \in I$, T_i is NTP₂ (or strong). Then any perfect $K \models \text{PIC}_0$ also is.*

The bound in Proposition 7.5 is actually tight (although the argument is surprisingly more involved than we expected at first).

Lemma 7.7. *If for some $i \in I$, $\text{bdn}_i(\mathbb{A}^d) \geq \kappa$ for some cardinal κ , then there exists an inp-pattern $(\phi_l(x, a_{lj}))_{l < \kappa, j < \omega}$ of depth κ in any τ_i -open $\mathcal{L}_i(K)$ -definable set $X \subseteq \mathbb{A}^d$ such that $a_{lj} \in K$ and $\phi_l(x, a_{lj})$ defines a τ_i -open set.*

Proof. Note first that, at the cost of reducing X , we may assume that $X = \prod_{k < d} X_k$ is a product of open (bounded) subsets of \mathbb{A}^1 . By [Lemma 7.3](#), we find $M_i \models T_{i,K}$ and $(\phi_l(x, a_{lj}))_{l < \kappa, j < \omega}$ an inp-pattern of depth κ in $X(M_i)$. Writing $\phi(M_i, a_{lj})$ as the union of its interior and a lower-dimensional set, by [Remark 2.4\(2\)](#), we may assume that either $\phi(M_i, a_{lj})$ is τ_i -open or $\dim(\phi(M_i, a_{lj})) < d$. In the latter case, it follows by [Remark 2.4\(6\)](#) that $\text{bdn}(\mathbb{A}^{d-1}) \geq \lambda$, where $\lambda + 1 = \kappa$. By induction, we find an inp-pattern of depth λ in $\prod_{k < d-1} X_k$ with $\psi_l(M_i, c_{lj})$ consisting of τ_i -open sets. Together with a uniformly definable family of disjoint τ_i -open subsets of X_{d-1} , they yield an inp-pattern of depth κ in X .

So we may assume that the $\phi_l(M_i, a_{lj})$ are τ_i -open and it only remains to prove that we can choose the a_{lj} in K . Let c_{lj} be generic in some \mathbb{A}^d such that $a_{lj} \in \text{acl}(c_{lj})$. We may assume that a_{lj} enumerates $\text{acl}_i(c_{lj})$, that $(c_{lj}a_{lj})_j$ are mutually indiscernible sequences and that it is the middle part of a sequence indexed by $\omega + \omega + \omega^*$. Let A_l consist of the union of K , the ω initial segment and the ω^* final segment of the sequence. For each j , let E_{lj} be the set of $\mathcal{L}_i(A_l c_{lj})$ -conjugates of a_{lj} .

Claim 7.7.1. *For every $j_0 < \omega$, E_{lj_0} is a complete type over $A_l(c_{lj})_j(a_{lj})_{j \neq j_0}$.*

Proof. Assume there exists $E \subset E_{lj_0}$ nonempty defined over $A_l(c_{lj})_j(a_{lj})_{j \neq j_0}$. Then, by indiscernibility, moving the parameters to the initial and final segment, such a set also exists with parameters in A . \square

Let $X_{lj} = \bigcup_{a \in E_{lj}} \phi_l(x, a)$. If the X_{lj} are consistent, then for every k and $j < k$ there exists $e_{lj} \in E_{lj}$ such that $\bigwedge_{j < k} \phi_l(x, e_{lj})$ is consistent. But, by [Claim 7.7.1](#), $\prod_{j < k} E_{lj}$ is a complete type. This would imply that $\bigwedge_{j < k} \phi_l(x, a_{lj})$ is consistent, contradicting that the $\phi_l(x, a_{lj})$ form an inp-pattern. It follows that the X_{lj} form an inp-pattern (of depth κ in X).

