# Imaginaries in equicharacteristic zero henselian fields

Silvain Rideau-Kikuchi* Mariana Vicaría

September 27, 2024

We prove an elimination of imaginaries results for most henselian valued fields of equicharacteristic zero. To do so, we consider a mix of sorts introduced in earlier works of the two authors and define a generalized version of the k-linear imaginaries. For a broad class of value groups containing all subgroups of $\mathbb{R}^n$ for some $n$, we prove that the imaginaries of such a valued field can be eliminated in the field, the k-linear imaginaries and the imaginaries of the value group.

2020 Mathematics Subject Classification. Primary: 03C45, 12J10, 12L12; Secondary: 03C10, 03C60.

## 1 Introduction

In the model theory of valued fields, one of the most striking results is a theorem by Ax, Kochen and, independently, Ershov which roughly states that the first-order theory of an unramified henselian valued field is completely determined by the first-order theory of its residue field k and of its value group $\Gamma$. A natural philosophy follows from this theorem: the model theory of a henselian valued field is controlled by its residue field and its value group.

In this paper we show that this philosophy also applies to the question of eliminating imaginaries: the classification of interpretable sets, that is, quotients of definable sets by definable equivalence relations; or equivalently, the description of moduli spaces for families of definable sets.

The study of imaginaries in various henselian valued fields has been ongoing in the past 20 years, starting with the case of algebraically closed valued field (ACVF) in the foundational work by Haskell, Hrushovski and Macpherson [HHM06]. This work laid the groundwork for a "geometric model theory" of valued fields. They proved that in ACVF, every quotient

*The first author was partially supported by GeoMod AAPG2019 (ANR-DFG), Geometric and Combinatorial Configurations in Model Theory

can be described as a subset of products of certain specific quotients, known as the *geometric sorts* : the main field K, and, for all $n \in \mathbb{Z}_{>0}$, the space $\mathrm{Gr}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ of free rank $n$ $\mathcal{O}$-submodules of $\mathrm{K}^n$ and the space $\mathrm{Lin}_n = \bigsqcup_{R \in \mathrm{Gr}_n} R/\mathfrak{m}R$, where $\mathcal{O}$ denotes the valuation ring and $\mathfrak{m} \subseteq \mathcal{O}$ is the unique maximal ideal. We say that ACVF eliminates imaginaries down to the geometric sorts. These results were later extended to other classes of henselian fields, potentially with additional structure [Mel06; HMR18; HKR18; Rid19].

In the early 2000, Hrushovski asked if such results could be explained by general result and proposed a classification reminiscent of the Ax-Kochen-Ershov principle. This paper provides a positive answer to this question for a broad class of henselian valued fields of equicharacteristic zero.

As can be expected, the natural obstructions to elimination of imaginaries in valued fields come in two flavors: those coming from the residue field, studied in work of Hils and the first author [HR21], and those coming from the value group, studied in work of the second author [Vic23a].

## 1.1 Obstructions arising from the residue field

In [HR21], Hils and the first author assume the value group to be definably complete — this only allows divisible ordered abelian groups and groups elementarily equivalent to $\mathbb{Z}$ — and classify the imaginaries that might arise. This includes the imaginaries of the residue field k, which might be arbitrarily complicated, but also linearly twisted versions.

Given a free rank $n$ $\mathcal{O}$-submodule $R \subseteq K^n$, the quotient module $R/\mathfrak{m}R$ is a k-vector space of dimension $n$, on which k induces a non-trivial structure. Once we name a basis, $R/\mathfrak{m}R$ is definably isomorphic to $\mathrm{k}^n$, but without that basis, imaginaries of $R/\mathfrak{m}R$ cannot be identified with imaginaries of k.

The structure $(\mathrm{k}, R/\mathfrak{m}R)$ can be seen as a structure in the language $\mathfrak{L}_{\mathrm{vect}}$ with two sorts:

- a field sort k with the ring language,
- a vector space sort V with the (additive) group language,
- A function $\lambda : \mathrm{k} \times \mathrm{V} \to \mathrm{V}$ interpreted as scalar multiplication.

Given a set $X$ interpretable (without parameters) in the $\mathfrak{L}_{\mathrm{vect}}$-theory of dimension $n$ vector spaces, the interpretable sets $X^{(\mathrm{k}, R/\mathfrak{m}R)}$ has to be accounted for. To do so, one introduces

$$\mathrm{Lin}_{n,X} = \bigsqcup_{R \in \mathrm{Gr}_n} X^{(k, R/\mathfrak{m}R)}$$

and the k-*linear sorts*

$$\mathrm{k}^{\mathrm{leq}} = \bigsqcup_{n,X} \mathrm{Lin}_{n,X}.$$

In fact, it suffices to consider interpretable sets $X$ that are quotients of V (and not some power of V and k). Note that if $X = \mathrm{V}$ then $\mathrm{Lin}_{n,X} = \mathrm{Lin}_n$, and if $X$ is the one element quotient of $V$ then $\mathrm{Lin}_{n,X} \simeq \mathrm{Gr}_n$.

One of the main result of [HR21], is that — under some mild hypothesis on k — these are the only obstructions to elimination of imaginaries in that case:

**Theorem 1.1** ([HR21, Theorem 6.1.1]). *Let $K$ be a henselian valued field of equicharacteristic zero such that:*

- *The value group is definably complete;*
- *The residue field eliminates $\exists^\infty$.*

*Then $K$ has weak elimination of imaginaries down to* $\mathrm{K} \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma$.

This result can be generalized to finite ramification and certain difference valued fields and it remains true when considering k-Γ-expansions of $K$ — that is when k is given additional structure and independently, so is Γ.

## 1.2 Obstructions arising from the value group

In [Vic23a], the second author studied imaginaries in equicharacteristic zero henselian valued fields with algebraically closed residue field. The complexity of the value group directly impacts the complexity of definable $\mathcal{O}$-modules and this needs to be taken in account.

This can be done by introducing the *stabilizer sorts* which provide codes for all the definable $\mathcal{O}$-submodules of $\mathrm{K}^n$, for any $n$. More precisely, let $\mathcal{C} = (\mathcal{C}_c)_{c \in \mathrm{Cut}^\star}$ be the (ind)-definable family of proper cuts in Γ. For every $c \in \mathrm{Cut}^\star$, let $\mathrm{I}_c$ denote the $\mathcal{O}$-submodule $\{x \in \mathrm{K} : \mathrm{v}(x) \in \mathcal{C}_c\}$. For every tuple $c \in \mathrm{Cut}^\star$, let $\Lambda_c$ be the module $\sum_i I_{c_i} e_i$, where $(e_i)_{i<n}$ is the canonical basis of $\mathrm{K}^n$.

The group $\mathrm{B}_n$ of upper triangular matrices acts on the set of all definable $\mathcal{O}$-submodules of $\mathrm{K}^n$, and we define

$$\mathrm{Mod}_c = \mathrm{B}_n / \mathrm{Stab}(\Lambda_c) \text{ and } \mathrm{Mod} = \bigsqcup_c \mathrm{Mod}_c.$$

In [Vic23a], the second author proved that, under some hypotheses on the value group, the stabilizer sorts are the only obstructions to elimination of imaginaries:

**Theorem 1.2** ([Vic23a, Theorem 5.12]). *Let $K$ be a valued field of equicharacteristic zero, such that:*

- *the residue field is algebraically closed;*
- *the value group has* bounded regular rank — i.e. *it has countably many definable convex subgroups.*

*Then $K$ admits weak elimination of imaginaries down to* $\mathrm{K} \cup \mathrm{Mod} \cup \Gamma^{\mathrm{eq}}$.

## 1.3 An imaginary Ax-Kochen-Ershov principle

In this paper, building on those previous works, we provide a common generalization of both Theorems 1.1 and 1.2, obtaining a general Ax-Kochen-Ershov principle for the classification of imaginaries, under a mild technical assumption on the value group (we refer the reader to Section 2.1 for notation related to imaginaries):

**Definition 1.3.** We say that an ordered group $G$ (potentially with additional structure) satisfies Property **D** if for every finite set of formulas $\Delta(x, y)$ containing the formula $x < y_0$, any $A = \mathrm{acl}(A) \subseteq G^{\mathrm{eq}}$ and any $\Delta$-type $p(x)$ that is $A$-definable, there is an $A$-definable complete type $q(x)$ containing $p$.

This is a stronger property than the density of definable types. It holds in ordered abelian groups of bounded regular rank (see the second half of the proof of [Vic23a, Theorem 5.3]), and it is an open question whether or not it holds in all ordered abelian groups.

*Dealing with both a complicated value group and a complicated residue field, requires introducing a version of the* k*-linear sorts adapted to this generalized setting where more definable* $\mathcal{O}$*-modules might be arise. These also required an encoding of the stabilizer sorts which is more alike the geometric sorts of [HHM06].*

Let $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^{\star} \smallsetminus \{0^+\}$ — unless the value group is discrete, in which case $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^{\star}$. A module $R$ is said to be $\mathfrak{m}$-avoiding if it is (coded) in $\mathrm{Mod}_c$, for some tuple $c \in \mathrm{Cut}^{\star\star}$. The dimension of $R/\mathfrak{m}R$ only depends on $c$ — it is equal to $r = |\{i : \mathcal{C}_{c_i} = \gamma^-$, for some $\gamma \in \Gamma\}|$.

For every quotient $X$ of V interpretable (without parameters) in the $\mathfrak{L}_{\mathrm{vect}}$-theory of dimension $r$ vector spaces, we define

$$\mathrm{Lin}_{c,X} = \bigsqcup_{R \in \mathrm{Mod}_c} X^{(\mathrm{k}, R/\mathfrak{m}R)}$$

and the *(generalized)* k*-linear imaginaries*:

$$\mathrm{k}^{\mathrm{leq}} = \bigsqcup_{c \in \mathrm{Cut}^{\star\star}, X} \mathrm{Lin}_{c,X}.$$

Among those, we denote $\mathrm{Gr} = \bigsqcup_{c \in \mathrm{Cut}^{\star\star}} \mathrm{Mod}_c$ and $\mathrm{Lin} = \bigsqcup_{c \in \mathrm{Cut}^{\star\star}} \mathrm{Lin}_{c,\mathrm{V}}$. Along with K, these form the *(generalized) geometric sorts*, and they encode all $\mathcal{O}$-definable submodules of $\mathrm{K}^n$, for any $n$.

Our main results are the following. Let $M$ be a model of $\mathrm{Hen}_{0,0}$ (potentially with additional structure on k and, independently $\Gamma$) such that the value groups is *either*:

- dense with property **D**;
- a pure discrete ordered abelian group of bounded regular rank — in which case, we add a constant for a uniformizer.

**Theorem** (Theorem 6.5)**.** *Assume that* either one *of the following conditions holds:*
*(a) for every* $n \in \mathbb{Z}_{>1}$ *one has* $[\Gamma : n\Gamma] < \infty$ *— in which case, we add constants in* RV *so that* $\Gamma/n\Gamma = \Gamma/n\mathrm{v}(\mathrm{RV}(\mathrm{acl}(\varnothing)))$*;*
*(b) or, the multiplicative group* $k^\times$ *is divisible.*
*Then* $M$ *weakly eliminates imaginaries down to* $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.

This result generalizes all previously known results in equicharacteristic zero (in particular, [HR21; Vic23a]) and provides a definitive answer for, among others, all equicharacteristic zero henselian valued fields with bounded Galois group — in fact a bounded ramification group suffices.

Note that without condition (a) or (b), the short exact sequence

$$1 \to \mathrm{k}^\star \to \mathrm{RV} = \mathrm{K}^\star/(1+\mathfrak{m}) \to \Gamma \to 0$$

does not, in general, eliminate imaginaries, creating further obstructions. This is not an issue in presence of an angular component, *i.e.* a section of this short exact sequence.

**Theorem** (Theorem 6.6)**.** *Let $M_{\mathrm{ac}}$ be an expansion of $M$ by angular components. Then $M_{\mathrm{ac}}$ weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

As a corollary (and an illustration) of these two results, when $\Gamma$ is dense, we give a complete classification of (almost) k-internal sets (see Corollary 6.7). In the case of ACVF this classification is a cornerstone of the study of stable domination and the subsequent work of Hrushovski and Loeser [HL16] on Berkovich spaces.

The complexity of dealing with both an arbitrary residue field and a very general value group introduces new issues that were not present in either earlier works of the authors [HR21; Vic23a]. We give details below on some of the new tools required to deal with these issues.

## 1.4 Overview of the paper

Section 2 provides some preliminary reminders on imaginaries and the model theory of equicharacteristic zero henselian fields.

In Section 3, we introduce the stabilizer sorts, which provide codes for the new definable $\mathcal{O}$-modules that arise from the value group, and we prove a unary decomposition for the stabilizer sorts (Proposition 3.9). We also introduce the generalized geometric sorts, and show that modules are codes in these sorts. The generalized geometric sorts — and the related notion of $\mathfrak{m}$-avoiding module — play a crucial role in classifying k-internal sets among the stabilizer sorts (Corollary 3.22), assuming that the value group is dense.

In Section 4, *we show that definable types in the structure induced from the maximal unramified algebraic extension are dense,* cf. *Theorem 4.1. This is the first main step of the proof.* Recent work on elimination of imaginaries relies on density of definable types. However this does not hold in all (equicharacteristic zero) henselian field $K$ because it might not hold in the residue field or the value group. In [HR21, Theorem 3.1.3] Hils and the first author showed that definable types in $K^{\mathrm{a}}$ are dense among definable sets in K, under the assumption that the residue field eliminates $\exists^{\infty}$. The second author ([Vic23a, Theorem 5.9]) proved density of definable types in the maximal unramified algebraic extension $K^{\mathrm{ur}}$, assuming the value group has bounded regular rank. In Section 4, we generalize both results and unify them by proving that the definable types in $K^{\mathrm{ur}}$ are also dense among the sets definable in $K$ (assuming Property **D**). This is the unique step where Property **D** is required and no further hypothesis on the residue field is required anymore. A significant new challenge in this construction is to relate the germs of functions definable in $K$ to those of functions definable in $K^{\mathrm{ur}}$ — see Section 4.3.

In Section 5, *we show that the partial definable types build in Section 4 have completions that are invariant over* RV *and* k*-vector spaces of the form $R/\mathfrak{m}R$, for some definable $\mathcal{O}$-modules $R$. This is the second main step of the proof.* The bulk of the work (Proposition 5.17) revolves around showing that (generalized) geometric points can be lifted to the valued field by a sufficiently invariant type. This, in turn, relies heavily on the technical computation of germs of function taking values in sets of the form $R/\mathfrak{m}R$ — *cf.* Proposition 5.4. We describe such germs in three steps: first we consider the case of valued fields with algebraically closed residue field (Section 5.1.1), then valued fields with dense value group (and arbitrary residue field) (Section 5.2.1) and, lastly, valued fields with discrete value group (and arbitrary residue field) (Section 5.2.2). This step is what distinguishes our treatment of the discrete case from the dense one, as it relies on the characterization of the k-internal sets (Corollary 3.22), which does not hold in the discrete case. In the discrete case, we circumvented this issue by considering a ramified extension with dense value group.

Finally, in Section 6, we wrap everything together and show our two main theorems.

### Acknowledgments

The authors wish to thank E. Hrushovski, T. Scanlon and P. Simon for many enlightening discussions on this topic.

## 2 Preliminaries

### 2.1 Model theoretic preliminaries

We refer the reader to [TZ12, Section 8.4] for a detailed exposition on elimination of imaginaries. Let $T$ be an $\mathcal{L}$-theory. Consider the language $\mathcal{L}^{\mathrm{eq}}$ obtained by adding to $\mathfrak{L}$ a new sort $S_X$ for every $\mathcal{L}$-definable set $X \subseteq Y \times Z$, where $Y$ and $Z$ are product of sorts, and a new symbol $f_X : Z \to S_X$. The $\mathcal{L}^{\mathrm{eq}}$-theory $T^{\mathrm{eq}}$ is then obtained as the union of $T$, the fact that the $f_X$ are surjective and that their fibers are the classes of the equivalence relation defined by $X_{z_1} = \{y \in Y : (y, z_1) \in X\} = X_{z_2}$.

Any $M \vDash T$ has a unique expansion to a model of $T^{\mathrm{eq}}$ denoted $M^{\mathrm{eq}}$ — whose points are called the *imaginaries*. Throughout this paper, when considering types, definable closures or algebraic closures, we will work in the $\mathfrak{L}^{\mathrm{eq}}$-structure, unless otherwise specified.

Given $M \vDash T$ and an $\mathcal{L}(M)$-definable set $X$, we denote by $\ulcorner X \urcorner \subseteq M^{\mathrm{eq}}$ the intersection of all $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ such that $X$ is $\mathcal{L}^{\mathrm{eq}}(A)$-definable. It is the smallest dcl-closed set of definition for $X$. Any dcl-generating subset of $\ulcorner X \urcorner$ is called *a code* of $X$. More generally, if $A \subseteq M^{\mathrm{eq}}$ is a set of parameters, any tuple $e$ such that $\mathrm{dcl}(Ae) = \mathrm{dcl}(A\ulcorner X \urcorner)$ is called a code of $X$ over $A$.

If $\mathcal{D}$ is a collection of sorts of $\mathcal{L}^{\mathrm{eq}}$ — equivalently, a collection of $\mathcal{L}$-interpretable sets — and $A \subseteq M^{\mathrm{eq}}$ is a set of parameters, we say that $X$ is *coded* in $\mathcal{D}$ over $A$ if it is $\mathcal{L}^{\mathrm{eq}}(A \cup \mathcal{D}(\ulcorner X \urcorner))$-definable — *i.e.*, it admits a code in $\mathcal{D}$ over $A$.

