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Valued fields, metastable groups

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Abstract

We introduce a class of theories called *metastable*, including the theory of algebraically closed valued fields (ACVF) as a motivating example. The key local notion is that of definable types dominated by their stable part. A theory is metastable (over a sort Γ) if every type over a sufficiently rich base structure can be viewed as part of a Γ-parametrized family of stably dominated types. We initiate a study of definable groups in metastable theories of finite rank. Groups with a stably dominated generic type are shown to have a canonical stable quotient. Abelian groups are shown to be decomposable into a part coming from Γ, and a definable direct limit system of groups with stably dominated generic. In the case of ACVF, among definable subgroups of affine algebraic groups, we characterize the groups with stably dominated generics in terms of group schemes over the valuation ring. Finally, we classify all fields definable in ACVF.

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1 Introduction

Let V be a variety over a valued field (F, val). By a val-*constructible set* we mean a finite Boolean combination of sets of the form $\{x \in U : \mathrm{val}(f(x)) \leq \mathrm{val}(g(x))\}$ where U is an open affine, and f, g are regular functions on U. Two such sets can be identified if they have the same points in any valued field extension of F, or equivalently, by a theorem of Robinson, in any fixed algebraically closed valued field extension K of F.

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In many ways these are analogous to constructible sets in the sense of the Zariski topology, and (more closely) to semi-algebraic sets over real fields. However while the latter two categories are closed under quotients by equivalence relations, the valuative constructible sets are not. For instance, the valuation ring $\mathcal{O} = \{x : \mathrm{val}(x) \geq 0\}$, is constructible, and has constructible ideals $\alpha\mathcal{O} = \{x : \mathrm{val}(x) \geq \alpha\}$ and $\alpha\mathfrak{M} = \{x : \mathrm{val}(x) > \alpha\}$. For any group scheme G over $\mathcal{O}$, one obtains corresponding congruence subgroups; but the quotients $\mathcal{O}/\alpha\mathcal{O}$ and $G(\mathcal{O}/\alpha\mathcal{O})$ are not constructible. We thus enlarge the category by formally adding quotients, referred to as the *imaginary sorts*; the objects of the larger category will be called the *definable sets*. It is explained in [6] that instead of this abstract procedure, it suffices to add sorts for the homogeneous spaces $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$, for all n, and also $\mathrm{GL}_n(K)/I$ for a certain subgroup I; we will not require detailed knowledge of this here. One such sort that will be explicitly referred to is the value group $\Gamma := \mathrm{GL}_1(K)/\mathrm{GL}_1(\mathcal{O})$; this is a divisible ordered Abelian group, with no additional induced structure. We will refer to *stable sorts* also. These include first of all the residue field $k = \mathcal{O}/\mathfrak{M}$, but also vector spaces over k of the form $L/\mathfrak{M}L$, where $L \subseteq K^n$ is an $\mathcal{O}$-lattice.

The paper [7], continuing earlier work, studied the category of quantifier-free definable sets over valued fields, especially with respect to imaginaries. As usual, the direct study of a concrete structure of any depth is all but impossible, if it is not aided by a general theory. We first tried to find a generalization of stability (or simplicity) in a similar format, capable of dealing with valued fields as stability does with differential fields, or simplicity with difference fields. To this we encountered resistance; what we found instead was not a new analogue of stability, but a new method of utilizing classical stability in certain unstable structures.

Even a very small stable part can have a decisive effect on the behavior of a quite "large," unstable type. This is sometimes analogous to the way that the (infinitesimal) linear approximation to a variety can explain much about the variety; and indeed, in some cases, casts tangent spaces and Lie algebras in an unexpected model theoretic role. Two main principles encapsulate the understanding gained:

(1) Certain types are dominated by their stable parts. They behave "generically" as stable types do.
(2) Uniformly definable families of types make an appearance; they are indexed by the linear ordering Γ of the value group, or by other, piecewise-linear structures definable in Γ. An arbitrary type can be viewed as a definable limit of stably dominated types (from principle (1)).

A general study of stably dominated types was initiated in [7]; it is summarized in Sect. 2. Principle (2) was only implicit in the proofs there. We state a precise version of the principle, and call a theory satisfying (2) *metastable*. We concentrate here on finite rank metastability.

Principle (1) is given a general group-theoretic rendering in Proposition 4.6. Stably dominated groups are defined, and it is shown that a group homomorphism into a stable group controls them generically. Theorem 5.16 clarifies the second principle in the context of Abelian groups. A metastable Abelian group of finite rank is shown to contain nontrivial stably dominated groups T_α, unless it is internal to Γ. Moreover, the groups T_α are shown to form a definable direct limit system, so that the group is described by three ingredients: Γ-groups, stably dominated groups, and definable direct limits of groups.

We then apply the theory to groups definable in algebraically closed valued fields. Already the case of Abelian varieties is of considerable interest; all three ingredients above occur, and the description beginning with a definable map into a piecewise linear definable group L takes a different aspect than the classical one. The points of L over a non-Archimedean local field will form a finite group, related to the group of connected components in the Néron model. On the other hand over $\mathbb{R}(t)$ the same formulas will give tori over $\mathbb{R}$.

Non-commutative groups definable in algebraically closed valued fields include examples such as $\mathrm{GL}_n(\mathcal{O})$, $\mathrm{GL}_n(\mathcal{O}/a\mathcal{O})$ and "congruence subgroups" such as the kernel of $\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/a\mathcal{O})$, which are all stably dominated. In Corollary 6.4 and Theorem 6.11 we relate stably dominated groups to group schemes over $\mathcal{O}$. Previous work in this direction used other theories, inspired by topology; see [20], [11] regarding the p-adics.

We now describe the results in more detail. Our main notion will be that of a stably dominated group. In a metastable theory, we will try to analyze arbitrary groups, and to some extent types, using them. In the case of valued fields, this notion is related to but distinct from compactness (over those fields where topological notions make sense, i.e. local fields). For groups defined over local fields, stable domination implies compactness of the group of points over every finite extension. Abelian varieties with bad reduction show that the converse fails; this failure is explained by another aspect of the theory, definable homomorphic quotients defined over Γ.

Let T be a first order theory. It is convenient to view a projective system of definable sets D_i as a single object, a pro-definable set; similarly for a compatible system of definable maps $\alpha_i : P \to D_i$. Precise definitions of the terminology and notations used in the following paragraphs can be found in Sects. 2.2 and 2.3.

Definition 1.1 (*Stable domination*) A type p over C is *stably dominated* if there exists, over C, a pro-definable map $\alpha : p \to D$, D stable and stably embedded, such that for any $a \models p$ and tuple b, $\alpha(a) \downarrow_C D \cap \mathrm{dcl}(b)$ implies:

$$\mathrm{tp}(b/C\alpha(a)) \models \mathrm{tp}(b/Ca).$$

Let Γ be a stably embedded definable set which is orthogonal to the stable part: there is no definable function with infinite image from a power of Γ to a stable definable set. In this paper we will mostly be focusing on the case where Γ is o-minimal, as is the case in algebraically closed valued fields. Many of the results presented here should remain true in Henselian valued fields with algebraically closed residue field and different value groups, such as $\mathbb{Z}$; where every definable subset of Γ is still a Boolean combination of $\varnothing$-definable sets and intervals.

Definition 1.2 (*Metastability*) A theory T is *metastable* (over Γ) if:

(B) Any set of parameters C_0 is included in a set C, called a metastability basis, such that for any a, there exists a pro-definable map γ (over C), from p to some power of Γ with $\mathrm{tp}(a/\gamma(a))$ stably dominated.

(E) Every type over an algebraically closed subset (including imaginaries) of a model of T has an automorphism-invariant extension to the model.

The main reason we require property (E) is because of descent (Proposition 2.11.(2)) which is not known to hold without the the additional hypothesis that $\mathrm{tp}(B/C)$ has a global C-invariant extension.

Question 1.3 (1) *Can descent be proved without the additional hypothesis that* $\mathrm{tp}(B/C)$ *has a global C-invariant extension?*
(2) *If we assume Γ to be o-minimal, does property (E) follow from the existence of metastability bases?*

Remark 1.4 (1) Instead of considering all C-definable stable sets, it is possible, in the definition of stable domination, to take a proper subfamily $\mathcal{S}_C$ with reasonable closure properties. The notion of $\mathcal{S}$-domination is meaningful even for stable theories.
(2) In the definition of metastability, one can also replace the single sort Γ with a family of sorts Γ_i, or with a family of parametrized families of definable sets $\mathcal{G}_A$, with no loss for the results of the present paper.

As in stability theory, a range of finiteness assumptions is possible. Let T be metastable over Γ.

Definition 1.5 (*cf. Definition* 2.17) The theory T has (FD) if:

(1) Γ is o-minimal.
(2) Morley dimension is uniformly finite and definable in families.
(3) Let D be a definable set. The Morley dimension of $f(D)$, where f ranges over all definable functions with parameters such that $f(D)$ is stable, takes a maximum value.
(4) Similarly, the o-minimal dimension of $g(D)$, where g ranges over all definable functions with parameters such that $g(D)$ is Γ-internal, takes a maximum value.

Let $A \leq M \models T$. We define St_A to be the family of all stable, stably embedded A-definable sets. Some statements will be simpler if we also assume:

Definition 1.6 (*cf. Definition* 2.21) The theory T has (FD_ω) if, in addition to (FD), any set is contained in a metastability basis M which is also a model. Moreover, for any acl-finitely generated $G \subseteq \Gamma$ and $S \subseteq \mathrm{St}_{M\cup G}$ over M, isolated types over $M \cup S$ are dense.

Remark 1.7 (1) (FD) and (FD_ω) both hold in ACVF, with all imaginary sorts included (cf. Proposition 2.33). (FD), at least, is valid for all C-minimal expansions of ACVF, in particular the rigid analytic expansions (cf. [10,14,15]).
(2) Existentially closed valued fields with a contractive derivative (cf. [26]) and separably closed valued fields (cf. [8]) are also metastable, but of infinite rank. The results presented here will only hold in finite rank definable groups.
(3) In practice, the main structural results will use finite weight hypotheses, see Definition 2.23; this is a weaker consequence of (FD).

A group is *stably dominated* if it has a generic type which is stably dominated (See Definitions 3.1 and 4.1). In this case, the stable domination is witnessed by a group homomorphism (cf. Proposition 4.6). One cannot expect every group to be stably dominated. But one can hope to shed light on any definable group by studying the stably dominated groups inside it. We formulate the notion of a *limit stably dominated* group: it is a direct limit of connected metastable groups by a pro-definable direct limit system (cf. Definition 5.6).

Theorem 1.8 (cf. Theorem 5.16) *Let T be a metastable theory with (FD_ω). Let A be a definable Abelian group. Then there exists a definable group $\Lambda \subset \Gamma^n$, and a definable homomorphism $\lambda : A \to \Lambda$, with $H := \ker(\lambda)$ limit stably dominated.*

In fact, under these assumptions, H is the union of a definable directed family of definable groups, each of which is stably dominated. Assuming only bounded weight (in place of (FD_ω)), we obtain a similar result but with H ∞-definable.

In the non-Abelian case the question remains open. The optimal conjecture would be a positive answer to:

Problem 1.9 *(FD_ω) Does any definable group G have a limit stably dominated definable subgroup H with $H\backslash G/H$ internal to Γ?*

Another goal of this paper is to relate definable groups in ACVF to group schemes over $\mathcal{O}$. We recall the analogous results for the algebraic and real semi-algebraic cases. Consider a field K of characteristic zero. Then the natural functor from the category of algebraic groups to the category of constructible groups is an equivalence of categories. This follows locally from Weil's group chunk theorem; nevertheless some additional technique is needed to complete the theorem. It was conjectured by Poizat and proved in [30] using definable topological manifolds, and in [12] using stability theoretic notions like definable types and germs. This methods will be explained in Sect. 3. Let us only remark here that an irreducible algebraic variety has a unique generic behavior, in that any definable subset has lower dimension or a complement of lower dimension; this is typical of stable theories.

In ACVF, we certainly cannot hope every definable group to be a subgroup of an algebraic group or even of the definable homomorphic image of an algebraic group. We can, however, hope for all definable groups to be towers of such groups:

Problem 1.10 *Let G be a definable group in $K \models$ ACVF. Do there exist definable normal subgroups $(1) = G_0 \leq \cdots \leq G_n = G$ of G and definable homomorphisms f_i, with kernel G_i, from G_{i+1} into the definable homomorphic image of an algebraic group over K?*

Call a definable set D *boundedly imaginary* if there exists no definable map with parameters from D onto an unbounded subset of Γ. If D is defined over a local field L, then D is boundedly imaginary if and only if $D(L')$ is finite for every finite extension L' of L. We prove:

Proposition 1.11 (cf. Corollary 6.4) *Let H be a stably dominated connected group definable in $K \models$ ACVF. Then there exists an algebraic group G over K and a definable homomorphism $f : H \to G(K)$ with boundedly imaginary kernel.*

Corollary 1.12 *Let H be a stably dominated group definable in* ACVF *with parameters from a local field L. Then there exists a definable homomorphism $f : H(L) \to G(L)$ with G an algebraic group over L, with finite kernel.*

Proposition 1.11 reduces, up to a boundedly imaginary kernel, the study of a stably dominated groups definable in ACVF to that of stably dominated subgroups of algebraic groups G. We proceed then to describe these.

There exists an exact sequence $1 \to A \to G \to_f L \to 1$, with A an Abelian variety and L an affine algebraic group. We show (cf. Lemma 4.3 and Corollary 4.5) that a definable subgroup H of G is stably dominated if and only if $H \cap A$ and $f(H)$ are. For linear groups, we have:

Theorem 1.13 (cf. Theorem 6.11) *Let G be an affine algebraic group and let H be a stably dominated definable subgroup of G. Then H is isomorphic to $\mathbb{H}(\mathcal{O})$, $\mathbb{H}$ a group scheme of finite type over $\mathcal{O}$.*

If H is Zariski dense in G, $\mathbb{H}$ can be taken to be K-isomorphic to G.

Finally, we show that enough of the structure of definable groups in ACVF is known to be able to classify all the fields: as expected, they are all isomorphic either to the valued field itself or to its residue field (cf. Theorem 6.23).

Let us conclude this introduction with a series of examples that illustrate some of the structure results proved in this paper. Since this structure is reflected in the generics of the groups, we also discuss these generics (cf. Section 3 for definitions).

Example 1.14 (Structure and generics of certain algebraic groups) Let K be an algebraically closed valued field.

(1) $\mathrm{SL}_n(\mathcal{O})$ is stably dominated. Its unique generic is stably dominated via the residue map.
(2) $\mathbb{G}_\mathrm{m}(K)$ has a largest stably dominated subgroup $\mathbb{G}_\mathrm{m}(\mathcal{O})$ and $\mathbb{G}_\mathrm{m}(K)/\mathbb{G}_\mathrm{m}(\mathcal{O}) \simeq \Gamma$. It has two generics: elements of small valuation and elements of large valuation (i.e. infinitesimals), both generics are attributable to Γ.
(3) $\mathbb{G}_\mathrm{a}(K)$ is a limit stably dominated group, it is the union of the stably dominated subgroups $\alpha\mathcal{O}$. It has a unique generic: elements of small valuation. Here the generic types correspond to cofinal definable types in the (partially) ordered set $(\Gamma, >)$ indexing the stably dominated subgroups $\alpha\mathcal{O}$. The fact that the poset has a unique cofinal definable type is however a phenomenon of dimension one.
The limit stably dominated group $\mathbb{G}_\mathrm{a}^2(K)$, however, has a large family of generics. For any definable curve E in Γ^2, cofinal in the sense that for any (a, b) there exists $(d, e) \in E$ with $d < a$ and $e < b$, there exists a generic type of $\mathbb{G}_\mathrm{a}^2(K)$ whose projection to Γ^2 concentrates on E.
(4) Let $B_n \leq \mathrm{SL}_n(K)$ be the solvable group of upper diagonal matrices. The valuation of the diagonal coefficients is a group homomorphism into Γ^{n-1} with limit stably dominated kernel U_n.
The group B_n has left, right as well as two sided generics. One of the latter is given as follows. Let $(x_{i,j})$ be the matrix coefficients of an element of $x \in \mathrm{GL}_n(K)$. A two-sided generic of G is determined by: $\mathrm{val}(x_{i,j}) << \mathrm{val}(x_{i',j'})$ when $(i, j) < (i', j')$ lexicographically (and $i < j$).

(5) For $n > 1$, $\mathrm{SL}_n(K)$ is simple but neither limit stably dominated nor Γ-internal. However, the double coset space of $U_n \backslash \mathrm{SL}_n(K)/U_n$ is a disjoint union of $n!$ copies of Γ^{n-1}. The group $\mathrm{SL}_n(K)$ has no generic types.

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2 Preliminaries

We recall some material from stability, stable domination, o-minimality and the model theory of algebraically closed valued fields, in a form suitable for our purposes.

We fix a theory T that eliminates imaginaries and a universal model $\mathbb{U}$ (a sufficiently saturated and homogeneous model). Any model of T that we consider will be an elementary submodel of $\mathbb{U}$. We write $\models \phi$ as a shorthand for $\mathbb{U} \models \phi$. By a *definable* set or function we mean one defined in T_A for some $A \subseteq \mathbb{U}$. If we wish to specify the base of definition, we say A-definable.

A pro-definable set X is a (small) projective filtered system $(X_i)_{i\in I}$ of definable sets and definable maps. We think of X as $\varprojlim_i X_i$. Pro-definable sets can equivalently be presented as a collection of formulas in potentially infinitely many variables. If all the maps in the system describing X are injective, we say that X is ∞-definable and we can identify X with a subset of any of the X_i. If all the maps in the system describing X are surjective, we say that X is strict pro-definable; equivalently, the projection of X to any X_i is definable. A relatively definable subset of X is a definable subset of one of the X_i; we identify it as a subset of X by pulling it back to X. Equivalently, if $x = (x_i)$ is a tuple of variables where each x_i ranges over X_i, a relatively definable subset of X is the set of realizations in X of some formula $\phi(x)$.

A pro-definable map $f : \varprojlim_i X_i \to Y$ where Y is definable is a definable map from some X_i to Y. A pro-definable map $f : \varprojlim_i X_i \to \varprojlim_j Y_j$ is a compatible collection of maps $f_j : \varprojlim_i X_i \to Y_j$. A pro-definable subset of a pro-definable set $X = \varprojlim_i X_i$ is a pro-definable set Y with an injective pro-definable map $f : Y \to X$; it can be identified with a compatible collection of ∞-definable subsets of the X_i, or with an intersection of relatively definable subsets of X.

More generally, by a *piecewise pro-definable* set, we mean a family $(X_i)_{i\in I}$ of pro-definable sets over a directed order I, with a compatible system of injective pro-definable maps $X_i \to X_j$, for all $i \leq j$, viewed as inclusion maps. The direct limit is thus identified with the union, and denoted $X = \bigcup_i X_i$. A pro-definable subset of X is a pro-definable subset of one of the X_i.

2.1 Definable types and filters

A type will mean a complete type in possibly infinitely many variables (equivalently an ultrafilter on the Boolean algebra of relatively definable sets). We will refer to possibly incomplete types as filters.

A filter $\pi(x)$ over $\mathbb{U}$ in the (possibly infinite) tuple of variables x is said to be C*-definable* if for all formulas $\phi(x, y)$, there exists a formula $\theta(y) =: (\mathrm{d}_\pi x)\phi(x, y)$ over C, such that $\pi = \{\phi(x, m) : \mathbb{U} \models (\mathrm{d}_\pi x)\phi(x, m)\}$. Note that π is completely determined by the map $\phi(x, y) \mapsto (\mathrm{d}_\pi x)\phi(x, y)$, its *definition scheme*. If π is a type over $\mathbb{U}$, then this map is a Boolean homomorphism.

If C is a set of parameters, we define $\pi|C := \{\phi(x, m) : \mathbb{U} \models (\mathrm{d}_\pi x)\phi(x, m)$ and $m \in C\}$. Note that if C is a model, $\pi|C$ completely determines π.

We say that π *concentrates* on a pro-definable set D if $\pi(x) \vdash x \in D$. By a *pro-definable function on* π, we mean a pro-definable map $f : D \to X$ such that π concentrates on D.

Definition 2.1 If f is a pro-definable function on a C-definable filter π, we define the push-forward $f_\star\pi$ by:

$$(\mathrm{d}_{f_\star\pi} u)\phi(u, v) = (\mathrm{d}_\pi x)\phi(f(x), v).$$

So that if $a \models \pi|C$ then $f(a) \models f_\star\pi|C$.

Definition 2.2 Let $\pi(x)$, $\mu(y)$ be C-definable filters. Define $\rho(x, y) =: \pi(x) \otimes \mu(y)$ by:

$$(\mathrm{d}_\rho xy)\phi(x, y, z) = (\mathrm{d}_\pi x)(\mathrm{d}_\mu y)\phi(x, y, z).$$

Then, if $a \models \pi|C$ and $b \models \mu|Ca$, $ab \models (\pi \otimes \mu)|C$. If π and μ are types, then $f_\star\pi$ and $\pi \otimes \mu$ are types.

We will occasionally use a more general construction. Assume $p(x)$ is a C-definable type. Let $a \models p$ and let $q_a(y)$ be a Ca-definable type of the theory T_{Ca}.

Lemma 2.3 *There exists a unique definable type $r(xy)$ such that for any $B \supseteq C$, if $ab \models r|B$ then $a \models p|B$ and $b \models q_a|Ba$.*

Proof Given a formula $\phi(xy, z)$, let $\psi(x, z)$ be a formula such that $\psi(a, z) = (\mathrm{d}_{q_a} y)\phi(a, y, z)$. The formula ψ is not uniquely defined, but if ψ and ψ' are two possibilities then $(\mathrm{d}_p x)(\psi \leftrightarrow \psi')$. Therefore we can define:

$$(\mathrm{d}_r xy)\phi(xy, z) = (\mathrm{d}_p x)\psi(x, y, z).$$

It is easy to check that this definition scheme works.

In fact, over $C = \mathrm{acl}(C)$, it suffices that q_a be definable over $\mathrm{acl}(Ca)$. This follows from:

Lemma 2.4 *Let M be a model and let $C = \mathrm{acl}(C) \subseteq M$. Let* $\mathrm{tp}(a/M)$ *be C-definable. Let $c \in \mathrm{acl}(Ca)$. Then* $\mathrm{tp}(ac/M)$ *is C-definable. Indeed,* $\mathrm{tp}(a/M) \cup \mathrm{tp}(ac/C) \vdash \mathrm{tp}(ac/M)$.

Proof Let $\phi(x, y)$ be a formula over C such that $\phi(a, c)$ holds, and such that $\phi(a, y)$ has k solutions. Let $\psi(x, y, z)$ be any formula. Then the equivalence relation $z_1 E z_2$ defined by $(\mathrm{d}_p x)(\forall y)\phi(x, y) \implies (\psi(x, y, z_1) \Leftrightarrow \psi(x, y, z_2))$ where $p := \mathrm{tp}(a/M)$ is C-definable and has at most 2^k classes. It follows that each of these classes, denote Z_i, are C-definable. Finally, for all m, $\models \psi(a, c, m)$ if and only if $m \in Z_i$ for some i such that $(\forall z \in Z_i)\psi(x, y, z) \in \mathrm{tp}(ac/C)$. □

See Proposition 2.10.(4) for a stronger statement in the stably dominated case.

Definition 2.5 (*Germs of definable functions*) Let p be a definable type. Two definable functions $f(x, b)$ and $g(x, b')$ are said to have the same *p-germ* if

$$\models (\mathrm{d}_p x) f(x, b) = g(x, b').$$

We say that *the p-germ of $f(x, b)$ is defined over C* if whenever $\mathrm{tp}(b/C) = \mathrm{tp}(b'/C)$, $f(x, b)$ and $f(x, b')$ have the same p-germ. Note that the equivalence relation $b \sim b'$ defined by "$f(x, b)$, $f(x, b')$ have the same p-germ" is definable; the p-germ of $f(x, b)$ is defined over C if and only if $b/{\sim} \in \mathrm{dcl}(C)$.

2.2 Stable domination

Definition 2.6 Let D be a C-definable set.

(1) D is said to be *stably embedded* if any definable $X \subseteq D^n$, for some n, is $C \cup D$-definable.
(2) D is said to be *stable* if every formula $\phi(x; y)$ with parameters in C, implying $x \in D^n$ for some n, is a stable formula.

Since being stable implies being stably embedded, the later is often referred to as being *stable, stably embedded* in the literature. See e.g. [18] for a treatment of basic stability and [1, Appendix] for the properties of stably embedded and stable sets.

For all C, let $(D_i)_i$ enumerate all C-definable stable sets and $\mathrm{St}_C := \prod_i D_i$. It is a strict pro-definable set. We often consider St_C as a (stable) structure, whose sorts are the D_i, with the full induced structure. By a pro-definable map into St_C, we mean (somewhat abusively) a collection of maps $f = (f_j)_j$ where the range of each f_j is stable. For all b, let $\mathrm{St}_C(b)$ denote $\mathrm{dcl}(Cb) \cap \mathrm{St}_C$.

Notation 2.7 For all a, b, C, we write $a \downarrow_C b$ if there exists an $\mathrm{acl}(C)$-definable type p such that $a \models p|\mathrm{acl}(Cb)$.

Note that if $a, b, C \in \mathrm{St}_D$ for some $D \subseteq C$ then $a \downarrow_C b$ if and only if they are (forking) independent in St_D over C. Let us now recall the definition of stable domination:

Definition 2.8 Let $p = \mathrm{tp}(a/C)$ and $\alpha : p \to \mathrm{St}_C$ be a pro-C-definable map. The type p is said to be *stably dominated* via $\alpha = (\alpha_i)_i$ if for any tuple b, if $\mathrm{St}_C(b) \downarrow_C \alpha(a)$, then $\mathrm{tp}(b/C\alpha(a)) \vdash \mathrm{tp}(b/Ca)$.

A type p over C is said to be stably dominated if it stably dominated via some C-definable map $\alpha : p \to \mathrm{St}_C$. For $a \models p$, let $\theta_C(a)$ enumerate $\mathrm{St}_C(a)$, then p is stably dominated if and only if it is stably dominated via θ_C.

Let us now recall some of the result from [7] regarding stable domination.

Proposition 2.9 [7, Corollary 3.31.(iii) and Proposition 3.13] *For all a and C,*

(1) $\mathrm{tp}(a/C)$ *is stably dominated if and only if* $\mathrm{tp}(a/\mathrm{acl}(C))$ *is.*
(2) *If $C = \mathrm{acl}(C)$ and $\mathrm{tp}(a/C)$ is stably dominated via f, then $\mathrm{tp}(a/C)$ has a unique C-definable extension p. Moreover, for all $B \supseteq C$, $a \models p|B$ if and only if $\mathrm{St}_C(B) \downarrow_C f(a)$.*

Proposition 2.10 *Assume* $\mathrm{tp}(a/C)$ *is stably dominated.*

(1) *If q is a global $\mathrm{acl}(C)$-definable type and $b \models q|\mathrm{acl}(C)$, then $a \downarrow_C b$ implies $b \downarrow_C a$. In particular, if $\mathrm{tp}(b/C)$ is stably dominated, $a \downarrow_C b$ if and only if $b \downarrow_C a$.*
(2) *$a \downarrow_C bd$ if and only if $a \downarrow_C b$ and $a \downarrow_{Cb} d$.*
(3) *If $\mathrm{tp}(b/Ca)$ is stably dominated, then so is $\mathrm{tp}(ab/C)$.*
(4) *If $b \in \mathrm{acl}(Ca)$, then $\mathrm{tp}(b/C)$ is stably dominated.*

Proof (1) and (2) are easy to check. (3) is [7, Proposition 6.11] and (4) is [7, Corollary 6.12].