So, at the cost of allowing $\phi_l(x, y)$ to be an $\mathcal{L}_i(A)$ -formula for some $A \subseteq M_i$ containing K , we may assume the a_{lj} to be generic in some \mathbb{A}^m (over A). Let C be a transcendence basis of A over K and $b \in A \subseteq \text{acl}_i(C)$ be such that $\phi_l(x, y) = \psi(x, y, b)$, where ψ is an $\mathcal{L}_i(K(C))$ -formula. As above, possibly enlarging C , we may assume that the set E of $\mathcal{L}_i(K(C))$ -conjugates of b is a complete type over $K(C, c_{lj})$. Then, for every $e \in E$, the $\psi(x, a_{lj}, e)$ are inconsistent. It follows that the

$$\theta(x, a_{lj}) = \bigvee_{e \in E} \psi(x, a_{lj}, e)$$

are inconsistent and hence form an inp-pattern (of depth κ in X). So we may assume that $A = K(C)$ is generated over K by a (finite) set of algebraically independent elements. Then $C(a_{lj})_{l < \kappa, j < \omega}$ is a tuple of algebraically independent elements over K ; in other words any finite subtuple is K -generic in the affine space of the right dimension. It follows, e.g., by [Proposition 4.3\(ii\)](#) that its $\mathcal{L}_i(K)$ -type can be realized in K . \square

Proposition 7.8. *Let X be $\mathcal{L}_\partial(K)$ -definable of dimension d . If for every $i \in I$, $\text{bdn}_i(\mathbb{A}^d) \geq \kappa_i$ for some cardinal κ_i , then*

$$\text{bdn}(X) \geq \sum_{i \in I} \kappa_i.$$

Proof. By [Remark 2.4\(6\)](#), we may assume that $X \subseteq \mathbb{A}^d$. By [Corollary 6.13](#), we may further assume that X is τ -dense in $\bigcap_i X_i$, where X_i is $\mathcal{L}_i(K)$ -definable and τ_i -open. By [Lemma 7.7](#), assuming that K is sufficiently saturated, for every $i \in I$ we find an inp-pattern $\phi_{i,l}(x, a_{i,l,j})_{l < \kappa_i, j < \omega}$ of τ_i -open sets in X_i , with $a_{i,l,j} \in K$.

Note that for any finite $Y_i \subseteq \kappa_i$ and $f_i : Y_i \rightarrow \omega$, $\bigcap_{l \in Y_i} \phi_{i,l}(M_i, a_{i,l,f(i,l)})$ is τ_i -open and hence, by τ density, $X \cap \bigcap_{i,l \in Y_i} \phi_{i,l}(M_i, a_{i,l,f(i,l)}) \neq \emptyset$. It follows that together, the $\phi_{i,l}(x, a_{i,l,j})$ form an inp-pattern of depth $\sum_i \kappa_i$ in X . \square

We have thus proved the following equality:

Theorem 7.9. *For every perfect $K \models \text{PIC}_\partial$ and $\mathcal{L}_\partial(K)$ -definable set X of dimension d , we have*

$$\text{bdn}(X) = \sum_i \text{bdn}_i(\mathbb{A}^d).$$

The equality holds in Card^* [[Touchard 2023](#), Definition 1.29].

Corollary 7.10. *Let $K \models \text{PAC}$ be perfect and bounded and v_i be n independent nontrivial valuations on K . Then*

$$\text{bdn}(K, v_1, \dots, v_n) = n.$$

A similar corollary holds for PRC and PpC fields, but we refer the reader to the more general statement in [Corollary A.3](#).

8. Forking and dividing

In this section we study the relationship between forking (and dividing) in the theories T_i and in the theory PIC_∂ .

Let us recall our conventions and notation. We assume that I is finite and that $T = \bigvee_{i \in I} T_i$. We also assume that (\mathbf{H}_i) holds for every $i \in I$ (see [Hypothesis 5.1](#)). We denote by τ the topology generated by the τ_i , and we fix $K \models \text{PIC}_\partial$ perfect, sufficiently saturated and homogeneous. We fix some $M_i \models T_{i,K}$.

Proposition 8.1. *Let $A = \text{acl}(A) \leq K$ and let X be an $\mathcal{L}_\partial(K)$ -definable set such that $X \subseteq \bigcap_i X_i$ is τ -dense, where X_i is $\mathcal{L}_i(K)$ -definable and τ_i -open. Then X does not divide over A if and only if for all i , X_i does not divide over A (in M_i).*

Proof. (\Leftarrow) We assume that, for every i , $X_i := \theta_i(M_i, b)$ does not divide over A in M_i , and we write X as $\psi(K, b)$. Let κ be a sufficiently large cardinal and $(b_j)_{j < \kappa}$ be an indiscernible sequence over A with $b_0 = b$. Let $X_{ij} = \theta_i(M_i, b_j)$.