The theory $T$ is said to *eliminate imaginaries* down to $\mathcal{D}$ if, for every $M \vDash T$, every $\mathcal{L}(M)$-definable set $X$ is coded in $\mathcal{D}$ — equivalently, for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{dcl}(e))$ such that $e \in \mathrm{dcl}(d)$. Finally, we say that the theory $T$ *weakly eliminates imaginaries* down to $\mathcal{D}$ if for every $e \in M^{\mathrm{eq}}$, there is some $d \in \mathcal{D}(\mathrm{acl}(e))$ such that $e \in \mathrm{dcl}(d)$.

Lastly, we refer the reader to [Rid17, Appendix A (Definition A.2 and A.5)] for a detailed presentation of expansions (called enrichments, there) and relative quantifier elimination. Note that in the present text, expansions do not allow adding new sorts.

## 2.2 Equicharacteristic zero henselian fields

Throughout this text, whenever $X$ is a definable set and $A$ is a subset of a structure, $X(A)$ denotes $X \cap A$. We change structures too often to not be explicit with the definable closures at play.

Let $\mathrm{Hen}_{0,0}$ be the theory of residue characteristic zero valued fields $(\mathrm{K}, \mathrm{v})$ in some language $\mathfrak{L}$. The exact language we use does not matter much since we really work in $\mathfrak{L}^{\mathrm{eq}}$. In this section, we recall some useful results about these structures. We denote by $\mathrm{RV}^{\star}$ the group $\mathrm{K}^{\star}/(1+\mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal of the valuation ring $\mathcal{O} \subseteq \mathrm{K}$ and $\mathrm{rv} : \mathrm{K} \to \mathrm{RV} = \mathrm{RV}^{\star} \cup \{0\}$ the canonical projection (extended by $\mathrm{rv}(0) = 0$). Let $M \vDash \mathrm{Hen}_{0,0}$ and $A \leq \mathrm{K}(M)$.

**Theorem 2.1** ([Bas91, Theorem B]). *Every $A$-definable subset of $\mathrm{K}^x \times \mathrm{RV}^y$ is of the form $\{(x,y) : (\mathrm{rv}(P(x)), y) \in X\}$, for some tuple $P \in A[x]$ and some $X \subseteq \mathrm{RV}^n$ which is $\mathrm{rv}(A)$-definable in the short exact sequence*

$$1 \to \mathrm{k}^{\star} \to \mathrm{RV}^{\star} \to \Gamma \to 0.$$

*where $\mathrm{k} = \mathcal{O}/\mathfrak{m}$ is the residue field and $\Gamma = \mathrm{v}(\mathrm{K})$ is the value group. Moreover, this remains true in RV-expansions* — i.e. *when* RV *comes with additional structure.*

From the result above, either by adding a section or proving a quantifier elimination result for short exact sequences, we can deduce the following:

**Proposition 2.2.** *The sets* k *and* $\Gamma$ *are stably embedded (with respectively the structure of a field and an ordered group) and they are orthogonal. In other words, any $M$-definable subset of $\mathrm{k}^x \times \Gamma^y$ is a finite union of products $X \times Y$ where $X$ is definable in the field* k *and $Y$ is definable in the ordered group $\Gamma$.*

*Moreover, any $A$-definable $X \subseteq \Gamma^n$ is $\mathrm{v}(A)$-definable. In particular,*

$$\Gamma^{\mathrm{eq}}(\mathrm{acl}(A)) \subseteq \mathrm{acl}(\mathrm{v}(A)).$$

These results remain true in k-$\Gamma$-expansions — *i.e.* when k comes with additional structure, and, independently so does $\Gamma$.

Theorem 2.1 can also be refined for unary sets — showing that $\mathrm{Hen}_{0,0}$ is 0-h-minimal:

**Proposition 2.3** ([Fle11, Proposition 3.6]). *Let $X \subseteq \mathrm{K} \times \mathrm{RV}^n$ be $A$-definable. There exists a finite set $C \subseteq A^{\mathrm{a}} \cap \mathrm{K}(M)$ such that for every $\xi \in \mathrm{RV}^n$, $X_{\xi} = \{x \in \mathrm{K} : (x,\xi) \in X\} = \mathrm{rv}_C^{-1}(\mathrm{rv}_C(X_{\xi}))$ where $\mathrm{rv}_C(x) = (\mathrm{rv}(x-c))_{c \in C}$.*

*In other words, for any ball $b$ that does not intersect $C$, $b \cap X_{\xi} = b$ or $b \cap X_{\xi} = \emptyset$.*

**Definition 2.4.** Let $a \in \mathrm{K}(M)$ and let $C$ be a cut in $\Gamma(M)$ — that is, an upwards closed subset. We define the generalized ball $b_C(a)$ of cut $C$ around $a$ to be $\{x \in \mathrm{K} : \mathrm{v}(x-a) \in C\}$. A generalized ball is open if its cut is not of the form $\Gamma_{\geq\gamma}$, for some $\gamma \in \Gamma(M)$.

Let $\mathrm{B_g}$ denote the set of (codes for) generalized balls.

Note that, for every $\gamma \in \Gamma(M)$, $b_{\Gamma_{>\gamma}}(a)$ is the open ball of radius $\gamma$ around $a$, $b_{\Gamma_{\geq\gamma}}(a)$ is the closed ball of radius $\gamma$ around $a$ — $b_\Gamma(a) = \mathrm{K}$ is also considered an open ball. Hence, a generalized ball is either a closed ball, an open ball or an open generalized ball which is not a ball.

**Corollary 2.5.** *Let $b$ be an $A$-algebraic generalized ball which is not an open ball. Then there exists a finite $C \subseteq A^{\mathrm{a}}$ such that $C \cap b(M) \neq \varnothing$ and the valuation between any two distinct points of the* $\mathrm{Gal}(A)$*-orbit of $C$ is smaller than the radius of $b$.*

Here we identify the radius of $b$ with its upwards closure in $\Gamma(M^{\mathrm{a}})$.

*Proof.* Let $B$ be the union of $A$-conjugates of $b$. Then, there exists $C \subseteq \mathrm{K}(\mathrm{acl}(A)) \subseteq A^{\mathrm{a}}$ be such that, for any ball $d$ avoiding $C$, either $d \subseteq B$ or $d \cap B = \varnothing$. If $b \cap C = \varnothing$, then $b$ is the largest ball around $b$ avoiding $C$, *i.e.* the open ball around $b$ with radius $\min_{c \in C} \mathrm{v}(x - c)$, for any $x \in b$. This contradicts the fact that $b$ is not an open ball. So $b(M) \cap C \neq \varnothing$.

Let $C_b \subseteq \mathrm{Gal}(A) \cdot C$ be the subset of points that are at a valuation larger than the radius of $b$ from $b(M) \cap C$. Since $M$ is henselian, the average $c$ of $C_b$ is in $M$ and, since we are in equicharacteristic zero, it is in $b$. By construction, each $\mathrm{Gal}(A)$-conjugates of $c$ is at a valuation smaller than the radius of $b$ from $c$. □

Finally, when the residue field is algebraically closed, Theorem 2.1 can be further simplified:

**Theorem 2.6** ([Vic23a, Corollary 2.33])**.** *Assume the residue field* $\mathrm{k}(M)$ *is algebraically closed. Every $A$-definable subset of $\mathrm{K}^x$ is of the form $\mathrm{v}(P(x)) \in X$ where $P \in A[x]$ is a tuple and $X \subseteq \Gamma^n$ is $\mathrm{v}(A)$-definable in the ordered group structure. Moreover, this remains true in $\Gamma$-expansions* — i.e. *when $\Gamma$ comes with additional structure.*

# 3 Codes of $\mathcal{O}$-modules

## 3.1 The stabilizer sorts

Let $M$ be an (enriched) valued field.

**Notation 3.1.** We fix an (ind-)definable family $\mathcal{C} = (\mathcal{C}_c)_{c \in \mathrm{Cut}}$ of cuts in $\Gamma$ such that any $M$-definable cut is of the of form $\mathcal{C}_c$ for some unique $c \in \mathrm{Cut}(M)$. We will further assume that $c$ is a canonical parameter for $\mathcal{C}_c$.

For every $c \in \mathrm{Cut}$, let $\mathrm{I}_c$ denote the $\mathcal{O}$-submodule $\{x \in \mathrm{K} : \mathrm{v}(x) \in \mathcal{C}_c\}$. Note that, by hypothesis, any $\mathfrak{L}(M)$-definable $\mathcal{O}$-submodule of $\mathrm{K}$ is of the form $a\mathrm{I}_c$ for some $a \in \mathrm{K}(M)$ and some unique $c \in \mathrm{Cut}(M)$. We also denote $\Delta_c = \{\gamma \in \Gamma : \gamma + \mathcal{C}_c = \mathcal{C}_c\}$ — it is a convex subgroup of $\Gamma$.

The following results are well-established and go back to Bauer's work on separated extensions.

**Definition 3.2.** A definable valuation $v$ on an interpretable $\mathrm{K}$-vector space $V$ is a map to some interpretable set $X$ with an order preserving action of $\Gamma$ such that

- for every $a \in \mathrm{K}$ and $x \in V$, $v(ax) = \mathrm{v}(a) + v(x)$;
- for every $x, y \in V$, $v(x+y) \geq \min\{v(x), v(y)\}$.

**Proposition 3.3.** *Assume that $M$ is definably spherically complete — that is, the intersection of any $M$-definable chain of balls is non empty.*

1. *For every $M$-definable valuation $v$ on $\mathrm{K}^n$, there exists a triangular basis $(a_i)_{i<n}$ of $\mathrm{K}^n$ such that, for all $i$, $v(a_i) \in \mathrm{dcl}(\ulcorner v \urcorner)$ and for every $\lambda_i \in \mathrm{K}$,*

$$v(\sum_i \lambda_i a_i) = \max_i \mathrm{v}(\lambda_i) \cdot v(a_i).$$

2. *Any $M$-definable $\mathcal{O}$-submodule $R$ of $\mathrm{K}^n$ is of the form $\sum_{i<n} \mathrm{I}_{c_i} a_i$, where $a_i$ is a triangular basis of $\mathrm{K}^n(M)$ and $c_i \in \mathrm{Cut}(\ulcorner R \urcorner)$.*

A basis as in the fist assertion is said to be separated. A module as in the second assertion is said to be of type $c = (c_i)_{i<n}$.

*Proof.* If $M$ is (elementarily equivalent to a) maximally complete field, the first assertion is [Vic23a, Lemma 5.7]. If $M$ is only definably spherically complete, the same proof works using [HR21, Claim 3.3.9] instead of [Vic23a, Fact 2.55].

Let us now prove the second assertion. For every $a \in \mathrm{K}^n$, we define $v_R(a) = \{v(x) : xa \in R\}$ a (non-empty) cut of $\Gamma$. We order them by inclusion (so $\Gamma = v_R(0)$ is the maximal element and $\{\infty\}$ is the minimal element). Note that, for every $x \in \mathrm{K}$, $v_R(xa) = v_R(a) - \mathrm{v}(x)$ and for this action of $\Gamma$, $v_R$ is an $M$-definable valuation.

By the first assertion, we can find a separated triangular basis $(a_i)_i$ of $\mathrm{K}^n(M)$, such that $v_R(a_i) \in \mathrm{dcl}(\ulcorner R \urcorner)$. Then $\sum_i x_i a_i \in R$ if and only if $0 = \mathrm{v}(1) \in v_R(\sum_i x_i a_i) = \min_i v_R(a_i) - \mathrm{v}(x_i)$, *i.e.* $\mathrm{v}(x_i) \in v_R(a_i)$ for all $i$. Let $c_i \in \mathrm{Cut}(\mathrm{dcl}(\ulcorner R \urcorner))$ be such that $v_R(a_i) = \mathcal{C}_{c_i}$. We then have $R = \sum_i \mathrm{I}_{c_i} a_i$, as required. □

**Notation 3.4.**

1. We write $B_n$ to denote the set of $n \times n$ upper triangular and invertible matrices. We write $\mathrm{D}_n \leq \mathrm{B}_n$ for the subgroup of diagonal matrices and $\mathrm{U}_n \leq \mathrm{B}_n$ for the subgroup of unipotent matrices, that is upper triangular matrices with ones on the diagonal.
2. For every $n$-tuple $c \in \mathrm{Cut}$, we define $\mathrm{Mod}_c$ to be the interpretable set of modules of type $c$ and $\Lambda_c = \sum \mathrm{I}_{c_i} e_i$, the canonical module of type $c$, where $e_i$ is the canonical basis of $K^n$. Then $\sum_{i<n} \mathrm{I}_{c_i} a_i = A \cdot \Lambda_c$ where $A \in \mathrm{B}_n$ is the upper triangular matrix of the $a_i$. In other words, $\mathrm{B}_n$ acts transitively on $\mathrm{Mod}_c$ and

$$\mathrm{Mod}_c \simeq \mathrm{B}_n / \mathrm{Stab}(\Lambda_c).$$

We will now identify $\mathrm{Mod}_c$ with this quotient of $\mathrm{B}_n$ and for every $s \in \mathrm{Mod}_c$, we write $R_s$ for the $\mathcal{O}$-module of type $c$ coded by $s$. Let $\mu_c : \mathrm{B}_n \to \mathrm{Mod}_c$ denote the natural quotient map.

If $\Delta \leq \Gamma$ is a (definable) convex subgroup, we write $\mathcal{O}_\Delta = \{x \in \mathrm{K} : \mathrm{v}(x) \in \Delta\}$ for the associated (definable) valuation ring. If $I, J \leq \mathrm{K}$ are two (definable) $\mathcal{O}$-submodules, let $(I : J)$ denote the (definable) $\mathcal{O}$-submodule $\{x \in \mathrm{K} : xJ \subseteq I\}$.

**Proposition 3.5.** *Let $c \in \mathrm{Cut}$ be a tuple. For every $a \in \mathrm{B}_n$, we have*

$$a \in \mathrm{Stab}(\Lambda_c) \text{ if and only if } \begin{cases} a_{i,i} \in \mathcal{O}^\times_{\Delta_{c_i}} & \text{for all } i < n, \text{ and} \\ a_{i,j} \in (\mathrm{I}_{c_i} : \mathrm{I}_{c_j}) & \text{for all } i < j < n. \end{cases}$$

*Proof.* We proceed by induction on $n$. Write $a$ as $\left(\begin{smallmatrix} a_{0,0} & b \\ 0 & e \end{smallmatrix}\right)$, with $e \in \mathrm{B}_{n-1}$, and $c$ as $(c_0, d)$, with $d \in \mathrm{Cut}^{n-1}$. If $a\Lambda_c \subseteq \Lambda_c$, then, considering the action on $\mathrm{I}_{c_0}$ and $\Lambda_d$, we see that $a_{0,0}\mathrm{I}_{c_0} \subseteq \mathrm{I}_{c_0}$, $b\Lambda_d \subseteq \mathrm{I}_{c_0}$ — so, considering the action on each $\mathrm{I}_{c_j}$, for every $j > 0$, $a_{0,j}\mathrm{I}_{c_j} \subseteq \mathrm{I}_{c_0}$ — and $e\Lambda_d \subseteq \Lambda_d$; and the converse also holds.

Since $a\Lambda_c = \Lambda_c$ if, moreover, $a^{-1}\Lambda_c = \left(\begin{smallmatrix} a_{0,0}^{-1} & -a_{0,0}^{-1}be^{-1} \\ 0 & e^{-1} \end{smallmatrix}\right)\Lambda_c \subseteq \Lambda_c$, it follows that we must further have $a_{0,0}\mathrm{I}_{c_0} = \mathrm{I}_{c_0}$, *i.e.* $\mathrm{v}(a_{0,0}) \in \Delta_{c_0}$ and $e\Lambda_d = \Lambda_d$. These conditions are sufficient since, in that case, $a_{0,0}^{-1}be^{-1}\Lambda_d = a_{0,0}^{-1}b\Lambda_d \subseteq a_{0,0}\mathrm{I}_{c_0} = \mathrm{I}_{c_0}$. The claim now follows by induction. ☐

**Definition 3.6.** Let $\mathrm{Cut}^\star = \mathrm{Cut} \smallsetminus \{\varnothing, \Gamma\}$ and Mod be the collection of all the $\mathrm{Mod}_c$ where $c$ is a tuple in $\mathrm{Cut}^\star$.

**Corollary 3.7.** *Any $M$-definable $\mathcal{O}$-submodule $R$ of $\mathrm{K}^n$ is coded in $\mathrm{K} \cup \mathrm{Mod}$.*

*Proof.* Let $V \subseteq \mathrm{K}^n$ be the K-span of $R$ and $W = \{x \in \mathrm{K}^n : \mathrm{K}x \subseteq R\}$. Then $V/W$ is $\mathrm{K}(\ulcorner R \urcorner)$-definably isomorphic to some $\mathrm{K}^r$ and $R$ is entirely determined by its image in $V/W$. So we may assume $V = \mathrm{K}^r$ and $W = 0$ and hence that $R$ is of type $c$ with $c \in \mathrm{Cut}^\star$. By definition, it is coded in $\mathrm{Mod}_c$. ☐

**Remark 3.8.** There is a lot of redundancy in Mod. If $c$ and $c'$ are tuples in Cut of the same length such that for every $i < n$, $c'_i$ is a translate of $c_i$, then there is a natural bijection between $\mathrm{Mod}_c$ and $\mathrm{Mod}_{c'}$ given by the action of a diagonal matrix.

If there exists an (ind-)definable subset $\mathrm{Cut}' \subseteq \mathrm{Cut}$ such that any definable cut is of the form $a + \mathcal{C}_c$ for a unique $c \in \mathrm{Cut}'$, it follows that every $M$-definable $\mathcal{O}$-submodule of $\mathrm{K}^n$ is coded in $\mathrm{K} \cup \bigcup_{c \in \mathrm{Cut}' \smallsetminus \{\varnothing, \Gamma\}} \mathrm{Mod}_c$. Similarly, we can replace Cut by $\mathrm{Cut}'$ in the definition of the geometric sorts (Definition 3.12).