Let p be a global C-invariant type. We say that p is stably dominated over C is $p|C$ is stably dominated.

Proposition 2.11 [7, Proposition 4.1 and Theorem 4.9] *Let $B \supseteq C$ and p be a global C-invariant type.*

(1) *Let f be a pro-C-definable function. If p is stably dominated over C via f, then it is also stably dominated over B via f.*
(2) *If p is stably dominated over B and $\mathrm{tp}(B/C)$ has a global C-invariant extension, then p is stably dominated over C.*

Proposition 2.12 (Strong germs, [7, Theorem 6.3]) *Let $C = \mathrm{acl}(C)$, p be a global C-invariant type stably dominated over C and f be a definable function defined at p.*

(1) *The p-germ of f is* strong*; i.e. there exists a $[f]_p C$-definable function g with the same p-germ as f.*
(2) *If $f(a) \in \mathrm{St}_C$ for $a \models p$, then $[f]_p \in \mathrm{St}_C$.*

Let us now recall the definition of metastability. Let Γ be an $\varnothing$-definable stably embedded set. We will also assume that Γ is orthogonal to the stable part: no infinite definable subset of Γ^{eq} is stable. For any C and a, let $\Gamma_C(a)$ denote $(C \cup \Gamma^{\mathrm{eq}}) \cap \mathrm{dcl}(Ca)$.

Definition 2.13 The theory T is *metastable* (over Γ) if, for any C:

(1) (Metastability bases) There exists $D \supseteq C$ such that for any tuple a, $\mathrm{tp}(a/\Gamma_D(a))$ is stably dominated.
(2) (Invariant extension property) If $C = \mathrm{acl}(C)$, then, for all tuple a, there exists a global C-invariant type p such that $a \models p|C$.

A D such as in (1) is called a metastability basis.

Remark 2.14 Since Γ is orthogonal to the stable part, if $\mathrm{tp}(a/C)$ is stably dominated, so is $\mathrm{tp}(a/C\gamma)$ for any tuple $\gamma \in \Gamma$. It follows that if C is a metastability basis, so is $C\gamma$.

Proposition 2.15 (Orthogonality to Γ, [7, Corollary 10.8]) *Assume that the theory T is metastable over Γ. A global type C-invariant type p is stably dominated if and only if for any definable map $g : p \to \Gamma^{\mathrm{eq}}$, the p-germ of g is constant (equivalently, $g_\star p$ is a realized type).*

Write $g(p)$ for the constant value of the p-germ. The property of p in the proposition above is referred to as *orthogonality of p to* Γ.

2.3 Ranks and weights

Let us now recall our "finite rank" assumptions. But, first let us recall the definition of internality.

Definition 2.16 A definable set X is Γ-*internal* if $X \subseteq \mathrm{dcl}(F\Gamma)$ for some finite set F; equivalently for any $M \prec M' \models T$, $X(M') \subseteq \mathrm{dcl}(M\Gamma(M'))$.

The same condition with acl replacing dcl is called *almost internality*.

Thus a definable X is almost Γ-internal if over some finite set F of parameters and for some finite m, X admits a definable m-to-one function $f : X \to Y/E$ for some definable $Y \subseteq \mathrm{dcl}(\Gamma^n)$ and definable equivalence relation E. When Γ eliminates imaginaries, E does not need to be mentioned. Note that X is Γ-internal whenever we can choose $m = 1$.

A definable set D is called *o-minimal* if there exists a definable linear ordering on D such that every definable subset of D (with parameters) is a finite union of intervals and points. There is a natural notion of dimension for definable subsets of D^m, such that D is dimension one (see [31]).

Definition 2.17 T has (FD) if:

(1) Γ is o-minimal.
(2) Morley rank (in the stable part, denoted MR) is uniformly finite and definable in families: if $(D_t)_t$ is a definable family of definable sets, then, when D_t is stable, the Morley rank of D_t is finite and bounded uniformly in t and for all $m \in \mathbb{Z}_{\geq 0}$, $\{t : \mathrm{MR}(D_t) = m\}$ is definable.
(3) For all definable D, the Morley rank of $f(D)$, where f ranges over all functions definable with parameters whose range is stable, takes a maximum value $\dim_{\mathrm{st}}(D) \in \mathbb{Z}_{\geq 0}$.
(4) For all definable D, the o-minimal dimension of $f(D)$, where f ranges over all functions definable with parameters whose range is Γ-internal, takes a maximum value $\dim_{\mathrm{o}}(D) \in \mathbb{Z}_{\geq 0}$.

Note that if Γ is an o-minimal group, it eliminates imaginaries. So, in (4), we could equally well ask that $f(D) \subseteq \Gamma^n$ for some n.

Definition 2.18 If $X = \varprojlim_i X_i$ is pro-definable, we define $\dim_{st}(X) = \max\{\dim_{st}(X_i) : i\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\dim_o(X) = \max\{\dim_o(X_i) : i\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

We define $\dim_{st}(a/B) := \min\{\dim_{st}(D) : a \in D$ and D is B-definable$\}$ and $\dim_o(a/B) := \min\{\dim_o(D) : a \in D$ and D is B-definable$\}$.

Note that if D is a stable definable set then $\dim_{st}(D) = \mathrm{MR}(D)$ and, similarly, for all tuples $a \in \mathrm{St}_C$, $\dim_{st}(a/C) = \mathrm{MR}(a/C)$. Note that, in general, we may have $\dim_{st}(d/B) > \dim_{st}(\mathrm{St}_B(d)/B)$. Similar statements hold for $\dim_o$.

A set A is acl-finitely generated over $B \subseteq A$ if $A \subseteq \mathrm{acl}(Ba)$ for some finite tuple a from A. If $c \in \mathrm{acl}(Ba)$ is a finite tuple, then $\dim_{st}(ca/B) = \dim_{st}(a/B)$. In particular, it makes sense to speak of the stable dimension (respectively the o-minimal dimension) over B of any set finitely acl-generated over B.

Lemma 2.19 *(FD) Let C be a set of parameters and a be a finite tuple. Then $\Gamma_C(a)$ and $\mathrm{St}_C(a)$ are* acl*-finitely generated over C.*

Proof Any tuple $d \in \mathrm{St}_C(a)$ can be written $d = h(a)$ for some C-definable function h. By the dimension bound in (FD), $\dim_{st}(d/C) \leq \dim_{st}(a/C)$. It follows that if $d_i \in \mathrm{St}_C(a)$ then for sufficiently large n, $d_n \in \mathrm{acl}(Cd_1, \ldots, d_{n-1})$. So $\mathrm{St}_C(a)$ is finitely acl-generated. The proof for $\Gamma_C(a)$ is the same.

Lemma 2.20 *(FD) Let $f : P \to Q$ be a definable map between definable sets P and Q. Let $P_a = f^{-1}(a)$. Then there exists m such that if P_a is finite, then $|P_a| \leq m$.*

Proof Say f, P, Q are $\varnothing$-definable. By compactness, it suffices to show that if P_a is infinite then there exists a $\varnothing$-definable set Q' with $a \in Q'$ and P_b infinite for all $b \in Q'$. □

Claim 2.20.1 *If P_a is infinite then either* $\dim_{st}(P_a) > 0$ *or* $\dim_o(P_a) > 0$.

Proof Let M be a metastability base, with $a \in M$. Let $c \in P_a \setminus M$. If $\Gamma_M(c) \neq \Gamma(M)$ then $\dim_o(P_a) > 0$. Otherwise, by metastability, $\mathrm{tp}(c/M)$ is stably dominated, say via f. If $f(c) \in M$, then $\mathrm{tp}(b/M) \vdash \mathrm{tp}(b/Mc)$ for all b, and taking $b = c$ it follows that $c \in M$. Thus $f(c) \in \mathrm{St}_M \setminus M$. It follows that $\dim_{st}(P_a) > 0$. □

If $\dim_{st}(P_a) > 0$, then there exists a definable family of stable definable sets D_t with $\dim_{st}(D_t) = k > 0$, and a definable function $f(x, u)$ such that for some b and t, $f(P_a, b) = D_t$. Then the formula $(\exists u)(\exists t) f(P_y, u) = D_t$ is true of $y = a$, and implies that P_y is infinite. The case $\dim_o(P_a) > 0$ is similar.

Definition 2.21 T has (FD_ω) if, in addition to (FD), any C is contained in a metastability basis M such that M is a model and such that for any finitely acl-generated $G \subseteq \Gamma$ and $S \subseteq \mathrm{St}_{M \cup G}$ over M, isolated types over $M \cup S$ are dense.

Say $\mathrm{tp}(a/C)$ is *strongly stably dominated* if there exists $\phi(x, c) \in \mathrm{tp}(a/\mathrm{St}_C(a))$ such that for any tuple b with $\mathrm{St}_C(b) \downarrow_C \mathrm{St}_C(a)$, $\mathrm{tp}(a/\mathrm{St}_C(a)b)$ is isolated via ϕ. In particular, $\mathrm{tp}(a/\mathrm{St}_C(a))$ is isolated via ϕ. Moreover, $\mathrm{tp}(a/C)$ is stably dominated, since $\mathrm{tp}(a/\mathrm{St}_C(a))$ implies ϕ which implies $\mathrm{tp}(a/\mathrm{St}_C(a)b)$ for any b as above.

When $\mathrm{tp}(a/C)$ is strongly stably dominated, the parameters c of ϕ may be written as $c_0 h(a)$ for some C-definable function h into St_C and tuple $c_0 \in C$. In this situation we say that $\mathrm{tp}(a/C)$ is strongly stably dominated via h. Conversely, if $\mathrm{tp}(a/C)$ is stably dominated via h and $\mathrm{tp}(a/Ch(a))$ is isolated, then $\mathrm{tp}(a/C)$ is strongly stably dominated via h.

Lemma 2.22 *(FD_ω) Let D be C_0-definable and $h : D \to S$ a definable map to a stable definable set of maximal possible dimension* $\dim_{\mathrm{st}}(S) = \dim_{\mathrm{st}}(D)$. *Then there exists M containing C_0 and $a \in D$ such that, with $C := \Gamma_M(a)$, we have* $\dim_{\mathrm{st}}(D) = \dim_{\mathrm{st}}(h(a)/C) = \dim_{\mathrm{st}}(\mathrm{St}_C(a)/C)$*, and* $\mathrm{tp}(a/C)$ *is strongly stably dominated via h.*

Proof We may assume that h is defined over C_0. Let $a_0 \in D$ be such that $\dim_{\mathrm{st}}(D) = \dim_{\mathrm{st}}(h(a_0)/C_0)$. Extend C_0 to a metastability basis M as in (FD_ω), with $h(a_0)$ independent from M over C_0. Then $\dim_{\mathrm{st}}(D) = \dim_{\mathrm{st}}(h(a_0)/M) = \dim_{\mathrm{st}}(\mathrm{St}_M(a_0)/M)$. Choose a_0 such that $\dim_{\mathrm{o}}(\Gamma_M(a_0)/M)$ is as large as possible—given the other constraints. Let $C := \Gamma_M(a_0)$ and $B := \mathrm{St}_C(a_0)$; then, by Lemma 2.19, C and B are finitely generated over M. Moreover, $\dim_{\mathrm{st}}(h(a_0)/C) = \dim_{\mathrm{st}}(D) \geq \dim_{\mathrm{st}}(B/C)$ so $B \subseteq \mathrm{acl}(Ch(a_0))$. Let D' be B-definable set contained in D, containing a_0, and such that for any $a \in D'$, $B \subseteq \mathrm{acl}(Ch(a))$. By ($\mathrm{FD}_\omega$), there exists $a \in D'$ such that $\mathrm{tp}(a/B)$ is isolated. By choice of D' we have $B \subseteq \mathrm{acl}(Ch(a))$. So $\mathrm{tp}(B/Ch(a))$ is atomic and, therefore, $\mathrm{tp}(a/Ch(a))$ is isolated.

Similarly, by maximality of $\dim_{\mathrm{o}}(\Gamma_M(a_0)/M)$ and the fact that $\Gamma_M(a_0) \subseteq B \cap \Gamma \subseteq \mathrm{acl}(Ca) \cap \Gamma \subseteq \Gamma_M(a)$, we have $\Gamma_M(a) = \Gamma_M(a_0) = C$. By metastability, $\mathrm{tp}(a/C)$ is stably dominated. But $\dim_{\mathrm{st}}(\mathrm{St}_C(a)/C) \leq \dim_{\mathrm{st}}(D) = \dim_{\mathrm{st}}(B/C)$ and $B \subseteq \mathrm{acl}(Ch(a)) \subseteq \mathrm{acl}(\mathrm{St}_C(a))$, so $\mathrm{St}_C(a) \subseteq \mathrm{acl}(B) \subseteq \mathrm{acl}(Ch(a))$ and $\mathrm{tp}(a/C)$ is stably dominated via h. It follows that $\mathrm{tp}(a/C)$ is strongly stably dominated via h. □

Finally, let us introduce weight.

Definition 2.23 Let p be a definable type and X be a pro-definable set. We say that X has *p-weight* smaller or equal to n if whenever $b \in X$, $(a_1, \ldots, a_n) \models p^{\otimes n}$, we have $a_i \models p|b$ for some i. The set X has bounded weight if for some n, for every stably dominated type p concentrating on X, X has p-weight smaller or equal to n.

For all definable D and stably dominated types p concentrating on D, the p-weight of D is bounded by $\dim_{\mathrm{st}}(D)$. It follows that if (FD) holds, every definable set has bounded weight. In ACVF, for example, the weight of any definable subset of a variety V is bounded by the dimension of V.

2.4 Some *o*-minimal lemmas

This subsection contains some lemmas on *o*-minimal partial orders and groups. The former will yield a generic type of limit stably dominated groups. The latter will be used to improve some statements from "almost internal" to "internal".

Let $(P, \leq)$ be a definable partial ordering, and p a definable type concentrating on P. We say p is cofinal if for any $c \in P$, $\models (\mathrm{d}_p x) x \geq c$. Equivalently, for every non-cofinal definable $Q \subseteq P$, $\models \neg(\mathrm{d}_p x) Q(x)$. The following result can also be found in [9, Lemma 4.2.18].

Lemma 2.24 *Let P be a definable directed partial ordering in an o-minimal structure Γ. Then there exists a definable type p cofinal in P.*

Proof We assume P is $\varnothing$-definable, and work with $\varnothing$-definable sets; we will find an $\varnothing$-definable type with this property.

Note first that we may replace P with any $\varnothing$-definable cofinal subset. Also if Q_1 and Q_2 are non-cofinal subsets of P, there exist a_1 and a_2 such that no element of Q_i lies above a_i; but by directedness there exists $a \geq a_1, a_2$; so no element of $Q_1 \cup Q_2$ lies above a, i.e. $Q_1 \cup Q_2$ is not cofinal. In particular if $P = P' \cup P''$, at least one of P', P'' is cofinal in P (hence also directed).

If $\dim_o(P) = 0$ then P is finite, so according to the above remarks we may assume it has one point; in which case the lemma is trivial. We use here the fact that in an o-minimal theory, any point of a finite $\varnothing$-definable set is definable.

If $\dim_o(P) = n > 0$, we can divide P into finitely many $\varnothing$-definable sets P_i, each admitting a map $f_i : P_i \to \Gamma$ with fibers of dimension strictly smaller than n. We may thus assume that there exists an $\varnothing$-definable $f : P \to \Gamma$ with fibers of dimension strictly smaller than n. For all $a < b \in \Gamma$, let $P(a) = f^{-1}(a)$, and $P(a, b) = f^{-1}(a, b)$.

Claim 2.24.1 *One of the following holds:*

(1) *For any $a \in \Gamma$, $P(a, \infty)$ is cofinal in P.*
(2) *For some $\varnothing$-definable $a \in \Gamma$, for all $b > a$, $P(a, b)$ is cofinal.*
(3) *For some $\varnothing$-definable $a \in \Gamma$, $P(a)$ is cofinal.*
(4) *For some $\varnothing$-definable $a \in \Gamma$, for all $b < a$, $P(b, a)$ is cofinal.*
(5) *For any $a \in \Gamma$, $P(-\infty, a)$ is cofinal.*

Proof Suppose (1) and (5) fail. Then $P(a, \infty)$ is not cofinal in P for some a; so $P(-\infty, b)$ must be cofinal, for any $b > a$. Since (5) fails, $\{b : P(-\infty, b) \text{ is cofinal}\}$ is a non-empty proper definable subset of Γ, closed upwards, hence of the form $[A, \infty)$ or (A, ∞) for some $\varnothing$-definable $A \in \Gamma$. In the former case, $P(-\infty, A)$ is cofinal, but $P(-\infty, b)$ is not cofinal for $b < A$, so $P(b, A)$ is cofinal for any $b < A$; thus (4) holds.

In the latter case, $(-\infty, b)$ is cofinal for any $b > A$, while $(-\infty, A)$ is not; so $P([A, b))$ is cofinal for any $b > A$. Thus either (2) or (3) hold.

Let p_1 be an $\varnothing$-definable type of Γ, concentrating on sets X with $f^{-1}(X)$ cofinal; for instance in case (1), p_1 concentrates on intervals (a, ∞).

Claim 2.24.2 *For any $c \in P$, if $a \models p_1|c$ then there exists $d \in P(a)$ with $d \geq c$.*

Proof Let $Y(c) = \{x : (\exists y \in P(x)) y \geq c\}$. Then $P^{-1}(\Gamma \setminus Y(c))$ is not cofinal in P, so it cannot be in the definable type p_1. Hence $Y(c) \in p_1|\{c\}$.

Now let $M \models T$. Let $a \models p_1|M$. By induction on n, let q_a be an a-definable type, cofinal in $P(a)$, and let $b \models q_a|Ma$. Then $\mathrm{tp}(ab/M)$, and hence $\mathrm{tp}(b/M)$ is definable (see Lemma 2.3). If $c \in M$ with $c \in P$, then by Claim 2.24.2, there exists $d \in P(a)$ with $d \geq c$. So $\{y \in P(a) : \neg(y \geq c)\}$ is not cofinal in $P(a)$. Therefore this set is not in q_a. Since $b \models q_a|Ma$, we have $b \geq c$. This shows that $\mathrm{tp}(b/M)$ is cofinal in P. □

Discussion If there exists a definable weakly order preserving map $j : \Gamma \to P$ with cofinal image, then we can use the definable type at ∞ of Γ, r_∞, to obtain a cofinal definable type of P, namely $j_\star r_\infty$.

When Γ admits a field structure, perhaps such a map j always exists. In general, it is not always possible to find a one-dimensional cofinal subset of P. For instance, when Γ is a divisible ordered Abelian group, consider the product of two closed intervals of incommensurable sizes; or the subdiagonal part of a square.

Lemma 2.25 *Let G be a definable group. Assume G is almost internal to a stably embedded definable set Γ. Then there exists a finite normal subgroup N of G with G/N internal to Γ.*

Proof The assumption implies the existence of a definable finite-to-one function $f : G \to Y$, where $Y \subseteq \mathrm{dcl}(\Gamma)$. Given a definable $Y' \subseteq Y$, let $m(Y')$ be the least integer m such that (possibly over some more parameters), there exists a definable m-to-one map $f^{-1}(Y') \to Z$, for some definable $Z \subseteq \mathrm{dcl}(\Gamma)$. We may assume that f is $m(Y)$-to-one. Let I be the family of all definable subsets Y' of Y with ($Y' = \varnothing$ or) $m(Y') < m(Y)$. This is clearly an ideal (closed under finite unions, and definable subsets). Let $F := \{Y \setminus Y' : Y' \in I\}$ be the dual filter. For $g \in G$, let $D(g)$ be the set of $y \in Y$ such that for some (necessarily unique) y', $g \cdot f^{-1}(y) = f^{-1}(y')$; and define $g_\star(y) = y'$. The function $x \mapsto (f(x), f(g \cdot x))$ shows that $\{y : |f(g \cdot f^{-1}(y))| > 1\} \in I$. Since $X := \{y : |f^{-1}(y)| = m\} \in F$ and $(Y \setminus I) \cap X \subseteq D(g)$, it follows that $D(g) \in F$.

For any definable $Z \subseteq Y$, let $F|Z = \{W \cap Z : W \in F\}$. Let G_0 be the set of bijections $\phi : Y' \to Y''$ with $Y', Y'' \in F$, carrying the filter $F|Y'$ to $F|Y''$. Note that $g_\star : D(g) \to D(g^{-1})$ lies in G_0. Write $\phi \sim \phi'$ if ϕ and ϕ' agree on some common subset of their domains, lying in F; and let $G' = G_0/\sim$. Composition induces a group structure on G' and we obtain a homomorphism $G \to G'$, $g \mapsto g_\star/\sim$. Let N be the kernel of this homomorphism. Let us prove that $|N| \leq m(Y)$. For suppose $n_0, \ldots, n_m$ are elements of N. Then for some $y \in \cap D(n_i)$ we have $(n_i)_\star(y) = y$. It follows that $n_i(f^{-1}(y)) = f^{-1}(y)$. Fix any $g \in f^{-1}(y)$ and since $|f^{-1}(y)| = m$, for some $i \neq j$ we have $n_i \cdot g = n_j \cdot g$ and hence $n_i = n_j$.

Let σ be an automorphism fixing Γ. Then for any $g \in G$, since $g_\star$ is a function between Γ-internal sets, $\sigma(g)_\star = \sigma(g_\star) = g_\star$ and hence $g \cdot N = \sigma(g) \cdot N$. As Γ is stably embedded, it follows that G/N is Γ-internal.

Lemma 2.26 *Let G be a definable group. Assume G is almost internal to an o-minimal definable set Γ. Then G is Γ-internal.*

Proof By Lemma 2.25, there exists a definable surjective homomorphism $f : G \to B$ with B a group definable over Γ, and $N = \ker(f)$ a group of finite size n. Let B^0 be the connected component of the identity in B; then B/B^0 is finite, and it suffices to prove the lemma for $f^{-1}(B^0)$. Assume therefore that B is connected.

If G has a proper definable subgroup G_1 of finite index, then $f(G_1) = B$ by connectedness of B. It follows that N is not contained in G_1, so $N_1 = N \cap G_1$ has smaller size than N. Hence using induction on the size of the kernel, G_1 is Γ-internal; hence so is G. Thus we may assume G has no proper definable subgroups of finite index. Since the action of G on N by conjugation has kernel of finite index, N must be central.

Let $Y = \{g^n : g \in B\}$. By [3, Theorem 7.2], there exists a definable function $\alpha : Y \to B$ with $\alpha(b)^n = b$ for all $b \in B$. Define $\beta : Y \to G$ by $\beta(b) = a^n$ where $f(a) = \alpha(b)$; this does not depend on the choice of a, and we have $f(\beta(b)) = f(a)^n = \alpha(b)^n = b$. It follows that $f^{-1}(Y) = N\beta(Y) \subseteq \mathrm{dcl}(N, \Gamma)$ is Γ-internal.

Similarly, let $[B, B] = \{[g, h] : g, h \in B\}$. As above, there exists a definable $\alpha_1 : [B, B] \to B$ such that $(\exists y)[\alpha_1(b), y] = b$, and $\alpha_2 : [B, B] \to B$ such that $[\alpha_1(b), \alpha_2(b)] = b$. Define $\beta' : [B, B] \to G$ by $\beta'(b) = [a_1, a_2]$ where $f(a_i) = \alpha_i(b)$. Again β' is definable and well-defined, and shows that $f^{-1}([B, B])$ is Γ-internal.

Hence for any k, letting $X^{(k)} = \{x_1 \dots x_k : x_1, \dots, x_k \in X\}$, $(f^{-1}(Y \cup [B, B]))^{(k)} = f^{-1}((Y \cup [B, B])^{(k)})$ is Γ-internal. So we are done once we show:

Claim 2.26.1 *Let B be any definably connected group definable in an o-minimal structure. Let $Y = Y_n(B) = \{g^n : g \in B\}$. Then for some $k \in \mathbb{Z}_{\geq 0}$, $(Y \cup [B, B])^{(k)} = B$.*

Proof If the Claim holds for a normal subgroup H of B with bound k', and also for B/H (with bound k''), then it is easily seen to hold for B (with bound $k' + k''$.)

We use induction on $\dim_{\mathrm{o}}(B)$. If B has a non-trivial proper connected definable normal subgroup H, then the statement holds for H and for B/H. We may thus assume B has no such subgroups H. Thus any definable normal subgroup of B is finite and by connectedness, central.

If B is centerless, it is definably simple. By [23], B is elementarily equivalent to a simple Lie group. In this case it is easy to see that every element is the product of a bounded number of commutators, by considering root subgroups.

Since the center Z of B is finite, then B/Z is definably simple and so the claim holds (say with $k_{B/Z}$). Let $W_l = (Y \cup [B, B])^{(l)}$, and let $\pi : B \to B/Z$ be the quotient homomorphism. Then $\pi(W_l) = B/Z$ for $l \geq k_{B/Z}$. For such l, $W_{5l} \cap Z = W_l \cap Z$ implies that $W_{4l} = W_{2l}$. Thus W_{2k} is a subgroup of B mapping onto B/Z, for some $k \leq k_{B/Z} 5^{|Z|}$. By connectedness $W_{2k} = B$. □

This concludes the proof. □

We conclude this section with a quick discussion of definable compactness. First, one can define the limit of a function along a definable type, generalizing the limit along a one-dimensional curve. Let X be a definable space over an o-minimal structure Γ.

Definition 2.27 Given a definable set D, a definable type p on D, and a definable function $g : D \to X$, we define $\lim_p g = x \in X$ if for any definable neighborhood U of x, $g_\star p$ concentrates on U.

When X is $\Gamma \cup \{-\infty, +\infty\}$, the limit always exists. Consider the definable type $g_\star p$. It must be of the form $r_\infty, r_{-\infty}, r_a := (x = a), r_a^+$ the type of elements infinitesimally bigger than a, or r_a^-. By definition $\lim_p g$ is $\infty, -\infty, a, a, a$ in the respective cases.

The following definition is equivalent to the one in [24].

Definition 2.28 The definable set X is *definably compact* if for any definable type p on Γ and any definable function $f : \Gamma \to X$, $\lim_p f$ exists.

Definition 2.29 Let A be a definable Abelian group.

- A definable set is called *generic* if finitely many translates of that set cover the group.
- Say A has the property (NG) if the non-generic definable sets form an ideal.

Proposition 2.30 [22, Corollary 3.9] *Any definably compact definable Abelian group A in an o-minimal theory has (NG).*

Moreover any definable subsemigroup of A is a group (cf. [22, Theorem 5.1]). We include the deduction of the latter fact in a case we will need.

