CORRIGENDUM TO "VALUED FIELDS, METASTABLE GROUPS"

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We wish to correct an error pointed out by the third author in the paper "Valued fields, Metastable groups" [HRK19] by the first two authors. Numbering and notation follow that of the original paper. In particular, we work in a large model $\mathbb{U}$ of complete theory T that eliminates imaginaries.

As far as we know, [HRK19, Lemma 2.24] does not suffice to prove the main theorem and we need a more general version (Corollary 2.45 of the present note). This requires a slightly strengthened version of the hypothesis (FD_ω) of [HRK19], namely:

Definition 2.21. *T has (FD_ω) if, in addition to (FD), any* countable *C is contained in a metastability basis M which is an $\aleph_1$-*saturated* model and such that for any finitely acl-generated $G \subseteq \Gamma$ and $S \subseteq \mathrm{St}_{M\cup G}$ over M, isolated types over $M \cup S$ are dense.*

This stronger hypothesis also holds in ACVF. Note that a maximally complete model of ACVF is $\aleph_1$-saturated if and only if its value group and residue fields are.

Definable directed partial orders and cofinal types. The main new results consist in showing that nicely behaved definable filters can be completed to a definable type (see Proposition 2.44). This in turn implies that definable directed orders admit definable cofinal types, generalizing [HRK19, Lemma 2.24].

Definition 2.42. A *definable filter basis* on a definable set X is a definable family $\mathcal{B}$ of definable subsets of X, forming a filter basis; i.e. if $U, V \in \mathcal{B}$ then there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$. We also assume $\varnothing \notin \mathcal{B}$.

This is a strengthening of the notion of definable filter (see [HRK19, Section 2.1]): if $\mathcal{B}$ is a definable filter basis, then the filter generated by $\mathcal{B}$ is definable. The converse does not hold in general.

We will be needing two operations on filters bases:

- Let $f : X \to Y$ be definable, and let $\mathcal{B}$ be a definable filter basis on X. Then the pushfoward $f_\star\mathcal{B} = \{f(U) : U \in \mathcal{B}\}$ is a definable filter basis.
- Let $\mathcal{B}$ be a filter basis on $X \times Y$ for some (pro)-definable set X and Y (over some model M) and let $a \in X$. Assume that, for every $U \in \mathcal{B}$, $U_a = U \cap \{a\} \times Y \neq \varnothing$ — we say that $\mathrm{tp}(a/M)$ is consistent with (the filter generated by) $\mathcal{B}$. Then $\mathcal{B}_a = \{U_a : U \in \mathcal{B}\}$ is a filter basis.

Let M be an $\aleph_1$-saturated model and let Γ be a stably embedded o-minimal M-definable set. Further assume that, for any finite tuple a, $\gamma = \Gamma(Ma)$ is countably dcl-generated over M and $\mathrm{tp}(a/M\gamma)$ is definable: for any formula $\phi(x,y,z)$, the set $\{c \in M : \phi(x,c,\gamma) \in \mathrm{tp}(a/M\Gamma(Ma))\}$ is definable.

Assuming (FD_ω), any countable set C is contained in such an M.

Lemma 2.43. *Let X be a countably pro-M-definable set. Let π be a filter on $X \times \Gamma$ which is generated by countably many M-definable filter bases. Let $a \in X$ be such that $\mathrm{tp}(a/M)$ is definable and consistent with π. Then there exists an M-definable type $q \supseteq p$ concentrating on $X \times \Gamma$ which is consistent with π.*

Proof. If there exists a $\gamma \in \mathrm{dcl}(Ma)$ such that $a\gamma \models \pi$, we can choose $q = \mathrm{tp}(a\gamma/M)$. Now, assume that there is not such γ. For every family of definable functions $g_m : X \to \Gamma$, we define the following equivalence relation : for all $m, n \in M$, we say that $m \sim_g n$ if there exists $U \in \pi$ such that $[g_m(a), g_n(a)] \cap U_a = \varnothing$.