This is the case, for example, in ordered abelian groups of bounded regular rank (*cf.* [Vic23a, Corollary 2.24]).

Let us now describe the structure of Mod. The solvability of the upper triangular invertible matrices will play a central role in this description.

We go through the elements of an upper triangular matrix diagonal by diagonal starting at the middle diagonal, and in each diagonal, we proceed from top to bottom. In other words, we order pairs $(i, j)$ such that $i \leq j < n$ first by $i - j$ and then by $i$. We will identify the set of such pairs with the set of non-negative integers smaller than $n(n+1)/2$, according to that order.

For every pair $(i, j)$, let $p_{i,j} : \mathrm{B}_n \to \mathrm{K}$ be the projection on coordinate $(i, j)$. Let also $\varepsilon_{i,j} = 1$ if $i = j$ and 0 otherwise. For every pair $\ell$, let $G_\ell = \{a \in \mathrm{B}_n : p_k(a) = \varepsilon_k,\ \forall k < \ell\}$. Then $G_0 = \mathrm{B}_n$, $G_n = \mathrm{U}_n$ and $G_{n(n+1)/2} = \{\mathrm{id}\}$. By choice of the order, for every $\ell$, $G_{\ell+1} \lhd G_\ell$ and $p_\ell$ induces an isomorphism from $G_\ell/G_{\ell+1}$ to $\mathbb{G}_\mathrm{m}$, if $\ell < n$, and to $\mathbb{G}_\mathrm{a}$

otherwise. Note also that $H_\ell = \{a \in G_\ell : p_k(a) = \varepsilon_k,\ \forall k > \ell\}$, is a section of $p_\ell$ restricted to $G_\ell$ and hence $G_\ell = G_{\ell-1} \rtimes H_\ell$.

Furthermore, we have $G_n = \mathrm{U}_n$, $\mathrm{B}_n = \mathrm{U}_n \rtimes \mathrm{D}_n$ and, for every $\ell \geq n$, $G_\ell$ is central in $G_n$ module $G_{\ell+1}$ — actually modulo the next upper triangular group $G_{0,j}$,— if $\ell$ is a pair $(i, i+j-1)$. In particular $G_\ell \trianglelefteq \mathrm{U}_n$.

We can now prove the following unary decomposition.

**Proposition 3.9.** *Let $s \in \mathrm{Mod}_c$. There exists a finite tuple $b = (b_\ell)_\ell \in M^{\mathrm{eq}}$ (identified with a subset of some $\mathrm{K}^{r_\ell}$) and $cb_{<\ell}$-interpretable sets $X_\ell$ such that:*

- *for every $\ell$, $b_\ell \in X_\ell$;*
- $\mathrm{dcl}(s) = \mathrm{dcl}(cb)$*;*
- *if $\ell < n$, then $X_\ell = \Gamma/\Delta_{c_i}$, where $\ell = (i,i)$;*
- *if $\ell \geq n$, then $X_\ell$ has a $cb_{<\ell}$-definable $\mathrm{K}/\mathrm{I}_\ell$-torsor structure where $\mathrm{I}_\ell$ is a $cb_{<\ell}$-definable multiple of $(\mathrm{I}_{c_i} : \mathrm{I}_{c_j})$ and $\ell = (i,j)$.*

*Moreover, for any choice of $a_k \in b_k$ there is a (uniformly) $ca_{<\ell}$-definable isomorphism $f_\ell : X_\ell \to \mathrm{K}/\mathrm{I}_\ell$ and a $ca_{<\ell}$-definable function $g_\ell : f_\ell(b_\ell) \to b_\ell$.*

*Proof.* Let $F = \mathrm{Stab}(\Lambda_c)$, we identify $s$ with a coset $gF$ for some $g \in \mathrm{B}_n$. Let $d \in \mathrm{D}_n$ and $u \in \mathrm{U}_n$ be such that $g = ud$. Note that, by Proposition 3.5, $F = F_\mathrm{U} \rtimes F_\mathrm{D}$ where $F_\mathrm{U} = F \cap \mathrm{U}_n$ and $F_\mathrm{D} = F \cap \mathrm{D}_n$, so, by (the proof of) [HHM08, Lemma 11.10], $\mathrm{dcl}(s) = \mathrm{dcl}(\ulcorner dF_\mathrm{D} \urcorner, \ulcorner uF_\mathrm{U}^d \urcorner)$, where $F_\mathrm{U}^d = dFd^{-1} \cap \mathrm{U}_n$. Then, by Proposition 3.5, $\mathrm{D}_n/F_\mathrm{D} \simeq \prod_{\ell<n} \mathrm{K}^\star/\mathcal{O}^\star_{\Delta_\ell} \simeq \prod_{\ell<n} \Gamma/\Delta_i$ and, for every $\ell < n$, we chose $b_\ell = v_{\Delta_\ell}(d_\ell)$ to be the $\ell$-th coordinate in this product.

Now, for every $\ell \geq n$, note that $F_\mathrm{U}^d G_\ell = G_\ell F_\mathrm{U}^d$ is a subgroup, since $G_\ell$ is normal in $\mathrm{U}_n$ and moreover, $F_\mathrm{U}^d G_{\ell+1} \trianglelefteq F_\mathrm{U}^d G_\ell$ since for every $g \in G_\ell$, $(F_\mathrm{U}^d)^g \subseteq F_\mathrm{U}^d G_{\ell+1}$ by centrality of the sequence. For every $\ell \geq n$, let $X_\ell = uF_\mathrm{U}^d G_\ell / F_\mathrm{U}^d G_{\ell+1}$ for the right regular action and $b_\ell = uF_\mathrm{U}^d G_{\ell+1} \in X_\ell$. Note that $X_\ell$ is a torsor for the group

$$F_\mathrm{U}^d G_\ell / F_\mathrm{U}^d G_{\ell+1} \simeq G_\ell/(F_\mathrm{U}^d \cap G_\ell)G_{\ell+1} \simeq \mathrm{K}/p_\ell(F_\mathrm{U}^d \cap G_\ell) \simeq \mathrm{K}/d_i d_j^{-1}(\mathrm{I}_{c_i} : \mathrm{I}_{c_j}),$$

by Proposition 3.5, if we have $\ell = (i,j)$. Let $\mathrm{I}_\ell = d_i d_j^{-1}(\mathrm{I}_{c_i} : \mathrm{I}_{c_j})$.

Now, any choice of $a_{\ell-1} \in b_{\ell-1} = G_\ell F_\mathrm{U}^d$ gives rise to an element of $X_\ell$ and hence to a $cda_{\ell-1}$-definable isomorphism $f_\ell : X_\ell \simeq \mathrm{K}/\mathrm{I}_\ell$ given by left multiplication by $a_{\ell-1}^{-1}$. Let $s : \mathrm{K} \to H_\ell$ be the section of $p_\ell$. Then for every $x \in f_\ell(b_\ell)$, we have $s(x) \in a_{\ell-1}^{-1} uF_\mathrm{U}^d G_{\ell+1}$ and hence $a_{\ell-1}s(x) \in b_\ell$. □

**Remark 3.10.** Looking at the proof, all the operation applied to any upper triangular matrix representation of $s$ are actually field operations. It follows that Proposition 3.9 can be refined as follows. If $A \leq \mathrm{K}(M)$ is a subfield, and $s \in \mu_c(\mathrm{B}_n(A))$, then for every $\ell$, $b_\ell(A) \neq \varnothing$. Furthermore, if $a_\ell \in b_\ell(A)$, then $f_\ell(b_\ell)(A) \neq \varnothing$ and $g_\ell$ sends $f(b_\ell)(A)$ to $b_\ell(A)$. Conversely, if $b_\ell(A) \neq \varnothing$, for all $\ell$, then $s \in \mu_c(\mathrm{B}_n(A))$.

We conclude this section with one of our main uses for Proposition 3.9: characterizing parameter sets over which every definable module has a (triangular) basis.

**Corollary 3.11.** *Let $A \subseteq M^{\mathrm{eq}}$ and assume that:*

1. *For every $\mathfrak{L}(A)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\mathrm{dcl}(A)) \subseteq v_\Delta(\mathrm{K}(A))$.*

2. *For every* $b \in \mathrm{B_g}(\mathrm{dcl}(A))$, $b(A) \neq \varnothing$.

*Then, for every tuple* $c \in \mathrm{Cut}$, $\mathrm{Mod}_c(\mathrm{dcl}(A)) \subseteq \mu_c(\mathrm{B}_n(A_\mathrm{K}))$, *where* $A_\mathrm{K}$ *is the field generated by* $\mathrm{K}(A)$.

*Proof.* Let $s \in \mathrm{Mod}_c(\mathrm{dcl}(A))$ — in particular $c \in \mathrm{Cut}(\mathrm{dcl}(A))$ — and $b = (b_\ell)_\ell$ be as in Proposition 3.9 applied in $M_1$. By Remark 3.10, it suffices to show that, by induction on $\ell$, $b_\ell(A_\mathrm{K}) \neq \varnothing$. For $\ell < n$, since $b_\ell \in \Gamma/\Delta_\ell$, this follows from the first assumption.

If $\ell \geq n$, by induction, $b_\ell$ is $\mathcal{L}_1(cA_\mathrm{K})$-interdefinable with some $b'_\ell \in \mathrm{K}/\mathrm{I}_\ell$. By Remark 3.10, $b'_\ell(M) \neq \varnothing$, so $b'_\ell$ is an $\mathfrak{L}(A)$-definable generalized ball in $M$ and hence $b'_\ell(A) \neq \varnothing$ by the second assumption. By Remark 3.10 again, we also have $b_\ell(A_\mathrm{K}) \neq \varnothing$. □

## 3.2 The geometric sorts

The goal of this section is to further simplify the codes of modules to something more akin to the geometric sorts of [HHM06]. This will be crucial to classify k-internal sets, when the value group is non discrete, in Corollary 6.7.

Let $M$ be an (enriched) valued field with non-discrete valued group.

**Definition 3.12.** 1. Let $\mathrm{Cut}^{\star\star}$ denote $\mathrm{Cut}^\star \smallsetminus \{0^+\}$ — unless $\Gamma$ is discrete, in which case, $\mathrm{Cut}^{\star\star} = \mathrm{Cut}^\star$. Any module of type a tuple $c \in \mathrm{Cut}^{\star\star}$ is said to be $\mathfrak{m}$-avoiding. Let Gr be the collection codes for all $\mathfrak{m}$-avoiding modules; that is $\mathrm{Gr} = \bigsqcup_{c\in\mathrm{Cut}^{\star\star}} \mathrm{Mod}_c$.

2. For every $\mathcal{O}$-module $R$, let $\mathrm{red}(R)$ denote the k-vector space $R/\mathfrak{m}R$ and let $\mathrm{red}_R : R \to \mathrm{red}(R)$ denote the canonical projection. We also define $\mathrm{Lin} = \bigsqcup_{s\in\mathrm{Gr}} \mathrm{red}(R_s)$.
3. Let $\mathcal{G} = \mathrm{K} \cup \mathrm{Gr} \cup \mathrm{Lin}$ be the (generalized) geometric sorts.

**Remark 3.13.** For every $c \in \mathrm{Cut}^\star$, we have $\mathfrak{m}\mathrm{I}_c = \mathrm{I}_c$ if and only if $\mathcal{C}_c \neq \Gamma_{\geq 0}$. Indeed, for every $x \in \mathrm{I}_c$, if $\mathcal{C}_c \neq \Gamma_{\geq 0}$, there exists $a \in \mathrm{I}_c$ such that $\mathrm{v}(a) < \mathrm{v}(x)$. Then $xa^{-1} \in \mathfrak{m}$ and hence $x = xa^{-1}a \in \mathfrak{m}\mathrm{I}_c$.

It follows that if $c \in \mathrm{Cut}^\star$ is some tuple and $R$ is an $\mathcal{O}$-module of type $c$. Then $\mathrm{red}(R)$ has dimension $|\{i : \mathcal{C}_{c_i} = \Gamma_{\geq 0}\}|$ over k.

**Lemma 3.14.** *Let* $R \subseteq \mathrm{K}^n$ *be an* $\mathcal{O}$*-module of type* $c$ *for some tuple* $c \in \mathrm{Cut}^\star$. *Then there exists an* $\ulcorner R \urcorner$*-definable* $\mathfrak{m}$*-avoiding module* $\overline{R}$ *containing* $R$ *and such that* $\mathfrak{m}R = \mathfrak{m}\overline{R}$. *In particular,* $\mathrm{red}(R) \subseteq \mathrm{red}(\overline{R})$ *is a subspace and* $\mathrm{dcl}(\ulcorner R \urcorner) = \mathrm{dcl}(\ulcorner \overline{R} \urcorner, \ulcorner \mathrm{red}(R) \urcorner)$.

*Proof.* Let $v_R$ be the valuation defined in the proof of Proposition 3.3. Recall that $R = \{x \in \mathrm{K}^n : \Gamma_{\geq 0} \subseteq v_R(x)\}$ and let $\overline{R} = \{x \in \mathrm{K}^n : \Gamma_{>0} \subseteq v_R(x)\}$. If $e_i$ is a (triangular) basis such that $R = \sum_i \mathrm{I}_{c_i} e_i$, then $\overline{R} = \sum_i \overline{I}_{c_i} e_i$. Since $\overline{\mathfrak{m}} = \mathcal{O}$ and $\overline{I}_{c_i} = \mathrm{I}_{c_i}$ otherwise, then, by Remark 3.13, $\mathfrak{m}\overline{\mathcal{O}} = \mathfrak{m}\mathcal{O} = \mathfrak{m} = \mathfrak{m}\overline{\mathfrak{m}}$ and $\mathfrak{m}\overline{I}_{c_i} = I_{c_i} = \mathfrak{m}\mathrm{I}_{c_i}$, otherwise. It follows that $\mathfrak{m}\overline{R} = \mathfrak{m}R$.

The last assertion follows from the fact that $\mathrm{red}_{\overline{R}}^{-1}(\mathrm{red}(R)) = R$. □

Let us now recall, following [HHM08, Lemma 2.6.4], how to code definable subspaces of $\mathrm{red}(\overline{R})$. The following abstract conditions were isolated in [Hru12].

**Proposition 3.15.** *Let* $T$ *be some theory,* k *be some* $\varnothing$*-definable field and* $\bigsqcup_s V_s$ *be a collection of finite dimensional* $\varnothing$*-definable* k*-vector spaces which*

1. *is closed under tensors: for every $s, r$, there is an $\emptyset$-definable injection from the interpretable set $V_s \otimes V_s$ into some $V_t$;*
2. *is closed under duals: for every $s$, there is an $\emptyset$-definable injection from the interpretable set $V_s^{\vee}$ into some $V_t$;*
3. *has flags: For every $s$, there exists $r, t$ and a $\emptyset$-definable exact sequence $0 \to V_r \to V_s \to V_t \to 0$, with $\dim(V_r) = 1$.*

*Then any definable subspace $W \subseteq V_s$, is coded in $\bigcup_s V_s$.*

*Proof.* By (the proof of) [Hru12, Proposition 5.2], $W$ is coded in some projective space $\mathbb{P}(V_r)$. By [Hru12, Lemma 5.6], given that the family has flags, $\mathbb{P}(V_r)$ is coded in $\bigcup_s V_s$. □

We consider once again an (enriched) valued field $M$.

**Definition 3.16.** For every $A \subseteq M^{\mathrm{eq}}$, let $\mathrm{Lin}_A = \bigsqcup_{s \in \mathrm{Gr}(\mathrm{acl}(A))} \mathrm{red}(R_s)$.

Before we prove that Proposition 3.15 can be applied to $\mathrm{Lin}_A$, let us prove the following useful computation:

**Lemma 3.17.** *Let $I, J \leq \mathrm{K}$ be ($M$-definable) $\mathcal{O}$-submodule. If $(I : J) = \mathfrak{m}$, then $I = a\mathfrak{m}$ and $J = a\mathcal{O}$, for some $a \in \mathrm{K}$.*

*Proof.* We first argue that $\mathrm{v}(I) \subsetneq \mathrm{v}(J)$. If $\mathrm{v}(J) \subseteq \mathrm{v}(I)$, then $J \subseteq I$ so $\mathrm{Stab}(J) \subseteq (I : J)$, where $\mathrm{Stab}(J) = \{x \in K : xJ = J\}$. Since $\mathrm{Stab}(J) = \mathcal{O}_{\Delta}^{\times}$ for some definable convex subgroup $\Delta$ of the value group, and $\mathfrak{m}$ does not contain $\mathcal{O}_{\Delta}^{\times}$ then $\mathrm{v}(I) \subsetneq \mathrm{v}(J)$.