Lemma 2.31 *Let A be an Abelian group with (NG). Let Y be a definable semi-group of A, such that $Y - Y = A$. Then $Y = A$.*

Proof Note first that $A \setminus Y$ is not generic. Otherwise, some finite intersection $\bigcap_{i=1}^{n}(c_i + Y) = \emptyset$. By assumption, there exist $b_i \in Y$ with $b_i + c_i \in Y$. If $b := \sum b_i$ then $b + c_i \in Y$; so by translating we may assume each $c_i \in Y$. But then $\sum c_i \in c_i + Y$ for each i, a contradiction.

As A is generic, Y cannot be in the non-generic ideal, so Y is generic and $\bigcup_{i=1}^{n} d_i + Y = G$ for some $d_i \in G$. Again we find $e \in Y$ with $e + d_i \in Y$. We have $\cup_{i=1}^{n}(e + d_i) + Y = G$. But $e + d_i + Y \subseteq Y$. So $Y = G$.

2.5 Valued fields: imaginaries and resolution

Let K be an algebraically closed valued field, with valuation ring $\mathcal{O}$, maximal ideal $\mathfrak{M}$ and value group Γ. The geometric language for valued fields has a sort for the valued field itself, and certain other sorts. In particular, there is a sort S_n such that $S_n(K)$ is the space of free $\mathcal{O}$-modules in K^n, equivalently $S_n(K) = \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$. Until the end of Section 2.5, we work in a model M of the theory ACVF of algebraically closed fields in the geometric language.

By a *substructure*, we mean a subset of M, closed under definable functions. For any geometric sort S and substructure $A \subseteq M$, $S(A)$ denotes $S \cap A$; in particular $K(A)$ is the set of field points of A. A substructure $A \subseteq M$ is called *resolved* if $A \subseteq \mathrm{dcl}(K(A))$. When $\Gamma(A) \neq 0$ and A is algebraically closed this just amounts to saying that $A \prec M$. Recall that, in any first order theory, if A is a substructure of a model M, M is *prime* over A if any elementary map $A \to N$ into another model, extends to an elementary map $M \to N$; and M is *minimal* over A if there is no $M' \prec M$ with $A \subset M'$. If a minimal model over A and a prime one exist, then any two minimal or prime models over A are isomorphic.

Proposition 2.32 *Let A be a substructure of M, finitely generated over a subfield L of K, and assume $\Gamma(A) \neq 0$. Then there exists a minimal prime model $\widetilde{A}$ over A which enjoys the following properties.*

(1) *$\widetilde{A}$ is a minimal resolution of A. Moreover it is the unique minimal resolution, up to isomorphism over A. It is atomic over A.*
(2) $\mathrm{St}_L(\widetilde{A}) = \mathrm{St}_L(A)$.

(3) *Let $A \le A'$, with A' finitely generated over A. Then $\widetilde{A}$ embeds into $\widetilde{A}'$ over A. If $A \le A' \le \widetilde{A}$, then $\widetilde{A}$ is the prime resolution of A'.*
(4) *For a valued field extension L' of L, let $L'(A)$ be the structure generated by $L' \cup A$. Then $L'(\widetilde{A})^{\mathrm{alg}}$ is a prime resolution of $L'(A)$.*
(5) *If $\mathrm{tp}(A/L)$ is stably dominated, then $\mathrm{tp}(\widetilde{A}/L)$ is stably dominated.*
(6) *If $\mathrm{tp}(A'/A)$ is stably dominated, and $A' \downarrow_A \widetilde{A}$, then there exists a prime resolution $\widetilde{A}'$ of A' such that $\mathrm{tp}(\widetilde{A}'/\widetilde{A})$ is stably dominated.*

Proof The existence, uniqueness and minimality of $\widetilde{A}$ are shown in [7, Theorem 11.14]. It is also shown there that $k(\widetilde{A}) = k(\mathrm{acl}(A))$ and $\Gamma(\widetilde{A}) = \Gamma(\mathrm{acl}(A))$, where k is the residue field; and that $\widetilde{A}/A$ is atomic, i.e. $\mathrm{tp}(c/A)$ is isolated for any tuple c from A.

(2) Since L is a field, for any $B = \mathrm{acl}(B)$ with $L \subset B$, $\mathrm{St}_L(B) = \mathrm{dcl}(B, k(B))$.
(3) This is immediate from the definition of prime resolution: since $\widetilde{A}'$ is a resolution of A, $\widetilde{A}$ embeds into $\widetilde{A}'$. If $A \le A' \le \widetilde{A}$, then $\widetilde{A}$ is clearly a minimal resolution of A'; hence by (1) it is the prime resolution.
(4) Let B be the prime resolution of $L'(A)$. Then $\widetilde{A}$ embeds into B. Within B, $L'(\widetilde{A})^{\mathrm{alg}}$ is a resolution of $L'(A)$; by minimality of B, $B = L'(\widetilde{A})^{\mathrm{alg}}$.
(5) We may assume $L = L^{\mathrm{alg}}$. Let p be an L-invariant extension of $\mathrm{tp}(\widetilde{A}/L)$. Let $\overline{L}$ be a maximal immediate extension of L. Choose it in such a way that $\widetilde{A} \models p|\overline{L}$. Since A/L is stably dominated, $\Gamma(\overline{L}(A)^{\mathrm{alg}}) = \Gamma(\overline{L})$. According to (4), $\overline{L}(\widetilde{A})^{\mathrm{alg}}$ is the prime resolution of $\overline{L}(A)^{\mathrm{alg}}$, and so $\Gamma(\overline{L}(\widetilde{A})^{\mathrm{alg}}) = \Gamma(\overline{L})$. Since $\overline{L}$ is a metastability basis, $\mathrm{tp}(\widetilde{A}/\overline{L})$ is stably dominated. Using descent (cf. Proposition 2.11.(2)), $\mathrm{tp}(\widetilde{A}/L)$ is stably dominated.
(6) Since $A' \downarrow_A \widetilde{A}$ and $\mathrm{tp}(A'/A)$ is stably dominated, $\mathrm{tp}(A'/\widetilde{A})$ is stably dominated. By (5), $\mathrm{tp}(B/\widetilde{A})$ is stably dominated, where B is a resolution of $\widetilde{A}(A')$. But B contains a resolution of A' whose type over $\widetilde{A}$ is therefore stably dominated.

□

Proposition 2.33 *The theory* ACVF *is metastable, with (FD) and (FD$_\omega$).*

Proof The fact that, in ACVF, maximally complete submodels are metastability bases is proved in [7, Theorem 12.18.(ii)] and the existence of invariant extensions is shown in [7, Corollary 8.16]. The fact that (FD) holds follows from the fact that the value group Γ is a divisible ordered Abelian group, that the stable part, being internal to an algebraically closed field, has finite Morley rank and that, for any definable subset D of the field, $\dim_{\mathrm{st}}(D)$ and $\dim_{\mathrm{o}}(D)$ are bounded by the dimension of the Zariski closure of D.

To see that (FD$_\omega$) holds, let $M \models$ ACVF and C be finitely generated over M—C might contain imaginaries. By Proposition 2.32, the resolution N of C is a model of ACVF which is atomic over M. Hence isolated types over C are dense.

Definition 2.34 (1) The (imaginary) element $e \in M$ is said to be *purely imaginary* over $C \subseteq M$ if $\mathrm{acl}(Ce)$ contains no field elements other than those in $\mathrm{acl}(C)$.
(2) An pro-definable set D is *purely imaginary* if there exists no pro-C-definable map (with parameters) from D onto an infinite subset of the field K.

Note that D is purely imaginary if and only if, for all C such that D is C-definable, any $e \in D$ is purely imaginary over C. For every $\alpha \in \Gamma$, let $\alpha\mathcal{O} := \{x \in K : \mathrm{val}(x) \geq \alpha\}$, and $\alpha\mathfrak{M} := \{x \in K : \mathrm{val}(x) > \alpha\}$.

Lemma 2.35 *The following are equivalent:*

(1) *e is purely imaginary over C.*
(2) $\mathrm{dcl}(Ce) \cap K \subseteq \mathrm{acl}(C)$.
(3) *For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma_C(e)$ and $d \in (\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n$, $e \in \mathrm{dcl}(\mathrm{acl}(C)\beta_0\beta_1 d)$.*

Proof Note that (1) implies (2) trivially. Let us now prove that (2) implies (1). Let $d \in \mathrm{acl}(Ce)$ be a field element. The finite set of conjugates of d over Ce is coded by a tuple d' of field elements and $d' \in \mathrm{dcl}(Ce)$. By (2), $d' \in \mathrm{acl}(C)$. Since $d \in \mathrm{acl}(d')$ we have $d \in \mathrm{acl}(C)$.

Let us now assume (3) and prove (2). Let $a = g(e) \in K$ for some C-definable map g. By (3), there exists an $\mathrm{acl}(C\beta_0\beta_1)$-definable map f with domain $(\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n$ and whose range contains e. Then $g \circ f : (\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n \to K$ is definable. However, its image cannot contain a ball since in some models of ACVF, $\beta_0\mathcal{O}/\beta_1\mathfrak{M}$ is countable while every ball is uncountable. Hence, the image of $g \circ f$ is finite and $a \in \mathrm{acl}(C)$.

Finally, let us prove that (2) implies (3): By [6, Theorem 1.0.2], and using (2), there exists an e-definable $\mathcal{O}$-submodule Λ of K^m (for some m), such that e is a canonical parameter for Λ, over $\mathrm{acl}(C)$. The K-vector space $V = K \otimes_{\mathcal{O}} \Lambda$ is coded by an element w of some Grassmanian $G_{m,l}$. Note that $w \in \mathrm{dcl}(Ce) \cap K = \mathrm{acl}(C) \cap K =: K_0$. Since K_0 is an algebraically closed field, V is K_0-isomorphic to K^l, so we may assume $V = K^l$. Dually, let $V' = \{v \in V : K \cdot v \subseteq \Lambda\}$. Then V' is a K_0-definable K-vector subspace of V. Replacing Λ by the image in V/V', we may assume $V' = (0)$. It follows, by [6, Lemma 2.2.4], that $V' \subseteq \beta_0\mathcal{O}^l$ for some $\beta_0 < 0$. The set of all such β_0 is $\mathrm{acl}(C)e$-definable. Since o-minimal groups have Skolem functions, we may choose β_0 in $\Gamma_C(e)$.

Let $v_1, \ldots, v_l$ be a standard basis for $V = K^l$. Since $v_i \in K \otimes_{\mathcal{O}} \Lambda$, we have $c_i v_i \in \Lambda$ for some $c_i \in \mathcal{O}$. So $\sum r_i c_i v_i \in \Lambda$ for all $r_1, \ldots, r_l \in \mathcal{O}$. Let $\Lambda'(\beta) = \{\sum r_i v_i : \mathrm{val}(r_i) > \beta\}$. Then $\Lambda'(\beta) \subseteq \Lambda$ for sufficiently large β; namely for $\beta \geq \max_i \mathrm{val}(c_i)$. The set $\{\beta : \Lambda'(\beta) \subseteq \Lambda\}$ is definable, hence it contains $\{\beta : \beta > \beta_1\}$ for some Ce-definable β_1. It follows that Λ is determined by its image D in $(\beta_0\mathcal{O}/\beta_1\mathfrak{M})^l$. Pick $c \in \beta_0\mathcal{O}/\beta_1\mathfrak{M}$; then $r \mapsto r \cdot c$ is an isomorphism $\mathcal{O}/(\beta_1 - \beta_0)\mathfrak{M} \to b_0\mathcal{O}/b_1\mathfrak{M}$. By [13], $\mathcal{O}/(\beta_1 - \beta_0)\mathfrak{M}$ is stably embedded; hence so is $\beta_0\mathcal{O}/\beta_1\mathfrak{M}$. Let $d \in (\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n$ be such that D is $\beta_0\beta_1 d$-definable. Then $e \in \mathrm{dcl}(\mathrm{acl}(C)\beta_0\beta_1 d)$, as required.

Remark 2.36 As we can see from the proof above, every geometric sort other than K is purely imaginary.

Definition 2.37 (1) An element $e \in M$ is said to be *boundedly imaginary* over $C \subseteq M$ if for any $\gamma \in \Gamma_C(e)$, $\mathrm{tp}(\gamma/C)$ is bounded, i.e. it is neither the type at $+\infty$ nor the one at $-\infty$.
(2) An ∞-definable set D is *boundedly imaginary* if there exists no definable map (with parameters) from D onto an unbounded subset of Γ.

Note that D is boundedly imaginary if and only if any $e \in D$ is boundedly imaginary over any C over which D is defined.

Lemma 2.38 *Any definable function $f : (\alpha\mathcal{O}/\beta\mathfrak{M})^n \to \Gamma$ is bounded.*

Proof Suppose first $n = 1$. Since parameters are allowed, we may assume $\alpha = 0$, and consider $f : \mathcal{O}/\beta\mathfrak{M} \to \Gamma$, defined over C. Let q be the type of elements of Γ greater than any element of $\Gamma(C)$. For $\gamma \models q$, let $X(\gamma)$ be the pullback to $\mathcal{O}$ of $f^{-1}(\gamma)$. This is a finite Boolean combination of balls of valuative radii $\delta_1(\gamma), \ldots, \delta_m(\gamma)$, with $0 \leq \delta_i(\gamma) \leq \beta$. But any C-definable function into a bounded interval in Γ is constant on q. Thus $X(\gamma)$ is a finite Boolean combination of balls of constant valuative radii $\delta_1, \ldots, \delta_m$. However, it is shown in [6, Proposition 2.4.4] that any definable function on a finite cover of Γ into balls of constant radius has finite image. Hence $X(\gamma)$ is constant on q. But if $X(\gamma) = X(\gamma') \neq \varnothing$ then for any $x \in X(\gamma)$, $\gamma = f(x + \beta\mathfrak{M}) = \gamma'$. It follows that $X(\gamma) = \varnothing$ for $\gamma \models q$, i.e. f is bounded. Now, given $f : (\alpha\mathcal{O}/\beta\mathfrak{M})^2 \to \Gamma$, let $F(x) = \sup\{f(x, y) : y \in \alpha\mathcal{O}/\beta\mathfrak{M}\}$. Then F is bounded, so f is bounded. This shows the case $n = 2$, and the general case is similar.

Corollary 2.39 *Let D be a C-definable set in* ACVF. *Then the following are equivalent:*

(1) *D is boundedly imaginary.*
(2) *There exists a definable surjective map $g : (\mathcal{O}/\beta\mathfrak{M})^n \to D$.*
(3) *There is an* $\mathrm{acl}(C)$*-definable surjective map $g : (\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n \to D$, where $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$.*

Note that infinite definable subsets of K contain balls and that balls are not boundedly imaginary: given any point a in a ball b, the set $\{\mathrm{val}(x - a) : x \in b\}$ is not bounded. It follows that boundedly imaginary definable sets are purely imaginary.

Proof By Lemma 2.38, (1) follows from (2) and (3) implies (2) easily. To prove that (1) implies (3), by compactness it suffices to fix $e \in D$ and find $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$ and an $\mathrm{acl}(C)$-definable map $g : (\beta_0\mathcal{O}/\beta_1\mathfrak{M})^n \to D$ whose range contains e. Since e is purely imaginary over C, by Lemma 2.35.(3), this does hold with $\beta_0, \beta_1 \in \Gamma_C(e)$. But $\beta_i = f_i(e)$ for some C-definable functions f_i; by (1), f_i is bounded on D, and the upper and lower bounds are C-definable. □

It follows, by compactness, that any ∞-definable boundedly imaginary D is contained in a definable boundedly imaginary D'.

Let us conclude this section with a characterization of independence of stably dominated types of field elements in ACVF in terms of a *maximum modulus* principle.

Proposition 2.40 (Maximum Modulus, [7, Theorem 14.12]) *Let p be a stably dominated C-definable type concentrating on an affine variety V defined over $K \cap C$. Let $P = p|C$ and F be a regular function on V over $L \supseteq C$. Then* $\mathrm{val}(F)$ *has an infimum $\gamma^F_{\min} \in \Gamma(L)$ on P. Moreover for $a \models P$, $a \models p|L$ if and only if* $\mathrm{val}(F(a)) = \gamma^F_{\min}$ *for all such F.*

Corollary 2.41 [7, Theorem 14.13] *Let U and V be varieties over the algebraically closed valued field C. Let p and q be stably dominated types over C of elements of U and V respectively. Let F be a regular function on $U \times V$. Then there exists $\gamma_F \in \Gamma$ such that:*

(1) *If* $(a, b) \models p \otimes q$ *then* $\mathrm{val}(F(a, b)) = \gamma_F$.
(2) *For any* $a \models p$, $b \models q$, *we have* $\mathrm{val}(F(a, b)) \geq \gamma_F$.
(3) *Assume* U *and* V *are affine. If* $a \models p$ *and* $b \models q$ *and* $\mathrm{val}(F(a, b)) = \gamma_F$ *for all regular* F *on* $U \times V$, *then* $(a, b) \models p \otimes q$.

Proof Since V admits a finite cover by open affines, and q concentrates on one of these affines, we may assume V is affine. Let $a \models p$. Then the statement follows from Proposition 2.40 applied to q over Ca. As p is stably dominated, the value of γ_F does not depend on a.

3 Groups with definable generics

A pro-definable group is a pro-definable set with a pro-definable group law. We will, in a few cases, need to consider a larger class of groups. A *piecewise pro-definable group* is a piecewise pro-definable set $G = \varinjlim_i G_i$ together with pro-definable maps $m_i : G_i \times G_i \to G_j$, for some $j \geq i$, compatible with the inclusions, and inducing a group structure $m : G \times G \to G$. By a pro-definable subgroup H of G we mean a pro-definable subset $H \subset G_i$ for some definable piece G_i of G, such that $m(H^2) \subset H$ and (H, m) is a subgroup of (G, m).

Let us now fix G a pro-definable group.

3.1 Definable generics

For all filters π concentrating on G and $g \in G$, let ${}^g\pi := \{\phi(x, a) : \pi(x) \vdash \phi(g \cdot x, a)\}$. Similarly, we define $\pi^g := \{\phi(x, a) : \pi(x) \vdash \phi(x \cdot g, a)\}$. Note that if π is C-definable, then ${}^g\pi$ and π^g are Cg-definable.

Definition 3.1 Let π be a filter concentrating on G. We say that π is *right generic*[1] in G over C if, for all $g \in G$ ${}^g\pi$ is C-definable. We say that it is *left generic* over C if, for all $g \in G$, π^g is C-definable.

We say that G admits a generic filter if there exists a filter π concentrating on G which is left or right generic in G (over some small set of parameters).

Lemma 3.2 *Let* π *be a definable filter concentrating on* G *then* π *is right generic if and only if* π *has boundedly many left translations by* G. *Moreover if* π *is right generic then for every formula* ϕ, *the set* $\{ ({}^g\pi)|_\phi : g \in G\}$ *is finite.*

Proof If π is right generic, then its orbit must be bounded since there are only boundedly many C-definable filters. Conversely, if the orbit of π is bounded we can find a small C such that every translate is C-definable.

Let us now assume that π is right generic and ϕ be some formula. Then the equivalence relation $g_1 \sim g_2$ if $({}^{g_1}\pi)|_\phi = ({}^{g_2}\pi)|_\phi$ is definable by $(\forall y)(\mathrm{d}_\pi x)\phi(g_1 \cdot x, y) \leftrightarrow (\mathrm{d}_\pi x)\phi(g_2 \cdot x, y)$. Since this equivalence relation has boundedly many classes, it must have finitely many. □

[1] When π is a complete type, this notion is usually referred to as a definable f-generic in the literature.

Remark 3.3 Let π be a right generic filter of G. Then it follows easily from the above that $\mu := \bigcap_{g\in G} {}^g\pi$ is a definable filter concentrating on G. It is obviously invariant under left translation by G.

Proposition 3.4 *Assume G is pro-C-definable and admits a (right) generic filter over C. Then G is pro-C-definably isomorphic to a pro-limit of C-definable groups. In particular, if G is ∞-C-definable, G is C-definably isomorphic to $\bigcap_i G_i$ where the G_i are all C-definable subgroups of some C-definable group H.*

Proof By Remark 3.3, there is a C-definable G-invariant filter π concentrating on G. Let us first assume that $G = \bigcap_i X_i$ is ∞-definable. By compactness, there exists $i_0 \in I$ such that if x, y and $z \in X_{i_0}$ then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot 1 = 1 \cdot x = x$. Because I is filtered, we may assume that i_0 is the smallest element of I. Let $Y_i = \{a \in X_{i_0} : \pi(x) \vdash a \cdot x \in X_i\}$, which is C-definable.

Claim 3.4.1 $G = \bigcap_{i\in I} Y_i$.

Proof Let $a \in G$ and $c \models \pi$. Then $a \cdot c \in G \subseteq X_i$ and $a \in Y_i$. Conversely, let $a \in \bigcap_i Y_i$. Then $\pi(x) \vdash a \cdot x \in \bigcap_i X_i = G$. Let $c \models \pi$, we have $b := a \cdot c \in G$ and hence $a = bc^{-1} \in G$. □

Let i_1 be such that $Y_{i_1} \cdot Y_{i_1} \subseteq X_{i_0}$ and $S_i := \{a \in Y_i : Y_i \cdot a \subseteq Y_i\}$, which is C-definable.

Claim 3.4.2 *For all $i \geq i_1$, $(S_i, \cdot)$ is a monoid containing G.*

Proof Pick any a and $c \in S_i$. For all $x \in Y_i$, $x \cdot a \in Y_i$ and $x \cdot c \in Y_i$, hence $x \cdot a \cdot c \in Y_i$ and, since $a \in Y_i$, $a \cdot c \in Y_i$, so $a \cdot c \in S_i$. Since we obviously have $1 \in S_i$, S_i is a monoid. Now, let $a \in G \subseteq Y_{i_1}$ and $c \in Y_i \subseteq Y_{i_1}$. We have $c \cdot a \in X_{i_0}$. Moreover, $\pi(x) \vdash c \cdot x \in X_i$ and, therefore, by G-invariance of π, ${}^a\pi = \pi \vdash c \cdot a \cdot x \in X_i$. It follows that $c \cdot a \in Y_i$ and $a \in S_i$. □

Then $G = \bigcap_{i\geq i_1} H_i$, where H_i is the group of invertible elements in S_i.

Let us now assume that $G = \varprojlim_{i\in I} X_i$. For all $i \in I$, let $G_i = \{a \in G : \pi^{\otimes 2}(x, y) \vdash \rho_i(x^{-1} \cdot a \cdot y) = \rho_i(x^{-1} \cdot y) = \rho_i(x^{-1} \cdot a^{-1} \cdot y)\}$, where $\rho_i : G \to X_i$ is the projection.

Claim 3.4.3 $G_i \trianglelefteq G$.

Proof By definition, if $a \in G_i$, then $a^{-1} \in G_i$. Let now $a, c \in G_i$. For all $(x, y) \models \pi^{\otimes 2}|Cac$, $c \cdot y \models {}^c\pi|Cacx = \pi|Cacx$. It follows that $(x^{-1} \cdot a \cdot (c \cdot y))_i = (x^{-1} \cdot (c \cdot y))_i = (x \cdot y)_i$. Similarly $(x^{-1} \cdot (a \cdot c)^{-1} \cdot y)_i = ((c \cdot x)^{-1} \cdot a^{-1} \cdot y)_i = (x \cdot y)_i$. Finally, if $a \in G_i$, $c \in G$ and $(x, y) \models \pi^{\otimes 2}|Cac$, then $(c \cdot x, c \cdot y) \models ({}^c\pi)^{\otimes 2}|Cac = \pi^{\otimes 2}|Cac$. It follows that $(x^{-1} \cdot c^{-1} \cdot a \cdot c \cdot y)_i = ((c \cdot x)^{-1} \cdot a \cdot (c \cdot y))_i = ((c \cdot x)^{-1} \cdot (c \cdot y))_i = (x \cdot y)_i$.

Since G_i is relatively C-definable, $H_i := G/G_i$ is an ∞-C-definable group.

Claim 3.4.4 *The natural map $G \to H := \varprojlim H_i$ is a group isomorphism.*

Proof Let $a \in \bigcap_i G_i$ and $(x, y) \models \pi^{\otimes 2}|Ca$. Then for all $i \in I$, $(x^{-1} \cdot a \cdot y)_i = (x^{-1} \cdot y)_i$ and thus $x^{-1} \cdot a \cdot y = x^{-1} \cdot y$. It follows that $a = 1$. Surjectivity follows, by compactness, from the fact that each map $G \to H_i$ is surjective. □

The image of π in H_i is generic, so we can conclude by the ∞-definable case. □
Similar results hold for rings:

Proposition 3.5 *Let R be an pro-C-definable ring. Assume that $(R, +)$ admits a generic filter over C and that there exists a C-definable filter π concentrating on $R^\star$ such that the stabilizer of π under (left) multiplication generates R. Then R is pro-C-definably isomorphic to a pro-limit of C-definable rings. In particular, if R is ∞-C-definable, R is C-definably isomorphic to an intersection of C-definable subrings of some C-definable ring.*

The above hypothesis on R hold in particular if both the additive group and the units of R admit a generic filter over C and that the units generate R.

Proof Let us first assume $R = \bigcap_{i \in I} P_i$ is ∞-definable. By Proposition 3.4, we may assume that the P_i are subgroups of some group $(P, +)$ and by compactness we may assume that multiplication is an associative bilinear map on P and that 1 is the identity. For all $i \in I$, let $Q_i = \{a \in P : \pi(x) \vdash a \cdot x \in P_i\}$ which is C-definable. It is easily checked that Q_i is a subgroup of P. Moreover, the multiplicative stabilizer of π is a subset of Q_i which therefore contains R. Conversely, if $a \in \bigcap_i Q_i$, then for any $c \models \pi|Ca$, $a \cdot c \in \bigcap_i P_i = R$ and hence $a \in Rc^{-1} = R$. By compactness, there exists i_0 such that $Q_{i_0} \cdot Q_{i_0} \subseteq P$. Let $R_i = \{a \in Q_i : Q_i \cdot a \subseteq Q_i\}$. One can check that, for all $i \geq i_0$, R_i is a ring, and proceeding as in Claim 3.4.2, we can show that if a stabilizes π multiplicatively, and $c \in Q_i$, then $c \cdot a \in Q_i$. So R_i contains R and $R = \bigcap_{i \geq i_0} R_i$.

Now, let $R = \varprojlim_{i \in I} P_i$. By, Proposition 3.4, we may assume that each P_i is an additive group. For all $i \in I$, let $J_i := \{a \in R : \pi^{\otimes 2}(x, y) \vdash (x^{-1} \cdot a \cdot y)_i = 0_i\}$, which is relatively C-definable. It is clear that J_i is an additive subgroup. Now pick any $a \in J_i$ and let $S_a := \{c \in R : c \cdot a \in J_i\}$. Then S_a is an additive subgroup. Moreover, if c stabilizes π, then so does c^{-1} and hence for any $(x, y) \models \pi^{\otimes 2}$, $(c^{-1} \cdot x, y) \models \pi^{\otimes 2}$. Since $a \in J_i$, it follows that $(x^{-1} \cdot c \cdot a \cdot y)_i = ((c^{-1} \cdot x)^{-1} \cdot a \cdot y)_i = 0_i$, so $c \in S_a$ and hence $S_a = R$. Similarly, we show that J_i is a two sided ideal. Bijectivity of the ring homomorphism $R \to \varprojlim_i R/J_i$ follows as in the group case and, since R/J_i is an ∞-C-definable ring, we conclude by the ∞-definable case.