By indiscernibility, $X_j = \psi(K, b_j) \subseteq \bigcap_i X_{ij}$ is τ -dense. Since X_i does not divide over A , we have $\bigcap_j X_{ij} \neq \emptyset$ and, by compactness, it contains some nonempty $\mathcal{L}_i(K)$ -definable τ_i -open U_i . Enlarging A and taking an appropriate subsequence of the b_j , we may assume that U_i is $\mathcal{L}_i(A)$ -definable.

Note that since $U := \bigcap_i U_i$ is τ -open, X is τ -dense in U . It follows by [Proposition 7.4](#) that $\bigcap_j (X_j \cap U) \neq \emptyset$, and hence X does not divide over A .

(\Rightarrow) We now assume that for some fixed i , $X_i := \theta_i(M_i, b)$ divides over A , and let $(b_j)_{j < \omega}$ be an A -indiscernible sequence such that $b_0 = b$ and $\bigwedge_j \theta(x, b_j)$ is inconsistent. We may assume that the b_j are algebraically independent over A . For all j , $A(B_j) \simeq A(b) \leq K$. Since $A \leq A(b_j)$ is regular, we have $A(b_{j+1}) \downarrow_A^I A(b_0 \dots b_j)$ and, by [Lemma 4.6](#) and induction, we see that $A(b_j : j < \omega) \models T_{i, \forall}$. By [Corollary 4.7](#), $K \leq K(b_j : j < \omega) \simeq K \otimes_A A(b_j : j < \omega)$ is \mathcal{L}_I -existentially closed, so we may assume that the b_j are in K . It follows that $X_i \cap K$ divides over A in K , and hence, so does $X \subseteq X_i \cap K$. \square

Corollary 8.2. *Let a be a tuple of K and $A \subseteq B \subseteq K$ such that $A = \text{acl}(A)$ and $B = \text{acl}(B)$. Then $\text{tp}(a/B)$ does not divide over A if and only if for all $i \in \{1, \dots, n\}$, $\text{tp}_i(a/B)$ does not divide over A (in M_i).*

Proof. This follows from [Proposition 8.1](#) and [Corollary 6.13](#). \square

Theorem 8.3. *Let $A = \text{acl}(A) \leq K$. If A is an extension base of T_i for all $i \in I$, then A is an extension base of K .*

Proof. By transitivity of forking, it suffices to show that no $\mathcal{L}_\delta(A)$ -definable set $X \subseteq K$ forks over A . So, if $X \subseteq \bigcup_{j < m} Y_j$, where the Y_j are $\mathcal{L}_\delta(K)$ -definable, we have to show that one of the Y_j does not divide over A . To do so, we may assume:

- (i) $X \subseteq \bigcap_i X_i$ is τ -dense, where the X_i are $\mathcal{L}_i(A)$ -definable.
- (ii) $X \setminus \bigcup_j Y_j$ is finite.
- (iii) $Y_j \subseteq \bigcap_i Y_{ji}$ is τ -dense, where $Y_{ji} \subseteq X_i$ is τ_i -open and $\mathcal{L}_i(K)$ -definable.
- (iv) For every i and j_1, j_2 , $Y_{j_1 i} = Y_{j_2 i}$ or $Y_{j_1 i} \cap Y_{j_2 i} = \emptyset$.
- (v) $X_i \setminus \bigcup_j Y_{ji}$ is finite and does not contain any element of A .

Indeed, by [Corollary 6.13](#), X is τ -dense in $\bigcup_i \bigcap_i X_{il} \cup Z$ where the X_{il} are $\mathcal{L}_i(A)$ -definable and Z is finite and $\mathcal{L}_\delta(A)$ -definable. If Z is nonempty, it does not fork over A , and we are done. Otherwise, it suffices to show that any of the $X \cap \bigcap_i X_{il}$ does not fork over A . Similarly, by [Corollary 6.13](#), Y_j is τ -dense in $\bigcup_i \bigcap_i Y_{jil} \cup Z_j$, where the Y_{jil} are $\mathcal{L}_i(K)$ -definable τ_i -open and Z_j is finite. Replacing the Y_j by the $Y_j \cap \bigcap_i (Y_{jil} \cap X_i)$ and setting the finite sets aside, we can ensure (ii) and (iii).