We aim to show that $\mathrm{v}(J)$ has a minimal element. Otherwise, given $\gamma \in \mathrm{v}(J) \setminus \mathrm{v}(I)$ there is some $\beta \in \mathrm{v}(J)$ such that $\beta < \gamma$, thus $\gamma = \beta + \delta$ where $\delta = \gamma - \beta > 0$. Take $x \in \mathfrak{m}$ and $y \in J$ such that $v(x) = \delta$ and $v(y) = \beta$. Consequently, $xy \notin I$ since $v(xy) = \gamma$, so $(I : J) \neq \mathfrak{m}$. Let $\gamma_0$ be the minimal element of $\mathrm{v}(J)$ and $a \in K$ such that $v(a) = \gamma_0$ thus $J = a\mathcal{O}$. To show that $I = a\mathfrak{m}$ it is sufficient to argue that $\mathrm{v}(J) \setminus \mathrm{v}(I) = \{\gamma_0\}$. If there is some $\beta \in \mathrm{v}(J) \setminus \mathrm{v}(I)$ such that $\beta \neq \gamma_0$, then $\beta - \gamma_0 > 0$. Take $x \in \mathfrak{m}$ such that $v(x) = \beta - \gamma_0$, then $x \in \mathfrak{m}$ but $ax \notin I$, hence $(J : I) \neq \mathfrak{m}$. Thus, $I = a\mathfrak{m}$, as required. □

**Proposition 3.18.** *For every $A \subseteq M^{\mathrm{eq}}$, $\mathrm{Lin}_A$ is a collection of finite dimensional $\mathrm{k}$-vector spaces which is closed under tensors, duals and has flags.*

*Proof.* Let $R_1 \subseteq \mathrm{K}^n$ and $R_2 \subseteq \mathrm{K}^m$ be two $\mathrm{acl}(A)$-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-modules. Let $f : \mathrm{K}^n \otimes \mathrm{K}^n \to \mathrm{K}^{nm}$ be the $\emptyset$-definable isomorphism induced by the canonical basis. By Lemma 3.14, we find an $A$-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-module $R = \overline{f(R_1 \otimes R_2)}$ inducing an inclusion $\mathrm{red}(f(R_1 \otimes R_2)) \subseteq \mathrm{red}(R)$. Since $\mathrm{red}(R_1) \otimes \mathrm{red}(R_2)$ is $A$-definably isomorphic to $\mathrm{red}(R_1 \otimes R_2)$, we conclude that $\mathrm{Lin}_A$ is closed under tensors.

As for duals, for every $\mathcal{O}$-submodule $I \subseteq \mathrm{K}$, let $I^{\vee} = (\mathcal{O} : I) = \{x \in \mathrm{K} : xI \subseteq \mathcal{O}\}$. By Lemma 3.17, $I^{\vee} \neq \mathfrak{m}$. Let $g : (\mathrm{K}^n)^{\vee} \to \mathrm{K}^n$ be the $\emptyset$-definable isomorphism induced by the canonical basis. Then, if $R = \sum_i \mathrm{I}_{c_i} a_i$ for some triangular basis $a_i$, then $g(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})) = \sum_i \mathrm{I}_{c_i}^{\vee} g(a_i)$, which is $\mathfrak{m}$-avoiding. This induces an $A$-definable isomorphism $\mathrm{red}(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})) \simeq \mathrm{red}(g(\mathrm{Hom}_{\mathcal{O}}(R_1, \mathcal{O})))$ which shows that $\mathrm{Lin}_A$ is closed under duals.

Finally, regarding flags, let $R = \sum_i \mathrm{I}_{c_i} a_i \subset \mathrm{K}^n$ be an $\mathrm{acl}(A)$-definable $\mathfrak{m}$-avoiding $\mathcal{O}$-module, with $a$ triangular. We find a flag for $\mathrm{red}(R)$ by induction on $n$. Let $\pi$ be the projection on the last $n-1$ variables. Then, we have an $A$-definable short exact sequence

$$0 \to \mathrm{I}_{c_0} a_0 \to R \to \pi(R) \to 0$$

which induces the following $A$-definable short exact sequence

$$0 \to \mathrm{I}_{c_0} a_0 / \mathfrak{m} \mathrm{I}_{c_0} a_0 \to R/\mathfrak{m}R \to \pi(R)/\mathfrak{m}\pi(R) \to 0.$$

If $\mathrm{I}_{c_0} = \mathfrak{m}\mathrm{I}_{c_0}$, then $\mathrm{red}(R) \simeq \mathrm{red}(\pi(R))$ and we conclude by induction on $n$. If not, $\mathrm{I}_{c_0} a_0 / \mathfrak{m}\mathrm{I}_{c_0} a_0$ is a dimension one k-vector space, and the above short exact sequence is a flag for $\mathrm{red}(R)$. □

**Remark 3.19.** Fix some $A \le M^{\mathrm{eq}}$.

1. $\mathrm{Lin}_A$ is stably embedded and its $A$-induced structure is definable in the structure with the field on k and the vector space structure on each sort $R_s/\mathfrak{m}R_s$. In fact, $\mathrm{Lin}_A \cup \mathrm{RV}$ is stably embedded by [HR21, Lemma 2.5.18]. Indeed, once we name a basis of every vector space in $\mathrm{Lin}_A$, every definable subset in $\mathrm{Lin}_A \cup \mathrm{RV}$ can be identified with a definable subset in RV, which is stably embedded.
2. Whenever k is algebraically closed, combining [Hru12, Lemma 5.6] with Proposition 3.18, $\mathrm{Lin}_A$, with its $A$-induced structure, eliminates imaginaries.

We can now improve Corollary 3.7:

**Corollary 3.20.** *Any $M$-definable $\mathcal{O}$-submodule $R$ of type $c \in \mathrm{Cut}^\star$ is coded in $\mathcal{G}$.*

Conversely, any element $a + \mathfrak{m}s \in \mathrm{Lin}$ is coded by the $\mathcal{O}$-submodule generated by $(a + \mathfrak{m}s) \times \{1\}$. So any element of $\mathcal{G}$ is coded in $\mathrm{K} \cup \mathrm{Mod}$.

*Proof.* By Lemma 3.14, $R$ is coded by $\ulcorner \overline{R} \urcorner \in \mathrm{Gr}$ for some $\mathfrak{m}$-avoiding module containing $R$ and such that $\mathfrak{m}R = \mathfrak{m}\overline{R}$ and $\ulcorner \mathrm{red}(R) \urcorner$ which is a subspace of $\mathrm{red}(\overline{R})$. By Propositions 3.15 and 3.18, $\mathrm{red}(R)$ is coded in $\mathrm{Lin}_{\ulcorner \overline{R} \urcorner} \subseteq \mathrm{Lin}$. □

One of the main reason for isolating the $\mathfrak{m}$-avoiding modules is the following result.

**Proposition 3.21.** *Let $M$ be an* RV*-expansion of a model of* $\mathrm{Hen}_{0,0}$ *with dense value group. If $X \subseteq \mathrm{Gr}$ is $M$-definable and orthogonal to $\Gamma$, then it is finite.*

*Proof.* Let us first consider the case of some $M$-definable $X \subseteq \mathrm{K}/\mathrm{I}_c$ for some $c \in \mathrm{Cut}^{\star\star}$. Let $Y \subseteq \mathrm{K}$ be the pre-image of $X$. By Proposition 2.3, there exists a (non-empty) finite set $C \subseteq \mathrm{K}(M)$ such that, any ball $b$ disjoint from $C$, is either contained in $Y$ or is disjoint from it. If $X$ is infinite, then there exists some $a \in Y$ such that $a + I$ is disjoint from $C$. Let $b$ be the maximal ball around $a$ that is disjoint from $C$ — *i.e.* the ball $a + (\min_{c \in C} \mathrm{v}(a - c))\mathfrak{m}$. Then $b \subseteq Y$. Since $\mathrm{I}_c \neq \mathfrak{m}$, the function $f : x \mapsto \mathrm{v}(x - a)$ induces a well-defined function on $b/I \subseteq X$ with infinite image, so $X$ is not orthogonal to $\Gamma$.

Let us now fix some $e \in X \subseteq \mathrm{Gr}$, in some elementary extension of $M$. Let $b_\ell$ be as in Proposition 3.9 and let us prove, by induction on $\ell$, that $b_\ell \in M^{\mathrm{eq}}$. For all $\ell < n$, we have $b_\ell \in \Gamma/\Delta_\ell$, for some convex subgroup $\Delta_\ell$. Since $e \in X$ and $X$ is orthogonal to $\Gamma$ and $b_\ell \in \mathrm{dcl}(e)$, we must have $b_\ell \in M^{\mathrm{eq}}$. Let now $\ell \geq n$ and let us assume that $b_{<\ell} \in M$. We have $b_\ell \in X_\ell$ which is an $M$-definable torsor for some $\mathrm{K}/\mathrm{I}_\ell$ where, by Lemma 3.17, $\mathrm{I}_\ell$ is not a multiple of $\mathfrak{m}$. Since $b_i \in \mathrm{dcl}(e)$, it is contained is an $M$-definable subset of $\mathrm{K}/\mathrm{I}_\ell$ which is orthogonal to $\Gamma$, and hence, by the first paragraph, finite. So $b_\ell \in M^{\mathrm{eq}}$.

Since $e \in \mathrm{dcl}(Mb)$, it follows that $e \in M^{\mathrm{eq}}$. As this holds for any $e \in X$ is some elementary extension, $X$ is finite. □

Recall that a definable set $X$ is (resp. almost) internal to another definable set $Y$ if, over a model, $X$ admits a one-to-one (resp. finite-to-one) map to $Y^{\mathrm{eq}}$.

**Corollary 3.22.** *Assume* k *and* $\Gamma$ *are orthogonal. Let* $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ *and let* $X$ *be an almost* k*-internal* $A$*-definable subset of* $\mathcal{G}$*, then* $X \subseteq \mathrm{K}(A) \cup \mathrm{Gr}(A) \cup \mathrm{Lin}_A$.

Note that it is necessary to assume that $\Gamma(M)$ is dense, otherwise this results does not hold: consider $\mathcal{O}/\mathfrak{m}^n$ for any $n \geq 2$.

*Proof.* Any almost k-internal set is orthogonal to $\Gamma$. By Proposition 2.3, any infinite definable subset of K contains a ball and hence is not orthogonal to $\Gamma$, so if $X \subseteq \mathrm{K}$, then $X \subseteq \mathrm{K}(A)$. By Proposition 3.21, if $X \subseteq \mathrm{Gr}$, then $X \subseteq \mathrm{Gr}(A)$. Finally, if $X \subseteq \mathrm{Lin}$, then the projection of $X$ to Gr is finite and hence $X \subset \bigsqcup_{s \in \mathrm{Gr}(A)} \mathrm{red}(R_s) = \mathrm{Lin}_A$. □

## 4 Density of quantifier free definable types

Let $M$ be an RV-expansion of a model of $\mathrm{Hen}_{0,0}$. Let $M_1 = M^{\mathrm{ur}}$ be its maximal algebraic unramified extension (with the full induced structure on $\Gamma$) and $M_0 = M^{\mathrm{a}}$ be its algebraic closure (as a pure valued field). In what follows, whenever, we want to refer to the structure in $M_0 = M^{\mathrm{a}}$ (resp. $M_1 = M^{\mathrm{ur}}$, resp. $M$), we will indicate this by a 0 (resp. 1, resp. nothing): *e.g.* $\mathrm{acl}_0$, $\mathrm{acl}_1$ or acl for the algebraic closure and $\mathrm{S}^0(M)$ or $\mathrm{S}^1(M)$ for the space of types over $M$. We also assume that the language $\mathfrak{L}_i$ of $M_i$ is morleyized, and we restrict ourselves to quantifier free $\mathfrak{L}_i$-formulas when interpreting them in a substructure.

The goal of this section is to prove the following density result:

**Theorem 4.1.** *Assume that* $\Gamma(M)$ *satisfies Property* ***D***. *Let* $A = \mathrm{acl}(A) \subseteq M$ *and* $X \subseteq \mathrm{K}^n$ *be* $\mathfrak{L}(A)$*-definable. Then there exists an* $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$*-definable type* $p \in \mathrm{S}^1(M_1)$ *consistent with* $X$ *— in the pair* $(M_1, M)$.

This statement was proved in [Vic23a, Theorem 5.9] when $M = M^{\mathrm{ur}}$ and $\Gamma$ is an abelian ordered group of bounded regular rank. It also generalizes [HR21, Theorem 3.1.3] in two ways. The first is that it provides a definable type in a stronger reduct (*i.e.* a type in $M^{\mathrm{ur}}$ and not just in $M^{\mathrm{a}}$). The second is that there is no hypothesis on the residue field — it is not required that the residue field eliminates $\exists^\infty$.

## 4.1 Codes of definable types

We start by proving the following useful fact allowing to compare types in $M$, $M_1 = M^{\mathrm{ur}}$ and $M_0 = M^{\mathrm{a}}$.

**Remark 4.2.** Let $c = (c_1, \dots, c_n)$ be a finite tuple in $\mathrm{Cut}^{**}$. Consider the natural map:

$$\mathrm{B}_n(M)/\operatorname{Stab}(\Lambda_c)(M) \to \mathrm{B}_n(M_1)/\operatorname{Stab}(\Lambda_c)(M_1).$$

Recall that $\Gamma(M_1) = \Gamma(M)$, so for every $c \in \mathrm{Cut}$, $\mathcal{C}_c(M)$ is definable in $M_1$. Then $I_c(M_1) = \mathcal{O}(M_1) \cdot I_c(M)$.

So the map above identifies the code of the $\mathcal{O}(M)$-module $R = \sum_{i \le n} \overline{a}_i I_{c_i}(M)$ in $M$ with the code of the $\mathcal{O}(M_1)$-module $\overline{R} = \sum_{i \le n} \overline{a}_i I_{c_i}(M_1) = \sum_{i \le n} \overline{a}_i \mathcal{O}(M_1) \cdot I_{c_i}(M)$ generated by $R$. Moreover, we have $\overline{R}(M) = R$.

**Proposition 4.3.** *Let $A = \operatorname{dcl}(A) \subseteq M^{\mathrm{eq}}$ and $p(x) \in \mathcal{S}^{\varepsilon}_{\mathrm{K}^{|x|}}(M)$ be finitely satisfiable in $M$ and $\mathfrak{L}(A)$-definable, for $\varepsilon = 0$ or $1$. Then $p$ has a unique extension $q_\varepsilon \in \mathcal{S}^{\varepsilon}(M_\varepsilon)$. Moreover, $q_1$ is $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable and $q_0|_{M_1}$ is $\mathfrak{L}_1(\mathcal{G}(A))$-definable (as a partial type in $M_1$).*

We follow the proof of [Vic23a, Theorem 5.9], *mutatis mutandis.*

*Proof.* The uniqueness of $q_\varepsilon$ follows from [HR21, Lemma 3.3.7]; note that $q_\varepsilon$ is finitely satisfiable in $M$.

For every integer $d \ge 0$, let $V_d \simeq \mathrm{K}^\ell$ be the space of polynomials in $\mathrm{K}(M)[x]_{\le d}$ of degree less or equal than $d$ (in each $x_i$). It comes with an $\mathfrak{L}(A)$-definable valuation defined by $v(P) \le v(Q)$ if $p(x) \vdash \mathrm{v}(P(x)) \le \mathrm{v}(Q(x))$.

By Proposition 3.3 there exists a separated basis $(P_i)_{i \le \ell} \in V_d$ such that, for every $i$, $\gamma_i = v(P_i) \in \operatorname{dcl}(\ulcorner v \urcorner) \subseteq A$. By [HR21, Claim 3.3.5], $(P_i)_i$ is also a separated basis of $V^1_d = \mathrm{K}(M_1)[x]_{\le d}$ with the valuation $v_1$ where $v_1(P) \le v_1(Q)$ if $q_1(x) \vdash \mathrm{v}(P(x)) \le \mathrm{v}(Q(x))$. It follows that $v_1(V^1_d) = v(V_d) = \bigcup_i \gamma_i + \Gamma(M)$, as this description is quantifier free $v_1$ is $\mathfrak{L}_1(M)$-definable.

By Corollary 3.7 the definable $\mathcal{O}$-modules $R^1_i = \{P \in V^1_d : v_1(P) \ge \gamma_i\}$ are coded by some tuple $e^1_i \in \operatorname{Mod}^1(M)$. We identify $e^1_i$ with the code $e_i \in \operatorname{Mod}(A)$ of $R^1_i \cap V_d$ via the map in remark 4.2. Furthermore, the $R^1_i$ entirely determine $v_1$ which is therefore coded in $\mathcal{G}(A)$ (*cf.* Corollary 3.20). Since $q_0|_{M_1}$ is entirely determined by the valuations $v_1$ on $V^1_d$, the proposition is proved in that case.

To conclude, let us prove the definability of $q_1$. By Theorem 2.6, any $\mathfrak{L}_1$-formula $\phi(x,y)$ (with variables in K) is equivalent to one of the form $\psi(\mathrm{v}(P(x,y)))$, where $P \in \mathbb{Z}[x,y]$ is a tuple. Let $X_\phi = \{v_1(P(x,a)) \in v_1(V^1_d) : q_1(x) \vdash \phi(\mathrm{v}(P(x,a)))\}$. Note that if $v_1(P(x,b)) = v_1(P(x,a))$, then $q_1(x) \vdash \mathrm{v}(P(x,b)) = \mathrm{v}(P(x,a))$ and hence $v_1(P(x,b)) \in X_\phi$ if and only if $v_1(P(x,a)) \in X_\phi$. So $q_1$ is entirely determined by the $X_\phi$ over $\mathcal{G}(A)$. Moreover, since $v_1(V^1_d) = v(V_d) = \bigcup_i \gamma_i + \Gamma(M)$ and $\gamma_i \in \mathcal{G}(A)$, this set can be identified, over $\mathcal{G}(A)$, to a disjoint union of copies of $\Gamma(M)$. It follows that $X_\phi$ is $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable, and hence so is $q_1$. $\square$

From now on, we will identify $\mathfrak{L}(A)$-definable types in $\mathcal{S}^1(M)$ with their unique extensions to $M_1 = M^{\mathrm{ur}}$ and $M_0 = M^{\mathrm{a}}$.

Coding definable $\mathfrak{L}_0$-types already allows us to code some imaginaries, namely certain germs of functions into the space of balls.