3.2 Stabilizers

The stabilizer of a definable filter can be viewed as an adjoint notion for the generic of a group. In this section, we will assume that G is a pro-$\varnothing$-definable group.

Let $\Delta(x)$ be a set of formulas $\phi(x, y)$ where x ranges over G, which is preserved by G (on the left): for all $\phi(x, m)$ instance of Δ and $g \in G$, $\phi(g \cdot x, m)$ is equivalent to an instance of Δ.

Definition 3.6 Let π be a definable filter concentrating on G. We define

$$\mathrm{Stab}_\Delta(\pi) := \{g \in G : {}^g\pi\big|_\Delta = \pi|_\Delta\}.$$

If Δ is the set of all formulas, we write $\mathrm{Stab}(\pi)$ for $\mathrm{Stab}_\Delta(\pi)$.

Note that since Δ is preserved by G, G acts on restrictions of filters to instances of Δ. It follows that $\mathrm{Stab}_\Delta(\pi)$ is a pro-definable subgroup of G.

Remark 3.7 Assume G is a pro-limit of $\varnothing$-definable groups. If $\phi(x, y)$ is any formula where x ranges over G, then $\phi'(x, y, t) = \phi(t \cdot x, y)$ where t also ranges over G is preserved by G. It follows that any finite set of formulas is contained in a finite set Δ preserved by G. For such a Δ, $\mathrm{Stab}_\Delta(\pi)$ is a relatively definable subgroup of G.

For infinite Δ, it follows that $\mathrm{Stab}_\Delta(\pi) = \bigcap_{\phi\in\Delta} \mathrm{Stab}_{\phi'}(\pi)$ is an intersection of relatively definable subgroups of G, i.e. a pro-limit of definable groups.

Remark 3.8 If π concentrates on $\mathrm{Stab}(\pi)$, then π is generic in $\mathrm{Stab}(\pi)$.

Lemma 3.9 *Assume G admits a (right) generic type, then there is a smallest pro-definable subgroup $G^0 \leq G$ of bounded index and G/G^0 is pro-finite. In fact, G^0 is the left stabilizer of any right generic of G (and the right stabilizer of any left generic).*

Proof Let p be a right generic type of G and $H \leq G$ be a pro-definable subgroup of bounded index. Then $H = \bigcap_i X_i$ is an intersection of relatively definable subsets and for each i, finitely many translates of X_i cover G. It follows that p concentrates on a coset of H and hence that $\mathrm{Stab}(p) \leq H$. Since p has boundedly many translates, $\mathrm{Stab}(p)$ is itself a pro-definable subgroup of bonded index and it is therefore the smallest one. In particular $G^0 = \mathrm{Stab}(p)$ does not depend on the choice of p.

Furthermore, for every formula ϕ preserved by G—which exist since G is a pro-limit of definable groups—G^0_ϕ has finite index in G and $G/G^0 = \varprojlim_\phi G/G^0_\phi$ is pro-finite.

Any generic type of G has a (unique) translate that concentrates on G^0. It is called a principal generic of G. If $G = G^0$, we say that G is connected.

If π and μ are definable filters, we also define $\mathrm{Stab}(\pi, \mu) := \{g \in G : {}^g\pi = \mu\}$. It might be empty, but when it is non-empty, it is a left torsor of $\mathrm{Stab}(\mu)$ and a right torsor of $\mathrm{Stab}(\pi)$. In particular, these two groups are then conjugate (by any element in $\mathrm{Stab}(\pi, \mu)$).

We will also consider the following more general variant of the stabilizer.

Definition 3.10 Let π be a definable filter. Let $\mathfrak{F}(\pi)$ be the semigroup under composition of $\mathbb{U}$-definable functions h such that $h_\star\pi = \pi$. This semigroup has a quotient consisting of the π-germs of elements of $\mathfrak{F}(\pi)$. The invertible germs form a group, denoted $\mathfrak{G}(p)$.

Note that given an $\varnothing$-definable family $(h_a)_a$ of functions defined on π, the set of a such that $(h_a)_\star\pi = \pi$ is a ∞-definable set. It follows that $\mathfrak{F}(\pi)$ and $\mathfrak{G}(\pi)$ are piecewise pro-definable. If π is a filter concentrating on a pro-definable group G, then the stabilizer $\mathrm{Stab}(\pi)$ embeds naturally into $\mathfrak{G}(p)$: c maps to the germ of left translation by c.

3.3 Symmetric generics

Let p be a definable type. We say that p is symmetric if for all definable type q and C such that p and q are defined over C, $a \models p|\mathrm{acl}(C)$ and $b \models q|\mathrm{acl}(Ca)$, then $a \models p|\mathrm{acl}(Cb)$. Proposition 2.10.(1) exactly states that stably dominated types are symmetric.

Lemma 3.11 *Assume G admits a symmetric right generic type, then left and right generics coincide, they are all symmetric and there is a unique left (respectively right) orbit of generics. In particular, there are only boundedly many generics in G.*

Proof Let p be a symmetric right generic. Then p^{-1} is a symmetric left generic. Let q be any left generic and C be a model over which p and q are defined, $a \models p|C$ and $b \models q|\mathrm{acl}(Ca)$. By definition $ba \models q^a|\mathrm{acl}(Ca)$. Since p is symmetric, $a \models p|\mathrm{acl}(Cb)$ and hence $ba \models {}^b p|\mathrm{acl}(Cb)$. So q^a and ${}^b p$ agree on C but, since both are C-definable and C is a model, $q^a = {}^b p$. Since right and left translation commute, it follows that p is left generic. Moreover if q_1 and q_2 are both left generic, then, by the previous argument, we find b_1 and b_2 and a such that $q_1^a = {}^{b_1} p$ and $q_2^a = {}^{b_2} p$. It follows that ${}^{b_2 b_1^{-1}} q_1 = q_2$ and all left generics are left translates of one another. Since p is a symmetric left generic, all left generics are symmetric.

By a similar argument starting with any (symmetric) left generic, we get that all left generics are right generic, that all right generics are right translates of each other and that all right generics are symmetric (and hence left generic). We have proved that left and right genericity coincide and that they are all left and right translates of each other. □

Lemma 3.12 *Assume G admits a symmetric generic type p. Then the following are equivalent:*

(1) *p is the unique generic type of G.*
(2) *For all $g \in G$, ${}^g p = p$.*
(3) *G is connected.*

Proof Assume (1). By definition of genericity, for any $g \in G$, ${}^g p$ is right generic. Hence by uniqueness, ${}^g p = p$. Conversely given (2), let q be generic. Then by Lemma 3.11, $q = {}^g p$ for some g. By (2), $p = q$. The equivalence with (3) is immediate, given that G^0 is the left stabilizer of any generic. □

In particular, if G admits a symmetric generic type, then there is a unique principal generic.

Lemma 3.13 *Assume G is pro-C-definable and admits a symmetric generic type. Then G^0 is pro-C_0-definable for some $C_0 \subseteq C$ of size at most $|\mathcal{L}| + |x|$ where x ranges over G.*

Proof Let p be the principal generic of G. Since p is fixed by every automorphism fixing G globally, it is C-definable. Moreover, it is determined by the function $\phi(x, y) \mapsto (\mathrm{d}_p x)\phi(x, y)$ and there are at most $|\mathcal{L}| + |x|$ formulas ϕ to consider. So p is C_0-definable for some $C_0 \subseteq C$ of size at most $|\mathcal{L}| + |x|$ and hence so is $G^0 = \mathrm{Stab}(p)$. □

3.4 Group chunks

This idea, in the context of algebraic groups, is due to Weil. Let G be a definable group, and π a left G-invariant definable filter concentrating on G. The $\pi \otimes \pi$-germ of multiplication is called the *group chunk* corresponding to (G, π).

Definition 3.14 An *abstract group chunk* over C is a C-definable filter π and a pro-C-definable map F defined on $\pi^{\otimes 2}$ (or $\pi \otimes \pi$-germ of such a function) such that:

(1) For all $a \models \pi|C$, $(F_a)_\star\pi = \pi$, where $F_a(x) = F(a, x)$.
(2) For all $a, b \models \pi^{\otimes 2}|C$, $a \in \mathrm{dcl}(C, b, F(a, b))$ and $b \in \mathrm{dcl}(C, a, F(a, b))$.
(3) $\pi^{\otimes 3}(x, y, z) \vdash F(x, F(y, z)) = F(F(x, y), z)$.

Proposition 3.15 *Let (π, F) be an abstract group chunk over C. Then (π, F) is pro-C-definably isomorphic to the group chunk of a pro-C-definable group G: there exists an injective pro-C-definable map $f : \pi \to G$ such that $\pi^{\otimes 2}(x, y) \vdash f(F(x, y)) = f(x) \cdot f(y)$ and $G = \mathrm{Stab}(f_\star\pi)$.*

Proof Let $P = \pi|C$. For any $a \in P$, by (2), the π-germ $f(a)$ of $F(a, x)$ is invertible. It is therefore an element of the group $\mathfrak{G}(\pi)$ (cf. Definition 3.10). By (2) also, the map f is injective. Let G be the subgroup of $\mathfrak{G}(\pi)$ generated by the elements $(f(a))_{a\in P}$ and their inverses. Let I enumerate all finite sets of the variables of π. For all $i \in I$, and $g \in \mathfrak{G}(\pi)$, let g_i denote the π-germ of $\rho_i \circ h$, where h is any map whose π-germ is g and ρ_i is the projection to the variables in i.

To show that G can be identified with a pro-definable group, it suffices to check that:

- Any element of G is a product $f(a) \cdot f(b)^{-1}$ for some $a, b \in P$.
- The set $\{(a_1, \ldots, a_6) \in P^6 : (f(a_1) \cdot f(a_2)^{-1} \cdot f(a_3) \cdot f(a_4)^{-1})_i = (f(a_5) \cdot f(a_6)^{-1})_i\}$ is relatively definable.

The second point is immediate from the definability of π. As for the first point, it suffices to show that a product $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1}$ has the required form. Note that, by (1), when $a \in P$ and $b \models \pi|Da$ for any $D \supseteq C$, $F(a, b) \models \pi|Da$ and there exists $c \models \pi|Da$ such that $F(a, c) = b$ and, by (3), $f(a)^{-1} \cdot f(b) = f(c)$. Thus given any $a, b, c, d \in P$, let $e_0 \models \pi|Cabcd$; then $f(d)^{-1} \cdot f(e_0) = f(e_1)$ for some $e_1 \models \pi|Cabcd$; so $f(c) \cdot f(e_1) = f(e_2)$ for some $e_2 \models \pi|Cabd$; continuing this way, we obtain that $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1} \cdot f(e_0) = f(e_4)$, for some $e_4 \models \pi|Cabd$. Therefore, $f(a) \cdot f(b)^{-1} \cdot f(c) \cdot f(d)^{-1} = f(e_4) \cdot f(e_0)^{-1}$ as required.

Finally, if $a \in P$ and $b \models \pi|Da$ for some $D \supseteq C$, $f(a) \cdot f(b) = f(F(a, b))$ where $F(a, b) \models \pi|Da$ so ${}^{f(a)}f_\star\pi = f_\star\pi$ and we do have $G = \mathrm{Stab}(f_\star\pi)$. This concludes the construction of G and the proof of its various properties. □

The uniqueness of G in Proposition 3.15 is guaranteed by the following:

Proposition 3.16 *Let G_1 and G_2 be pro-C-definable groups, π be a left G_1-invariant C-definable filter concentrating on G_1 and $f : \pi \to G_2$ be a pro-C-definable map such that $\pi^{\otimes 2}(x, y) \vdash f(x \cdot y) = f(x) \cdot f(y)$. Then there exists a unique pro-C-definable homomorphism $g : G_1 \to G_2$ such that $\pi(x) \vdash f(x) = g(x)$.*

Proof Uniqueness of g is clear, since $\pi|C$ generates G: if $g \in G$, $a \models \pi|Cg$, then $g \cdot a \models \pi|Cg$ and $g = (g \cdot a) \cdot a^{-1}$.

Existence is also clear, provided we show that, for all $a, b, c, d \models \pi|C$ such that $a \cdot b^{-1} = c \cdot d^{-1}$, $f(a) \cdot f(b)^{-1} = f(c) \cdot f(d)^{-1}$. It suffices to show that $f(a) \cdot f(b)^{-1} \cdot f(e) = f(c) \cdot f(d)^{-1} \cdot f(e)$ for any $e \models \pi|Cabcd$. But it follows from our hypothesis on f that $f(b)^{-1} \cdot f(e) = f(b^{-1} \cdot e)$ and $f(d)^{-1} \cdot f(e) = f(d^{-1} \cdot e)$. Moreover, $b^{-1} \cdot e \models p|Cabcd$, so $f(a) \cdot f(b^{-1} \cdot e) = f(a \cdot b^{-1} \cdot e)$. Similarly, $f(c) \cdot f(d^{-1} \cdot e) = f(c \cdot d^{-1} \cdot e)$. Since $a \cdot b^{-1} \cdot e = c \cdot d^{-1} \cdot e$, the equality holds.

Since $g(a) = b$ if and only if $(\mathrm{d}_p x) f(a \cdot x) = b \cdot f(x)$, g is pro-C-definable. □

Analogous statements for group actions exist.

3.5 Products of types in groups

Let G be a piecewise pro-definable group, let $p_1, \ldots, p_n$ be definable types of elements of G, and let w be an element of the free group F on generators $\{1, \ldots, n\}$. The words of the free group will be denoted by expressions such as $1\bar{2}3$; $\bar{2}$ is the inverse of the generator 2.

We construct a definable type $p_w = w_\star(p_1, \ldots, p_n)$. Let $a_i \models p_i|\mathrm{acl}(Ca_1 \ldots a_{i-1})$. Let $a_w = w(a_1, \ldots, a_n)$ be the image of w under the homomorphism $F \to G$ with $i \mapsto a_i$. Let $p_w|C = \mathrm{tp}(a_w/C)$.

If w is the product of the generators 1, 2, we write $p_1 \star p_2$ for p_w. We denote $p_{123\ldots n}$ as $\bigotimes_{i=1}^n p_i$. Note that if G is Abelian and p is symmetric, then the order of enumeration does not actually matter. If a single type p is given, rather than a sequence p_w will refer to the sequence $(p, p, \ldots, p)$. Let $p^{\star n}$ denote $p_{123\cdots n}$, $p^{\pm 2n} = p_{\bar{1}2\bar{3}4\cdots(2n)}$ and $p^{\pm 2n+1} = p_{1\bar{2}3\bar{4}\cdots\overline{(2n)}(2n+1)}$.

4 Stably dominated groups

In this section, T is assumed to be metastable.

Definition 4.1 A pro-definable group G is *stably dominated* if G has a stably dominated generic type.

Since stably dominated types are symmetric, it follows from Lemma 3.11 that all generics of G are both left and right generics and they are all stably dominated.

Remark 4.2 The class of stably dominated pro-definable groups is closed under Cartesian products and image under a definable group homomorphism. If G and H are stably dominated pro-definable groups and p (respectively q) is a stably dominated generic of G (respectively H), then $p \otimes q$ is a stably dominated generic of $G \times H$. If $f : G \to H$ is a pro-definable surjective group homomorphism, then $f_\star p$ is a stably dominated generic of H.

Lemma 4.3 *Let G be a pro-definable group and $N \leq G$ be a stably dominated pro-definable subgroup. Assume that there exists a stably dominated type concentrating on G/N whose orbit under G-translations is bounded. Then G is stably dominated.*

In particular, if $N \trianglelefteq G$ and N, G/N are stably dominated, then so is G.

Proof Let p be the generic type of N^0 and q be a stably dominated type of G/N, with bounded orbit under G-translations. Let C be such that G, N are defined over C and p, q are stably dominated over C.

Assume first that $N = N^0$. For all $n \in N$, ${}^n p = p$. Thus, if $c = dn$ for some $d \in G$ and $n \in N$, then ${}^c p = {}^{dn} p = {}^d p$. So the type ${}^c p$ depends only on the coset $s = cN$. We denote it p_s. Let us now define the type r as follows: a realization of r over $B \supseteq C$ is a realization of $p_s|Bs$ where $s \models q|B$. Note that, for any $c \in G$, a realization of ${}^c r|Bc$ is a realization of $p_s|Bcs$ where $s \models {}^c q|Bc$. It follows that r has at most as many G-translates as q.

Since p is stably dominated over C, for any $c \in s$, p_s is stably dominated over Cc. By descent (Proposition 2.11.(2)), *is, in fact, stably dominated over Cs. It follows, by transitivity (Proposition 2.10.(3)), that r is stably dominated over C. That concludes the proof in the case $N = N^0$.*

In general, the map $\pi : G/N^0 \to G/N$ has pro-finite fibers and hence for any $s \in G/N^0$, $s \in \mathrm{acl}(C\pi(s))$. It follows, by Proposition 2.10.(4), that any q_0 such that $\pi_\star q_0 = q$ is stably dominated over C. Fix such a q_0, then q_0 has bounded orbit under G-translations and, by the above, G is stably dominated. □

Lemma 4.4 *Let G be a stably dominated pro-definable group and $N \leq G$ be a pro-definable subgroup. Let $\eta : G \to G/N$ be the map $\eta(g) = g \cdot N$. Assume that there exists a pro-definable $Y \subset G$ such that $\eta|_Y$ is surjective, bounded to one. Then N is stably dominated.*

Proof Let p be the principal generic of G, C be such that G, N, Y, p are C-definable. Let Q be the set of types $\mathrm{tp}(h^{-1} \cdot g/C)$ where $g \models p|C$, $h \in Y$ and $h \cdot N = g \cdot N$. Because there are boundedly many choices for h, the set Q is bounded. Moreover, for all $g \models p|C$, $h^{-1} \cdot g \in \mathrm{acl}(Cg)$ and hence, by Proposition 2.10.(4), the types in Q are all stably dominated types concentrating on N.

Let us now show that Q is $N \cap G^0$-invariant. Pick $n \in N \cap G^0$, $g \models p|Cn$, $h \in Y$ such that $h \cdot N = g \cdot N$ and $q := \mathrm{tp}(h^{-1} \cdot g/C) \in Q$. We have that $h^{-1} \cdot g \cdot n \models q^n|Cn$, but since $g \cdot n \models p^n = p$ and $g \cdot n \cdot N = g \cdot N = h \cdot N$, $q^n|C = \mathrm{tp}(h^{-1} \cdot g \cdot n/C) \in Q$.

It follows that the size of the orbits of the types in Q is bounded by $|Q| \cdot |N/N \cap G^0|$. Recall that Q is bounded and that $N/N \cap G^0$ can be embedded in G/G^0 whose size is bounded. So all the types in Q are generic. □

Corollary 4.5 *Let G be an algebraic group, $N \leq G$ an algebraic subgroup. Let $H \leq G$ be definable in* ACVF *and stably dominated. Then $H \cap N$ is stably dominated.*

Proof By elimination of imaginaries in algebraically closed fields, the coset space $X := G/N$ can be identified with a subset of some Cartesian power of the field sort. Let η be the projection $G \to X$. Let $M_0 \models$ ACVF be such that G, N and H are defined over M_0. For all $h \in H$, $M = M_0(\eta(h))^{\mathrm{alg}} \models$ ACVF, so there exists $y \in H(M) \cap h \cdot N$. By compactness, we find a definable subset Z of the graph of $\eta|_H$ with finite projection to X. Let $Y \subseteq H$ be the projection of Z on H. Then, applying Lemma 4.4 to H, $H \cap N$ and Y, we get that $H \cap N$ is stably dominated. □

4.1 Domination via a group homomorphism

Proposition 4.6 *Let G be a stably dominated pro-definable group. There exists a pro-definable stable group $\mathfrak{g}$, and a pro-definable homomorphism $\theta : G \to \mathfrak{g}$, such that the generics of G are stably dominated via θ.*

If (FD) holds and G is definable then $\mathfrak{g}$ can be taken to be definable.

We say that such a G is stably dominated via θ.

Proof Let p be generic in G, C be such that G and p are C-definable. Let $\theta(a)$ enumerate $\mathrm{St}_C(a)$, and, for all $a, b \in G$, let $f_a(b) = \theta(a \cdot b)$. Fix $g \in G$. Since ${}^g p$ is stably dominated, by Proposition 2.12, the ${}^g p$-germ of f_a is strong and is in $\mathrm{St}_C(a) = \theta(a)$. It follows that for all $b \models {}^g p|Ca$, $f_a(b) = h_{\theta(a)}(b)$ for some C-definable map h. Since St_C is stably embedded, h factors through $\theta(b)$. Indeed, let $c = f_a(b) = h_{\theta a}(b)$. Since $c \in \mathrm{St}_C$, $\mathrm{tp}(\theta(a)c/C\theta(b)) \vdash \mathrm{tp}(\theta(a)c/Cb)$. Thus $c \in \mathrm{dcl}(\theta(a)\theta(b))$. We have proved that, for all $g_1, g_2 \in G$, there exists a C-definable map F_{g_1,g_2} such that, for all $a \models {}^{g_1} p|C$, $b \models {}^{g_2} p|Ca$, $\theta(ab) = F_{g_1,g_2}(\theta(a), \theta(b))$. By compactness, we may assume that $F = F_{g_1,g_2}$ does not depend on g_1 or g_2.

Recall that for all $g \in G$, ${}^g p$ is stably dominated over C. It follows that, for all g_1, $g_2 \in G$, $a \models {}^{g_1} p|C$, $b \models {}^{g_2} p|Ca$ and $c = a \cdot b$, $\mathrm{tp}(\theta(b)\theta(c)/C\theta(a)) \vdash \mathrm{tp}(bc/Ca)$ and hence $\theta(b) \in \mathrm{dcl}(\theta(a), \theta(c)) = \mathrm{dcl}(\theta(a), F(\theta(a), \theta(b)))$. Symmetrically, we have $\theta(a) \in \mathrm{dcl}(\theta(b), F(\theta(a), \theta(b)))$.

Thus, F is a group chunk on $\pi := \bigcap_{g \in G} \theta_\star {}^g p$ and, by Proposition 3.15, F is the restriction of the multiplication map of some pro-definable group $\mathfrak{g}$ in St_C. By Proposition 3.16, θ extends to a pro-definable group homomorphism from G to $\mathfrak{g}$.

Let us now assume (FD). By Proposition 3.4, $\mathfrak{g} := \varprojlim_{i \in I} \mathfrak{g}_i$ where the $\mathfrak{g}_i$ form a (filtered) projective system of definable groups. Let $\theta_i : G \to \mathfrak{g}_i$ be the canonical projection. Since there are no descending chains of definable subgroups of $\mathfrak{g}_i$, every ∞-definable subgroup is definable, in particular $\theta_i(G)$ is definable. Then $\mathfrak{g} = \varprojlim_{i \in I} \theta_i(G)$ and we may assume that every θ_i is surjective.

Let a realize the principal generic of G over C. By Lemma 2.19, $\mathrm{St}_C(a)$ is acl-finitely generated over C; say $\mathrm{St}_C(a) \subseteq \mathrm{acl}(Cd)$ for some tuple $d \in \mathrm{St}_C$. Since $\dim_{\mathrm{st}}(d/C(\theta_i(a))$ decreases as i increases, it stabilizes at some i_0 (Note that, if C is sufficiently large, i_0 does not depend on C). But all $\theta_i(a)$ are in $\mathrm{acl}(Cd)$, and hence $\theta_i(a) \in \mathrm{acl}(C(\theta_{i_0}(a)))$ for all $i \geq i_0$. It follows $\mathrm{tp}(a/C)$ is dominated via θ_{i_0} and so are all of its left translates.

Remark 4.7 If (FD) holds, we have also proved the following. If $\theta : H \to \mathfrak{g} = \varprojlim_i \mathfrak{g}_i$ is a pro-definable surjective group homomorphism and that $\dim_{\mathrm{st}}(\mathfrak{g}) = n < \infty$, then there exists i_0 such that $\dim_{\mathrm{st}}(\mathfrak{g}_{i_0}) = n$. Moreover, since there are no descending chains of definable subgroups of $\mathfrak{g}_{i_0}$, $\theta_{i_0}(G)$ is definable. So we can actually find a pro-definable surjective group homomorphism onto a *definable* stable group of Morley rank n.

In such a situation, we say that H has stable homomorphic image of Morley rank n.

The homomorphism θ of Proposition 4.6 is not uniquely determined by G; even if G is stable, there may be a non-trivial homomorphism of this kind. We do however have a maximal one.

Remark 4.8 There exists a pro-C-definable stable group $\mathfrak{g}$, and a pro-C-definable homomorphism $\theta : G \to \mathfrak{g}$, maximal in the sense that any pro-C-definable homomorphism $\theta' : G \to \mathfrak{g}'$ into a pro-C-definable stable group factors through θ.

The kernel of this maximal θ is uniquely determined. If G is stably dominated it will be stably dominated via this maximal homomorphism.

Lemma 4.9 *Let G be a pro-C-definable group stably dominated via some pro-C-definable surjective group homomorphism $\theta : G \to \mathfrak{g}$. Then* $\mathrm{tp}(a/C)$ *is generic in G (i.e. $a \models r|\mathrm{acl}(C)$ for some generic r of G) if and only if* $\mathrm{tp}(\theta(a)/C)$ *is generic in* $\mathfrak{g}$.

Proof Assume $\mathrm{tp}(\theta(a)/C)$ is generic in $\mathfrak{g}$, then so is $\mathrm{tp}(\theta(a)/\mathrm{acl}(C))$. So we may assume $C = \mathrm{acl}(C)$. Let p be a generic type of G, stably dominated via $\theta : G \to \mathfrak{g}$. Then $q = {}^a p$ is a generic in G and hence stably dominated via θ. Let $b \models p|Ca$. Note that $a \cdot b \models q|Ca$. Since $\theta(a)$ is generic in $\mathfrak{g}$, we have $\theta(a \cdot b) = \theta(a) \cdot \theta(b) \downarrow_C \theta(b)$. Since $\theta|C$ is stably dominated via θ, $b \models p|C\theta(a \cdot b)$ and hence $\mathrm{St}_C(b) \downarrow_C \theta(a \cdot b)$. Since $q|C$ is stably dominated via θ, $a \cdot b \models q|Cb$. So $a \models q^{b^{-1}}|Cb$, and in particular $a \models q^{b^{-1}}|C$, where $q^{b^{-1}}$ is generic in G.

The converse is obvious since for any generic r of G, ${}^{\theta(x)}\theta_\star r = \theta_\star({}^x r)$. □

Corollary 4.10 *Let G be a pro-definable group stably dominated via some pro-definable group homomorphism $\theta : G \to \mathfrak{g}$. Let $H \leq G$ be a relatively definable. If $\theta(H) = \mathfrak{g}$, then H has finite index in G.*

Proof Work over some C over which G and H are definable. Let $b \in H$ be such that $\theta(b)$ is generic in $\mathfrak{g}$ over C. By Lemma 4.9, $\mathrm{tp}(b/C)$ is generic in G. Thus a generic of G lies in H, so H has finite index in G.