Replacing the Y_{ji} by the τ_i -interior of the atoms of Boolean algebra that they generate (for fixed i), we can further assume (iii). Now, for some finite set Z , we have $X \subseteq \bigcup_j Y_{ji} \cup Z \subseteq X_i$, and since X is τ_i -dense in X_i , $Z_i := X_i \setminus (\bigcup_j Y_{ji} \cup Z)$

has empty τ_i -interior and is therefore finite. Now, if $(Z_i \cup Z) \cap A \neq \emptyset$, we can remove those points from X and the X_i without losing any of the other properties.

It remains to show that one of the Y_j does not divide over A . Note first that for every $f : I \rightarrow m$, by τ -density, there exists an $a \in X \cap \bigcap_i Y_{f(i)i}$. Let j be such that $a \in Y_j \subseteq \bigcap_i Y_{ji}$. Then, by (iv), for all i we have $Y_{f(i)i} = Y_{ji}$.

Now, by hypothesis, X_i does not fork over A , and since any finite set that does not divide over $A = \text{acl}(A)$ must contain a point from A , there is an $f(i) < m$ such that $Y_{f(i)i}$ does not divide over A . As noted above, we may assume that $f(i) = j$ is constant. It now follows from [Proposition 8.1](#) that Y_j does not divide over A . \square

Definition 8.4. Let a be a tuple of K and $A, B \subseteq K$. We write $a \perp_A B$ if $\text{tp}(a/AB)$ does not fork over A (in K) and $a \perp_A^i B$ if $\text{tp}_i(a/AB)$ does not fork over A (in M_i).

Corollary 8.5. Suppose that T_i is NTP_2 for all $i \in I$ and that forking equals dividing in T_i . Then forking equals dividing in K and if a, A, B are in K , then $a \perp_A^i B$ for all $i \in I$ if and only if $a \perp_A B$.

Proof. Since K is NTP_2 ([Corollary 7.6](#)), to show that forking equals dividing in K it's enough to show that any set in K is an extension base. But this is trivial by [Theorem 8.3](#).

The last equivalence is clear from [Proposition 8.1](#) and the fact that forking equals dividing in K . \square

Appendix: Dependent topologies

Following the approach of [\[Johnson 2019\]](#) we can also understand dependent valuations and orders by reduction to the case of independent topologies.

We now fix a finite tree I — that is, a filtered partial order such that for all $i \in I$, the set $\{j \in I : i \leq j\}$ is totally ordered. Let r be its maximal element — its root. To every leaf i — that is, every minimal element — we associate:

- An integer p_i which is either prime or zero.
- A complete theory T_i of fields of characteristic p_i which eliminates quantifiers in a relational expansion of the ring language such that (\mathbf{H}_i) holds (see [Hypothesis 5.1](#)).

To every nonleaf $i \in I$, we associate:

- An integer p_i which is either prime or zero. We assume that for all daughters j of i , the p_j are equal. Moreover, they are equal to p_i if it is nonzero. We also assume that p_r is equal to its daughter p_j .
- A complete theory $T_{i,\Gamma}$ of (enriched) ordered abelian groups. We assume that it is the theory of the trivial group if and only if $i = r$. If p_i is nonzero, we assume that models of $T_{i,\Gamma}$ are p_i -divisible. If p_i is zero and its daughters p_j

are nonzero, we require $T_{i,\Gamma}$ to come with a constant c_i for a positive element. We then assume that $[-c_i, c_i]$ is either p_j -divisible or finite—in the latter case, we say that i is finitely ramified.