Claim 2.43.1. *The relation $\sim_g$ has finitely many classes which are all definable.*

Proof. Let $U = (U_b)_b$ be a definable filter basis contained in π. We say that $m \sim_{g,U} n$ if there exists a $b \in M$ with $[g_m(a), g_n(a)] \cap U_{b,a} = \varnothing$. This is an equivalence relation on the set of m such that there exists a b with $g_m(a) \notin U_{b,a}$ — whose complement we can see as another class. Since Γ is *o*-minimal, each $U_{b,a}$ is union of at most n intervals. Then, since the $U_{b,a}$ form a filter basis, there are at most $n+2$ classes. Moreover, the relation $\sim_g$ is the intersection of all the $\sim_{g,U}$. Choosing (countably many) representatives for all the classes of the $\sim_{g,U}$, we see that all the classes of $\sim_g$ are represented in the $\aleph_1$-saturated model M.

Now, for any $m \in M$, the class of m is the union, as U ranges over definable filter bases contained in π, of the sets $\{m' : \exists b\ [g_m(a), g'_m(a)] \cap U_{b,a} = \varnothing\}$ which are definable. So all the classes are (countably) ind-M-definable. Hence, by compactness and $\aleph_1$-saturation of M, there are finitely many classes and they are all definable. □

Let E_g denote the $\sim_g$-class of tuples m such that there exists a $U \in \pi$ such that $(-\infty, g_m(a)] \cap U_a = \varnothing$ — or $E_g = \varnothing$ if no such m exists. Let $q \supseteq p$ be the type such that $q(x,y) \vdash g_m(x) < y$ if and only of $m \in E_g$. By construction, it is definable and consistent with π. □

Proposition 2.44. *Let $\mathcal{B}$ be an M-definable filter basis on some M-definable set X. Then $\mathcal{B}$ is consistent with an M-definable type.*

Proof. Let $\mathfrak{F}$ consist of all M-definable functions $X \to \Gamma$, seen as a pro-definable function. Using Lemma 2.43 iteratively, we find an M-definable type $\mathrm{tp}(\gamma/M)$ of tuples from Γ which is consistent with $\mathfrak{F}_\star\pi$.

Note that, at stage α, we have a definable type $\mathrm{tp}(\gamma/M)$ of tuples from Γ consistent with $\mathfrak{F}_\star\pi$. So there exists some $a \models \pi|_M$ with $\gamma \in \Gamma(\mathrm{acl}(Ma))$ (via the first α functions in $\mathfrak{F}$). So γ is countably dcl-generated over M, we may assume that γ is a countable tuple to apply Lemma 2.43. In the end, we have $a \models \pi|_M$ with $\mathfrak{F}(a) = \gamma$. Then $\mathrm{tp}(a/M\gamma)$ is definable and, by transitivity, $\mathrm{tp}(a/M)$ is definable. □

Corollary 2.45. *Let $\leq$ be an M-definable directed partial order on an M-definable set P. Then there exists an M-definable type p cofinal in P : for any $c \in P$, we have $\models (\mathrm{d}_p x) x \geq c$.*

Proof. Consider the M-definable filter basis of all cones $\{x : b \leq x\}$. □

Discussion. If there exists a definable weakly order preserving map $j : \Gamma \to P$ with cofinal image, then we can use the definable type at ∞ of Γ, r_∞, to obtain a cofinal definable type of P, namely $j_\star r_\infty$.

In general, it is not always possible to find a one-dimensional cofinal subset of P. For instance, when Γ is a divisible ordered Abelian group, consider the product of two closed intervals of incommensurable sizes; or the subdiagonal part of a square.

4. Stably dominated groups

We also take this opportunity to correct an error in the statement of [HRK19, Corollary 4.12].

Corollary 4.12. *Let G be a stably dominated pro-C-definable group, $\dim_{\mathrm{st}}(G) < \infty$. Then the generic types of G over C are precisely the types $\mathrm{tp}(c/C)$ such that for h generic in G over Cc,* $\dim_{\mathrm{st}}(\mathrm{St}_C(h\cdot c)/Ch)$ is maximal (as c varies).

Proof. Let θ and $\mathfrak{g}$ be as in [HRK19, Proposition 4.6]. Since $h{\cdot}c$ is generic in G over C, $\mathrm{acl}(\mathrm{St}_C(h\cdot c)) = \mathrm{acl}(C\theta(h\cdot c)) = \mathrm{acl}(C\theta(h)\theta(c))$. So $\dim_{\mathrm{st}}(\mathrm{St}_C(h\cdot c)/Ch) = \dim_{\mathrm{st}}(\theta(h)\cdot\theta(c)/Ch) = \dim_{\mathrm{st}}(\theta(c)/Ch) \le \dim_{\mathrm{st}}(\theta(c)/C) \le \dim_{\mathrm{st}}(\mathfrak{g})$.