Corollary 4.11 *(FD_ω) Let G be a C-definable group stably dominated via some C-definable group homomorphism $\theta : G \to \mathfrak{g}$. Then G^0 is definable, i.e. $[G : G^0]$ is finite.*

Proof We may assume θ surjective. Note that $\theta^{-1}(\mathfrak{g}^0)$ is a definable subgroup of G with finite index, so we may assume that $\mathfrak{g}$ is connected. Since $[G : G^0]$ is bounded, we can find C such that every coset of G^0 in G contains a point from C. We can also assume that C is a metastability basis as in the definition of (FD_ω). Let a_0 be such that $\theta(a_0)$ is generic in $\mathfrak{g}$ over C. By (FD_ω), there exists $a \in G$ such that $\theta(a) = \theta(a_0)$ and $\mathrm{tp}(a/C\theta(a))$ is isolated. Since $\theta(a)$ is generic in $\mathfrak{g}$ over C, by Lemma 4.9, $\mathrm{tp}(a/C)$ is generic in G. Let p be the principal generic of G. Then there exists $c \in G(C)$ such that $b := c \cdot a \models p|C$. It follows that $\mathrm{tp}(c \cdot a/C\theta(a)) = \mathrm{tp}(b/C\theta(b))$ is isolated.

On the other hand, let $G^0 = \bigcap_i G_i$ where $\{(G_i)_{i \in I}\}$ is a bounded family of definable subgroups of finite index, closed under intersections. Note that by Lemma 4.9, $\{x \in G_i : i \in I\} \cup \{\theta(x) = \theta(b)\}$ generates $\mathrm{tp}(b/C\theta(b))$. This type being isolated, there exists $i_0 \in I$ such that it is generated by $\theta(x) = \theta(b)$ and $x \in G_{i_0}$. Since $\mathfrak{g}$ is connected, any generic of G_i has a realization d such that $\theta(d) = \theta(b)$ and hence is equal to p. So $G_i = G^0$, and G^0 is definable. □

Here is a characterization of generics that does not explicitly mention θ and $\mathfrak{g}$.

Corollary 4.12 *Let G be a stably dominated pro-C-definable group,* $\dim_{st}(G) < \infty$. *Then the generic types of G over C are precisely the types* $tp(c/C)$ *such that for h generic in G over Cc,* $\dim_{st}(St_C(h \cdot c)/Ch) = \dim_{st}(G)$.

Proof Let θ and $\mathfrak{g}$ be as in Proposition 4.6. Since $h \cdot c$ is generic in G over C, $acl(St_C(h \cdot c)) = acl(C\theta(h \cdot c)) = acl(C\theta(h)\theta(c))$. So $\dim_{st}(G) = \dim_{st}(St_C(h \cdot c)/Ch) = \dim_{st}(\theta(h) \cdot \theta(c)/Ch) = \dim_{st}(\theta(c)/Ch) \leq \dim_{st}(\theta(c)/C)$. It follows that $\theta(c)$ is generic in $\mathfrak{g}$ over C and, by Lemma 4.9, c is generic in G over C.

Conversely, if $c \models p|acl(C)$ for some stably dominated generic p, by symmetry, $c \models p|acl(Ch)$ and hence $h \cdot c$ is generic in G over Ch. Thus $\theta(h \cdot c)$ is generic in $\mathfrak{g}$ and $\dim_{st}(St_C(h \cdot c)/Ch) \geq \dim_{st}(\mathfrak{g}) = \dim_{st}(G)$. □

Note that the proof uses, rather than proves, the existence of a generic.

Remark 4.13 If G is piecewise-definable, in a superstable theory, and p is a type of maximal rank in G, then p is a translate of a generic of $Stab(p)$, a definable subgroup of G.

For stably dominated types, the analog need not hold. In ACVF, consider $G = \mathbb{G}_a(\mathcal{O})^2$, fix γ a positive element of the value group and let p be the stably dominated type of elements such that x is generic in $\mathcal{O}$ and y is generic (over x) in the closed ball of radius γ centered at x^2. Then $\dim_{st}(G) = 2$ and the stable part of p has dimension 2. But, although p has maximal rank, it is not a generic type. However, we do have that $p^{\star 2}$ is a stably dominated generic of G.

Let us now prove a certain converse to Proposition 4.6:

Proposition 4.14 *Let G be a pro-definable group. Assume G admits a surjective pro-definable homomorphism $\theta : G \to \mathfrak{g}$, with $\mathfrak{g}$ stable and with* $\dim_{st}(\mathfrak{g}) = \dim_{st}(G) = n < \infty$. *Then, there exists a pro-definable subgroup T of G with:*

(1) *T connected stably dominated;*
(2) *T normal and G/T almost internal to Γ;*
(3) $\theta(T) = \mathfrak{g}^0$*;*

and T is uniquely determined by (1,2) or by (1,3). Moreover, T is stably dominated via $\theta|_T$.

Proof Let Z be the collection of types q of elements of G, definable over some set of parameters, with $\theta_\star q$ generic in $\mathfrak{g}$, and q stably dominated.

Claim 4.14.1 *$Z \neq \varnothing$.*

Proof Let $C = acl(C)$ be a metastability basis over which G and θ are pro-definable. Let $a \in G$ be such that $\theta(a)$ generic in $\mathfrak{g}$ over $acl(C)$. Let $\gamma := \Gamma_C(a)$. Any relatively $acl(C\gamma)$-definable subset of $\mathfrak{g}$ is already relatively C-definable, so $\theta(a)$ is generic in $\mathfrak{g}$ over $acl(C\gamma)$. By metastability, $tp(a/acl(C\gamma))$ is stably dominated and extends to a unique $acl(C\gamma)$-definable type which is in Z. □

Claim 4.14.2 *Any two elements of Z are left and right translates of each other.*

Proof Pick any $q, r \in Z$; say both are B-definable for some $B = \mathrm{acl}(B)$ over which G and θ are also defined. Let $a \models q|B, b \models r|Ba$ and let $c = a \cdot b^{-1}$. Then $\theta(a)$ and $\theta(b)$ are B-independent generic elements of $\mathfrak{g}$. Hence so are $\theta(a)$ and $\theta(a) \cdot \theta(b)^{-1} = \theta(c)$. Since $\dim_{\mathrm{st}}(\theta(a)/B) = n = \dim_{\mathrm{st}}(G) \geq \dim_{\mathrm{st}}(\mathrm{St}_B(a)/B) \geq \dim_{\mathrm{st}}(\theta(a)/B)$, we have $\mathrm{St}_B(a) \subseteq \mathrm{acl}(B\theta(a))$. Similarly $\mathrm{St}_B(c) \subseteq \mathrm{acl}(B\theta(c))$. Thus $\mathrm{St}_B(c) \downarrow_B \mathrm{St}_B(a)$. By stable domination, $a \models q|Bc$. Similarly $b \models r|Bc$. It follows that $q = {}^c r$. By the symmetric argument, $q = r^d$ where $d = b^{-1}a$. □

It follows that the pro-definable subgroup $T := \mathrm{Stab}(q) = \mathrm{Stab}(q^c)$ does not depend on the choice of $q \in Z$. Also, since one can easily check that Z is closed under left translation, T is normal. Moreover if a and b are independent realizations of q over B and $c = a \cdot b^{-1}$, then the proof of Claim 4.14.2 shows that ${}^c q = q$ and hence $r := q^{b^{-1}}$ is the unique generic type of T. Thus T is connected stably dominated. Since $\theta(T)$ contains a generic of $\mathfrak{g}$ and is connected, we have $\theta(T) = \mathfrak{g}^0$. Also, as in the above claim, we have that $\mathrm{St}_C(c) \subseteq \mathrm{acl}(\theta(c))$. It follows that $\mathrm{tp}(c/C)$, and hence r, is stably dominated via $\theta|_T$.

Since $T = \mathrm{Stab}(q)$, it is the intersection of a family of relatively definable subgroups $(T_i)_{i \in I}$ of G.

Claim 4.14.3 *For all $i \in I$, G/T_i is almost Γ-internal.*

Proof Pick any $a \in G$ and let C be a metastability basis over which G, T and θ are defined. If $c \models r|Ca$, then $a \cdot c \in a \cdot T$ and $\theta(a \cdot c) = \theta(a) \cdot \theta(c)$ is generic in $\mathfrak{g}$ over $\mathrm{acl}(C)$. So we may assume that $\theta(a)$ is generic in $\mathfrak{g}$ over $\mathrm{acl}(C)$ without changing the coset $a \cdot T$. By the proof of Claim 4.14.1, there exists a generic q, $\mathrm{acl}(\Gamma_C(a))$-definable, such that $a \models q|\Gamma_C(a)$. By the proof of Claim 4.14.2, q concentrates on $a \cdot T$. It follows that for all $i \in I$, $a \cdot T_i \in \mathrm{acl}(\Gamma_C(a))$ and hence G/T_i is almost Γ-internal. □

Let us now prove the uniqueness of T. Let T' be a connected stably dominated pro-definable subgroup of G. Then $T'/(T \cap T') \simeq T \cdot T'/T \leq G/T$ is both almost Γ-internal and connected stably dominated, hence trivial. So $T' \leq T$. If G/T' is almost Γ-internal, the symmetric proof shows that $T \leq T'$. So T is indeed characterized by (1,2). Now assume that $\theta(T') = \mathfrak{g}^0$ and let $T' = \bigcap_j T'_j$ where the T'_j are relatively definable. By Corollary 4.10, $[T : T'_j \cap T]$ is finite and therefore $T \leq T'_j$. It follows that $T \leq T'$ and T is also characterized by (1,3). □

4.2 Stably dominated subgroups of maximal rank

While not all stably dominated subgroups are definable, we will show that subgroups whose residual rank is $\dim_{\mathrm{st}}(G)$ are.

Lemma 4.15 *Let T be an ∞-C-definable connected and stably dominated normal subgroup of some definable group G with G/T almost internal to Γ.*

(FD) There exists a C-definable subgroup stably dominated S of G with $S^0 = T$.

(FD$_\omega$) T itself is definable.

Proof Let us assume (FD). Let p be the principal generic of T, then $T = \mathrm{Stab}(p)$ is an intersection of C-definable subgroups S_i of G; and G/S_i is almost Γ-internal. By Proposition 4.6, there exists a definable homomorphism $\theta : T \to \mathfrak{g}$, with $\mathfrak{g}$ stable, and T stably dominated via θ. By compactness, θ extends to a definable homomorphism on some S_i; replacing G by this S_i, we may assume $\theta : G \to \mathfrak{g}$.

By Lemma 2.25, some quotient of G/S_i by a finite normal subgroup is Γ-internal. Define $\dim_{\mathrm{o}}(G/S_i)$ to be the o-minimal dimension of any such quotient. By (FD), for any $a \in G$ and any B, $\dim_{\mathrm{o}}(\Gamma_B(a)/B)$ is finite, so $\dim_{\mathrm{o}}(G/S_i)$ is bounded independently of i. Thus for some i, for all $j \geq i$, $\dim_{\mathrm{o}}(G/S_j) = \dim_{\mathrm{o}}(G/S_i)$. It follows that the natural map $G/S_j \to G/S_i$ has zero-dimensional fibers, hence it has finite kernel. This shows that for some i, for all $j \geq i$, S_i/S_j is finite. Thus $[S_i : T]$ is bounded, S_i is stably dominated, $S_i^0 = T$, and (1) holds.

If we assume (FD$_\omega$), definability of T follows from Corollary 4.11. □

Corollary 4.16 *Let G be a definable group, H be a connected ∞-C-definable stably dominated subgroup of G, with a stable homomorphic image of Morley rank $n :=$* $\dim_{\mathrm{st}}(G)$.

(FD) There exists a C-definable stably dominated subgroup S of G with $S^0 = H$.
(FD$_\omega$) H itself is definable.

Note that the group S is stably dominated and has a stable homomorphic image of Morley rank $n := \dim_{\mathrm{st}}(G)$.

Proof Assume (FD). Let $\theta : H \to \mathfrak{g}$ be a definable surjective homomorphism with $\mathfrak{g}$ definable stable and $\dim_{\mathrm{st}}(\mathfrak{g}) = n$. Since H is the stabilizer of its principal generic, $H = \bigcap_i H_i$ for some definable subgroups H_i of G. By compactness, θ extends to a definable surjective group homomorphism $H_{i_0} \to \mathfrak{g}$. Note that $n = \dim_{\mathrm{st}}(\mathfrak{g}) \leq \dim_{\mathrm{st}}(H_{i_0}) \leq \dim_{\mathrm{st}}(G) = n$ so we may assume that θ extends to G. Note also that $\theta(H) = \mathfrak{g} = \mathrm{Stab}(\theta_\star p) = \mathfrak{g}^0$. By the uniqueness in Proposition 4.14, H is normal in G and G/H is almost Γ-internal. We can now conclude using Lemma 4.15. □

5 Abelian groups

We will use the notation of Sects. 3.2 and 3.5.

Lemma 5.1 *Let A be a piecewise pro-definable Abelian group and p a symmetric definable type concentrating on A. Assume that A has p-weight strictly smaller than $2n$. Then there exists a pro-definable connected $H \leq A$ with generic $p^{\pm 2n}$ such that p concentrates on a coset of H.*

Proof Let $(a_1, a_2, \ldots, a_{2n}) \models p^{\otimes 2n}|C$, where C is such that A is pro-C-definable and p is C-definable. Let $b = a_1^{-1} \cdot a_2 \cdot \ldots \cdot a_{2n} \models p^{\pm 2n}|C$. By the weight assumption, $a_i \models p|Cb$ for some i. Say i is odd. Since p is symmetric and A is Abelian, $\mathrm{tp}(a_1, a_2, \ldots, a_{2n}/b)$ is $\mathrm{Sym}(n)$-invariant (for the action by permutation on the pairs

$a_{2i-1}, a_{2i})$, so $a_1 \models p|Cb$ and hence $b \models p^{\pm 2n}|Ca_1$. Since $a_1 \cdot b \models {}^{a_1}p^{\pm 2n}|Ca_1$ and $a_1 \cdot b = a_2 \cdot \ldots a_{2n} \models p^{\pm 2n-1}|Ca_1$, it follows that $a_1 \in X := \mathrm{Stab}(p^{\pm 2n}, p^{\pm 2n-1})$. If i is even, then $a_{2n} \models p|Cb^{-1}$ and hence $a_{2n} \in X$.

In both cases, p concentrates on X, a coset of $H := \mathrm{Stab}(p^{\pm 2n})$. It follows that $p^{\pm 2}$ and therefore $p^{\pm 2n}$, concentrate on $H = \mathrm{Stab}(p^{\pm 2n})$. So $p^{\pm 2n}$ is the generic of H. □

Remark 5.2 If p is stably dominated, so is $p^{\pm 2n}$ and H is connected stably dominated.

For the rest of this section, we assume T metastable.

Proposition 5.3 *Let A be a pro-definable Abelian group of bounded weight. Let $(B_i)_{i\in I}$ be connected stably dominated pro-definable subgroups of A. Then there exists a stably dominated connected pro-definable subgroup B containing all the B_i.*

Proof Let C be a metastability basis such that A and all the B_i are defined over C. Let $\mathfrak{F}$ be the collection of all C-definable functions on A into Γ, seen as a pro-definable function. Then, for any $c \in A$: $\mathrm{tp}(c/C, \mathfrak{F}(c))$ is stably dominated.

Let p_i be the principal generic of B_i. Consider the partial type:

$$q_0 = \{(\mathrm{d}_{p_i} y) f(x) = f(y \cdot x) : i \in I, f \in \mathfrak{F}\}$$

Claim 5.3.1 *q_0 is consistent.*

Proof It suffices to show that any finite number of formulas, concerning the types $(p_i)_{i\in I_0}$ for some finite $I_0 \subseteq I$, can be satisfied. Let $B = \sum_{i\in I_0} B_i$. Note that $p := \bigotimes_{i\in I_0} p_i$ is the unique stably dominated generic of B.

If $(a, c) \models p_i \otimes p|C$, then by genericity of p, $a \cdot c \models p|Ca$. By symmetry, $a \models p_i|Cc$. Now, for any $f \in \mathfrak{F}$, as p is stably dominated, and thus orthogonal to Γ, there exists γ_f such that for any $c' \models p|C$, $f(c') = \gamma_f$. In particular, $f(c) = \gamma_f = f(a\cdot c)$. Thus $\models (\mathrm{d}_{p_i} y) f(c) = f(y \cdot c)$. Since this is true for each $i \in I_0$ and $f \in \mathfrak{F}$, q_0 is consistent. □

Let $c \models q_0$, $C' = C \cup \mathfrak{F}(c)$ and $C'' = \mathrm{acl}(C')$. Recall that $\mathrm{tp}(c/C')$ is stably dominated. Let p be the unique C''-definable extension of $\mathrm{tp}(c/C'')$.

Claim 5.3.2 *For all $a \models p_i$, ${}^a p|C' = p|C'$.*

Proof Let $a \models p_i|\mathrm{acl}(Cc)$ and $X \subseteq A$ be some C-definable subset. Let $f_X \in \mathfrak{F}$ be the function sending x to $0 \in \Gamma$ if $x \in X$ and x to $\infty \in \Gamma$ if $x \notin X$. Since $c \models q_0$, $f_X(a \cdot c) = f_X(c)$. It follows that $\mathrm{tp}(a \cdot c/C) = \mathrm{tp}(c/C)$ and thus $\mathrm{tp}(a \cdot c, \mathfrak{F}(a \cdot c)/C) = \mathrm{tp}(c, \mathfrak{F}(c)/C)$. But, for all $f \in \mathfrak{F}$, $f(a \cdot c) = f(c)$, so $\mathrm{tp}(a \cdot c, \mathfrak{F}(c)/C) = \mathrm{tp}(c, \mathfrak{F}(c)/C)$. Since, by symmetry, $c \models p|\mathrm{acl}(C'a)$, this shows that $p|C' = \mathrm{tp}(c/C') = \mathrm{tp}(a \cdot c/C') = {}^a p|C'$. □

Note also that since p and p_i are invariant over C'', so is ${}^a p$. It is therefore one of the boundedly many C''-definable extensions of $\mathrm{tp}(c/C')$.

Moreover, by Lemma 5.1, there exist a stably dominated pro-definable group B with generic $p^{\pm 2n}$, for some n, such that p concentrates on a coset of B. Since the orbit of $p^{\pm 2n}$ under left translation by $B_i B$ is bounded, it follows that $B_i/(B \cap B_i) \simeq (B_i B)/B$ is bounded and, as B_i is connected, $B_i \leq B$.

In the previous proof, the subgroup B we constructed is, *a priori*, pro-C''-definable, but we can assume it is pro-C'-definable:

Lemma 5.4 *Let A be a pro-limit of C-definable Abelian groups and $B \leq A$ be a stably dominated connected pro-*$\mathrm{acl}(C)$*-definable group. Then, there exists a stably dominated pro-C-definable $B' \leq A$ containing B.*

Proof Let p be the principal generic of B and $(p_i)_{i\in I}$ be its conjugates over C. For all finite sets Δ of formulas (with parameters in C) preserved by left translation by A, the set of codes for the defining schemes of the types $p_i|_\Delta$, for $i \in I$, is equal to the set of C-conjugates of $p|_\Delta$. It is therefore finite. Let $I_\Delta \subseteq I$ be a finite set such that for any $i \in I$, there exists $j \in I_\Delta$ with $p_i|_\Delta = p_j\big|_\Delta$. Let $q_\Delta = \bigotimes_{i\in I_\Delta} p_i$. Note that, since A is a pro-limit of definable groups, any formula is contained in such a Δ.

Claim 5.4.1 *For any $i_0 \in I$, p_{i_0} concentrates on* $\mathrm{Stab}_\Delta(q_\Delta)$.

Proof Let $c \models p_{i_0}|M$ for some model M containing C and $(a_i)_i \models (\otimes_{i\in I_\Delta} p_i)|Mc$. Also, pick $i_1 \in I_\Delta$ such that $p_{i_0}\big|_\Delta = p_{i_1}\big|_\Delta$. For every $\phi(x, y) \in \Delta$ and $m \in M$, we have that $\phi(x, m) \in q_\Delta$ if and only if $\models \phi(\sum_i a_i, m)$, which is equivalent to $\phi(\sum_{i\neq i_1} a_i + x, m) \in p_{i_1}$, by symmetry of stably dominated types. Recall that Δ is preserved by left translation by elements of A, so that last formula is also an instance of Δ and hence $\phi(\sum_{i\neq i_1} a_i + x, m) \in p_{i_1}$ if and only if $\phi(\sum_{i\neq i_1} a_i + x, m) \in p_{i_0}$. Since p_{i_0} concentrates of $\mathrm{Stab}_\Delta(p_{i_0})$, this is equivalent to $\phi(\sum_{i\neq i_1} a_i + c + x, m) \in p_{i_0}$ and hence to $\models \phi(c + \sum_i a_i, m)$, i.e. $\phi(x, m) \in {}^c q_\Delta$. □

It immediately follows that q_Δ concentrates on $\mathrm{Stab}_\Delta(q_\Delta)$ and that, for any $J \supseteq I_\Delta$, $(\bigotimes_{i\in J} p_i)\big|_\Delta = q_\Delta|_\Delta$. Let q be the unique type such that for all Δ, $q|_\Delta = q_\Delta|_\Delta$. For all $i \in I$, p_i concentrates on $\mathrm{Stab}_\Delta(q) = \mathrm{Stab}_\Delta(q_\Delta)$ and hence so does q_Δ.

Note that for all finite set Δ, $\mathrm{Stab}_\Delta(q)$ is relatively C-definable and thus $B' = \mathrm{Stab}(q)$ is pro-C-definable. There remains to show that q concentrates on B'. For all Δ, let $\Delta' \supseteq \Delta$ be a finite set of formulas preserved by left translation by A such that $\mathrm{Stab}_\Delta(q)$ is defined by an instance of Δ'. Since $q_{\Delta'}$ concentrates on $\mathrm{Stab}_{\Delta'}(q_{\Delta'}) \leq \mathrm{Stab}_\Delta(q_{\Delta'}) = \mathrm{Stab}_\Delta(q)$, so does q.

Lemma 5.5 *(FD) Let C be a metastability basis and A be a pro-limit of C-definable Abelian groups with* $\dim_{\mathrm{st}}(A) = n < \infty$. *Then A contains a stably dominated pro-$C\Gamma$-definable subgroup S with stable homomorphic image of dimension n.*

Proof Let $g \in A$ with $\dim_{\mathrm{st}}(\mathrm{St}_C(g)/C) = n$, $C' = \Gamma_C(g)$, $C'' = \mathrm{acl}(C')$ and $p = \mathrm{tp}(g/C'')$. So p is stably dominated. Moreover, for all $B \supseteq C''$ and $a \models p|B$, $\dim_{\mathrm{st}}(\mathrm{St}_B(a)/B) = n$.

Claim 5.5.1 *For all C, a, b,* $\dim_{\mathrm{st}}(\mathrm{St}_C(a, b)/C) \leq \dim_{\mathrm{st}}(\mathrm{St}_{Ca}(b)/Ca) + \dim_{\mathrm{st}}(\mathrm{St}_C(a)/C)$.

Proof We have that $\dim_{st}(\mathrm{St}_C(a,b)/C) = \dim_{st}(\mathrm{St}_C(a,b)/\mathrm{St}_C(a)) + \dim_{st}(\mathrm{St}_C(a)/C)$ and that $\dim_{st}(\mathrm{St}_C(a,b)/\mathrm{St}_C(a)) = \dim_{st}(\mathrm{St}_C(a,b)/Ca) \leq \dim_{st}(\mathrm{St}_{Ca}(b)/Ca)$.

Claim 5.5.2 *If p and q are stably dominated over C concentrating on A, $a \models p|C$, $b \models q|Ca$, $c = a \cdot b$ and $\dim_{st}(\mathrm{St}_C(a)/C) = \dim_{st}(\mathrm{St}_C(b/C)) = n$, then $\dim_{st}(\mathrm{St}_C(c)/C) = n$.*

Proof Since $\mathrm{tp}(a,b/C)$ is stably dominated, so is $\mathrm{tp}(c/C)$. Since $\mathrm{St}_C(a) \downarrow_C \mathrm{St}_C(b)$,

$$\dim_{st}(\mathrm{St}_C(a,b)/C) \geq \dim_{st}(\mathrm{St}_C(a)\mathrm{St}_C(b)/C) = 2n.$$

Moreover, $\dim_{st}(\mathrm{St}_C(a,b)/C) = \dim_{st}(\mathrm{St}_C(b,c)/C) \leq \dim_{st}(\mathrm{St}_{Cc}(b)/Cc) + \dim_{st}(\mathrm{St}_C(c)/C) = n + \dim_{st}(\mathrm{St}_C(c)/C)$. So $\dim_{st}(\mathrm{St}_C(c)/C) \geq n$. But $\dim_{st}(A) = n$, so $\dim_{st}(\mathrm{St}_C(c)/C) = n$.

By Lemma 5.1, A contains a pro-C''-definable stably dominated group R, with generic type $p^{\pm 2m}$ for some m. By Claim 5.5.2, for any $a \models p^{\pm 2m}|C''$, $\dim_{st}(\mathrm{St}_{C''}(a)/C'') = n$. By Lemma 5.4, we find $S \geq R$ pro-C'-definable with stably dominated type q—we can check that, by construction and Claim 5.5.2, for all $a \models q|C'$, $\dim_{st}(\mathrm{St}_{C'}(a)/C') = n$. We can now conclude with Proposition 4.6.

5.1 Limit stably dominated groups

Definition 5.6 Let G be a pro-C-definable group and q be a (potentially infinitary) type of Γ over C. For all $t \models q$, let S_t be a pro-Ct-definable subgroup of G (uniformly in t). We call $(S_t)_{t\models q}$ a *limit stably dominated family* for G if:

(1) S_t is a connected stably dominated subgroup of G.
(2) If $W \leq G$ is connected and stably dominated, pro-C-definable, then $W \leq S_t$ for some $t \models q$.
(3) The family $(S_t)_{t\models q}$ is directed: any small set of realizations of q has an upper bound in the order defined by $t_1 \leq t_2$ if $S_{t_1} \leq S_{t_2}$.

The group $H = \bigcup_{t\models q} S_t$ is called the *limit stably dominated subgroup* of G. If $G = H$, we say that G is limit stably dominated.

We view H as the limit of a Γ-internal directed system of stably dominated groups. It is clearly independent of the particular choice of limit stably dominated family $(S_t)_{t\models q}$.