For every nonleaf $i \in I$, we define the theories $T_{i,k}$ and T_i by induction:

- The theory $T_{i,k}$ (the residual theory of i) is the theory of pseudo I_i -closed fields without finite extensions of degree divisible by p_i (if it is nonzero), where $I_i = \{T_j : j \text{ daughter of } i\}$.
- The theory T_i is a Morleyization of the theory of characteristic p_i algebraically maximal complete fields with residue field a model of $T_{i,k}$ and value group a model of $T_{i,\Gamma}$. If p_i is zero and the daughters p_j are nonzero, then we also require that $c_i = \text{val}(p_j)$.

Note that, by [Halevi and Hasson 2019, Corollary A.3], if i is not finitely ramified, the reduct of T_i to the valued field language eliminates quantifiers relative to RV_1 ; and it eliminates quantifiers relative to $\bigcup_n \text{RV}_n$ if i is finitely ramified by [Basarab 1991, Theorem B]. Moreover, each completion admits maximally complete models. In particular, (\mathbf{H}_i) holds in every T_i .

Definition A.1. We say that a field is pseudo I -closed if it is pseudo I_r -closed.

This is coherent with our earlier notation, provided we identify a set of independent valuations to the tree of height one with trivial valuation at the root, and the independent valuations as leaves.

We also define:

$$\kappa_i = \begin{cases} \text{bdn}(T_i) & \text{if } i \text{ is a leaf,} \\ \max(\aleph_0 \cdot \sum_{j \in I_j} \kappa_j, \text{bdn}(T_{i,\Gamma})) & \text{if } i \text{ is finitely ramified,} \\ \max(\sum_{j \in I_j} \kappa_j, \text{bdn}(T_{i,\Gamma})) & \text{otherwise.} \end{cases}$$

Once again, for a finer estimate, κ_i can be computed in Card^* [Touchard 2023, Definition 1.29].

Theorem A.2. Let K be a bounded perfect pseudo I -closed field. Then

$$\text{bdn}(K) = \kappa_r.$$

Proof. We proceed by induction on i . By Theorem 7.9, $\text{bdn}(T_{i,k}) = \sum_{j \in I_i} \kappa_j$ (by Theorem 7.9) and, by Fact 2.6, $\text{bdn}(T_i) = \kappa_i$. \square

Corollary A.3. Let K be a bounded perfect PAC, m -PRC or m -PpC field (setting $m = 0$ if K is PAC) and v_1, \dots, v_l be valuations on K . Let n be the maximal number of nondependent valuations among the v_i that are also not dependent from the m orders (resp. p -adic valuations) if K is PRC (resp. PpC). Then

$$\text{bdn}(K, v_1, \dots, v_l) = m + n.$$

Proof. We assume K is PAC. Let I be the tree of valuations generated by the v_i ; in other words, the nodes of I are the valuation rings $\prod_{i \in J} \mathcal{O}_i$, where $J \subseteq I$ and \mathcal{O}_i is the valuation ring of v_i (and the trivial valuation at the root). For any node $i \in I$ we also write v_i for the associated valuation. Since K is PAC, the residue field of any of the v_i is algebraically closed and its value group is divisible — this follows, for example, from [Proposition 3.12](#). We annotate I with a copy of ACVF on each leaf and with the theory of divisible ordered abelian groups on each nonleaf node (except for the root), as well as the relevant characteristics and constants.

It follows from [Theorem 5.11](#) that $K \models \text{PIC}$ and this structure is bi-interpretable with (K, v_1, \dots, v_n) . By induction on i we see that κ_i is the number of leaves below the node i . The corollary follows.

The PRC and PpC cases follow in a similar manner, noting that orders (or p -adic valuations) can only be leaves in the tree. \square

Acknowledgment

We thank the referee for careful reading and insightful remarks on an earlier version of this text.