If $c \vDash p|\mathrm{acl}(C)$ for some stably dominated generic p, by symmetry, $c \vDash p|\mathrm{acl}(Ch)$ and hence $h{\cdot}c$ is generic in G over Ch. Thus $\theta(h{\cdot}c)$ is generic in $\mathfrak{g}$ and $\dim_{\mathrm{st}}(\mathrm{St}_C(h{\cdot}c)/Ch) = \dim_{\mathrm{st}}(\mathfrak{g})$ is maximal. Conversely, if $\dim_{\mathrm{st}}(\mathrm{St}_C(h\cdot c)/Ch)$ is maximal, $\dim_{\mathrm{st}}(\mathrm{St}_C(h\cdot c)/Ch) = \dim_{\mathrm{st}}(\theta(c)/C) = \dim_{\mathrm{st}}(\mathfrak{g})$. It follows that $\theta(c)$ is generic in $\mathfrak{g}$ over C and, by [HRK19, Lemma 4.9], c is generic in G over C. □

5. Abelian groups

We can now amend [HRK19, Section 5] using the new material.

Lemma 5.5. *(FD) Let C be a metastability basis* which is a model *and A be a pro-limit of C-definable Abelian groups with* $\dim_{\mathrm{st}}(A) = n < \infty$*. Then A contains a stably dominated pro-$C\Gamma$-definable subgroup S with stable homomorphic image of dimension n.*

Proof. Since C is a model and Morley rank is definable, we can find a $g \in A$ such that $\dim_{\mathrm{st}}(\mathrm{St}_C(g)/C) = n$. The proof now follows as in [HRK19, Lemma 5.5].

Note that, since Morley rank is definable, it is subadditive. Indeed, it suffices to check that, by induction, for every definable function $f : X \to Y$ whose fibers have constant Morley rank n, $\mathrm{MR}(X) = \mathrm{MR}(f(Y)) + n$. Assume not, then there exist $(S_i)_{i\in\omega} \subseteq X$ definable pairwise disjoint with $\mathrm{MR}(S_i) \ge \mathrm{MR}(f(Y)) + n$. We may assume that, for all i, the Morley rank of the fibers of f restricted to S_i is constant. If, for some i, $\mathrm{MR}(f(S_i)) < \mathrm{MR}(f(Y))$, then, by induction, $\mathrm{MR}(S_i) < \mathrm{MR}(f(Y)) + n$, a contradiction. So, for all i, we have $\mathrm{MR}(f(S_i)) = \mathrm{MR}(f(Y))$ and hence we may assume that there is some $a \in \bigcap_i f(S_i) \ne \emptyset$. Then, the fibers $S_{i,a} = S_i \cap f^{-1}(a)$ are all disjoint and $\mathrm{MR}(S_{i,a}) < n$. By induction, it follows that $\mathrm{MR}(S_i) < \mathrm{MR}(f(Y)) + n$, a contradiction once again. □

5.1. Limit stably dominated groups.

Definition 5.6. Let G be a pro-C-definable group and q be *a (potentially infinitary) type over* C. For all $t \vDash q$, let S_t be a pro-Ct-definable subgroup of G (uniformly in t). We call $(S_t)_{t\vDash q}$ a *limit stably dominated family* for G if:

1. S_t is a connected stably dominated subgroup of G.
2. If $W \le G$ is connected and stably dominated, pro-C-definable, then $W \le S_t$ for some $t \vDash q$.
3. The family $(S_t)_{t\vDash q}$ is directed: any small set of realizations of q has an upper bound in the order defined by $t_1 \le t_2$ if $S_{t_1} \le S_{t_2}$.

The group $H = \bigcup_{t\vDash q} S_t$ is called the *limit stably dominated subgroup* of G. If $G = H$, we say that G is limit stably dominated.

Proposition 5.9. *Let A be a pro-limit of C-definable Abelian groups. Assume A has bounded weight. Then, a limit stably dominated family for A exists over C.*

Proof. Let C_1 be a sufficiently saturated model containing C and let $C_2 \supseteq C_1$ be a metastability basis which is a model. Let $(A_i)_{i\in I}$ be the family of all connected stably dominated pro-C_1-definable subgroups of A. Let S be as in [HRK19, Proposition 5.3]. By [HRK19, Lemma 3.13], $S = S_t$ is pro-Ct-definable for some small tuple t. Let $q = \mathrm{tp}(t/C)$.