Lemma 5.7 *Let G be a pro-definable group and $H \leq G$ be its limit stably dominated subgroup, if it exists. Then H is pro-definable.*

Proof Since the S_t are pro-definable, there exists a directed family of relatively definable sets $S_t^i \subseteq G$ such that $S_t = \bigcap_i S_t^i$. By compactness, we have that $a \in H$ if and only if, for all i and all finite $q_0 \subseteq q$, there exists t such that $t \models q_0$ and $a \in S_t^i$. So H is an intersection of relatively definable subsets of G, i.e. a pro-definable subset. □

Lemma 5.8 *Let G be a definable group and $(S_t)_{t\models q}$ a limit stably dominated family for G. Assume that S_t is the connected component of some definable group R_t, then the order $t_1 \leq t_2$ is relatively definable.*

Proof If $S_{t_1} \leq R_{t_2}$, then $S_{t_1} \leq S_{t_2}$. It follows that $t_1 \leq t_2$ if and only if $S_{t_1} \leq R_{t_2}$, equivalently $(\mathrm{d}_{p_{t_1}} x)x \in R_{t_2}$ where, p_t denotes the principal generic of S_t. □

Two different behaviors are possible, according to whether or not the direct limit system has a maximal element. The latter is equivalent to the existence of a maximal connected stably dominated subgroup of G.

Proposition 5.9 *Let A be a pro-limit of definable Abelian groups. Assume A has bounded weight. Then, the limit stably dominated subgroup of A exists.*

Proof Let C_1 be an sufficiently saturated model over which A is defined and let $C \supseteq C_1$ be a metastability basis. Let $\mathfrak{F}$ be the family of all C-definable functions into Γ. Let $(A_i)_{i\in I}$ be the family of all connected stably dominated pro-C-definable subgroups of A. Let S be as in Proposition 5.3. Note that, by Lemma 5.4, we may assume $S = S_t$ to be pro-Ct-definable for some tuple t in Γ. By Lemma 3.13, we can take t to be a small tuple and we can find a small $C_0 \subset C$ such that S_t is pro-C_0t-definable. Let $q = \mathrm{tp}(t/C_0)$.

If $W \leq A$ is a connected stably dominated group, we must show that $W \leq S_{t'}$, for some $t' \models q$. For this purpose we can replace W by a conjugate, under the group of automorphisms of the universal domain over C_0. Thus we may assume W is defined over C. In this case, $W \leq S_t$. This proves (2) of the definition of a limit stably dominated family. Directness of the family $(S_t)_{t\models q}$ follows from Proposition 5.3, together with (2). □

Lemma 5.10 *Let C be a metastability basis. Let A be a C-definable Abelian group and let H be a connected stably dominated ∞-C-definable subgroup of A.*

(FD) H is contained in a $C\Gamma$-definable stably dominated subgroup.
(FD_ω) H is contained in a $C\Gamma$-definable connected stably dominated subgroup.

Proof Assume (FD). Since H is the stabilizer of its principal generic, $H = \bigcap_i H_i$ for some directed system of C-definable subgroups H_i of A. Then $\dim_{\mathrm{st}}(A/H_i)$ is non-decreasing with i, and eventually stabilizes; we may assume it is constant, $\dim_{\mathrm{st}}(A/H_i) = n$. Clearly $\dim_{\mathrm{st}}(A/H) = n$. By Lemma 5.5, A/H contains a stably dominated pro-$C\Gamma$-definable subgroup S with stable homomorphic image of dimension n. Let S_i be the image of S in A/H_i. Then S_i is stably dominated and $S = \varprojlim_i S_i$. The stable homomorphic image of dimension n of S factorizes through S_i for large enough i, so we may assume that, for all i, S_i has stable homomorphic image of dimension $n = \dim_{\mathrm{st}}(A/H_i)$. By Corollary 4.16, there exists a $C\Gamma$-definable stably dominated subgroup $R_i \leq A$ such that $S_i = (R_i/H_i)^0$. Note that for any $j \geq i$, R_i/H_j has a stable homomorphic image of dimension $n = \dim_{\mathrm{st}}(A/H_j) = \dim_{\mathrm{st}}(R_i/H_j)$. So replacing A by any of the R_i, we may assume that for all i, A/H_i has a stable homomorphic image of dimension $n = \dim_{\mathrm{st}}(A/H_i)$.

By Proposition 4.14 there exists a unique ∞-definable group T_i with $H_i \leq T_i \leq A$, A/T_i is almost Γ-internal and T_i/H_i is connected stably dominated. Note that, since it is unique, T_i is ∞-$C\Gamma$-definable. By Lemma 4.15, there exists $W_i \leq A$, $C\Gamma$-definable, such that $T_i/H_i = (W_i/H_i)^0$.

Let $i \leq j$. Since T_j/H_j is connected stably dominated, so is is homomorphic image $T_j/(T_i \cap T_j)$ and, since A/T_i is almost Γ-internal, so is its subgroup $T_iT_j/T_i \simeq T_j/(T_i \cap T_j)$. It is therefore trivial and $T_j \leq T_i$. Since A/T_j is (almost) Γ-internal, the argument of Lemma 4.15 shows that for large enough i_0, for all $i \geq i_0$, $[T_{i_0} : T_i]$ is bounded. Let $T = \bigcap_i T_i$. Then $[T_{i_0} : T]$ is bounded. Moreover, $T/T \cap H_i$ is (isomorphic to) a bounded index subgroup of T_i/H_i, it is therefore stably dominated. It follows that $T/H = \varprojlim_i T/T \cap H_i$ is also stably dominated. By Lemma 4.3, so is T and hence so are the T_i. Since $[W_i : T_i]$ is bounded, any of the W_i is $C\Gamma$-definable and stably dominated.

If (FD_ω) holds, since $H \leq T^0$, the second statement follows from Corollary 4.11. □

Definition 5.11 A definable family H_t of subgroups of some definable group G is said to be *certifiably stably dominated* if for all t, H_t is stably dominated and there exists t' such that $H_t \leq H_{t'}$ and $H_{t'}$ is connected.

Proposition 5.12 *(FD_ω) Let A be a definable Abelian group. Then for some C, there exist C-definable families H^ν of definable subgroups H^ν_t of A, where t is a Γ-tuple, such that:*

(1) *Any H^ν_t is stably dominated.*
(2) *Any connected stably dominated ∞-definable subgroup of A (over any set of parameters) is contained in some H^ν_t.*

Proof Let $(S_t)_{t\in q}$ be a limit stably dominated family for A; it exists by Proposition 5.9. Let C be a metastability basis such that A and the family S_t are defined over C. To prove (2), it suffices to consider the groups S_t, in particular, it suffices to consider ∞-$C\Gamma$-definable connected stably dominated subgroups of A. By Lemma 5.10 (and Remark 2.14), it suffices to consider $C\Gamma$-definable connected stably dominated subgroups of A.

For a $C\Gamma$-definable Abelian group B, define invariants n and k as follows: $n = \dim_{\mathrm{st}}(B)$. Let $Z(B)$ be the collection of $C\Gamma$-definable subgroups $S \leq B$ with stable homomorphic images of dimension n; by Lemma 5.5 and Corollary 4.16, $Z(B) \neq \varnothing$. Let $Z_2(B) = \{(S, T) : S \in Z(B), T \leq S\, C\Gamma\text{-definable}, S/T\, \Gamma\text{-internal}\}$. Let $k = \max\{\dim_{\mathrm{o}}(S/T) : (S, T) \in Z_2(B)\}$; by (FD), such a maximum exists. The pairs (n, k) are ordered lexicographically.

Pick any definable connected stably dominated $C\Gamma$-definable $B \leq A$. If $(S/B, T/B)$ attains the maximum for A/B, then the pullbacks to A show that $(n, k)(A) \geq (n, k)(A/B)$. Thus increasing B has the effect of decreasing $(n, k)(A/B)$. Let B_0 be such that $(n, k)(A/B_0)$ is minimal. For any stably dominated $H \leq A$, $H + B_0$ is also stably dominated and if S_t is a family of stably dominated subsets of A/B_0, its lifting to A is a family of stably dominated subsets of A; so it suffices to find families $\{S^\nu_t\}$ for A/B_0. Thus, we may assume $(n, k)(A/B) = (n, k)(A)$ for any connected stably dominated $C\Gamma$-definable $B \leq A$. Let $(n, k) = (n, k)(A)$.

Claim 5.12.1 *Let S be a $C\Gamma$-definable subgroup of A admitting a $C\Gamma$-definable surjective homomorphism $\theta : S \to \mathfrak{g}$ to a $C\Gamma$-definable stable group of Morley rank n and* *a $C\Gamma$-definable surjective homomorphism $\xi : S \to W$ to a $C\Gamma$-definable Γ-internal group of o-minimal dimension k. Then there exists a C-definable family H_t of stably dominated subgroups of A, with t a Γ-tuple, such that* $\ker(\xi) = H_t$ *for some t.*

Proof Clearly $S, \xi, \mathfrak{g}, \theta, W$ lie in a family $(S_t, \xi_t, \mathfrak{g}_t, \theta_t, W_t)$ such that $\mathfrak{g}_t$ is stable, $\dim_{\mathrm{st}}(\mathfrak{g}_t) = n$, W_t is a Γ-internal, $\dim_{\mathrm{o}}(W_t) = k$, $\theta_t : S_t \to \mathfrak{g}$ is a surjective homomorphism, $\xi_t : S_t \to W_t$ is a surjective homomorphism. Let $H_t = \ker(\xi_t)$. Let $T_t \leq S_t$ be as in Proposition 4.14. By Lemma 4.15, T_t is Ct-definable. By Lemma 2.26, S_t/T_t is Γ-internal. The group $T_t/H_t \cap T_t$ is Γ-internal and stably dominated, it is therefore finite and hence trivial, since T_t is connected. So $T_t \leq H_t$. By maximality of k, H_t/T_t is Γ-internal of o-minimal dimension 0 so it is finite. It follows that H_t is stably dominated.

Let B be any $C\Gamma$-definable stably dominated connected subgroup of A. Let $(S, T) \in Z_2(A/B)$ be such that S has a stable homomorphic image of dimension n and S/T is Γ-internal of o-minimal dimension k. Let (S', T') be the pullbacks of (S, T) to A. Then $B \leq T'$. Clearly S' admits a stable homomorphic image of dimension n and S'/T' is Γ-internal of o-minimal dimension k. By the Claim 5.12.1, there exists a C-definable family H_t of stably dominated groups, with t a Γ-tuple, such that $T' = H_t$ for some t. □

Corollary 5.13 *(FD_ω) Let A be a definable Abelian group. There exists a definable certifiably stably dominated family H_t, where t is a Γ-tuple, such that any connected stably dominated ∞-definable subgroup of A is contained in some H_t.*

Proof Let C and H_t^ν be as in Proposition 5.12.

Claim 5.13.1 *There exists a type p over C and a ν such that any ∞-definable connected stably dominated subgroup of A is contained in some H_t^ν, with $t \models p$.*

Proof Assume not, then for any ν and p, there exists a ∞-definable stably dominated $B_{p,\nu} \leq A$ which is not contained in $\bigcup_{t\models p} H_t^\nu$. By Proposition 5.3, there exists an ∞-definable stably dominated $B \leq A$ containing all the $B_{p,\nu}$. This B is contained in some H_t^ν for some t and ν. Let $p = \mathrm{tp}(t/C)$. Then $B_{p,\nu} \leq H_t^\nu$, a contradiction. □

Fix p and ν as in the Claim and $\gamma \models p$. Then $(H_\gamma^\nu)^0$ is definable and has finite index, say l, in H_γ^ν. Let L_t be a C-definable family of subgroups of H_t such that $[H_t : L_t] \leq l$ and whenever $t \models p$, $L_t = H_t^{\nu,0}$. Then every L_t is stably dominated and any ∞-definable connected stably dominated subgroup of A is contained in some L_t with $t \models p$, which is connected by construction. □

Before we can state the next theorem, we need to give an explicit definition of Γ-internality for certain hyperimaginaries.

Definition 5.14 Let D be a pro-definable set and E be a pro-definable equivalence relation on D. We say that D/E is *almost Γ-internal* if there exists a a pro-definable map $f : D \to X$ such that:

(1) X is a pro-definable set in Γ^{eq}.

(2) Each fiber of f intersects only boundedly many classes of E.

Remark 5.15 Note that if E is the intersection of relatively definable equivalence relations on D, then D/E is a pro-definable set and the above definition is equivalent to the usual notion of almost Γ-internality.

In particular, if G is a pro-definable group and H is a pro-definable subgroup admitting a generic and G/H is almost Γ-internal, then G/H is a pro-limit of almost Γ-internal definable groups. If, moreover, Γ is o-minimal, then, by Lemma 2.26, each of these almost Γ-internal definable groups is in fact Γ-internal and is therefore (isomorphic to) a group definable in Γ. It follows that G/H is (pro-definably isomorphic to) a group pro-definable in Γ.

Theorem 5.16 *Let A be a pro-limit of definable Abelian groups. Assume A has bounded weight. Then the limit stably dominated subgroup H exists and A/H is almost internal to Γ.*

If (FD_ω) holds and A is definable, then H is definable, admits a generic type p, is connected and A/H is Γ-internal.

Proof The existence of H is proved in Proposition 5.9. Let C be a metastability basis over which A and H are defined. Let f be a pro-definable function enumerating all C-definable maps from A into Γ.

Claim 5.16.1 *For all $c, d \in A$, if $\mathrm{tp}(c/\mathrm{acl}(Cf(c))) = \mathrm{tp}(d/\mathrm{acl}(Cf(c)))$, then $c \cdot d^{-1} \in H$.*

Proof By definition of a metastability basis, $\mathrm{tp}(c/Cf(c))$ is stably dominated. Let p be the unique $\mathrm{acl}(Cf(c))$-definable extension of $\mathrm{tp}(c/\mathrm{acl}(Cf(c)))$. By Lemma 5.1, there exists a stably dominated pro-$\mathrm{acl}(Cf(c))$-definable connected stably dominated subgroup $S \leq A$ such that p concentrates on a coset of S. Note that, by definition of H, $S \leq H$. Since p is $\mathrm{acl}(Cf(c))$-definable, it follows that $p|\mathrm{acl}(Cf(c))$ concentrates on a coset of S; i.e. $c \cdot d^{-1} \in S \leq H$. □

It immediately follows that each fiber of f intersects boundedly many cosets of H and, since the image of f is a pro-definable set in Γ, A/H is almost Γ-internal.

Let us now assume (FD_ω). Let $(S_t)_{t \models q}$ be a limit stably dominated family for H. By Corollary 5.13 we can also find a certifiably stably dominated family $(H_{t'})_{t' \in Q}$ where $Q \leq \Gamma^n$ for some n, such that any ∞-definable connected stably dominated subgroup of A is contained in some $H_{t'}$. It is easy to see that $\bigcup_{t \models q} S_t = H = \bigcup_{t' \in Q} H_{t'}$ and hence that H is definable.

Define a partial order on Q by $t'_1 \leq t'_2$ if $H_{t'_1} \leq H_{t'_2}$. Note that this order is directed. By Lemma 2.24, Q has a cofinal definable type q'. For any $t' \models q'$, let $p_{t'}$ be the principal generic of $H^0_{t'}$. Let r be the type such that, for all $D \supseteq C$ over which q is defined, $a \models r|D$ if and only if $a \models p_{t'}|Dt'$ for some $t' \models q'|D$. By Lemma 2.3, r is definable. Let D be such that A, H and r are D-definable and pick any $g \in H$. Then there exists $t \models q$ and $t' \models q'|Dg$ such that $g \in S_t \leq H^0_{t'}$. Let $a \models p_{t'}|Dgt'$, then $g \cdot a \models p_{t'}|Dgt'$ and hence $g \cdot a \models r|Dg$. It follows that ${}^g p = p$ and thus that $H = \mathrm{Stab}(p)$ is connected.

The fact that A/H is Γ-internal now follows from the fact that it is almost Γ-internal and Lemma 2.26 (cf. Remark 5.15).

Remark 5.17 The only use of (FD_ω) in the proof above is to apply Corollary 5.13. It follows that, to get the second part of the statement in Theorem 5.16, it suffices to assume that there exists a certifiably stably dominated family H_t, where t is a Γ-tuple, such that any ∞-definable connected stably dominated subgroup is contained in some H_t.

Corollary 5.18 *(FD_ω) Let A be a definable Abelian group. There exists a universal pair (f, B) with B a Γ-internal definable group, and $f : A \to B$ a definable homomorphism. In other words for any (f', B') of this kind there exists a unique definable homomorphism $h : B \to B'$ with $f' = h \circ g$.*

Equivalently, there is a smallest definable subgroup H with A/H Γ-internal.

Proof Let H be the limit stably dominated subgroup of A. Then H is definable. For any pair (f', B') as above, f' vanishes on any definable connected stably dominated group. Since H is a union of such groups, f' vanishes on H. But A/H is Γ-internal, so the canonical homomorphism $A \to A/H$ clearly solves the universal problem. □

Corollary 5.19 *(FD_ω) Let A be a definable Abelian group. There exists a smallest definable subgroup A^0 of A of finite index.*

Proof Let H be the limit stably dominated subgroup of A and $B \leq A$ be any subgroup of finite index. Since $H/H \cap B$ is finite and H is connected, it follows that $H \leq B$. The question thus reduces to the o-minimal group A/H, where it is known (cf. [19, Proposition 2.12]). □

Question 5.20 *In* ACVF*, is G/G^0 always finite, where G is any definable group and G^0 denotes the intersection of all definable finite index subgroups of G?*

The answer to this question is presumably yes (cf. Problem 1.9 and the proof of Corollary 5.19) and it is already known to be true of stably dominated groups (cf. Corollary 4.11).

Corollary 5.21 *(FD_ω) Let A be a definable Abelian group. There exists a universal pair (f, B) with B a stable definable group, and $f : A \to B$ a definable homomorphism. Equivalently, there exists a smallest definable subgroup H with A/H stable.*

Proof Clearly if A/H and A/H' are stable, so is $A/(H \cap H')$. It suffices to show that there are no strictly descending chains $A \supset H_1 \supset H_2 \ldots$ of such subgroups. By (FD), $\dim_{st}(A/H_n)$ is bounded; so we may assume it is constant, $\dim_{st}(A/H_n) = d$. Then H_n/H_{n+1} is finite. By Corollary 5.19, the chain stabilizes.

As one can see, Corollary 5.21 follows formally from the existence of a smallest definable subgroup of finite index and does not actually depend on the existence of the limit metastable subgroup. Using the fact that in a metastable theory, extensions of Γ-internal groups are Γ-internal, Corollary 5.18 can similarly be deduced formally from the existence of a smallest definable subgroup of finite index. In particular, a positive answer to Question 5.20 would imply that Corollaries 5.18 and 5.21 hold of every definable group in ACVF.

Remark 5.22 (FD_ω) Since a pro-definable stable group always has a generic and hence, by Proposition 3.4 is a prolimit of groups, any ∞-definable normal subgroup of a definable group A such that A/H is stable is the intersection of definable subgroups H_i such that A/H_i is stable. It follows that if A is Abelian, the smallest ∞-definable subgroup H with A/H stable exists and is definable.

It is then straightforward to prove that any pro-limit A of Abelian groups—in particular, any pro-definable Abelian group with a generic—has a smallest pro-definable subgroup H such that A/H is stable.

Corollary 5.23 *(FD_ω) Let A be a connected definable Abelian group. Then for almost all primes p, $p \cdot A = A$.*

Proof Let H be the limit stably dominated subgroup of A. We have $H = \cup_{t \models q} S_t$ where q is a complete type over some C and S_t is a definable connected stably dominated group. Let A_t be the maximal stable quotient of S_t and $m = \dim_{\mathrm{st}}(A_t)$. Let $A_t(p) = \{x \in A_t : p \cdot x = 0\}$. Then $A_t(p)$ can be infinite for at most m values of p. For any other p, $A_t(p)$ is finite and hence $p \cdot A_t = A_t$. It follows that $p \cdot S_t$ has finite index in S_t (Corollary 4.10); so $p \cdot S_t = S_t$. Thus $pH = H$.

On the other hand, A/H is an o-minimally definable group. So its p-torsion is finite (eg. [29, Proposition 6.1]) and hence $\dim_{\mathrm{o}}((A/H)/p(A/H)) = 0$. Since A/H is connected, $p(A/H) = A/H$. So $p \cdot A = A$. □

5.2 Metastable fields

By a rng we mean an Abelian group with a commutative, associative, distributive multiplication (but possibly without a multiplicative unit element.) If a rng has no zero-divisors, the usual construction of a field of fractions makes sense.

Proposition 5.24 *Let F be a definable field with $0 < \dim_{\mathrm{st}}(F) = n < \infty$. Then there exists an ∞-definable subrng D of F, with $(D, +)$ connected stably dominated. Moreover F is the field of fractions of D.*

Proof Using Lemma 5.5, we find a connected ∞-definable subgroup M of $F^\star$ with a stably dominated generic type p of stable dimension n.

Recall that the p-weight of F is bounded by $\dim_{\mathrm{st}}(F)$. We can therefore apply Lemma 5.1 additively to find D a connected subgroup of $(F, +)$ with principal generic type $p^{\pm 2n}$ and such that p concentrates on an additive coset of D. As in Lemma 5.5, we see that $\dim_{\mathrm{st}}(D) \geq n$. Let C be such that F, M, D and p are C-definable. If $(a, b_1, b_2, \ldots, b_{2n}) \models p^{\otimes 2n+1}|C$, then $a \cdot (\sum_i -b_{2i} + b_{2i+1}) = \sum_i -(a \cdot b_{2i}) + (a \cdot b_{2i+1}) \models p^{\pm 2n}|Ca$, i.e. if $a \models p|C$ and $b \models p^{\pm 2n}|Ca$, $a \cdot b = p^{\pm 2n}|Ca$. Let now $a \models p|C$ and $d \in D$. Since $p^{\pm 2n}$ is the principal generic of D, we find $b, c \models p^{\pm 2n}|Ca$ such that $d = b - c$. So $a \cdot d = a \cdot b - a \cdot c \in -p^{\pm 2n}|C + p^{\pm 2n}|C \subseteq D$. Similarly, since M is multiplicatively generated by $p|C$, $M \cdot D \subseteq D$. Finally, since D is generated by $M - M$, $D \cdot D \subseteq D$; so D is a subrng of F.

Let $F' \subseteq F$ be the fraction field of D. If $a \in F \setminus F'$, then the map $(x, y) \mapsto x + a \cdot y$ takes $D^2 \to F$ injectively—if $x + a \cdot y = x' + a \cdot y'$ and $(x, y) \neq (x', y')$ then $y \neq y'$ and $a = (x' - x)/(y - y')$. Thus $D + a \cdot D \subseteq F$ is definably isomorphic to D^2. But

$\dim_{st}(D^2) \geq 2n > n = \dim_{st}(F)$, a contradiction. We have proved that $F = F'$ is the field of fractions of D. □

Remark 5.25 Assuming (FD_ω), in the previous proposition, if D is a ring, i.e. $1 \in D$, then the generic type of $(D, +)$ is also generic in $(D^\star, \cdot)$.

Proof Note first that by construction, the generic type q of $(D, +)$ has stable dimension $n = \dim_{st}(F)$, so, by Corollary 4.16, D is definable. Let N be the smallest definable subgroup of $(D, +)$ with D/N stable (cf. Corollary 5.21). Clearly N is multiplicatively invariant under the units of D. Thus N is an ideal of D and D/N is a definable stable ring. Then D/N has only finitely many maximal ideals which are all definable (cf. [2, Theorem 2.6]). It follows that the generic of D/N avoids all these maximal ideals and therefore concentrates on $(D/N)^\star$.

Let $a \models q$. The image in D/N of the ideal $a \cdot D$ contains the generic $a + N$ which is invertible. So that image equals D/N. By Corollary 4.10, $a \cdot D = D$—recall that D is connected—and a is invertible. So the additive generic of D concentrates on $D^\star$ and since, for any $c \in D^\star$, the map $x \mapsto c \cdot x$ is a definable isomorphism, it must preserve q. □

Remark 5.26 If F is a definable field of finite weight, then the limit stably dominated subgroup H of $(F, +)$ is an ideal so it is either trivial—in which case F is Γ-internal—or coincides with F. If $(F, +)$ is stably dominated then its minimal definable subgroup N with F/N stable is a proper ideal and hence is trivial—in which case F is stable.

So, unless F is stable or Γ-internal, $(F, +)$ is properly limit stably dominated.

Proposition 5.27 *(FD_ω) Let F be an infinite definable field which is neither stable nor internal to Γ. Then $F^\star$ has a definable homomorphic Γ-internal non definably compact image.*

Proof Let H be a stably dominated ∞-definable subgroup of $F^\star$ with generic p. As in the proof of Proposition 5.24, we find a subrng R, additively generated by $p^{\pm 2}$, invariant under multiplication by H, whose additive group is connected stably dominated and such that p concentrates on an additive coset of R. Recall that, since we built $p^{\pm 2}$ by looking at the additive structure, realizations of $p^{\pm 2}$ are of the form $-a + b$ for some $(a, b) \models p^{\otimes 2}$.

If $H_1 \leq H_2$ are two ∞-definable stably dominated subgroups of $F^\star$, let p_i be their generics and R_i the corresponding subrng. Assume that everything is defined over some C. If $a, b \in H_1$ and $e \models p_2|Cab$, then both $a \cdot e$ and $b \cdot e$ realize $p_2|Cab$. It follows that $a - b = e^{-1}(a \cdot e - b \cdot e) \in R_2$. In particular, the set of realizations of $p_1^{\pm 2}$ is contained in R_2. Since R_1 is additively generated by $p_1^{\pm 2}$, $R_1 \subseteq R_2$.

Let $M = \bigcup_{t \models q} M_t$ be the limit stably dominated subgroup of $F^\star$ and R_t the subrng corresponding to M_t. Note that, since M_t can be—and is—chosen definable and that $M_t - M_t$ generates R_t in boundedly many steps, R_t is definable. Note that if $\dim_{st}(F) = 0$, working over a metastability basis C, for any $a \in F$, $\mathrm{tp}(a/\Gamma_C(a))$ has a unique realization and F is internal to Γ. So $\dim_{st}(F) > 0$. Let R' (and M') be as in Proposition 5.24. In particular, F is the fraction field of R'. By definition, there exists $t \models q$ such that $M' \leq M_t$. By the previous paragraph, $R' \subseteq R_t$. It follows that for any $t \models q$, F is the fraction field of R_t. Let N_t be the minimal quotient of $(R_t, +)$ such that R_t/N_t is stable (cf. Corollary 5.21). Since N_t is characteristic, it is invariant under multiplication by elements of M_t. It is therefore an ideal of R_t. By Proposition 4.6, the principal generic of R_t is dominated by a group morphism, so if $N_t = R_t$, $R_t = \{0\}$. On the other hand, if $N_t = (0)$, R_t, and hence F is stable. It follows that N_t is a proper non-trivial ideal. If $t \leq s$, $R_t/N_s \cap R_t$ is stable as it embeds in R_s/N_s. So $N_t \subseteq N_s$. If $a \in N_t$ is not 0, then $a^{-1} \notin R_s$ for any $s > t$ and it follows that $R := \bigcup_t R_t \neq F$.

Let $V := F^\star/M$. It is Γ-internal. Let $V^+ := (R_t \smallsetminus \{0\})/M$ for some choice of t. It is a definable subsemigroup of V. Since R is stabilized by M, $MR_t \subseteq R \neq F$, so $V^+ \neq V$. Also, since F is the fraction field of R_t, $V = V^+ - V^+$. By Lemma 2.31, V does not have (NG). By Proposition 2.30, it is not definably compact.