References

- [Anscombe et al. 2020] S. Anscombe, P. Dittmann, and A. Fehm, “Approximation theorems for spaces of localities”, *Math. Z.* **296**:3–4 (2020), 1471–1499. [MR](#) [Zbl](#)
- [Ax 1968] J. Ax, “The elementary theory of finite fields”, *Ann. of Math.* (2) **88** (1968), 239–271. [MR](#) [Zbl](#)
- [Basarab 1991] Ş. A. Basarab, “Relative elimination of quantifiers for Henselian valued fields”, *Ann. Pure Appl. Logic* **53**:1 (1991), 51–74. [MR](#) [Zbl](#)
- [Chatzidakis 2002] Z. Chatzidakis, “Properties of forking in ω -free pseudo-algebraically closed fields”, *J. Symbolic Logic* **67**:3 (2002), 957–996. [MR](#) [Zbl](#)
- [Chernikov 2014] A. Chernikov, “Theories without the tree property of the second kind”, *Ann. Pure Appl. Logic* **165**:2 (2014), 695–723. [MR](#) [Zbl](#)
- [Chernikov and Kaplan 2012] A. Chernikov and I. Kaplan, “Forking and dividing in NTP_2 theories”, *J. Symbolic Logic* **77**:1 (2012), 1–20. [MR](#) [Zbl](#)
- [Cubides Kovacsics and Point 2023] P. Cubides Kovacsics and F. Point, “Topological fields with a generic derivation”, *Ann. Pure Appl. Logic* **174**:3 (2023), art. id. 103211. [MR](#) [Zbl](#)
- [van den Dries 1978] L. van den Dries, *Model theory of fields: decidability, and bounds for polynomial ideals*, Ph.D. thesis, Rijksuniversiteit Utrecht, 1978.
- [Engler and Prestel 2005] A. J. Engler and A. Prestel, *Valued fields*, Springer, 2005. [MR](#) [Zbl](#)
- [Fehm 2013] A. Fehm, “Elementary geometric local-global principles for fields”, *Ann. Pure Appl. Logic* **164**:10 (2013), 989–1008. [MR](#) [Zbl](#)
- [Fried and Jarden 2008] M. D. Fried and M. Jarden, *Field arithmetic*, 3rd ed., *Ergebnisse der Math.* (3) **11**, Springer, 2008. [MR](#) [Zbl](#)