**Lemma 4.4.** *Assume that $\Gamma(M)$ is dense. Let $a \in \mathrm{K}(N)$, for some $N \succ M$, be such that $p_0(x) = \mathrm{tp}_0(a/M)$ is $\mathfrak{L}(M)$-definable. Let $b(a)$ be an $\mathfrak{L}_0(Ma)$-definable open ball. Then $[b]_p$ is coded in $\mathcal{G}(M)$ over $\mathcal{G}(\ulcorner p \urcorner)$.*

*Proof.* Let $q(x,y)$ be the $\mathfrak{L}(M)$-definable $\mathfrak{L}_0(M)$-type $\int_{p(x)} \eta_{b(x)}(y)$ whose realizations are the tuples $ac$ such that $a \models p$ and $c$ is generic in $b(a)$ over $Ma$ (in $N^{\mathrm{a}}$). Note that, since we are in equicharacteristic zero, $b(a)$ has point in $N$ and, in fact, since $b(a)$ is open and $\Gamma(N)$ is dense, the generic of $b(a)$ in $N^{\mathrm{a}}$ is finitely satisfiable in $N$. It follows that $q|_M$ is finitely satisfiable in $M$.

By Proposition 4.3, $p$ and $q$ are coded in $\mathcal{G}(M)$. Moreover, for every $\sigma \in \mathrm{Aut}(M/\ulcorner p \urcorner)$ and $ac \models q$, we have $\sigma(q) = q$ if and only if $b(a) = b(c) = b^{\sigma}(a)$. So $[b]_p$ is coded by $\mathcal{G}(\ulcorner q \urcorner)$ over $\mathcal{G}(\ulcorner p \urcorner)$. □

## 4.2 Unary sets

We first consider the case of Theorem 4.1 when $X \subseteq \mathrm{K}$ is unary. The proof proceeds as in [Vic23a, Theorem 5.3] where, in the unary case, the hypothesis that $M = M^{\mathrm{ur}}$ is not used.

**Lemma 4.5.** *Let $A = \mathrm{acl}(A) \subseteq M$ and $X \subseteq \mathrm{K}$ be $\mathfrak{L}(A)$-definable. There exists an $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable type $p_0 \in \mathrm{S}^0(M_1)$ consistent with $X$.*

This follows from [HR21, Section 3] which states a relative version of that statement. However, since the machinery set up for the relative version of the statement is rather heavy, let us sketch a proof. A version of this proof can also be found in [Vic23a, Theorem 5.5].

*Proof.* Let $\mathrm{B}(M_1)$ be the set of all open and closed balls (including points and K itself) in $M_1$. Given $b_1, b_2 \in \mathrm{B}$ we write $b_1 \trianglelefteq b_2$ if $b_1 \cap X \subseteq b_2 \cap X$. This is a pre-order with associated equivalence $\equiv$ and the associated order is a tree $\mathcal{T}$ if we remove the class $E_\varnothing$ of balls that don't intersect $X$.

Note that any $\equiv$-class $E \neq E_\varnothing$, the generalized ball $b_E = \bigcap_{b \in E} b$ is defined by knowing a point in $b_E$ and the set $\{\mathrm{v}(x-y) : x, y \in B\}$ which is definable in $\Gamma(M) = \Gamma(M_1)$. So $b_E$ is $\mathfrak{L}_1(M)$-definable. It follows that $E$ is coded in $\mathrm{B_g}(M)$ and that the generic type $\eta_E(x) \in \mathcal{S}^0(M_1)$ generated by

$$\{x \in b : b \in E\} \cup \{x \notin \bigcup_i b_i : b_i \in \mathrm{B} \text{ and } \forall b \in E,\ b_i \subset b\}$$

is $\mathfrak{L}_1(\ulcorner E \urcorner)$-definable. If the type $\eta_E$ is not consistent with $X$, by compactness, $E$ has finitely many direct predecessors for $\trianglelefteq$, each of them in $\mathrm{acl}(A, \ulcorner E \urcorner)$. So either the lemma holds or the tree $\mathcal{T}$ has an initial discrete finitely branching tree of $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable $\equiv$-classes.

By Proposition 2.3, there exists a finite set $C \subseteq \mathrm{K}(M)$ preparing $X$ and, we can find an $\equiv$-class $E$ in the initial discrete tree such that $b_E \cap C = \varnothing$. Then $b_E \subseteq X$ and hence $X$ is consistent with the $\mathfrak{L}_1(\mathrm{B_g}(A))$-definable type $\eta_E$. □

The type $p_0 = \eta_E$ can then be completed to an $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p_1 \in \mathcal{S}^1(M_1)$ consistent with $X$:

**Lemma 4.6.** *Assume that $\Gamma(M)$ satisfies Property **D**. Let $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ and $b$ be an $\mathfrak{L}(A)$-definable generalized ball. Then, there exists an $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(\mathrm{acl}(A)))$-definable type $p \in \mathcal{S}^1(M_1)$ containing $\eta_b$.*

*Proof.* For any $a \in b(M)$ and $c \vDash \eta_b$, $\gamma = \mathrm{v}(c - a)$ does not depend on $a$. If $\gamma \in \Gamma(M)$, then $\eta_b$ generates a complete type in $\mathrm{S}^1(M_1)$. If not, let $C$ be the $\mathfrak{L}_1(A)$-definable cut of $\gamma$ over $\Gamma(A)$ and $Y_a = \mathrm{v}(X - a)$, for every $a \in b(M)$. Note that $\gamma \in C \cap \bigcap_{a \in b(M)} Y_a$. By property **D**, there exists an $\mathfrak{L}(\mathrm{acl}(A))$-definable type $q \in \mathcal{S}(\Gamma(M))$ consistent with $C \cap \bigcap_{a \in b(M)} Y_a$. Then, the type $p_1(x) = \eta_b(x) \wedge q(\mathrm{v}(x - a))$ is complete by Theorem 2.6, and it is $\mathfrak{L}_1(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable. □

**Corollary 4.7.** *Assume that $\Gamma(M)$ satisfies Property **D**. Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$. Then any $\mathfrak{L}(A)$-definable subset of $\mathrm{K}$ is consistent with an $\mathfrak{L}(\mathrm{B_g}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p \in \mathrm{S}^1(M)$.*

## 4.3 Germs of functions

To prove density of definable types in general, we now wish to proceed by transitivity. However, since we are working with definable $\mathfrak{L}_1$-types, we first need to address the potential difference between acl and $\mathrm{acl}_1$. For every tuple $a \in M$, let $a_{\mathrm{K}}$ enumerate $\mathrm{K}(a)$.

**Proposition 4.8.** *Assume that $\Gamma$ has property **D**. Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$, let $X$ be pro-$\mathfrak{L}(A)$-definable and $p(x) \in \mathcal{S}^1(M)$ be $\mathfrak{L}(A)$-definable and consistent with $X$. Assume that for any $a \vDash p$, $a \subseteq \mathrm{acl}_1(Ma_{\mathrm{K}})$. Then, for every $\mathfrak{L}(A)$-definable one-to-finite correspondence $F$ into $\mathcal{G} \cup \Gamma^{\mathrm{eq}}$ (defined at realizations of $p$), there exists an $\mathfrak{L}(A)$-definable $q(xy) \in \mathcal{S}^1(M)$ containing $p(x)$ and $ac \vDash q$ with $a \in X$, $c \in \mathrm{acl}_1(Ma_{\mathrm{K}}c_{\mathrm{K}})$ and $F(a) \cap \mathrm{dcl}_1(ac) \neq \varnothing$.*

*Proof.* We first consider the case where $F$ is almost definable in $M_1$.

**Claim 4.8.1.** *If there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence $G$ such that $p(x) \wedge x \in X \vdash F(x) \subseteq G(x)$, then the proposition holds.*

*Proof.* Since $p \in \mathrm{S}^1(M)$, $|G(x)|$ is constant when $x$ varies over realizations of $p$; and we may assume that it is minimal. Then for every other such $G'$, $p(x) \wedge x \in X \vdash F(x) \subseteq G(x) \cap G'(x)$ and hence $p(x) \wedge x \in X \vdash G(x) = G'(x)$. In other words, for some (and hence for every) $a \vDash p$, $G(a) = G'(a)$. This holds in particular of any $G' = \sigma(G)$, where $\sigma \in \mathrm{Aut}(M/A)$ and hence $[G]_p \in \mathrm{dcl}(A) = A$. Let $G_0 \subseteq G$ be minimal $\mathfrak{L}_1(M)$-definable such that $p(x) \vdash \varnothing \neq G_0(x) \subseteq G(x)$. Then $[G_0]_p \in \mathrm{acl}(A) = A$. So the type $q(xy) = p(x) \wedge y \in G_0(x)$ is $\mathfrak{L}(A)$-definable and, by construction, for any $ac \vDash q$, $c \in G_0(a) \subseteq \mathrm{acl}_1(Aa) \subseteq \mathrm{acl}_1(Ma_K)$. Moreover, if $q(xy)$ is not consistent with

$x \in X \wedge y \in F(x)$, then $p(x) \wedge x \in X \vdash F(x) \subseteq G(x) \setminus G_0(x)$, contradicting the minimality of $G$. So the type $q$ is as required. □

Let us now assume that the co-domain of $F$ is $\Gamma^{\text{eq}}$. By Proposition 2.2, for every $a \models p$,

$$\Gamma^{\text{eq}}(\text{acl}(Aa)) \subseteq \Gamma^{\text{eq}}(\text{acl}(Ma_{\text{K}})) \subseteq \text{acl}_1(\text{v}(M(a_{\text{K}})))$$

and hence, by compactness, there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence $G$ such that $p(x) \vdash F(x) \subseteq G(x)$. We now conclude with Claim 4.8.1. Iterating this case of the proposition, we may assume that $a$ contains all of $\Gamma^{\text{eq}}(\text{acl}(Aa))$.

**Claim 4.8.2.** *Let $f : p \to \text{B}_{\text{g}}$ be $\mathfrak{L}_1$-definable. Then, there exists an $\mathfrak{L}(A)$-definable $q(xy) \in \mathcal{S}^1(M)$ containing $\int_{p(x)} \eta_{f(x)}(y)$ — that is, the type of tuples $ac$ such that $a \models p$ and $c \models \eta_{f(a)}|_{Ma}$.*

*Proof.* Let $a \models p$ and $b = f(a)$. Since $\Gamma^{\text{eq}}(\text{acl}(b)) \subseteq \Gamma^{\text{eq}}(\text{acl}(Aa)) \subseteq a$, by Lemma 4.6, there exists an $\mathfrak{L}_1(a)$-definable type $r_a(y)$ containing $\eta_b(y)$. Then $\int_{p(x)} r_x(y)$ is as required. □

Iterating Claim 4.8.2, we can further assume that for every $\mathfrak{L}(Aa)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\text{acl}(Aa)) \subseteq v_\Delta(a_{\text{K}})$.

If the co-domain of $F$ is k, let $\eta_k(y) \in \mathcal{S}^1(M)$ be the $\mathfrak{L}$-definable generic of k — that is, the only non-algebraic type concentrating on k. If $p(x) \otimes \eta_{\text{k}}(y)$ is not consistent with $x \in X \wedge y \in G(y)$, then there exists an $\mathfrak{L}_1(M)$-definable one-to-finite correspondence $G$ such that $p(x) \wedge x \in X \vdash F(x) \subseteq G(x)$. Once again, we conclude with Claim 4.8.1. On the other hand, if $p(x) \otimes \eta_{\text{k}}(y)$ is consistent with $x \in X \wedge y \in F(y)$, let $q(xz) = p(x) \otimes \eta_{\mathcal{O}}(z)$, then, by hypothesis, $q(xz)$ is consistent with $x \in X \wedge \text{res}(z) \in F(y)$. Applying Claim 4.8.2 and iterating, we can thus assume that $\text{k}(\text{acl}(Aa)) \subseteq \text{res}(a_{\text{K}})$. Then, we also have $\text{RV}(\text{acl}(Aa)) \subseteq \text{rv}(a_K)$.

Let us now assume that the co-domain of $F$ is the set of generalized balls that are not open balls. For any $a \models p$ and $b \in F(a) \subseteq \text{acl}(Aa) \subseteq \text{acl}(Ma_{\text{K}})$, by Corollary 2.5, we have $b(\text{acl}_1(Ma_{\text{K}})) \supseteq b(\text{acl}(Ma_{\text{K}})) \neq \varnothing$. Moreover, $\text{cut}(b) \in \Gamma^{\text{eq}}(\text{acl}(Aa)) \subseteq \text{dcl}_1(a)$, by Proposition 2.2. Hence $b \in \text{acl}_1(Ma)$ and we conclude with Claim 4.8.1. So we may assume that $b \in a$. Applying Claim 4.8.2 and iterating, we may assume that any generalized ball $b \in \text{acl}(Aa)$, which is not an open ball, has a point in $a$.

Now, if $b \in \text{acl}(Aa)$ is an open ball, the smallest closed ball around $b$ has a point $c \in a$ and $b - c \in \text{RV}(\text{acl}(Aa)) \subseteq \text{rv}(a_{\text{K}})$ also has a point in $a$, hence so does $b$. Recall that we already assumed that, for every $\text{acl}(Aa)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\text{acl}(Aa)) \subseteq v_\Delta(a_{\text{K}})$.
By Corollary 3.11, it follows that $\mathcal{G}(\text{acl}(Aa)) \subseteq \text{dcl}_1(a)$, concluding the proof of the proposition. □

Enumerating $\mathcal{G}(\text{acl}(Aa)) \cup \Gamma^{\text{eq}}(\text{acl}(Aa))$ and iterating Proposition 4.8, we get:

**Corollary 4.9.** *There exists $a \models p$ in $X$ such that*

$$\text{tp}_1(\mathcal{G}(\text{acl}(Aa)) \cup \Gamma^{\text{eq}}(\text{acl}(Aa))/M)$$

*is $\mathfrak{L}(A)$-definable.*

The proof of Theorem 4.1 is now a standard induction.

*Proof of Theorem 4.1.* Recall that, by Proposition 4.3, we can identify $\mathfrak{L}(A)$-definable types in $\mathcal{S}^1(M)$, with their unique $\mathfrak{L}(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable extension to $M_1$.

We may assume that $X \subseteq \mathrm{K}^n$ and we proceed by induction on $n$. Let $\pi : \mathrm{K}^n \to \mathrm{K}$ be the projection on the first coordinate. By Corollary 4.7, there exists an $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable type $p \in \mathcal{S}^1(M_1)$ consistent with $\pi(X)$. Let $a \vDash p(x) \wedge x \in X$, $c$ enumerate $\mathcal{G}(\mathrm{acl}(Aa)) \cup \Gamma^{\mathrm{eq}}(\mathrm{acl}(Aa))$ and $X_a = \{y \in \mathrm{K}^{n-1} : (a, y) \in X\}$. By Corollary 4.9, we may assume that $q(y) = \mathrm{tp}_1(c/M_1)$ is $\mathfrak{L}(A)$-definable — and hence, by Proposition 4.3, it is $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable. By induction, there exists an $\mathfrak{L}_1(c)$-definable $r_c(z)$ consistent with $X_a$. The type $\int_{q(y)} r_y(z)$ is then as required. ☐

In the case $M = M^{\mathrm{ur}}$, we can deduce a slight generalization of [Vic23a, Theorem 5.12]:

**Corollary 4.10.** *Assume that:*

- $\Gamma(M)$ *satisfies Property* ***D****;*
- $\mathrm{k}(M)$ *is algebraically closed.*

*Then $M$ weakly eliminates imaginaries down to $\mathcal{G} \cup \Gamma^{\mathrm{eq}}$.*

# 5 Invariant Extensions

In this section, we will consider the invariance of types over large subsets of our model (which are points of some stably embedded definable set *e.g.* RV). This gives rise to several notions of invariance isolated in [HR21, Section 4.2].

Whenever $D = \bigcup_i D_i$ is an ind-definable, we denote by $D^{\mathrm{eq}}$ the ind-definable union of all interpretable sets $X$ that admit a definable surjection $\prod_j D_{i_j} \to X$.

**Definition 5.1.** Let $M$ be an $\mathfrak{L}$-structure, $C \subseteq M$, $D$ be an (ind-)definable set and $p$ be a partial type over $M$. We say that:

1. $p$ is $\mathrm{Aut}(M/C)$-invariant if for every $\sigma \in \mathrm{Aut}(M/C)$, $p = \sigma(p)$;
2. $p$ has $\mathrm{Aut}(M/C)$-invariant $D$-germs if it is $\mathrm{Aut}(M/C)$-invariant and so is the $p$-germ of every $M$-definable map $f : p \to D^{\mathrm{eq}}$;
3. $p$ is $\mathrm{Aut}(M/D)$-invariant if it has $\mathrm{Aut}(M/D(M))$-invariant $D$-germs.

A nice property of the stronger notion is that it is transitive — *cf.* [HR21, Lemma 2.4.2]:

**Lemma 5.2.** *Let $M \prec N$ be $\mathfrak{L}$-structures with $N$ saturated and sufficiently large, let $C \subseteq M$ be potentially large, let $D$ be an (ind-)$\mathfrak{L}$-definable stably embedded set, let $p \in S(M)$ have* $\mathrm{Aut}(M/C)$*-invariant $D$-germs, let $a \vDash p$ in $N$ and let $q \in S(N)$ be* $\mathrm{Aut}(N/CD(N)a)$*-invariant. Then $q|_M$ is* $\mathrm{Aut}(M/C)$*-invariant.*

*Moreover, if $q$ has* $\mathrm{Aut}(N/CD(N)a)$*-invariant $E$-germs, for some (ind-)$\mathfrak{L}$-definable set $E$, then $q|_M$ has* $\mathrm{Aut}(M/C)$*-invariant $E$-germs.*

The main goal of this section is to prove the following statement. Recall that $\mathrm{Lin}_A = \bigsqcup_{s \in \mathrm{Gr}(\mathrm{acl}(A))} \mathrm{red}(R_s)$ (Definition 3.16).