If $W \le A$ is a connected stably dominated group, we must show that $W \le S_{t'}$, for some $t' \vDash q$. For this purpose we can replace W by a conjugate, under the group of automorphisms of the universal domain over C. Thus, we may assume W is defined over C_1. In this case, $W \le S_t$.

This proves (2) of the definition of a limit stably dominated family. Directness of the family $(S_t)_{t \models q}$ follows from [HRK19, Proposition 5.3], together with (2). $\square$

Note that if $C = C_2$ can be assumed to be a sufficiently saturated metastability basis, then we can choose q to be a type concentrating on some power of Γ as is the case in [HRK19, Definition 5.6].

Lemma 5.10. *Let C be a metastability basis* which is a model*. Let A be a C-definable Abelian group and let H be a connected stably dominated ∞-C-definable subgroup of A.*

(FD) *H is contained in a $C\Gamma$-definable stably dominated subgroup.*
(FD_ω) *H is contained in a $C\Gamma$-definable connected stably dominated subgroup.*

The proof is the same as [HRK19, Lemma 5.10], except that we use the new version of Lemma 5.5, requiring C to be a model.

Proposition 5.12. *(FD_ω) Let A be a C-definable Abelian group. Then there exist C*-definable families H^ν of definable subgroups H^ν_t of A*, such that:*

1. *Any H^ν_t is stably dominated.*
2. *Any connected stably dominated ∞-definable subgroup of A (over any set of parameters) is contained in some H^ν_t.*

Proof. By Lemma 5.10, to prove (2), it suffices to consider definable connected stably dominated subgroups of A.

For a definable Abelian group B, define invariants n and k as follows. First, set $n = \dim_{\mathrm{st}}(B)$. Let $Z(B)$ be the collection of definable subgroups $S \leq B$ with stable homomorphic images of dimension n; by Lemma 5.5 and [HRK19, Corollary 4.16], $Z(B) \neq \varnothing$. Let $Z_2(B) = \{(S,T) : S \in Z(B), T \leq S\ C\Gamma\text{-definable}, S/T\ \Gamma\text{-internal}\}$. Let $k = \max\{\dim_{\mathrm{o}}(S/T) : (S,T) \in Z_2(B)\}$; by (FD), such a maximum exists. The pairs (n,k) are ordered lexicographically.

Pick any definable connected stably dominated definable $B \leq A$. If $(S/B, T/B)$ attains the maximum for A/B, then the pullbacks to A show that $(n,k)(A) \geq (n,k)(A/B)$. Thus increasing B has the effect of decreasing $(n,k)(A/B)$. Let B_0 be such that $(n,k)(A/B_0)$ is minimal. For any stably dominated $H \leq A$, $H + B_0$ is also stably dominated and if S_t is a family of stably dominated subsets of A/B_0, its lifting to A is a family of stably dominated subsets of A; so it suffices to find families $\{S^\nu_t\}$ for A/B_0. Thus, we may assume $(n,k)(A/B) = (n,k)(A)$ for any connected stably dominated $C\Gamma$-definable $B \leq A$. Let $(n,k) = (n,k)(A)$.

Claim 5.12.1. *Let S be a definable subgroup of A admitting a definable surjective homomorphism $\theta : S \to \mathfrak{g}$ to a definable stable group of Morley rank n and a definable surjective homomorphism $\xi : S \to W$ to a definable Γ-internal group of o-minimal dimension k. Then there exists a C-definable family H_t of stably dominated subgroups of A such that* $\ker(\xi) = H_t$ *for some t.*

Proof. Clearly $S, \xi, \mathfrak{g}, \theta, W$ lie in a C-definable family $(S_t, \xi_t, \mathfrak{g}_t, \theta_t, W_t)$ such that $S_t \leq A$, $\mathfrak{g}_t$ is stable, $\dim_{\mathrm{st}}(\mathfrak{g}_t) = n$, W_t is a Γ-internal, $\dim_{\mathrm{o}}(W_t) = k$, $\theta_t : S_t \to \mathfrak{g}$ is a surjective homomorphism, $\xi_t : S_t \to W_t$ is a surjective homomorphism. Let $H_t = \ker(\xi_t)$. Let $T_t \leq S_t$ be as in [HRK19, Proposition 4.14]. By [HRK19, Lemma 4.15], T_t is Ct-definable. By [HRK19, Lemma 2.26], S_t/T_t is Γ-internal. The group $T_t/H_t \cap T_t$ is Γ-internal and stably dominated, it is therefore finite and hence trivial, since T_t is connected. So $T_t \leq H_t$. By maximality of k, H_t/T_t is Γ-internal of o-minimal dimension 0, so it is finite. It follows that H_t is stably dominated. $\square$