- [Geyer and Jarden 2013] W.-D. Geyer and M. Jarden, “Model completeness of valued PAC fields”, pp. 173–214 in *Travaux mathématiques*, vol. 23, edited by S. Arias-de-Reyna et al., Fac. Sci. Technol. Commun. Univ. Luxemb., Luxembourg, 2013. [MR](#) [Zbl](#)
- [Grob 1987] C. Grob, *Die entscheidbarkeit der theorie der maximalen pseudo p -adisch abgeschlossenen körper*, Ph.D. thesis, Universität Konstanz, 1987.
- [Halevi and Hasson 2019] Y. Halevi and A. Hasson, “Eliminating field quantifiers in strongly dependent henselian fields”, *Proc. Amer. Math. Soc.* **147**:5 (2019), 2213–2230. [MR](#) [Zbl](#)
- [Halevi et al. 2020] Y. Halevi, A. Hasson, and F. Jahnke, “Definable V -topologies, Henselianity and NIP”, *J. Math. Log.* **20**:2 (2020), art. id. 2050008. [MR](#) [Zbl](#)
- [Haran and Jarden 1988] D. Haran and M. Jarden, “The absolute Galois group of a pseudo p -adically closed field”, *J. Reine Angew. Math.* **383** (1988), 147–206. [MR](#) [Zbl](#)
- [Heinemann and Prestel 1984] B. Heinemann and A. Prestel, “Fields regularly closed with respect to finitely many valuations and orderings”, pp. 297–336 in *Quadratic and Hermitian forms* (Hamilton, ON, 1983), edited by C. R. Riehm and I. Hambleton, CMS Conf. Proc. **4**, Amer. Math. Soc., Providence, RI, 1984. [MR](#) [Zbl](#)
- [Hong 2023] J. Hong, “Quantifier elimination on some pseudo-algebraically closed valued fields”, *Ann. Pure Appl. Logic* **174**:1 (2023), art. id. 103170. [MR](#) [Zbl](#)
- [Jarden and Razon 1998] M. Jarden and A. Razon, “Rumely’s local global principle for algebraic PSC fields over rings”, *Trans. Amer. Math. Soc.* **350**:1 (1998), 55–85. [MR](#) [Zbl](#)
- [Johnson 2019] W. Johnson, “Finite burden in multivalued algebraically closed fields”, preprint, 2019. [arXiv 1905.04991](#)
- [Johnson 2022] W. Johnson, “Forking and dividing in fields with several orderings and valuations”, *J. Math. Log.* **22**:1 (2022), art. id. 2150025. [MR](#) [Zbl](#)
- [Kollár 2007] J. Kollár, “Algebraic varieties over PAC fields”, *Israel J. Math.* **161** (2007), 89–101. [MR](#) [Zbl](#)
- [Kruckman et al. 2021] A. Kruckman, C.-M. Tran, and E. Walsberg, “Interpolative fusions”, *J. Math. Log.* **21**:2 (2021), art. id. 2150010. [MR](#) [Zbl](#)
- [Montenegro 2017a] S. Montenegro, “Imaginaries in bounded pseudo real closed fields”, *Ann. Pure Appl. Logic* **168**:10 (2017), 1866–1877. [MR](#) [Zbl](#)
- [Montenegro 2017b] S. Montenegro, “Pseudo real closed fields, pseudo p -adically closed fields and NTP_2 ”, *Ann. Pure Appl. Logic* **168**:1 (2017), 191–232. [MR](#) [Zbl](#)
- [Montenegro and Rideau-Kikuchi 2021] S. Montenegro and S. Rideau-Kikuchi, “Imaginaries, invariant types and pseudo p -adically closed fields”, *Trans. Amer. Math. Soc.* **374**:2 (2021), 803–828. [MR](#) [Zbl](#)
- [Montenegro et al. 2020] S. Montenegro, A. Onshuus, and P. Simon, “Stabilizers, NTP_2 groups with f -generics, and PRC fields”, *J. Inst. Math. Jussieu* **19**:3 (2020), 821–853. [MR](#) [Zbl](#)
- [Pop 2003] F. Pop, “Classically projective groups and pseudo classically closed fields”, pp. 251–283 in *Valuation theory and its applications* (Saskatoon, SK, 1999), vol. II, edited by F.-V. Kuhlmann et al., Fields Inst. Commun. **33**, Amer. Math. Soc., Providence, RI, 2003. [MR](#) [Zbl](#)
- [Prestel 1981] A. Prestel, “Pseudo real closed fields”, pp. 127–156 in *Set theory and model theory* (Bonn, 1979), edited by R. B. Jensen and A. Prestel, Lecture Notes in Math. **872**, Springer, 1981. [MR](#) [Zbl](#)
- [Prestel and Ziegler 1978] A. Prestel and M. Ziegler, “Model-theoretic methods in the theory of topological fields”, *J. Reine Angew. Math.* **299/300** (1978), 318–341. [MR](#) [Zbl](#)

- [Robinson 1977] A. Robinson, *Complete theories*, 2nd ed., North-Holland Publishing Co., Amsterdam, 1977. [MR](#) [Zbl](#)
- [Schmid 2000] J. Schmid, “Regularly T -closed fields”, pp. 187–212 in *Hilbert’s tenth problem: relations with arithmetic and algebraic geometry* (Ghent, 1999), edited by J. Denef et al., *Contemp. Math.* **270**, Amer. Math. Soc., Providence, RI, 2000. [MR](#) [Zbl](#)
- [Stacks] “The Stacks project”, electronic reference, available at <http://stacks.math.columbia.edu>. [Zbl](#)
- [Touchard 2023] P. Touchard, “Burden in Henselian valued fields”, *Ann. Pure Appl. Logic* **174**:10 (2023), art. id. 103318. [MR](#) [Zbl](#)
- [Tressl 2005] M. Tressl, “The uniform companion for large differential fields of characteristic 0”, *Trans. Amer. Math. Soc.* **357**:10 (2005), 3933–3951. [MR](#) [Zbl](#)

Received 20 Apr 2023. Revised 29 May 2024.

SAMARIA MONTENEGRO:

samaria.montenegroguzman@ucr.ac.cr

Escuela de Matemática-CIMPA, Universidad de Costa Rica, San José, Costa Rica

SILVAIN RIDEAU-KIKUCHI:

silvain.rideau-kikuchi@ens.fr

Département de mathématiques et applications, École normale supérieure, Université PSL, CNRS, Paris, France