**Theorem 5.3.** *Let $M$ be sufficiently saturated and homogeneous* $\mathrm{RV}$*-expansion of a model of* $\mathrm{Hen}_{0,0}$ *such that the value group* $\Gamma$ *is orthogonal to* $\mathrm{k}$ *and either:*

- *dense with property* ***D****;*
- *a pure discrete ordered abelian group of bounded regular rank — in that case we also add a constant for an uniformizer $\pi$.*

*Let* $M_0 = M^{\mathrm{a}}$*,* $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ *and* $a \in N \succ M$ *a tuple such that* $\mathrm{tp}_0(a/M)$ *is* $\mathrm{Aut}(M/\mathcal{G}(A))$*-invariant. Then* $\mathrm{tp}(a/M)$ *is* $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$*-invariant.*

We follow the general strategy of [HR21, Section 4]. The main new challenge is to prove the equivalent (Proposition 5.17) of [HR21, Corollary 4.4.6] in the present setting since the geometric sorts are now larger.

## 5.1 Germs of functions into the linear sorts

One important ingredient of the proof of Proposition 5.17 is a description of the germ of certain functions into the linear sorts (*cf.* Lemmas 5.10 and 5.14). We proceed in three steps. First, we consider the case of valued fields with algebraically closed residue field. Then we consider valued fields with dense value groups (and arbitrary residue fields). Finally, we consider valued fields with discrete value groups for which a serious obstruction arises: the classification of k-internal sets given in Corollary 3.22 does not hold for discrete value groups. This can be circumvented by considering a ramified extension with dense value group.

### 5.1.1 Algebraically closed residue fields

Let $M$ be (a $\Gamma$-expansion of) a model of $\mathrm{Hen}_{0,0}$, with algebraically closed residue field and NIP value group. We first prove that, in that case, germs of functions into the linear sorts are internal to the residue field:

**Proposition 5.4.** *Let* $A \subseteq M^{\mathrm{eq}}$*. Let* $p(x) \in \mathcal{S}(M)$ *be $A$-definable concentrating on* $\mathrm{K}^n$ *for some $n$. Let $f$ be an $M$-definable function. Assume that for every* $a \models p$*,* $f(a) \in \mathrm{Lin}_{Aa}$*, then* $[f]_p$ *lies in a* $\mathrm{k}$*-internal $A$-definable set.*

We follow the ideas underlying the proof of [HHM08, Proposition 6.9]. A key ingredient of this proof is that there cannot be large dcl-closed chains inside $\mathrm{Lin}_A$ — see Corollary 5.8. We start by describing the growth of dcl in $\mathrm{Lin}_A$.

**Lemma 5.5.** *Let $a$ be a finite tuple in* $M^{\mathrm{eq}}$ *and $A$ any base structure. There is some set* $Z_a \subseteq \mathrm{acl}(A)$ *such that* $|Z_a| \leq |T|$ *and* $\mathrm{tp}(a/\mathrm{acl}(A))$ *is* $\mathrm{dcl}(A, Z_a)$*-invariant.*

*Proof.* The structure $M$ is NIP, *e.g.* see [JS20, Theorem 3.3]. The statement then follows immediately from [She90, Chapter III, Theorem 7.5]. ☐

**Lemma 5.6.** *Let $U$ be an open $Ac$-$\infty$-definable generalized ball. Let $a$ be a generic element of $U$ over $Ac$. Then* $\mathrm{Lin}_A(\mathrm{dcl}(Aac)) \subseteq \mathrm{acl}(Ac)$*.*

*Proof.* Let $f : U \to \mathrm{Lin}_A$ be some $Ac$-definable function and let $\mathrm{cut}(U)$ be the cut of $U$. Fix some $e \in U(M)$. For now, we work over $M$, so we can assume that $e = 0$ and we can identify the sorts in $\mathrm{Lin}_A$ with some $\mathrm{k}^n$. For every $\gamma \in \mathrm{cut}(U)$ and $d \in \mathrm{Lin}_A$, let $X_{d,\gamma} = \{x \in U : \mathrm{v}(x) = \gamma \text{ and } f(x) = d\}$. Then by Proposition 2.3, there exists a finite set $C \subseteq U(M)$ which does not depend on $\gamma$ or $d$, such that for every ball $b$, if $b \cap C \neq \varnothing$, then $b \cap X_{d,\gamma} \neq \varnothing$ implies that $b \subseteq X_{d,\gamma}$.

If $a \in U$ is not in the smallest ball $b$ containing $C$ and 0, then the open ball of radius $\mathrm{v}(a)$ around $a$ — that is $\mathrm{rv}(a)$ — is entirely contained in $X_{f(a),\mathrm{v}(a)}$. In other words, $f$ induces a well-defined function $\overline{f} : \mathrm{rv}(U \smallsetminus b) \to \mathrm{k}^n$.

**Claim 5.6.1.** *Let $f : \mathrm{RV} \to \mathrm{k}$ be $M$-definable. Then there are finitely many $\gamma_i \in \Gamma(M)$ such that $f(\{x \in \mathrm{RV} : \mathrm{v}(x) \neq \gamma_i\})$ is finite.*

*Proof.* For any choice of $\alpha \in \mathrm{k}$ and $\gamma \in \Gamma$ (in some $N > M$), we find an automorphism $\sigma \in \mathrm{Aut}(\mathrm{RV}(N)/\mathrm{RV}(M), \mathrm{k}(N))$ such that, if $\mathrm{v}(x) = \gamma$, then $\sigma(x) = x + \alpha$. First, we find a group morphism $f : \Gamma \to \mathrm{k}$ sending $\gamma$ to $\alpha$ and $\Gamma(M)$ to 0. Indeed, $\gamma \notin \mathbb{Q} \otimes \Gamma(M)$ and hence, it suffices to choose images for $n^{-1}\gamma$, for every $n \in \mathbb{Z}_{>0}$, which we can do (coherently) because k is algebraically closed. This $f$ induces an automorphism $\sigma \in \mathrm{Aut}(\mathrm{RV}(N)/\mathrm{RV}(M), \mathrm{k}(N))$ defined by $\sigma(x) = x + f(\mathrm{v}(x))$.

Let $x, y \in \mathrm{RV}$ be such that $\mathrm{v}(x) = \mathrm{v}(y) \notin \Gamma(M)$. Then, by the above paragraph, there is an automorphism $\sigma$ fixing k and $M$, and hence $f$, and such that $\sigma(x) = \sigma(y)$. It follows that $f(x) = \sigma(f(x)) = f(\sigma(x)) = f(y)$. By compactness, it follows that there are finitely many $\gamma_i \in \Gamma(M)$ such that $f$ induces a function $\Gamma \smallsetminus \bigcup_i \gamma_i \to \mathrm{k}$. This function has finite image by orthogonality of k and $\Gamma$. ☐

Thus we have found an $M$-definable closed ball $b \subset U$ such that $f(U \smallsetminus b)$ is finite. Let us conclude the proof by showing that $b$ can be replaced by a generalized $Ac$-definable ball. If there are two such $M$-definable closed balls with empty intersection, then $f(U)$ is finite. If not, they form a chain and their intersection is an $Ac$-definable generalized sub-ball $B$ of $U$ such that $f(U \smallsetminus B)$ is finite. In both cases, if $a$ is generic in $U$ over $Ac$, $f(a)$ is in a finite $Ac$-definable set. In other words, $\mathrm{Lin}_A(\mathrm{dcl}(Aac)) \subseteq \mathrm{acl}(Ac)$. ☐

**Proposition 5.7.** *Let $a \in \mathrm{K}^n(M)$ and $A = \mathrm{dcl}(A)$ be a set of parameters. Then there is some set $Z \subseteq \mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{dcl}(A, Z)$ and $|Z| \leq |T|$.*

*Proof.* We adapt the proof of [HHM08, Corollary 9.6] and start by proving the following claim:

**Claim 5.7.1.** *There is a countable tuple $c \in \mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(Ac)$.*

*Proof.* By backwards induction, we construct an increasing sequence of tuples $d_n \subseteq \cdots \subseteq d_1$ in $\mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that, for every $i \leq n$, we have $d_i \in \mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{>i}))$ and $\mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{>i})) \subseteq \mathrm{acl}(\mathrm{Lin}_A(a_{<i}, d_i))$. Fix $i \leq n$ and suppose $d_{>i}$ has been constructed.

Let $W = \{B : a_i \in B$ and $B$ is a $Aa_{<i}d_{>i}$-definable generalized ball$\}$. Then $a_i$ is generic in the $\infty$-$Aa_{<i}d_{>i}$-definable generalized ball

$$U = \bigcap_{b \in W} b.$$

If $U$ is open, let $d_i = d_{i+1}$. Then, by Lemma 5.6,

$$\mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{i>})) = \mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{i+1})) \subseteq \mathrm{Lin}_A(\mathrm{acl}(a_{<i}, d_i)).$$

If $U$ is closed, let $y_0 = \mathrm{res}_U(a_i) \in \mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{>i}))$. Let $U_0$ be intersection of all $Aa_{<i}d_{>i}y_0$-definable generalized ball containing $a_i$. Either $U_0$ is open, or we set $y_1 = \mathrm{res}_{U_0}(a_i) \in \mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_{>i}))$. We continue this process unless $U_i$ is open and we set $U_\omega = \bigcap_j U_j$. Then $a_i$ is generic in the $Aa_{<i}d_{i>i}y_{\geq 0}$-definable open generalized ball $U_\omega$. By Lemma 5.6, $\mathrm{Lin}_A(\mathrm{dcl}(Aa_{\leq i}, d_i)) \subseteq \mathrm{acl}(\mathrm{Lin}_A(\mathrm{dcl}(Aa_{<i}, d_{>i}, y_{\geq 0})))$, so we set $d_i = d_{>i}y_{\geq 0}$. Taking $c = d_1$, we have $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(Ac)$. □

We might replace $A$ by $\mathrm{dcl}(Ac)$ and assume that $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(A)$. By Lemma 5.5, there is $Z_a \subseteq \mathrm{acl}(A)$ such that $|Z_a| \leq |T|$ and $\mathrm{tp}(a/\mathrm{acl}(A))$ is $\mathrm{dcl}(A, Z_a)$-invariant. In particular,

$$\mathrm{dcl}(Aa) \cap \mathrm{acl}(A) \subseteq \mathrm{dcl}(A, Z_a). \tag{1}$$

For any tuple $z \in Z_a$, $\mathrm{tp}(z/A\mathrm{Lin}_A(\mathrm{dcl}(Aa)))$ is isolated, because $z \in \mathrm{acl}(A)$. So there is some $z' \in \mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that $\mathrm{tp}(z/A, z') \vdash \mathrm{tp}(z/A, \mathrm{Lin}_A(\mathrm{dcl}(Aa)))$. Let $Z = \{z' | z \in Z_a^m, m = 1, 2, \dots\}$. By construction $Z \subseteq \mathrm{Lin}_A(\mathrm{dcl}(Aa))$, $|Z| \leq |T|$ and $\mathrm{tp}(Z_a/A, Z) \vdash \mathrm{tp}(Z_a/A, \mathrm{Lin}_A(\mathrm{dcl}(Aa)))$.
Because $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{acl}(A)$, by Eq. (1), $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{dcl}(A, Z_a)$. Consequently, $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{dcl}(A, Z)$, as required. □

**Corollary 5.8.** *Let* $\mathrm{dcl}(A) = A \subseteq M$ *be a small set of parameters. Let* $e \in M^{eq}$, *then there is no strictly ascending chain of length* $(2^{|T|})^+$ *of sets* $B = \mathrm{dcl}(B) \cap \mathrm{Lin}_A$ *between* $\mathrm{Lin}_A$ *and* $\mathrm{Lin}_A(\mathrm{dcl}(Ae))$.

*Proof.* The proof follows ideas from [HHM08, Lemma 6.5], replacing $St_C$ by $\mathrm{Lin}_A$ and applying Proposition 5.7. We include details for sake of completeness.

Let $a \in K$ be a tuple such that $e \in \mathrm{dcl}(a)$. By Proposition 5.7 there is some subset $Z \subseteq \mathrm{Lin}_A(\mathrm{dcl}(Aa))$ such that $|Z| \leq |T|$ and $\mathrm{Lin}_A(\mathrm{dcl}(Aa)) \subseteq \mathrm{dcl}(A, Z)$. Let $(B_\alpha : \alpha \in (2^{|T|})^+)$ be an increasing chain of definably closed subsets of $\mathrm{Lin}_A(\mathrm{dcl}(Ae)) \subseteq \mathrm{dcl}(A, Z)$ ordered by inclusion with $B_0 = A$. Since the residue field is algebraically closed, $\mathrm{Lin}_A$ is stable. Thus $\mathrm{tp}(Z/\bigcup_{\alpha \in (2^{|T|})^+} B_\alpha)$ is definable. By local character, it is definable over $B_\mu$ for some $\mu$. Let $c \in B_{\mu+1}$. Then there is an $A$-definable function and a tuple $z \in Z$ such that $c = f(z)$. Because $\mathrm{tp}(Z/AB_{\mu+1})$ is $B_\mu$-definable then $\{x \in AB_{\mu+1} : f(z) = x\}$ is $B_\mu$-definable and is a singleton. Since $B_\mu$ is definably closed $c \in B_\mu$ and the chain stabilizes. □

*Proof of Proposition 5.4.* We work in a sufficiently saturated and homogeneous elementary extension of $M$. Let $f_d$ be an $A$-definable family of functions such that, for all $d$ and $a \vDash p$, $f_d(a)$ is defined and in $\mathrm{Lin}_{Aa}$. Let $\Theta$ be the $A$-interpretable set of all $[f_m]_p$.

We first argue that for a given $d$, there is an $n \in \mathbb{Z}_{\geq 0}$ and $A$-definable function $\gamma$ such that for any realization $a_{<n} \models p^{\otimes n-1}|_{Ad}$ we have $\gamma(a_{<n}, b_{<n}) = [f_d]_p$ where $b_i = f_d(a_i)$.

Let $I$ be a linear order without endpoints. Take a Morley sequence in $p$ of order type $I$ over $Ad$, i.e. $a_i \models p|_{Ada_{<i}}$. Let $D = A(a_i)_{i \in I}$ and $b_i = f_d(a_i)$, for $i \in I$. If $I$ is sufficiently large, we can find an increasing strict sequence of initial segments $(I_\alpha)_{\alpha \in (2^{|T|})^+}$. For each $\alpha \in (2^{|T|})^+$ we let $E_\alpha = \mathrm{Lin}_D(\mathrm{dcl}(D(b_i)_{i \in I_\alpha})) \subseteq \mathrm{Lin}_D(\mathrm{dcl}(Dd))$. By Corollary 5.8 (applied to $\mathrm{Lin}_D(Dd)$) the sequence $(E_\alpha)_{\alpha \in (2^{|T|})^+}$ is eventually constant, thus there is some $\ell \in I$ such that $b_\ell \in \mathrm{dcl}(Db_{<\ell})$.

**Claim 5.8.1.** *There is some $n \in \mathbb{Z}_{>0}$ such that for any $i_1 < \cdots < i_n$ in $I$, we have $b_{i_n} \in \mathrm{dcl}(A, a_{i_1}, \ldots, a_{i_n}, b_{i_1}, \ldots, b_{i_{n-1}})$.*

*Proof.* Since the sequence $(a_i)_{i \in I}$ is $Ad$-indiscernible there are $n, m \in \mathbb{Z}_{>0}$ such that $i_1 < \cdots < i_n < j_1 < \cdots < j_m$ and:

$$b_{i_n} \in \mathrm{dcl}(Aa_{i_1}, \ldots, a_{i_n}, b_{i_1}, \ldots, b_{i_{n-1}}, a_{j_1}, \ldots, a_{j_m}).$$

To simplify the notation we write $I_{\leq s}$ for $i_1 < \cdots < i_s$, and $J_{\leq k}$ for $j_1 < \cdots < j_k$. Let $b'_{i_n} \equiv_{Aa_{I_{\leq n}} b_{I_{<n}}} b_{i_n}$, we aim to show that $b'_{i_n} \equiv_{Aa_{I_{\leq n}}, b_{I_{<n}} a_{J_{\leq n}}} b_{i_n}$, and therefore $b_{i_n} \in \mathrm{dcl}(Aa_{I_{\leq n}} b_{I_{<n}})$. For each $k \leq m$, we can assume $a_{j_k} \models p|_{Aa_{I_{\leq n}} b_{I_{<n}} b_{i_n} b'_{i_n} a_{J_{\leq k-1}}}$. Suppose that there is a formula $\phi(x, a_{I_{\leq n}}, b_{I_{\leq n-1}}, a_{J_{\leq m}})$ such that $\phi(b_{i_n}, a_{I_{\leq n}}, b_{I_{\leq n-1}}, a_{J_{\leq m}})$ holds while $\neg\phi(b'_{i_n}, a_{I_{\leq n}}, b_{I_{\leq n-1}}, a_{J_{\leq m}})$ holds. Since $p$ is $A$ definable, so it is $p^{\otimes n+m}$ thus there is a $A$-formula $\psi(x, y)$ such that $\psi(b_{i_n}, b_{I_{\leq n-1}})$ holds while $\psi(b'_{i_n}, b_{I_{\leq n-1}})$ does not. This contradicts that $b_{i_n}$ and $b'_{i_n}$ elements have the same type over $Aa_{I_{\leq n}} b_{I_{<n}}$. $\square$

By indiscernability over $Ad$, for any $a_{\leq n} \models p^{\otimes n}|_{Ad}$ and $b_i = f_d(a_i)$, it follows that $b_n \in \mathrm{dcl}(Aa_{\leq n} b_{<n})$. So there is an $Aa_{<n} b_{<n}$-definable function $h_{a_{<n} b_{<n}}$ sending $a_n$ to $b_n$. Note that $a_n \models p|_{Ada_{<n} b_{<n}}$, since $a_n \models p|_{Aa_{<n} d}$ and $b_{<n} \in \mathrm{dcl}(Ada_{<n})$. Summarizing, we have that $h_{a_{<n} b_{<n}}(a_n) = b_n$, $f_d(a_n) = b_n$ and $a_n \models p|_{Ada_{<n} b_{<n}}$, consequently $[h_{a_{<n} b_{<n}}]_p = [f_d]_p$. Let $\gamma$ be an $A$-definable function such that $\gamma(a_{<n}, b_{<n}) = [h_{a_{<n} b_{<n}}]_p$.