Let B be any definable stably dominated connected subgroup of A. Let $(S,T) \in Z_2(A/B)$ be such that S has a stable homomorphic image of dimension n and S/T is Γ-internal of o-minimal dimension k. Let (S', T') be the pullbacks of (S, T) to A. Then $B \leq T'$. Clearly S' admits a stable homomorphic image of dimension n and S'/T' is Γ-internal of o-minimal dimension k. By the Claim 5.12.1, there exists a C-definable family H_t of stably dominated groups such that $T' = H_t$ for some t. □

Corollary 5.13. *(FD_ω) Let A be a C-definable Abelian group. There exists a* definable certifiably stably dominated family H_t *such that any connected stably dominated ∞-definable subgroup of A is contained in some H_t.*

Proof. Let H_t^ν be as in Proposition 5.12 and let p' and $S_{t'}$ be as in Proposition 5.9. Let $a \vDash p'$, then, for some ν and t, we have $S_a \leq H_t^\nu$. Let $p = \mathrm{tp}(t/C)$. Then, for $t' \vDash p'$, for some $t \vDash p$, $S_{t'} \leq H_t^\nu$. In particular, any ∞-definable connected stably dominated subgroup of A is contained in some H_t^ν, with $t \vDash p$.

For any $t \vDash p$, $H_t^{\nu,0}$ is definable and has finite index, say l, in H_t^ν. Let L_t be a C-definable family of subgroups of H_t^ν such that $[H_t^\nu : L_t] \leq l$ and whenever $t \vDash p$, $L_t = H_t^{\nu,0}$. Then every L_t is stably dominated and any ∞-definable connected stably dominated subgroup of A is contained in some L_t with $t \vDash p$.

Finally, let Q be the set of t such that for every s, if $L_t/L_s \cap L_t$ is finite then it is trivial. This set is definable by [HRK19, Lemma 2.20]. If $t \vDash p$, then L_t is connected and hence $t \in Q$. Moreover, for any $s \in Q$, $L_s^0 \leq L_t$ for some $t \vDash p$. But then $L_s^0 \leq L_s \cap L_t$, so $L_s/L_s \cap L_t$ is finite and hence trivial, by definition of Q. It follows that $L_s \leq L_t$, concluding the proof. □

Theorem 5.16. *Let A be a pro-limit of interpretable Abelian groups. Assume A has bounded weight. Then the limit stably dominated subgroup H exists and A/H is almost internal to Γ.*

If (FD_ω) holds and A is interpretable, then A/H is Γ-internal and H is definable, it admits a generic type p and it is connected.

Proof. The existence and properties of H are all established as in [HRK19, Theorem 5.16], expect for the existence of the definable cofinal type on Q (which might not be a subset of some Γ^n) which follows from Corollary 2.45. □

6. Valued fields: stably dominated groups and algebraic groups

We conclude by pointing out the cursory modifications induced on [HRK19, Section 6].

Lemma 6.18. *Let G be a bounded C-definable subgroup of an algebraic group $\tilde{G}$ over F. Let $(H_t)_t$ be a certifiably stably dominated C-definable family of subgroups of G forming a directed system under inclusion and let $H := \bigcup_t H_t$. Assume G/H is Γ-internal. Then H is stably dominated. Moreover G/H is definably compact.*

The proof is identical as that of [HRK19, Lemma 6.18], except that we now use Corollary 2.45 instead of [HRK19, Lemma 2.24].

The following is an immediate consequence of Theorem 5.16 and Lemma 6.18.

Corollary 6.19. *Let A be an Abelian variety over K. Then there is a definably compact group C defined over Γ, and a definable homomorphism $\phi : A \to C$ with stably dominated kernel H. In particular A has a unique maximal stably dominated connected ∞-definable subgroup—which is definable.*

References

[HRK19] Ehud Hrushovski and Silvain Rideau-Kikuchi. Valued fields, metastable groups. *Sel. Math., New Ser.*, 25(3):58, 2019. Id/No 47.

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