6 Valued fields: stably dominated groups and algebraic groups

In this section, we work in ACVF. Let K be an algebraically closed valued field; $\mathcal{O}$ denotes the valuation ring. Occasionally we will assume K to be sufficiently saturated.

Recall that a definable set is purely imaginary if there are no definable functions (with parameters) from that set onto an infinite subset of K, cf. Definition 2.34.

Proposition 6.1 *Let $C = \mathrm{acl}(C)$ and H be a connected ∞-C-definable group with a (right) generic. Then there exists an algebraic group G over $C \cap K$ and a C-definable homomorphism $f : H \to G$, with purely imaginary kernel.*

Proof Let $F = C \cap K$. Let p be the principal generic type of H; let $(a_1, a_2) \models p^{\otimes 2}|C$ and $a_3 = a_1 \cdot a_2$. Let $\tau = \{a_1, a_2, a_3, a_{12} = a_1 \cdot a_2, a_{23} = a_2 \cdot a_3, a_{123} = a_1 \cdot a_2 \cdot a_3\}$. We view this six-element set as a *matroid*, given by specifying the collection of algebraically dependent subsets of τ. This data is called a group configuration.

For $c \in \tau$, let $A(c) = \mathrm{acl}(Cc) \cap K$; this is the set of field elements in the algebraic closure of c over C. Pick a tuple $\alpha(c) \in A(c)$ such that $A(c) = F(\alpha(c))^{\mathrm{alg}}$. We view $c \mapsto \alpha(c)$ as a map on τ into the matroid of algebraic dependence in the algebraically closed field K over F. Then α preserves both dependence and independence. For independence this is clear. Preservation of dependence follows from:

Claim 6.1.1 *Let E_1 and E_2 be two algebraically closed substructures of a model of* ACVF, *all sorts allowed. Let L_i be the set of field elements of E_i. If $c \in \mathrm{acl}(E_1 \cup E_2) \cap K$, then $c \in (L_1 L_2)^{\mathrm{alg}}$.*

Proof This follows immediately from the fact that there are no definable maps with infinite range from a geometric sort other than K to K (cf. Remark 2.36).

According to the group configuration theorem for stable theories, applied to the theory ACF over the model F, there exists a group G, ACF-definable over F, such that $a(\tau)$ is a group configuration for G; in particular there exist $b_1, b_2 \in G$, $b_{12} = b_1 b_2$ such that $A(a_i) = F(b_i)^{\mathrm{alg}}$. Since ACF-definable groups are (definably isomorphic to) algebraic groups, we can take G to be an algebraic group over F (cf. [11, Proposition 3.1]).

We work in the group $H \times G$. Let $c_i = (a_i, b_i)$. Since $a_i \models p|C$, p is C-definable and $C = \mathrm{acl}(C)$, $\mathrm{tp}(c_i/C)$ has a C-definable extension q_i. Let $Z = \mathrm{Stab}(q_2, q_3)$. This is a coset of $S := \mathrm{Stab}(q_2)$ and $q := {}^{c_1^{-1}}q_1$ is a generic type of S. Let $J = \{h \in H : (h, 1) \in S\}$.

Claim 6.1.2 *J is purely imaginary.*

Proof Pick any $h \in J$, $D \supseteq C$ and $(a, b) \models q_2|\mathrm{acl}(Dh)$. Then $(h \cdot a, b) \models q_2|Dh$. Since all geometric sorts but K are purely imaginary and $a \cap K \subseteq \mathrm{acl}(b)$, we have $\mathrm{acl}(Da) \cap K \subseteq \mathrm{acl}(Db)$ and hence $\mathrm{acl}(D, h \cdot a) \cap K \subseteq \mathrm{acl}(Db)$. So $\mathrm{acl}(Dh) \cap K \subseteq \mathrm{acl}(D, a, h \cdot a) \cap K \subseteq \mathrm{acl}(Db)$. This inclusion holds for any $(a, b) \models q_2|\mathrm{acl}(Dh)$. Since q_2 is D-definable, in particular D-invariant, it follows that $\mathrm{acl}(Dh) \cap K \subseteq \mathrm{acl}(D)$ and h is purely imaginary over any $D \supseteq C$. Therefore J is purely imaginary.

Note that S projects onto H; this is because the projection contains a realization of $p_1|C$ and p_1 generates H. Similarly, S projects dominantly onto G. Let $J' = \{g \in G : (1, g) \in S\}$. By an argument similar to the proof of the above claim, for all $h \in J'$ and $(a, b) \models q_2|Ch$, since $b \in \mathrm{acl}(a)$, we have that $h \in \mathrm{acl}(Ca)$. Since q_2 is C-invariant, it follows that $h \in \mathrm{acl}(C)$ and hence J' is finite. Note also that J' is normal in the projection of S to G and hence is a normal subgroup of G.

Now S can be viewed as the graph of a homomorphism $f : H \to G/J'$ with kernel J. Since G/J' is isomorphic to an algebraic group defined over F, replacing G by G/J' we may assume that $f : H \to G$. Since $S \subseteq H \times G$ is the ∞-C-definable graph of a function, it is relatively definable—we have $(a, b) \notin S$ if and only if $(a, b') \in S$ for some $b' \neq b$—and f is a C-definable function.

Remark 6.2 Assume that H is a connected ∞-definable Abelian group with a (right) generic, then there is a universal map f into an algebraic group; i.e. for any algebraic group G' and any definable homomorphism $f' : H \to G'$, there exists a unique definable homomorphism $g : f(H) \to G'$ with $f' = g \circ f$.

Instead of H being Abelian, it suffices to assume that Question 5.20 has a positive answer for all definable subgroups of H.

Proof Let f be as in Proposition 6.1, then, since $J := \ker(f)$ is purely imaginary, $f'(J)$ is finite. Thus $\ker(f') \cap J$ is a finite index subgroup of J. If we choose $\ker(f)$ to be as small as possible, i.e. $\ker(f)/\ker(f)^0$ to be least possible, then $J \leq \ker(f')$. □

The following example shows that the assumption that the group has a generic cannot be eliminated. Without it, it seems likely that one can find an isogeny between a quotient of H by a purely imaginary group and a quotient of an ind-algebraic group.

Example 6.3 (Versions of tori) The value group Γ of an algebraically closed valued field is a divisible ordered Abelian group. So every group definable in Γ is locally definably isomorphic to the group $V := \Gamma^n$. Nevertheless, interesting variants of V are known. If Δ is a finitely generated subgroup of V, the convex hull $C(\Delta)$ of Δ

(with respect to the product partial ordering of V) is an ind-definable subgroup (a direct limit of definable sets) which is itself not definable. But the quotient $C(\Delta)/\Delta$ is canonically isomorphic to a group definable in Γ (cf. [25]). One example of such a group is, given any $\tau \in \Gamma_{>0}$, the group $\Gamma(\tau) = [0, \tau)$ with the law $x \star y = x + y$ if it stays in $\Gamma(\tau)$ and $x \star y = x + y - \tau$ otherwise.

Note that a non realized definable type concentrating on $\Gamma(\tau)$ corresponds to a definable cut and hence has a trivial stabilizer. It follows that $\Gamma(\tau)$ does not have generics.

These examples lift to the tori $T = \mathbb{G}_{\mathrm{m}}^n$ over algebraically closed valued fields. Let $r : T \to V$ be the homomorphism induced by the valuation map. Let Δ be a finitely generated subgroup of T. Then $C(\Delta) := r^{-1}(C(r(\Delta)))$ is ind-definable, and $C(\Delta)/\Delta$ is definable in ACVF. For example, fix $t \in K$ with $\mathrm{val}(t) = \tau > 0$, and define a group structure on $A(t) := \{x : 0 \leq \mathrm{val}(x) < \tau\}$ by: $x \star y = x \cdot y$ if $x \cdot y \in A(t)$, $x \star y = x \cdot y/t$ otherwise. This group admits a homomorphism onto the definably compact Γ-internal group $\Gamma(\tau)$, with stably dominated kernel. Since $\Gamma(\tau)$ does not have generics, neither does $A(t)$.

Note that $A(t)$ contains no infinite purely imaginary subgroup. Moreover, if, for some finite group J, $A(t)/J$ is a Zariski-dense subgroup of an algebraic group G, then the group configuration of G is inter-algebraic with that of $\mathbb{G}_{\mathrm{m}}$. It follows that G is isogenous to $\mathbb{G}_{\mathrm{m}}$. But $A(t)$ has too many torsion points for this to be possible.

Corollary 6.4 *Let $C = \mathrm{acl}(C)$ and H be a connected ∞-C-definable stably dominated group. Then, there exists an algebraic group G over $C \cap K$ and a C-definable homomorphism $f : H \to G$, with boundedly imaginary kernel.*

Proof We follow the notation and proof of Proposition 6.1. Note that, by Proposition 2.10.(4) and Proposition 2.9, q_i is stably dominated over C. There only remains to show that J is boundedly imaginary. So pick an $h \in J$. Then for any $(a, b) \models q_2|Ch$, $(h \cdot a, b) \models q_2|Ch$. Now a is purely imaginary over Cb. By Lemma 2.35, there exists $\beta_0 \leq 0 \leq \beta_1 \in \Gamma_C(a, b)$ and a tuple $d \in \beta_0\mathcal{O}/\beta_1\mathfrak{M}$ such that $a \in \mathrm{dcl}(C, b, \beta_0, \beta_1, d)$. Since $\mathrm{tp}(ab/C)$ is stably dominated, we have $\beta_0, \beta_1 \in \Gamma(C)$, and $a \in \mathrm{dcl}(C, b, d)$. Since $\mathrm{tp}(h \cdot a, b/C) = \mathrm{tp}(a, b/C)$, we also have $h \cdot a \in \mathrm{dcl}(C, b, d')$, for some tuple $d' \in \beta_0\mathcal{O}/\beta_1\mathfrak{M}$. Thus $h \in \mathrm{dcl}(a, h \cdot a, C) \subseteq \mathrm{dcl}(C, b, d, d')$. Since q_2 is C-invariant and $\beta_0\mathcal{O}/\beta_1\mathfrak{M}$ is stably embedded C-definable, it follows that $h \in \mathrm{dcl}(C, d'')$ for some tuple $d'' \in \beta_0\mathcal{O}/\beta_1\mathfrak{M}$. By compactness and Corollary 2.39, J is boundedly imaginary. □

A very similar proof yields the following:

Remark 6.5 Let H be a purely imaginary stably dominated definable group. Then H is boundedly imaginary.

Proof Let p be the principal generic type of H; with H and p defined over $C = \mathrm{acl}(C)$. Then, for any $a \models p|C$, $\Gamma_C(a) \subseteq C$. Since a is purely imaginary over C, it follows from Lemma 2.35, stable domination of $p|C$ and Corollary 2.39 that a is boundedly imaginary over C. Any element of H^0 is a product of two realizations of $p|C$, so H^0 is boundedly imaginary. It follows that some definable set H' containing H^0 is boundedly imaginary. Finitely many translates of H' cover H, so H is also boundedly imaginary. □

6.1 Stably dominated subgroups of algebraic groups

Our goal in this section is to relate stably dominated subgroups of affine algebraic groups to algebraic geometric objects. To do so, we consider schemes over $\mathcal{O}$. Note that schemes only appear this section and that this section is independent from the rest of the paper. Note also that the technical hypotheses are meant to insure that every thing works as one would expect. All of the required definitions regarding schemes can be found in [17].

All the schemes over $\mathcal{O}$ that we consider will be assumed to be flat and reduced over $\mathcal{O}$. By an $\mathcal{O}$-variety (or variety over $\mathcal{O}$), we mean a (flat and reduced) scheme of finite over $\mathcal{O}$; it admits a finite open covering by schemes isomorphic to $\mathrm{Spec}(\mathcal{O}[X_1, \ldots, X_n]/I)$. In particular, an affine variety over $\mathcal{O}$ is of the form $\mathrm{Spec}(\mathcal{O}[X_1, \ldots, X_n]/I)$. Note that we do not require varieties to be irreducible. Since $\mathcal{O}$ is a valuation ring, an affine scheme $\mathrm{Spec}(R)$ is flat over $\mathrm{Spec}(\mathcal{O})$ if and only if no non-zero element of $\mathcal{O}$ is a 0-divisor in R. So $\mathrm{Spec}(\mathcal{O}[X_1, \ldots, X_n]/I)$ is flat if and only if $I = IK[X_1, \ldots, X_n] \cap \mathcal{O}[X_1, \ldots, X_n]$. Hence there are no infinite descending chains of $\mathcal{O}$-subvarieties.

For any scheme V over $\mathcal{O}$, we write $V(\mathcal{O})$ for the set of its $\mathcal{O}$-points; the set of morphisms $\mathrm{Spec}(\mathcal{O}) \to V$ over $\mathrm{Spec}(\mathcal{O})$. When $V = \mathrm{Spec}(\mathcal{O}[X_1, \ldots, X_n]/I)$ is an affine variety over $\mathcal{O}$, $V(\mathcal{O}) = \{x \in \mathcal{O}^n : \forall f \in I, f(x) = 0\}$. Conversely, for any set $Z \subseteq \mathcal{O}^n$, let $I := \{f \in \mathcal{O}[X_1, \ldots, X_n] : f(Z) = 0\}$ and $R = \mathcal{O}[X_1, \ldots, X_n]/I$. Then $V = \mathrm{Spec}(R)$ is flat over $\mathrm{Spec}(\mathcal{O})$ and if $Z = W(K) \cap \mathcal{O}^n$ for some affine variety W over K, then $V(\mathcal{O}) = Z$. Note that the set $V(\mathcal{O})$ is pro-definable in ACVF. When V is of finite type, $V(\mathcal{O})$ is definable.

If V is a scheme over $\mathcal{O}$, we write $V_K := V \times_{\mathcal{O}} K$ for the generic fiber and $V_k := V \times_{\mathcal{O}} k$ for the special fiber. Note that if V is of finite type, both the generic fiber and the special fibers are, non necessarily reduced, varieties over, respectively, K and k. Let $\rho : V(\mathcal{O}) \to V_k(k)$ be the natural (pro-)definable map. When V is an affine variety over $\mathcal{O}$ it simply consists in coefficient-wise reduction modulo $\mathfrak{M}$. If W' is a subvariety of V_K, there exists a unique $\mathcal{O}$-subvariety W of V such that $W_K = W'$, and $W(\mathcal{O}) = W'(K) \cap V(\mathcal{O})$. For example, if $V' \subset \mathbb{A}^n$ is an affine variety over K, defined by a radical ideal $P \subset K[X]$, we let $V = \mathrm{Spec}(\mathcal{O}[X]/P \cap \mathcal{O}[X])$. Then $V_K = V'$ and V_k is the zero set of the image of $P \cap \mathcal{O}[X]$ in $k[X]$. In this case, we denote the affine coordinate ring $K[X]/P$ by $K[V]$, and $\mathcal{O}[V] = \mathcal{O}[X]/(P \cap \mathcal{O}[X])$.

Lemma 6.6 *Let V be an irreducible affine $\mathcal{O}$-variety and assume $V(\mathcal{O}) \neq \varnothing$. Then $V(\mathcal{O})$ is Zariski dense in V_K.*

Proof Let L be a sufficiently saturated algebraically closed field extending K. Let $a \in V_K(L)$ be a K-generic point. Thus for any $b \in V(K)$ there exists a K-algebra homomorphism $h : K[a] \to K$ with $h(a) = b$. In particular, there exists such an h with $h(a) \in V(\mathcal{O})$. It follows that the maximal ideal $\mathfrak{M}$ of K generates a proper ideal of $\mathcal{O}[a]$: otherwise, for some $m \in \mathfrak{M}$ and $f \in \mathcal{O}[X]$, $m \cdot f(a) = 1$; but then applying h, we would have $m \cdot f(h(a)) = 1$. Thus $\mathfrak{M}$ extends to a maximal ideal $\mathfrak{M}'$ of $\mathcal{O}[a]$, and thence to a maximal ideal $\mathfrak{M}''$ of some valuation ring $\mathcal{O}_L$ of L, with $\mathcal{O}[a] \subset \mathcal{O}_L$. Thus the valuation on K can be extended to L in such a way that $a \in \mathcal{O}(L)$. By model completeness of ACVF, there exists $a' \in V_K(K)$, with coordinates in $\mathcal{O}$, outside any given proper K-subvariety of V. □

We denote the Krull dimension of schemes by dim. Note that when V is a variety over k, $\dim(V) = \mathrm{MR}(V) = \dim_{\mathrm{st}}(V)$.

Lemma 6.7 *Let V be a scheme over $\mathcal{O}$, with $\dim(V_K) = n$. Let q be a K-definable type of elements of $V(\mathcal{O})$ with $\dim_{\mathrm{st}}(\rho_\star q) = n$. Assume V is defined over $B = \mathrm{acl}(B)$; then q is stably dominated over B.*

Proof Say q is defined over $B' = \mathrm{acl}(B') \supset B$. Let $a \models q|B'$. Since $\mathrm{trdeg}(k(B')(\rho(a))/k(B')) \geq n$, we have $\mathrm{trdeg}(k(B)(\rho(a))/k(B)) \geq n$. It follows that $B(a)$ is an extension of B of transcendence degree n and residual transcendence degree n. The type $\mathrm{tp}(a/B)$ is therefore stably dominated via ρ. Indeed, a is in the algebraic closure over B of n independent generics of $\mathcal{O}$. Since $\rho(a) \downarrow_B B'$, q is its unique B-definable extension. □

Recall that a group scheme G over $\mathcal{O}$ is a scheme G over $\mathcal{O}$ with three morphisms of schemes over $\mathcal{O}$: the multplication $\mu : G \times_{\mathcal{O}} G \to G$, the inverse map $\iota : G \to G$ and the identity $\epsilon : \mathrm{Spec}(\mathcal{O}) \to G$, such that the obvious diagrams commute. When $G = \mathrm{Spec}(R)$ is affine, these maps correspond to three operations: the comultiplication $R \to R \otimes_{\mathcal{O}} R$, the coinversion $R \to R$, and the counit $R \to \mathcal{O}$, making R into a Hopf algebra. In fact, Spec induces a correspondance between affine group schemes and Hopf algebras.

Let G be a group scheme over $\mathcal{O}$, with generic fiber G_K and special fiber G_k. Then $G(\mathcal{O})$ is a (pro-)definable subgroup of the (pro-)definable group $G_K(K)$, and the previously defined map $\rho : G(\mathcal{O}) \to G_k(k)$ is a group homomorphism. Note that, when G is not a scheme of finite type, $\dim(G_k)$ might be smaller than $\dim(G_K)$; cf. Example 6.12.

Proposition 6.8 *Let p be a definable type concentrating on $G(\mathcal{O})$. Assume that $\rho_\star p$ is a generic type of G_k and that $\dim(G_K) = \dim(G_k) < \infty$. Then p is a stably dominated generic type of $G(\mathcal{O})$. When G is a scheme of finite type, $G(\mathcal{O})$ has finitely many generic types.*

Proof Consider translates $q = {}^g p$ of p, for some $g \in G(\mathcal{O})$. Clearly $\rho_\star q = {}^{\rho(g)}(\rho_\star p)$. By Lemma 6.7, q is stably dominated over $\mathrm{acl}(B)$, where B is a base of definition of G. So p itself is stably dominated and since all of its translates are defined over B, which is small, p is generic.

When G is of finite type, by Corollary 4.11, G^0 is definable and hence, the orbit of generic types is finite. □

6.1.1 Linear groups

Our goal is to show that all stably dominated ∞-definable subgroups of affine algebraic groups, may be obtained as the $\mathcal{O}$-points of a group scheme G over $\mathcal{O}$. This result has since been generalized in [5] to the non-affine setting. Note that Example 6.12 shows that we cannot always assume G to be of finite type.

We start with a version of the maximum modulus principle for generics of subgroups:

Proposition 6.9 *Let G be an affine algebraic group over K, H be a Zariski dense definable subgroup of $G(K)$ and p be a definable type of elements of H. Then the following are equivalent:*

(1) *p is the unique stably dominated generic of H.*
(2) *For any regular function f on G, p attains the highest modulus of f on H; i.e for some γ_f, for any $x \in H$, $\operatorname{val} f(x) \geq \gamma_f$ and*

$$\models (\mathrm{d}_p x)(\operatorname{val} f(x) = \gamma_f).$$

Note that, because any function constant on connected components of G is regular, connectedness of G follows from these assumptions.

Proof Let us first prove that (1) implies (2). Since p is stably dominated, for any regular f there exists γ_f with $(\mathrm{d}_p x)(\operatorname{val}(f(x)) = \gamma_f)$. If $(a, b) \models p \otimes p$ then $a \cdot b \models p$; so $\operatorname{val}(f(a \cdot b)) = \gamma_f$. By Corollary 2.41, for any $a, b \models p$ we have $\operatorname{val}(f(a \cdot b)) \geq \gamma_f$. But since p is the unique generic, any element c of H is a product of two realizations of p. Thus $\operatorname{val}(f(c)) \geq \gamma_f$.

Let us now prove that (2) implies (1). Note that, using quantifier elimination for algebraically valued fields, (2) characterizes p uniquely. Also, since γ_f does not depend on x, p is orthogonal to Γ, hence stably dominated, by Proposition 2.15. On the other hand, for any $h \in H$, replacing f by $f(h \cdot x)$, we see that (2) is invariant under H-translations. Thus if (2) holds of p, it holds of every translate, so every translate of p equals p. By Remark 3.12, p is the unique generic type of H.

Proposition 6.10 *Let G be an affine group scheme over $\mathcal{O}$. Let p be a stably dominated type of elements of $G(\mathcal{O})$. Assume:*

($\star$) *if $f \in K[G]$ and $\operatorname{val}(f(x)) \geq 0$ for $x \models p$, then $f \in \mathcal{O}[G]$.*

Then p is the unique generic of $H := G(\mathcal{O})$ if and only if $\rho_\star p$ is the unique generic type of $\rho(H)$.

Proof Let us assume that p is the unique generic of H. In an algebraic group, to show that a type is the unique generic is to show that any regular function f vanishing on the type, vanishes on the whole group. Let $f \in k[\rho(H)]$ vanish on $\rho_\star p$. Lifting to $\mathcal{O}$, we have $F \in \mathcal{O}[G]$ with $\operatorname{val}(F(a)) > 0$ for $a \models p$. By Proposition 6.9.(2), $\operatorname{val}(F(a')) > 0$ for all $a' \in H$. So f vanishes on $\rho(H)$.

The converse uses ($\star$). Assume that $\rho_\star p$ is generic in $\rho(H)$. We want to show that Proposition 6.9.(2) holds to be able to conclude. Let $F \in K[G]$. Since p is stably dominated, for some γ, for any $a \models p$, $\operatorname{val}(F(a)) = \gamma$. If $\gamma = \infty$, then, by ($\star$), any K-multiple of F lies in $\mathcal{O}[G]$, so $F = 0$. Otherwise, we may assume $\gamma = 0$. By ($\star$), it follows that $F \in \mathcal{O}[G]$. Suppose there exists $a' \in G(\mathcal{O})$ with $\operatorname{val}(F(a')) = \operatorname{val}(c) < 0$—we may take $a' \in G(\mathcal{O}_0)$, a fixed submodel. Then $c^{-1}F \in \mathcal{O}[G]$, and $\operatorname{val}(c^{-1}F(a)) > 0$ for $a \models p$, i.e. $\operatorname{res}(c^{-1}F(a)) = 0$. By genericity of $\rho_\star p$, $\operatorname{res}(c^{-1}F)$ vanishes on $\rho(H)$; so $\operatorname{val}(c^{-1}F(a')) > 0$ for all $a' \in G(\mathcal{O})$; a contradiction. ☐

Theorem 6.11 *Let G be an affine pro-algebraic group over K. Let H be a Zariski dense pro-definable connected stably dominated subgroup of $G(K)$. Then there exists a prolimit $\mathbb{H}$ of affine group schemes of finite type over $\mathcal{O}$ and an isomorphism $\phi : G \to \mathbb{H}_K$, such that $\phi(H) = \mathbb{H}(\mathcal{O})$.*

If G is an affine algebraic group and H is definable, $\mathbb{H}$ can be chosen to be an affine group scheme of finite type over $\mathcal{O}$.

In the proof below, the duality between affine schemes and algebras, which induces a duality between affine group schemes and Hopf algebras, is used to rewrite the above statement as a purely algebraic result on Hopf $\mathcal{O}$-subalgebras of $K[G]$. We then prove this purely algebraic result.

Proof Let $K_0 = K_0^{\mathrm{alg}}$ be some subfield of K over which G, H are defined, $\mathcal{O}_0 = \mathcal{O} \cap K_0$ its valuation ring, $R_0 := K_0[G]$ be the affine coordinate ring of G and p be the stably dominated generic of H. Note that since p generates H, p is Zariski dense in G. It follows that equalities in $R_0 = K_0[G]$ can be determined by evaluating at any realization of $p|K_0$. We define $S_0 = \{r \in K_0[G] : (\mathrm{d}_p x)\mathrm{val}(r(x)) \geq 0\}$. This is an $\mathcal{O}_0$-subalgebra of R_0.

Claim 6.11.1 $S_0 \otimes_{\mathcal{O}_0} K_0 = R_0$.

Proof Pick any $0 \neq r \in R_0$. Since p is stably dominated, it is orthogonal to Γ. Thus for some $c \in K_0$, for $a \models p|K_0$, $\mathrm{val}(r(a)) = \mathrm{val}(c)$. If $c = 0$, then r vanishes on p and hence on G, i.e. $r = 0 \in R_0$, contradicting the choice of r. So $c \neq 0$, and $c^{-1}r \in S_0$. This shows that the natural map $S_0 \otimes_{\mathcal{O}_0} K_0 \to R_0$ is surjective. Injectivity is clear since S_0 has no $\mathcal{O}_0$-torsion.

Let $\mathbb{H} = \mathrm{Spec}(S_0)$. We have $\mathbb{H}_K := \mathbb{H} \times_{\mathcal{O}_0} \mathrm{Spec}(K) \simeq \mathrm{Spec}(S_0 \otimes_{\mathcal{O}_0} K) = G$. We identify $\mathbb{H}_K$ with G. So p is the type of elements of $G(K)$ and in fact, by definition of S_0, of $G(\mathcal{O}) = \mathbb{H}(\mathcal{O})$.

The morphisms $x \mapsto x^{-1} : G \to G$ and $(x, y) \mapsto x \cdot y : G^2 \to G$ correspond to two operations:

$$i : R_0 \to R_0,\ i(r)(a) = r(a^{-1})$$

and

$$c : R_0 \to R_0 \otimes_{K_0} R_0,\ c(r) = \sum_{i=1}^{n} r_i \otimes s_j,\ r(a \cdot b) = \sum_{i=1}^{n} r_i(a)s_i(b).$$

Note that any $\mathcal{O}_0$-subalgebra S_0' of R_0 is $\mathcal{O}_0$-torsion-free, hence a flat $\mathcal{O}_0$-module. Thus the maps $S_0' \otimes_{\mathcal{O}_0} S_0' \to S_0' \otimes_{\mathcal{O}_0} S_0 \to S_0 \otimes_{\mathcal{O}_0} S_0$ are injective. We identify $S_0' \otimes_{\mathcal{O}_0} S_0'$ with its image in $S_0 \otimes_{\mathcal{O}_0} S_0$. Let us say that an $\mathcal{O}_0$-subalgebra S_0' of R_0 is *Hopf* if $i(S_0') \subseteq S_0'$ and $c(S_0') \subseteq S_0' \otimes_{\mathcal{O}_0} S_0'$. Note that Hopf $\mathcal{O}_0$-subalgebras of R_0 correspond, via Spec, to affine group schemes whose base change to K is a quotient of G.