For each fixed realization $a_{<n} \models p^{<n}$ let $\Theta_{a_{<n}} = \{\gamma(a_{<n}, b'_{<n}) : \text{for all choices of } b'_{<n}\}$, which is internal to the residue field as it is $\mathrm{Lin}_{Aa_{<n}}$-internal. In particular $\Theta_{a_{<n}}$ is stable. Given $d \in M$, if $a_{<n} \models p^{\otimes n-1}|_{Ad}$ then $[f_d]_p = \gamma(a_{<n}, b_{<n}) \in \Theta_{a_{<n}}$. Then $\Theta(M) \subseteq \Theta_{a_{<n}}$ and $\Theta$ is stable.

**Claim 5.8.2.** *Finitely many $\Theta_a$ cover $\Theta$.*

*Proof.* We proceed by contradiction. By induction we construct $a_i$ and $e_i$ such that:

- $a_{i+1} \models p^{\otimes n-1}|_{Aa_{\leq i} e_{\leq i}}$.
- $e_i \in \Theta \backslash (\Theta_{a_0} \cup \cdots \cup \Theta_{a_i})$.

Then $e_i \in \Theta_{a_j}$ if and only if $j > i$, but this contradicts the stability of $\Theta$. $\square$

Since each $\Theta_{a_j}$ is internal to the residue field, $\Theta$ also is. $\square$

## 5.2 Coding germs of $\mathfrak{L}_0$ definable open balls in the linear part

Let $M$ be a sufficiently saturated and homogeneous RV-expansion of a model of $\mathrm{Hen}_{0,0}$, whose value group $\Gamma$ has property **D** and is orthogonal to k. As before, let $M_0 = M^{\mathrm{a}}$ and $M_1 = M^{\mathrm{ur}}$.

We start with a lemma.

**Lemma 5.9.** *We have $\mathcal{G}(\mathrm{dcl}_1(M)) = \mathcal{G}(M)$.*

*Proof.* Let $A = \mathrm{K}(M)$. Note that $\Gamma(M_1) = \mathrm{v}(A)$. Also, any $\mathfrak{L}_1(A)$-definable ball $b$ contains a point in $a \in \mathrm{K}(M_1) \subseteq A^{\mathrm{a}}$. Since $A$ is henselian, the Galois-conjugates of $a$ in $M_1$ over $A$ are all in the generalized ball $b$ and their mean $d$ is fixed by $\mathrm{Gal}(M_1/A)$. Since the extension $A \leq \mathrm{K}(M_1)$ is normal, $d \in A$. So we can apply Corollary 3.11 (in $M_1$) to see that $\mathcal{G}(\mathrm{dcl}_1(M)) \subseteq \bigcup_{c\in\mathrm{Cut}} \mu_c(\mathrm{B}_n(A)) = \mathcal{G}(M) \subseteq \mathcal{G}(\mathrm{dcl}_1(M))$. □

### 5.2.1 Dense value groups

If we further assume that the value group is dense, what we have done so far is enough to show that germ of $\mathfrak{L}_0$-definable open balls are coded in the linear part:

**Lemma 5.10.** *Let $A \subseteq \mathcal{G}(M)$ and let $a \in N > M$ be a tuple of* K*-points such that $p = \mathrm{tp}_0(a/M)$ is $\mathfrak{L}(A)$-definable. Let $b(a)$ be an open $\mathfrak{L}_0(Ma)$-definable ball whose radius is in $\Gamma(N)$. Then, in the structure $M_1$, $[b]_p$ is coded in $\mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$ over $A$.*

*Proof.* By Proposition 4.3 and property **D**, we may assume that $tp_1(a/M_1)$ is $\mathfrak{L}_1(\mathrm{acl}_1(A))$-definable. We have $b(a) \in \mathrm{Lin}_{Aa}$, so, by Proposition 5.4 applied in $M_1$, $[b]_p$ lies in an $\mathfrak{L}_1(A)$-definable k-internal set. On the other hand, by Lemma 4.4, it is coded by some $e \in \mathcal{G}(M_1)$ over $A$. It now follows from Corollary 3.22 that $e \in \mathcal{G}(\mathrm{acl}_1(A)) \cup \mathrm{Lin}_{\mathrm{acl}_1(A)}(M_1)$. Since $e \in \mathrm{dcl}_1(M)$, by Lemma 5.9, we have in fact $e \in \mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$, concluding the proof. □

### 5.2.2 Discrete value groups

We now also assume that $\Gamma(M)$ is pure, discrete, of finite bounded regular rank and orthogonal to k. We add a constant $\pi$ for a uniformizer in $M$. As before, $M_0 = M^{\mathrm{a}}$, let $M_1 = M^{\mathrm{ur}}$ and we introduce $M_1' = M_1[\pi^{1/\infty}]$ be the extension of $M_1$ obtained by adding $n$-th roots of $\pi$ for all $n > 0$. We assume the language $\mathfrak{L}_i'$ of $M_1'$ is Morleyized and we restrict ourselves to quantifier free $\mathfrak{L}_i$-formulas when interpreting them in a substructure. We write $\mathrm{acl}_1'$ and $\mathrm{dcl}_1'$ to indicate the algebraic and definable closure in $M_1'$.

**Lemma 5.11.** *The definable convex subgroups of $\Gamma(M_1')$ are exactly the convex hulls of definable convex subgroups of $\Gamma(M_1)$ and $\Gamma(M_1')$ has bounded regular rank. Furthermore, the definable cuts in $\Gamma(M_1')$ are exactly the upward closures of definable cuts in $\Gamma(M_1)$ and the cuts above or below a point of $\Gamma(M_1')$.*

*Proof.* Fix some $n \in \omega$. Since $\Gamma(M)$ has bounded regular rank, for each $n$ there is a finite sequence of convex subgroups $0 < \Delta_1 < \cdots < \Delta_k = \Gamma(M)$ such that $\Delta_{i+1}/\Delta_i$ is $n$-regular. Note that $\Gamma(M_1') = \mathbb{Q}v(\pi) + \Gamma(M)$ and $\overline{\Delta}_i = \mathbb{Q}v(\pi) + \Delta_i$ is also a convex subgroup of $\Gamma(M_1')$. Then $\overline{\Delta}_1$ is $n$-divisible and $\overline{\Delta}_{i+1}/\overline{\Delta}_i$ is $n$-regular as it is isomorphic to $\Delta_{i+1}/\Delta_i$. Consequently, for each $n < \omega$, $\Gamma(M_1')$ has the same $n$-regular rank than $\Gamma(M)$, thus by [Far17, Proposition 2.3] $\Gamma(M_1')$ is of bounded regular rank and each $\overline{\Delta}_i$ is definable in $\Gamma(M_1')$. Furthermore, the map $\Delta$ to $\overline{\Delta}$ is a one to one correspondence between the convex subgroups of $\Gamma(M)$ and $\Gamma(M_1')$.

Let $S \subseteq \Gamma(M)$ be a definable cut and $\Delta_S = \{\gamma \in \Gamma(M) : \gamma + S = S\}$. By [Vic23b, Fact 3.2] $\Delta_S$ is a convex definable subgroup of $\Gamma(M)$, and it is the maximal convex subgroup such that $S$ is a union of $\Delta_S$-cosets. If $\Delta_S = \{0\}$, then there exists a $\gamma \in S$ such that $\gamma - \mathrm{v}(\pi) \notin S$. It follows that $S$ is the cut below $\gamma$ and so is its upwards closure in $\Gamma(M_1')$. If $\Delta_S \neq \{1\}$, then $S$ can be identified with a subset of $\Gamma(M)/\Delta_S$ which is isomorphic to $\Gamma(M_1')/\overline{\Delta}_S$ and hence the upwards closure of $S$ in $\Gamma(M_1')$ is definable.

Conversely, let $S' \subseteq \Gamma(M_1')$ be a definable cut and $\overline{\Delta}_{S'} = \{\gamma \in \Gamma(M_1') : \gamma + S' = S'\}$. If $\overline{\Delta}_{S'} \neq \{0\}$, then, as above, $S'$ is the upward closure of $S' \cap \Gamma(M)$ which is definable. If $\overline{\Delta}_{S'} = \{0\}$, then, by [Vic23b, Proposition 3.3], $S'$ is of the form $nx \,\square\, \beta$ for some $\beta \in \Gamma(M_1')$ and $\square \in \{>, \geq\}$. Growing $n$, we may assume that $\beta \in \Gamma(M)$. Moreover, since $\overline{\Delta}_{S'} = \{0\}$, for some $\gamma \in S'$, $\gamma - \mathrm{v}(\pi) \notin S'$. As $(\gamma - \mathrm{v}(\pi), \gamma] \cap \Gamma(M) \neq \varnothing$, we may assume that $\gamma \in \Gamma(M)$. Then $\beta = n\gamma - i\mathrm{v}(\pi)$, for some $i$, and $S'$ is the cut above or below $\gamma - n^{-1}i\mathrm{v}(\pi)$. □

We can therefore identify the sets $\mathcal{G}(M)$ in $M$, in $M_1$ and in $M_1'$. We do, however, have to code the imaginaries of $M$ that $M_1'$ believes to be geometric:

**Lemma 5.12.** *Let $R \in \mathrm{Gr}(\mathrm{dcl}_1'(M_1))$.*

1. *There is a $Q \in \mathrm{Gr}(M)$ such that $R \in \mathrm{dcl}_1'(Q)$ and $Q$ is definable from $R$ in the pair $(M_1', M)$.*
2. *Moreover, for every $e \in \mathrm{red}(R)(\mathrm{dcl}_1'(M))$, for some choice of such $Q$, there exists $\varepsilon \in \mathrm{red}(Q)(M)$ such that $e \in \mathrm{dcl}_1'(\varepsilon)$ and $\varepsilon$ is definable from $e$ in the pair $(M_1', M)$.*

*Proof.* By Lemma 5.9, we may assume that $M = M_1$. Then, for some $n \geq 1$, $R \in \mu_c(\mathrm{B}_m(\mathrm{K}(M)[\varpi]))$, where $c$ is a tuple in $\mathrm{Cut}^{\star\star}$ and $\varpi^n = \pi$. Let $f_\varpi : \mathrm{K}(M)^n \to \mathrm{K}(M)[\varpi]$ send $a$ to $\sum_{i<n} a_i \varpi^i$. Then for every $a \in \mathrm{K}(M)^n$, $\mathrm{v}(f_\varpi(a)) = \min_i \mathrm{v}(a_i) + n^{-1}i\mathrm{v}(\pi)$. It follows that the pre-image of $R(\mathrm{K}(M)[\varpi])$ by $f_\varpi$ is an $\mathfrak{L}(M)$-definable $\mathcal{O}$-submodule $Q(M)$ with $Q \in \mathrm{Gr}(M)$. If $\varpi'$ is another $n$-th root of $\pi$, then $\varpi' = \sigma(\varpi)$ for some $\sigma \in \mathrm{Aut}(M_1'/M)$. Then, since $\sigma(R) = R$, we have

$$f_{\varpi'}^{-1}(R(\mathrm{K}(M)[\varpi'])) = \sigma(f_\varpi^{-1}(R(\mathrm{K}(M)[\varpi]))) = \sigma(Q(M)) = Q(M).$$

So $Q$ does not depend on the choice of $\varpi'$ and it is definable from $R$ in the pair $(M_1', M)$. Also, since $f_\varpi$ is linear, it induces a surjective map $Q(M_1') \to R(M_1')$, whose image does not depend on $\varpi$. It follows that $R \in \mathrm{dcl}_1'(Q)$.

Let us now consider some $e \in \mathrm{red}(R)(\mathrm{dcl}_1'(M))$. Growing $n$, we may assume that $e \in \mathrm{red}(R)(\mathrm{K}(M)[\varpi])$. Let $\varepsilon$ be the pre-image of $e$ under the bijection $\mathrm{red}(Q)(M) \to \mathrm{red}(R)(\mathrm{K}(M)\varpi)$ induced by $f$. As above, $\varepsilon$ does not depend on the choice of $\varpi$ and it has the required properties. □

Let us now prove this variant of Proposition 4.3:

**Lemma 5.13.** *Let* $A = \mathrm{dcl}(A) \subseteq M^{\mathrm{eq}}$ *and let* $a \in N \succ M$ *be such that* $\mathrm{tp}_0(a/M)$ *is* $\mathfrak{L}(A)$*-definable. Then* $\mathrm{tp}_0(a/M_1')$ *is uniquely determined and* $\mathfrak{L}_1'(\mathcal{G}(A))$*-definable.*

*Proof.* The uniqueness follows from Proposition 4.3 — in fact, there is a unique extension to $M_0$. Let $d \geq 0$, $V_d = \mathrm{K}[x]_{\leq d}$ and $v$ be the valuation on $V_d$ defined by $v(P) \leq v(Q)$ if $\mathrm{v}(P(a)) \leq \mathrm{v}(Q(a))$. By Proposition 3.3, the space $V_d(M)$ admits a separated basis $(P_i)_{i\leq \ell} \in V_d(M)$. By [HR21, Claim 3.3.5], it is also a separated basis of $V_d(M_1')$.

For every $i, j$, let $C_{i,j} = \{\gamma \in \Gamma : v(P_i) + \gamma v(P_j)\}$. If the stabilizer $\Delta$ of $C_{i,j}$ is not 0, then, since $\Gamma/\Delta(M) = \Gamma/\Delta(M_1')$ (by Lemma 5.11), $C_{i,j}(M)$ is co-initial in $C_{i,j}(M_1')$ which is indeed definable. If this stabilizer is 0, since $\Gamma(M_1)$ is discrete, $C_{i,j}$ has a minimal element $\gamma_{i,j} \in \Gamma(M)$ and $v(P_j) = v(P_i) + \gamma_{i,j}$. So $v$ is indeed definable in $M_1'$. Moreover, the $\mathcal{O}_1'$-module $R_i(M_1') = \{P \in V_d(M_1') : v(P) \geq v(P_i)\}$ is the $\mathcal{O}_1'$-module generated by $R_i(M) = \{P \in V_d(M) : v(P) \geq v(P_i)\}$ whose codes we identify as in Proposition 4.3 via the natural inclusion map.

□

We can now recover the equivalent of Lemma 5.10 in the case of a discrete value groups:

**Lemma 5.14.** *Let* $A \subseteq \mathcal{G}(M)$ *and let* $a \in N \succ M$ *be a tuple of* K*-points such that* $p = \mathrm{tp}_0(a/M)$ *is* $\mathfrak{L}(A)$*-definable. Let* $b(a)$ *be an open* $\mathfrak{L}_0(Ma)$*-definable ball whose radius is in* $\Gamma(N)$*. Then, in the structure* $M_1$*,* $[b]_p$ *is coded in* $\mathcal{G}(\mathrm{acl}(A)) \cup \mathrm{Lin}_A(M)$ *over* $A$*.*

*Proof.* By Proposition 4.3 and Lemma 5.9, we may assume that $M = M_1$. Growing $M_1$, we may also assume that $M_1$ is sufficiently saturated and homogeneous. By Lemma 5.13, $\mathrm{tp}_0(a/M_1')$ is $\mathfrak{L}_1'(A)$-definable. Now, applying Lemma 5.10 in $M_1'$, $[b]_p$ is coded in $\mathcal{G}(\mathrm{acl}_1'(A)) \cup \mathrm{Lin}_{\mathrm{acl}_1'(A)}(M_1')$ over $A$. In other words, there are some tuple $t \in \mathrm{K}(\mathrm{acl}_1'(A)) \cap \mathrm{dcl}_1'(M) = \mathrm{K}(\mathrm{acl}(A))$, some $R \in \mathrm{Gr}(\mathrm{acl}_1'(A)) \cap \mathrm{dcl}_1'(M)$ and some $e \in \mathrm{red}(R)(\mathrm{dcl}_1'(M))$ which code $[f]_p$ over $A$. Let $Q$ and $\varepsilon$ be as in Lemma 5.12. Now, any automorphism of $\sigma \in \mathrm{Aut}(M/A)$ (extended in any way to $M_1'$) fixes $Q$ if and only if it fixes $R$ — so $Q \in \mathrm{Gr}(\mathrm{acl}(A))$ — and $\sigma$ fixes $[f]_p$ if and only if it fixes $t$, $R$ and $e$, if and only if it fixes $t$,$Q$ and $\varepsilon$. □

## 5.3 Invariant resolutions

Let $M$ be as in Theorem 5.3. As before, let $M_0 = M^{\mathrm{a}}$. Given a subset $A$ of $\mathcal{G}$, our goal is now to find a subset $C$ of K, with a definable type, which generates $A$ and "canonical" generators of $\mathrm{rv}(M(C))$. By the following lemma, this will imply that $\mathrm{tp}(C/M)$ is invariant over some large (stably embedded definable) set:

**Lemma 5.15.** *Let* $N \succ M$*, let* $D \subseteq M$ *be potentially large, let* $a \in \mathrm{K}(N)$ *be a tuple and let* $\rho$ *be a pro-*$\mathfrak{L}_1(M)$*-definable map. Assume that* $\mathrm{rv}(M(a)) \subseteq \mathrm{dcl}_1(D\rho(a))$ *and that* $p_1 = \mathrm{tp}_1(a/M)$ *and* $[\rho]_{p_1}$ *are* $\mathrm{Aut}(M/D)$*-invariant. Then* $p = \mathrm{tp}(a/M)$ *has* $\mathrm{Aut}(M/D)$*-invariant* RV*-germs.*

This is essentially [HR21, Lemma 4.2.5] in a slightly different context and the proof is identical. The main ingredient is elimination of quantifier down to RV — see Theorem 2.1.