Claim 6.11.2 S_0 *is Hopf*

Proof Let us first consider co-inversion. Let $s = i(r)$ for some $r \in S_0$. Clearly $\mathrm{val}(s(x)) = \mathrm{val}(r(x^{-1})) \geq 0$ for any $x \models p$. Hence $s \in S_0$.

Let us now consider co-multiplication. Pick any $(a, b) \models p^{\otimes 2}$ and $r \in S_0$. Write $c(r) = \sum_j r_j \otimes s_j$ for finitely many $r_j, s_j \in R_0$. By [7, Lemma 12.4], and definability of p, we may assume that $\mathrm{val}(r(a \cdot b)) = \min_j(\mathrm{val}(r_j(a)) + \mathrm{val}(s_j(b)))$. We may also assume that no $r_j(a)$ is zero. As in Claim 6.11.1, we renormalize so that $\mathrm{val}(r_j(a)) = 0$ for all j. Since $a \cdot b \models p$, it follows that $\min_j \mathrm{val}(s_j(b)) = \mathrm{val}(r(a \cdot b)) \geq 0$. Since both a and b realize $p|K_0$, for any $c \models p|K_0$ we have $\mathrm{val}(r_j(c)) = 0$ and $\mathrm{val}(s_j(c)) \geq 0$, i.e. $r_j, s_j \in S_0$. □

It follows that $\mathbb{H} = \mathrm{Spec}(S_0)$ is an affine group scheme over $\mathcal{O}_0$ whose base change to K is G.

Claim 6.11.3 *The type p is the unique generic of $\mathbb{H}(\mathcal{O})$.*

Proof Note that, by definition, for all $x \in \mathbb{H}(\mathcal{O})$ and $r \in S_0$, $\mathrm{val}(r(x)) \geq 0$. Pick any $r \in S_0$. As in Claim 6.11.1, we find $c \in K_0$ such that $(\mathrm{d}_p x)\mathrm{val}(r(x)) = \mathrm{val}(c)$ and $c^{-1}r \in S_0$. Thus, for all $x \in \mathbb{H}(\mathcal{O})$, $\mathrm{val}(r(x)) \geq \mathrm{val}(c)$. The claim now follows by Proposition 6.9.

Since they have the same generic, it follows that $\mathbb{H}(\mathcal{O}) = H$.

Let us now prove that $\mathbb{H}$ is a prolimit of affine group schemes of finite type. Let $\mathcal{F}$ be the family of finitely generated $\mathcal{O}_0$-subalgebras of S_0 that are Hopf. If $S_0' \in \mathcal{F}$ then $\mathrm{Spec}(S_0')$ is an affine group scheme of finite type over $\mathcal{O}_0$, which is a quotient of $\mathbb{H}$. So, to prove that $\mathbb{H}$ is the prolimit of its quotients of finite type, it suffices, dually, to show that $\mathcal{F}$ is filtered and that S_0 is the direct limit of the $S_0'\mathcal{F}$. Note that if S is generated by S_0' and S_0'' as an $\mathcal{O}_0$-algebra, then S is closed under c if S_0' and S_0'' are. Indeed $c : S_0 \to S_0 \otimes_{\mathcal{O}} S_0$ is an $\mathcal{O}_0$-algebra homomorphism, so $\{r : c(r) \in S \otimes_{\mathcal{O}_0} S\}$ is an $\mathcal{O}_0$-subalgebra of S, hence equal to S since it contains S_0' and S_0''. Moreover, by the commutativity rules between c and i, the $\mathcal{O}_0$-algebra $i(S_0')$ is also closed under c and hence, since the $\mathcal{O}_0$-algebra generated by $S_0' \cup i(S_0')$ is closed under i, it is Hopf. It follows that it suffices, given $r \in S_0$, to find a finitely generated $\mathcal{O}_0$-subalgebra S_0' of S_0 closed under c with $r \in S_0'$.

Fix $r \in S_0$ and write $c(r) = \sum_{i=1}^n r_i \otimes s_i$, with n least possible, and $r_1, \ldots, r_n, s_1, \ldots, s_n \in S_0$.

Claim 6.11.4 *Let $(a_1, \ldots, a_n) \models p^{\otimes n}|K_0$. The matrix $s = (s_i(a_j))_{1 \leq i,j \leq n}$ is invertible over K.*

Proof If not, there exists a non-zero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ in K with $\alpha \cdot s = 0$. We may assume $\alpha_n = 1$. Since the a_i are independent and α has weight at most $n - 1$, some a_j must be independent (in the underlying algebraically closed field) from α. Therefore, for any $a \models p|K_0\alpha$, $\sum_i \alpha_i s_i(a) = 0$ and hence $\sum_i \alpha_i s_i = 0$. Since p is K_0-definable and the s_i are in $R_0 = K_0[G]$, we may assume that $\alpha_i \in K_0$. Thus $s_1, \ldots, s_n$ are K_0-linearly dependent, contradicting the minimality of n.

The expression $r(x \cdot y) = \sum_{i=1}^n r_i(x) \cdot s_i(y)$ shows that $\{r(g \cdot y) : g \in G(K_0)\}$ spans a finite-dimensional K_0-subspace of $K[G]$. Similarly $\{r(g \cdot y \cdot h) : g, h \in G(K_0)\}$ spans a finite-dimensional K_0-subspace V_0 of R_0. Let S_0' be the $\mathcal{O}_0$-algebra generated by $V_0 \cap S_0$. □

Claim 6.11.5 $c(V_0) \subseteq V_0 \otimes_{\mathcal{O}_0} V_0$.

Proof Let $(g_1, \ldots, g_n, x) \models p^{\otimes n+1}|K_0$. The $r(x \cdot g_j) = \sum_i a_i(x)b_i(g_j)$ are in V_0 by definition. By Claim 6.11.4, $a_i \in V_0$. By the symmetric argument, we also have $b_i \in V_0$ and hence $r \in A_0 := \{r \in V_0 : c(r) \in V_0 \otimes_{\mathcal{O}_0} V_0\}$. Note that, by definition, A_0 is closed by left and right multiplication by $G(K_0)$, so $V_0 \subseteq A_0$ and thus $c(V_0) \subseteq V_0 \otimes_{\mathcal{O}_0} V_0$. □

By the renormalization argument of Claim 6.11.1, $V_0 \otimes_{\mathcal{O}_0} V_0 \subseteq V_0 \otimes_{\mathcal{O}_0} S_0 = S_0 \otimes_{\mathcal{O}_0} V_0$ and hence $c(V_0 \cap S_0) \subseteq (V_0 \otimes_{\mathcal{O}_0} V_0) \cap (S_0 \otimes_{\mathcal{O}_0} V_0) \cap (S_0 \otimes_{\mathcal{O}_0} V_0) \cap (S_0 \otimes S_0) = (S_0 \cap V_0) \otimes_{\mathcal{O}_0} (S_0 \cap V_0)$. It immediately follows that $c(S_0') \subseteq S_0' \otimes S_0'$. Since $r \in S_0'$, we are done.

The finite type statement follows by compactness. □

In Theorem 6.11, we cannot always assume G to be finite type:

Example 6.12 Let $K_0 = \mathbb{C}(t)^{\mathrm{alg}}$, with $\mathrm{val}(t) > 0$. Let $H_n := \{(x, y) \in (\mathbb{G}_a \times \mathbb{G}_m)(\mathcal{O}) : \mathrm{val}(y - \sum_{i=1}^n 1/i!(t \cdot x)^i) \geq \mathrm{val}(t^{n+1})\}$, a definable subgroup of $(\mathbb{G}_a \times \mathbb{G}_m)(\mathcal{O})$, isomorphic to $(\mathbb{G}_a \times \mathbb{G}_m)(\mathcal{O})$. The group $H := \cap H_n$ is connected stably dominated; but its generic is dominated via the map $(x, y) \mapsto \mathrm{res}(x)$. The Zariski closure of H has dimension 2, but the generic type of H has a residual part of transcendence degree one.

It follows that H cannot be the $\mathcal{O}$ points of a group scheme of finite type, nor can it be the connected component of a definable group.

Lemma 6.13 *Let G be a group scheme over $\mathcal{O}$. For each n, let $\phi_n(g) := g^n$, and assume $\phi_n : G \to G$ is a finite morphism. Then $G(\mathcal{O})$ is connected.*

Proof By properness, $\phi_n : G(\mathcal{O}) \to G(\mathcal{O})$ is surjective. So any finite quotient has order prime to n. This holds for all n, so $G(\mathcal{O})$ has no finite quotients.

6.2 Abelian varieties

Definition 6.14 Let V be an affine variety over K. A definable subset W is *bounded* if for any regular function f on V, $\{\mathrm{val}(f(x)) : x \in W\}$ has a lower bound. For a general K-variety V, a definable subset W is *bounded* if there exists an open affine covering $V = \bigcup_{i=1}^m U_i$, and a bounded $W_i \subseteq U_i$, with $W = \bigcup_i W_i$.

This definition is due to [27, §6.1]. The assumption there that the valuation is discrete is inessential.

If V' is a closed subvariety of V and W is bounded in V, then clearly $W \cap V'$ is bounded in V'. In the affine case, if V has coordinate ring $K[f_1, \ldots, f_n]$, for V to be bounded (as a subset of itself), it suffices that the f_i have bounded valuation. In the case of projective space $\mathbb{P}^n$, a standard covering by bounded sets is given in projective coordinates by: $U_i = \{(x_0 : \cdots : x_n) : x_i = 1 \text{ and } (\forall j \neq i)\mathrm{val}(x_j) \geq 0\}$. Complete varieties are bounded as subsets of themselves.

Let $C \models \mathrm{ACVF}$, so $C = \mathrm{dcl}(F)$ where $F = C \cap K$; F is an algebraically closed valued field. Let $\mathfrak{F}$ be the family of all C-definable functions on V into Γ. Recall (see

Proposition 2.15) that a C-definable type r concentrating on V is stably dominated if and only if for any $f \in \mathfrak{F}$ there exists $f(r) := \gamma \in \Gamma(M)$ such that if $c \models r|C$ then $f(c) = \gamma$.

Definition 6.15 Let q be a C-definable type extending a type q_0 over C, let $(p_t : t \models q_0)$ be a family of stably dominated Ct-definable types concentrating on some variety V over F and U be an open affine of V. We say that the family $(p_t)_{t \models q_0}$ is *uniformly bounded at q (on U)* if for any regular function f on U defined over $D \supseteq C$, there exists $\alpha \in \Gamma$ such that if $t \models q|D\alpha$ and $c \models p_t|Dt\alpha$ then $c \in U$ and $\mathrm{val}(f(c)) \geq \alpha$.

Lemma 6.16 *Let p_t and q be as above. Assume p_t concentrates on a bounded $W \subseteq V$. Then $(p_t)_{t \models q_0}$ is uniformly bounded at q.*

Proof The types p_t (when $t \models q_0$) concentrate on one of the bounded affine sets W_i in Definition 6.14. Let U be the corresponding affine U_i. Any regular function on U_i is bounded on all of W_i, hence in particular on realizations of the p_t. □

Recall Definition 2.27.

Lemma 6.17 *Let p_t and q be as above. Assume $(p_t)_{t \models q_0}$ is uniformly bounded at q on U. Then there exists a unique C-definable type $p_\infty := \lim_q p_t$ such that for any regular function f on U, if $a \models p_\infty$ then $\mathrm{val}(f(a)) = \lim_q \mathrm{val}(f(p_t))$.*

Moreover, p_∞ is stably dominated and if $h : V \to W$ is an isomorphism of varieties, then $\lim_q h_\star p_t = h_\star \lim_q p_t$.

Proof By assumption, we cannot have $\lim_q \mathrm{val}(f(p_t)) = -\infty$. The set $\{f : \lim_q \mathrm{val}(f(p_t)) = +\infty\}$ is a prime ideal I; part of the condition on p_∞ is that $(f = 0) \in p_\infty$ if and only if $f \in I$. Let $V' \subseteq U$ be the zero set of I. The affine coordinate ring of V' is $\mathcal{O}_V(U)/I$. An element of $F(V')$ can, therefore, be written g/h with $g, h \in \mathcal{O}_V(U)$ and $h \notin I$; equivalently $\lim_q p_t(h) \neq \pm\infty$. Hence we can define a valuation $\overline{\mathrm{val}}$ on $F(V')$ by $\overline{\mathrm{val}}(g/h) = \lim_q g(p_t) - \lim_q h(p_t)$. This determines a valued field extension F^+ of F and hence gives a complete type p_∞ of elements of V'. It is clear that p_∞ is definable; and that that $\Gamma(F^+) = \Gamma(F)$, so p_∞ is orthogonal to Γ, and hence, by Proposition 2.15, it is stably dominated. Note that $\mathrm{val}(f(p_\infty)) = \lim_q \mathrm{val}(p_t(f))$ for any $f \in F(V')$. In particular the choice of U is immaterial. The functoriality is evident. □

Recall Definition 2.28.

Lemma 6.18 *Let G be a bounded C-definable subgroup of an algebraic group $\tilde{G}$ over F. Let $(H_t)_t$ be a certifiably stably dominated C-definable family of subgroups of G, where t is a Γ-tuple, forming a directed system under inclusion and let $H := \bigcup_t H_t$. Assume G/H is Γ-internal. Then H is stably dominated. Moreover G/H is definably compact.*

Proof Let $q(t)$ be a C-definable type cofinal in the partial ordering $H_t \subseteq H_{t'}$, given by Lemma 2.24. Let p_t be the principal generic type of H_t. Note that $H = \bigcup_{t \models q|C} H_t^0$. By Lemma 6.16, using the boundedness of G, the family (p_t) is uniformly bounded at q, so $p_\infty = \lim_q p_t$ exists.

Claim 6.18.1 *Let H and H' be connected stably dominated definable subgroups of G. Let p and p' be their generic types. Then $H \subseteq H'$ if and only if $p \star p' = p'$.*

Proof If $H \subseteq H'$, then $p \star p' = p'$ by definition of genericity for p. Conversely if $p \star p' = p'$ then a generic of H is a product of two realizations of p'; in particular it lies in H'. But any element of H is a product of two generics, hence any element of H lies in H'. □

By the functoriality of Lemma 6.17, we have $\lim_q {}^a p_t = {}^a(\lim_q p_t)$ for any $a \in G$. Let $s \models q|C$ and $t \models q|Cs$. If $a \models p_s|Cs$, then ${}^a p_t = p_t$, and hence ${}^a p_\infty = p_\infty$. Thus $H_s^0 \subseteq \mathrm{Stab}(p_\infty)$ and thus $H \subseteq \mathrm{Stab}(p_\infty)$. On the other hand, as p_∞ is stably dominated and G/H is Γ-internal, the function $x \mapsto x \cdot H$ is constant on p_∞, so p_∞ lies in a single coset $x \cdot H$. It follows that $p_\infty \star p_\infty^{-1}$ is generic in H, so H is stably dominated.

Now let r be a definable type on Γ, and let $h : \Gamma \to G/H$ be a definable function. For any $t \in \Gamma$, let p'_t be the generic type of $h(t)$, viewed as a coset of H; this is a translate of the generic type of H. Let $p'_\infty = \lim_r p'_t$. Then p'_∞ is stably dominated, so it concentrates on a unique coset of H, corresponding to an element $e \in G/H$. Tracing through the definitions we see that $e = \lim_r h$. Since r and h are arbitrary, G/H is definably compact.

The following is an immediate consequence of Theorem 5.16 and Lemma 6.18. Recall that Abelian varieties are complete and hence bounded.

Corollary 6.19 *Let A be an Abelian variety over K. Then there is a definably compact group C defined over Γ, and a definable homomorphism $\phi : A \to C$ with stably dominated kernel H. In particular A has a unique maximal stably dominated connected ∞-definable subgroup—which is definable.*

If A has good reduction, i.e $A = \mathcal{A}_K$ where $\mathcal{A}$ is some Abelian scheme over $\mathcal{O}$, then, since A is the zero locus of homegeneous polynomials in some projective space, $A(K) = \mathcal{A}(\mathcal{O})$. By Proposition 6.8, $A_K(K)$ is stably dominated. Since $A_K(K)$ is divisible, it is connected. Thus, in this case, the definably compact quotient C is trivial.

In general, if F is locally compact, then the set $C(F)$ of points of C lifting to F-points will be a *finite* subgroup of the definably compact group C. On the other hand if $F = \mathbb{Q}_p((t))$, $C(F)$ can be a finite extension of $\mathbb{Z}$.

The stably dominated group H is dominated via a group homomorphism $h : H \to \mathfrak{h}$ to a stable—i.e. a k-internal—group $\mathfrak{h}$. After base change to a finite extension, $\mathfrak{h}$ becomes isomorphic to an algebraic group over k. It would be interesting to compare this with the classical theory of semi-stable reduction.

6.3 Definable fields

Before we prove the classification of fields definable in algebraically closed valued fields, let us recall a result of Zilber, cf. [32]. Note that we are not only finding a K'-vector space structure on A for some definable field K', but we are also identifying K' with the field K from which we started. As far as we know, the latter only appears in later work of Poizat, cf. [21].

Lemma 6.20 *Let $(G, \cdot)$ and $(A, +)$ be infinite Abelian groups definable in an algebraically closed field K. Assume that G acts definably on A by group automorphisms and that the action is irreducible: i.e. there are no proper non-trivial G-invariant subgroups. Then A has a definable K-vector space structure and G acts linearly on A.*

Proof Quotienting by the kernel of the action, we may assume that G acts faithfully. By Schur's lemma $R := \mathrm{End}_G(A)$ is a division ring. Since G is Abelian, we can identify G with a multiplicative subgroup of $R^\star$. As in the proof of [16, Theorem 7.8.9], we can show that the ring generated by G—which is a field—is definable and hence by [21] it is definably isomorphic to K. So we have obtained a definable action of $K^\star$ on A by group automorphisms—a definable K-vector space structure— and the action of G factorizes through the action of $K^\star$.

We will also need an notion of dimension than generalizes Krull dimension to all definable subsets of the field:

Definition 6.21 Let D be a definable subset of K^n. We write $\dim_{\mathrm{K}}(D)$ for the Krull dimension of the Zariski closure of D (inside $\mathbb{A}^n$).

Remark 6.22 (1) For any $K_0 = K_0^{\mathrm{alg}} \subseteq K$, $\dim_{\mathrm{K}}(D) = \max\{\mathrm{trdeg}(a/K_0) : a \in D\}$.
(2) $\dim_{\mathrm{K}}$ is preserved under definable bijection.
(3) Let $f : D \to X$ be some definable map with X purely imaginary. One can easily show from the above that $\dim_{\mathrm{K}}(D) = \max\{\dim_{\mathrm{K}}(f^{-1}(a)) : a \in X\}$.
(4) Any definable set D has non empty interior in its Zariski closure. It follows that if $D_1 \subseteq D_2$ are definable and $\dim_{\mathrm{K}}(D_1) = \dim_{\mathrm{K}}(D_2)$, then D_1 has non empty interior in the Zariski closure of D_2.
(5) In fact, $\dim_{\mathrm{K}}(D)$ is exactly the C-minimal dimension of D.

Theorem 6.23 *Let F be an infinite field definable in* ACVF. *Then F is definably isomorphic to the residue field or the valued field.*

Proof If F is stable, then by [6, Lemma 2.6.2], it is definable in the residue field (over some new parameters), which is a pure algebraically closed field. Hence, by [21], it is definably isomorphic to the residue field. There are no infinite fields definable over Γ, since Γ is a pure ordered divisible Abelian groups.

Let us assume that F is unstable. By Proposition 5.27, there exists a non-definably compact Γ-internal definable homorphic image V of $F^\star$. By [24, Theorem 1.2], V contains a definable one-dimensional torsion-free group. Since Γ is a pure ordered divisible Abelian group, such V contains a group isomorphic to $(\Gamma, +)$—see [4, Theorem 1.5] for a more general statement. Hence F is not boundedly imaginary.

Let D be a subrng of F, with $(D, +)$ connected stably dominated, and such that F is the field of fractions of D (cf. Proposition 5.24). There exists a surjective definable map $D \times D \to F$—namely $(x, y) \mapsto x/y$ for non-zero y, $(x, 0) \mapsto 0$. Since F is not boundedly imaginary, neither is D. Let f be the universal homomorphism of Remark 6.2 from $(F, +)$ into an algebraic group. Let I be the purely imaginary kernel. For any $c \in F$, $d \mapsto f(c \cdot d)$ is another homomorphism into an algebraic group, so it must factor through f; thus if $f(d) = 0$ then $f(c \cdot d) = 0$, i.e. I is an ideal of F. If

$I = F$, then $(D, +)$ is a purely imaginary stably dominated and by Remark 6.5, D is boundedly imaginary, a contradiction. It follows that $I \neq F$, so $I = (0)$. Hence f is an isomorphism onto a subgroup of an algebraic group G.

Let L be the limit stably dominated subgroup of $(F^\star, \cdot)$ and let $H := L \ltimes (F, +)$. Let $h : H \to G$ be the homomorphism of Proposition 6.1. The kernel is a purely imaginary subset of $H \subseteq F^2$ so it is finite. But H has no finite non-trivial normal subgroups, so h is an embedding and we identify H with its image in G.

We now proceed as in [20], with some local changes of reasoning. The Zariski closure $\overline{F}$ of F is a (connected) commutative algebraic group and the Zariski closure $\overline{L}$ acts on F by conjugation. Let $B \leq \overline{F}$ be an $\overline{L}$-invariant subgroup of $\overline{F}$. If $a \in B \cap F$, let Z be the L-orbit of a, which is in definable bijection with L. Since $F^\star/L$ is Γ-internal, we have $\dim_{\mathrm{K}}(Z) = \dim_{\mathrm{K}}(L) = \dim_{\mathrm{K}}(F^\star) = \dim_{\mathrm{K}}(F)$. So the Zariski closure of K, which is contained in B, is equal to $\overline{F}$; and $B = \overline{Z}$. If $B \cap F = \varnothing$ and B is not trivial, quotienting by B, we would be able to embed F in a variety of strictly lower dimension than $\overline{F}$, a contradiction, so B is trivial.

Applying Lemma 6.20, we see that $\overline{F}$ is a vector space over the valued field K and that the action of $\overline{L}$ is linear. We also denote this action by $\cdot$ since it extends the action of L on F by multiplication. If $r_i \in L$ and $\sum r_i = 0$, then $\sum r_i \cdot y = (\sum r_i) \cdot y = 0$ for any $y \in F$, and by Zariski density of F, for $y \in \overline{F}$. Thus the action may be extended to an action of the ring R generated by L on $\overline{F}$, again extending the action by multiplication of R on F. Note that R contains the subrng D constructed in Proposition 5.24 and hence its fraction field is F. Finally, if $0 \neq r \in R$ then r acts on F as an invertible linear transformation, since the image contains $r \cdot F = F$ and hence, by Zariski density, $\overline{F}$. Thus we can extend the action to a K-linear action of F on $\overline{F}$ which extends the action by multiplication. We also denote that action by $\cdot$.

Let Z be some non-zero orbit of L on F. Then Z is definably isomorphic to L, and, since $F^\star/L$ is Γ-internal, it has the same C-minimal dimension as F. So Z contains a non-empty open subset U of $\overline{F}$. Pick any $c \in U$, then for any $\alpha \in \mathcal{O}$ with $\mathrm{val}(\alpha - 1)$ sufficiently large, $\alpha c \in U$ and thus there exists $h \in L$ such that $h \cdot c = \alpha c$. Since F is a field, h is uniquely defined. For any $b \in L$, we have $\alpha(b \cdot c) = b \cdot (\alpha c) = b \cdot (h \cdot c) = h \cdot (b \cdot c)$ so the action by h and scalar multiplication by α agree on Z. Since Z has the same C-minimal dimension as $\overline{F}$, the linear span of Z is $\overline{F}$ and those two linear functions actually agree on $\overline{F}$.

Let $E = \{\alpha \in K : (\exists b \in F)(\forall x \in \overline{F})\alpha x = b \cdot x\}$. This is clearly a subfield of K, and it contains a neighborhood of 1. Hence it contains a neighborhood N of 0. If $0 \neq x \in K$ then $x^{-1}N \cap N$ is open and contains some non-zero u, so $x \cdot u \in N$ and $x = (x \cdot u)/u \in E$. It follows that $E = K$. Moreover, the map sending any $\alpha \in K$ to $b \in F$, such that the action by b coincides with scalar multiplication by α, is an embedding of rings. Since F has bounded dimension, for some m no definable subset of F admits a definable map onto K^m; so $\dim_K F < m$. Since K is algebraically closed we have $K \simeq F$. □

We naturally expect any infinite non-Abelian definably simple group definable in ACVF to be isomorphic to an algebraic group defined over the residue field or the valued field. A proof along the above lines may be possible assuming a positive solution to Problem 1.9 for ACVF, along with an interpretation of an ordered proper semi-group structure on $H\backslash G/H$.

The existing metastable technology yields a proof of the Abelian case:

Proposition 6.24 *Let A be an non-trivial Abelian group definable in* ACVF. *Then there exist definable subgroups $B < C \leq A$ with C/B definably isomorphic (with parameters) to an algebraic group over the residue field or a definable group over the value group.*

Proof Let H be the limit stably dominated subgroup of A, cf. Theorem 5.16. If $H < A \neq 0$ we can take $C = A$ and $B = H$. If $A = H$, then A is limit stably dominated. In particular, A contains a non-zero stably dominated definable Abelian group C and, by Proposition 4.6, C has a non trivial stable quotient. Since all stable definable sets in ACVF are internal to k and k is a pure algebraically closed field, it follows that this quotient is (isomorphic to) an algebraic group over k. □

6.4 Residually Abelian groups

Example 6.25 In ACVF, there exists connected stably dominated non-Abelian groups, with Abelian stable part:

(1) Let $A = \mathbb{G}_a^2$, and let $\beta : A^2 \to \mathbb{G}_a$ be a non-symmetric bilinear map defined over the prime field, e.g. $\beta((a_1, a_2), (b_1, b_2)) = a_1 b_2 - a_2 b_1$. Let t be an element with $\mathrm{val}(t) > 0$ and consider the following group law on A^2: $(a, b) \star (a', b') = (a + a', b + b' + t\beta(a, a'))$.
(2) If we work in $\mathbb{G}_a(\mathcal{O}/t^2\mathcal{O})$ where $\mathrm{val}(t) > 0$, we can also just take $a \star b = a + b + t\beta(a, b)$. This dimension 1 example is due to Simonetta [28]. This group does not lift to an algebraic group over $\mathcal{O}$.

However, we may ask if any connected stably dominated group of weight 1 definable is nilpotent.

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