*Proof.* Let $N_1$ be a large saturated elementary extension of $M_1$ containing $N$. Fix $\sigma \in \mathrm{Aut}(M/D)$. Since $p = \mathrm{tp}_1(a/M)$ is $\mathrm{Aut}(M/D)$ invariant, there is an $\mathfrak{L}_1$-elementary embedding $\tau : M(a) \to M(a)$ extending $\sigma$. Because $[\rho]_{p_1}$ is $\mathrm{Aut}(M/D)$-invariant, we have $\rho(a) = \sigma(\rho)(a)$. Consequently, since $\mathrm{rv}(M(a)) \subseteq \mathrm{dcl}_1(D\rho(a))$, $\tau|_{\mathrm{rv}(M(a))}$ is the identity map. By Theorem 2.1 in $N_1$, extending $\tau$ by the identity on $\mathrm{RV}(N_1)$ yields an $\mathfrak{L}_1$-elementary embedding. Since RV is stably embedded, this embedding further extends to an element $\tau$ of $\mathrm{Aut}(N_1/D, \mathrm{RV}(N_1), a)$ — *cf.* [TZ12, Lemma 10.1.5].

By Theorem 2.1 (in $M$ now), $\tau|_{M(a)\cup\mathrm{RV}(N)}$ is $\mathfrak{L}$-elementary. Consequently, $\mathrm{tp}(a, M) = \mathrm{tp}(a, \sigma(M))$ and we conclude that $\sigma(p) = p$, as required. Lastly, we argue that $\mathrm{tp}(a/M)$ has $\mathrm{Aut}(M/C)$-invariant RV-germs. Let $X \subseteq \mathrm{RV}^n$ be $\mathfrak{L}(Ma)$-definable. Then, by Theorem 2.1, it is $\mathfrak{L}(\mathrm{rv}(M(a)))$-definable and hence $X(N) = \tau(X(N)) = \tau(X)(N)$. Equivalently, $\sigma$ fixes the $p$-germ of any $\mathfrak{L}(M)$-definable function $f : p \to \mathrm{RV}^{\mathrm{eq}}$. □

Let us now describe how RV grows when adding one field element:

**Lemma 5.16.** *Let $A \subseteq \mathcal{G}(M) \cup \Gamma^{\mathrm{eq}}(M)$ contain $\mathcal{G}(\mathrm{acl}(A))$ and let $a \in N \succ M$ be a tuple of* K*-points such that $p = \mathrm{tp}_1(a/M_1)$ is $\mathfrak{L}(A)$-definable. Let $B(a)$ be a finite set of $\mathfrak{L}_1(Aa)$-definable generalized ball such that no proper subset is $\mathfrak{L}_1(Aa)$-definable. Let $c \in N$ realize the generic $\eta_{B(a)}|_{M_1a}$ — that is, $c$ is in a ball of $B(a)$ but in no proper generalized sub-ball $b \in \mathrm{acl}_1(M_1a)$. Let $q = \mathrm{tp}_1(ac/M_1)$. Then there is a (pro-)$\mathfrak{L}_1(M)$-definable map $\rho$ into some power of* RV *such that $[\rho]_q \in \mathrm{dcl}_1(A, \mathrm{Lin}_A(M))$ and $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \rho(ac))$.*

In this paper we only need $B(a)$ to be a single ball.

*Proof.* We proceed by cases. If the balls of $B(a)$ are not closed balls, we can apply [HR21, Lemma 4.3.10] — in equicharacteristic zero, condition (2) of [HR21, Lemma 4.3.10] is verified as soon as the balls of $B(a)$ are not closed balls. So, there exists a (pro-)$\mathfrak{L}_0(M)$-definable map $\rho$ into some power of RV such that $[\rho]_q \in \mathrm{dcl}(A)$ and such that $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_0(\mathrm{rv}(M(a)), \rho(ac))$. By Lemmas 5.10 and 5.14, $[\rho]_q \in \mathrm{dcl}_1(A)$.

Now assume that the balls of $B(a)$ are closed ball. Let $b \in B(a)$ be the ball containing $c$. By [HR21, Lemma 4.3.4], there exists a tuple $\nu(ac) \in \mathrm{RV}(\mathrm{dcl}_1(Aac))$ such that $b$ is $\mathfrak{L}_1(Aa\nu(ac))$-definable. Also, by Corollary 2.5, there is an $\mathfrak{L}_0(Ma)$-definable finite set $G(a) \in \mathrm{K}$ such that $G(a) \cap b = \{g\}$ is a singleton. By [HR21, Lemma 4.3.13], we have

$$\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \nu(ac), \mathrm{rv}(c - g)).$$

Let $h(ac) = \mathrm{rv}(c - g) \in \mathrm{dcl}_0(Mac)$ and $\rho(ac) = (\nu(ac), h(ac))$. We have $\gamma = \mathrm{v}(c - g) \in \Gamma(\mathrm{acl}_1(Aa))$ as it is the radius of $b$. So $[h]_q \in \mathrm{dcl}_1(A, \mathrm{Lin}_A(M))$, by Lemmas 5.10 and 5.14. Then $[\rho]_q \in \mathrm{dcl}_1(A, \mathrm{Lin}_A(M))$ and $\mathrm{rv}(M(ac)) \subseteq \mathrm{dcl}_1(\mathrm{rv}(M(a)), \rho(ac))$, as required. □

We can now prove the existence of sufficiently invariant resolutions of geometric points:

**Proposition 5.17.** *Let $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$. There exists $C \subseteq \mathrm{K}(N)$, for some $N \succ M$, with:*

1. $\mathcal{G}(A) \subseteq \mathrm{dcl}_1(C, \Gamma(M))$*;*
2. $\mathrm{tp}_1(C/M)$ *is* $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$*-definable;*
3. $\mathrm{tp}(C/M)$ *has* $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$*-invariant* RV*-germs.*

*Proof.* By transfinite induction, we construct a tuple $c \in N \succ M$ and a (pro-)$\mathfrak{L}_1(M)$-definable function $\rho$ such that:

- $p_1 = \mathrm{tp}_1(c/M)$ is finitely satisfiable in $M$ and $\mathfrak{L}(A)$-definable;
- $\mathrm{rv}(M(c)) \subseteq \mathrm{dcl}_1(\mathrm{RV}(M), \rho(c))$;
- $[\rho]_{p_1} \in \mathrm{dcl}_1(\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$;
- any $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable generalized ball $b$ has a point in $C$;
- for all $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable convex subgroup $\Delta \leq \Gamma$, $\Gamma/\Delta(\mathrm{dcl}_1(Ac)) \subseteq v_\Delta(\mathrm{K}(C))$.

Note that $\mathrm{tp}_1(\mathrm{acl}_1(Ac) \cap N/M)$ is definable over $\mathcal{G}(\mathrm{acl}(A)) \cup \Gamma(\mathrm{acl}(A))^{\mathrm{eq}} \subseteq A$ (*cf.* Proposition 4.3). Given an $\mathfrak{L}_1(\mathcal{G}(A)c)$-definable generalized ball $b(c)$, by property **D**, the generic $\eta_{b(c)}$ can be extended to a complete $\mathfrak{L}(\mathrm{acl}_1(Ac) \cap N)$-definable $\mathfrak{L}_1$-type — and this type is finitely satisfiable in $N$. Using Lemma 5.16, we can thus add a generic of $b(c)$ to $c$. We then iterate this construction.

Given such a tuple $c$, by Corollary 3.11 applied in $M_1$, we have $\mathcal{G}(A) \subseteq \mathrm{dcl}_1(c, \Gamma(A)^{\mathrm{eq}}) \subseteq \mathrm{dcl}_1(c, \Gamma(M))$. Moreover, by Lemma 5.15, $\mathrm{tp}(c/M)$ has $\mathrm{Aut}(M/A, \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant RV-germs. □

We now deduce Theorem 5.3 from Proposition 5.17 and the machinery of [HR21, Section 4].

*Proof of Theorem 5.3.* Fix $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$ and $a$ in some elementary extension of $M$ such that $p = \mathrm{tp}_0(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A))$-invariant.

- By Proposition 5.17, we find $C \subseteq \mathrm{K}(N_1)$, for some $N \succ M$ (sufficiently saturated and homogeneous), such that $\mathrm{tp}(C/M)$ has $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant RV-germs and $\mathcal{G}(A) \subseteq \mathrm{dcl}_1(C\gamma)$ for some (infinite) tuple $\gamma \in \Gamma(M)$.
- By [HR21, Corollary 4.4.1] and transitivity (Lemma 5.2), growing $C$, we may assume that $\gamma \in \mathrm{v}(C)$. By [HR21, Corollary 4.4.3] and transitivity, we can further assume that $C \prec N$ contains a realization of every type over $\mathcal{G}(A)$.
- We may assume that $a \vDash p|_N$ — *cf.* [HR21, Claim 4.4.7]. Then $\mathrm{tp}_0(a/N)$ is $\mathrm{Aut}(N/C)$-invariant. By [HR21, Corollary 4.3.17], $\mathrm{tp}(a/N)$ is $\mathrm{Aut}(N/C, \mathrm{RV})$-invariant.

By transitivity $\mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant. □

# 6 Eliminating imaginaries

Following the general strategy of [HR21, Theorem 6.1.1], we can now deduce elimination of imaginaries. Let $M$ be a sufficiently saturated and homogeneous and as in Theorem 5.3. Let $M_0 = M^{\mathrm{a}}$ and $M_1 = M^{\mathrm{ur}}$.

**Proposition 6.1.** *Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(e)$. Then*

$$e \in \mathrm{dcl}(\mathcal{G}(A), (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)).$$

*Proof.* We may assume $M$ is sufficiently saturated and homogeneous. There is an $\mathfrak{L}$-definable map $f$ and a tuple $m \in K(M)$ such that $f(m) = e$. Let $X = f^{-1}(e)$. By Theorem 4.1 we can find a type $p \in \mathcal{S}^1(M)$ such that:

- $p \cup X$ is consistent;
- $p$ is $\mathfrak{L}_1(\mathcal{G}(A) \cup \Gamma^{\mathrm{eq}}(A))$-definable.

Take $a \models p \cup X$. Then $\mathrm{tp}_0(a/M)$ is $\mathfrak{L}(\mathcal{G}(A))$-definable. By Theorem 5.3, the type $q = \mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$-invariant. So, for every automorphism $\sigma \in \mathrm{Aut}(M/\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M))$, we have $e = \sigma(e)$ since $q = \sigma(q) \vdash \sigma(e) = f(x) = e$.

As $\mathrm{RV} \cup \mathrm{Lin}_A$ is stably embedded (*cf.* Remark 3.19), if follows (*e.g.* [HR21, Lemma 4.2.3]) that

$$e \in \mathrm{dcl}(\mathcal{G}(A), \mathrm{RV}(M), \mathrm{Lin}_A(M)).$$

So there is a $\mathfrak{L}(\mathcal{G}(A))$-definable function $g$ and a tuple $c \in \mathrm{RV}^m(M) \times \mathrm{Lin}_A^n(M)$ such that $g(c) = e$. Let $Y = g^{-1}(e)$. This is an $A$-definable subset of $\mathrm{RV}^m \times \mathrm{Lin}_A^n$. Consequently, $\ulcorner Z \urcorner \in (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)$ and

$$e \in \mathrm{dcl}(\mathcal{G}(A), \ulcorner Z \urcorner) \subseteq \mathrm{dcl}(\mathcal{G}(A) \cup (RV \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)),$$

as required. □

We now want to describe the imaginaries in $\mathrm{RV} \cup \mathrm{Lin}_A$. This amounts to describing imaginaries in short exact sequences (with auxiliary sorts) as in [HR21, Proposition 5.2.1]. Let us first proof a version of that result under alternative finiteness assumptions that focus on the kernel of the sequence.

**Proposition 6.2.** *Let $\mathcal{L}$ be a language with sorts $\mathcal{A} \sqcup \{\mathrm{B}, \mathrm{C}\}$. Let $R$ be an integral domain. Let $M$ be an $\mathcal{L}$-structure (with potentially additional structure on C and, independently $\mathcal{A}$) of the pure (in the sense of model theory) sequence of $R$-modules*

$$0 \to \mathrm{A} \to \mathrm{B} \xrightarrow{v} \mathrm{C} \to 0 \tag{2}$$

*where $\mathrm{A} \in \mathcal{A}$ and $\mathrm{C} \in \mathcal{C}$. Assume that the following properties hold:*

1. *For any $l \in R \smallsetminus \{0\}$, $A = lA$ — in particular,* A *is a pure $R$-submodule of* B *(in the sense of module theory);*
2. C *is a torsion free $R$-module.*

*Let $e \in M^{\mathrm{eq}}$, $E = \mathrm{acl}(e)$ and $\Delta = \mathrm{C}(E)$. Then*

$$e \in \mathrm{dcl}(\mathrm{C}^{\mathrm{eq}}(E) \cup (\mathcal{A} \cup \mathrm{B}_\Delta)^{\mathrm{eq}}(E)),$$

*where $\mathrm{B}_\Delta$ denotes the union of all $\mathrm{B}_\delta$ for $\delta \in \Delta$.*

*Proof.* We follow the proof of [HR21, Theorem 5.1.5] with slight modifications. Note that $\mathcal{A}$ and C are orthogonal in this structure.

Let $X \subseteq \mathrm{B}^n$ be $\mathcal{L}(M)$-definable, $E = \mathrm{acl}(\ulcorner X \urcorner)$ and $\Delta = \mathrm{C}(E)$. We proceed by induction on $r = \dim_R(X)$, see [HR21, p. 59]. By [HR21, Lemma 5.1.3] we may assume that there exists an $R$-linear map $L : \mathrm{B}^r \to \mathrm{B}^{n-r}$, $\delta \in \mathrm{C}^{n-r}(E)$ and $m \in R \setminus \{0\}$ such that for every $(x', x'') \in X$, with $|x'| = r$, we have $mv(x'') = L(v(x')) + \delta$. Let $X_m = \{(x', x'') : ((x')^m, x'') \in X \text{ and } |x'| = r\}$, then $\ulcorner X_m \urcorner = \ulcorner X \urcorner$ and for every $(x', x'') \in X_m$ with $|x'| = r$, we have $v(x'') = L(v(x')) + \delta/m$. So we may assume that $m = 1$.

We consider the action of $\mathrm{A}^r$ on $\mathrm{B}^n$ given by $a \cdot (x', x'') = (x' + a, x'' + L(a))$. By [HR21, Claim 5.1.6], and since $l\mathrm{A} = \mathrm{A}$, for every $l \in R \setminus \{0\}$, we may assume that $\mathrm{A}^r \cdot X = X$. For every $c \in \mathrm{C}^r$ and $x' \in \mathrm{B}^r$ with $v(x') = c$, let $Y_c = X_{x'} + L(x') = \{x'' - L(x') : (x', x'') \in X\} \subseteq B_\delta$. This set $Y_c$ does not depend on the choice of $x'$. Indeed, if $v(y') = c$ and $(x', x'') \in X$, then $a = y' - x' \in \mathrm{A}^r$ and hence $a \cdot (x', x'') = (y', x'' - L(x') + L(y')) \in X$. So $x'' - L(x') \in X_{y'} - L(y')$ and hence, by symmetry, $X_{x'} - L(x') = X_{y'} - L(y')$. Then $X$ and $Y = \{(c, b) : b \in Y_c\} \subseteq \mathrm{C} \times \mathrm{B}_\delta$ are inter-definable, and we conclude by orthogonality of $\mathrm{B}_\Delta$ and C.

Now, if $X \subseteq \mathrm{A}' \times \mathrm{B}^n$ where $\mathrm{A}'$ is a product of sorts in $\mathcal{A}$, for every $a \in A'$, the fiber $X_a \subseteq \mathrm{B}^n$ is coded in $\mathrm{C}^{\mathrm{eq}} \cup (\mathcal{A} \cup \mathrm{B}_\Delta)^{\mathrm{eq}}$ where $\Delta = \mathrm{C}(\mathrm{acl}(\ulcorner X \urcorner a)) = \mathrm{C}(\mathrm{acl}(\ulcorner X \urcorner))$, by orthogonality. It follows, by orthogonality again, that the graph of the function $a \mapsto \ulcorner X_a \urcorner$, and hence $X$ itself, is coded also in $\mathrm{C}^{\mathrm{eq}} \cup (\mathcal{A} \cup \mathrm{B}_\Delta)^{\mathrm{eq}}$. □

We deduce the following variant of [HR21, Proposition 5.3.1]. This covers new cases since there are no conditions on $\Gamma$ when $\mathrm{k}^\star$ is divisible.

**Corollary 6.3.** *Further assume that $M$ is a* k-Γ-*expansion of* $\mathrm{Hen}_{0,0}$ *and that* either one *of the following conditions holds:*

*(a) For every $n \in \mathbb{Z}_{\geq 0}^{\geq 2}$ one has $[\Gamma : n\Gamma] < \infty$ and the pre-image in* RV *of any coset of $n\Gamma$ contains a point which is algebraic over $\varnothing$,*

*(b) The group $\mathrm{k}^\star$ is divisible.*

*Let $A \subseteq M^{\mathrm{eq}}$ and $e \in (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(M)$ and $E = \mathrm{acl}(e)$. Then*

$$e \in \mathrm{dcl}(\Gamma^{\mathrm{eq}}(E) \cup (\mathrm{Lin}_A \cup \mathrm{RV}_{\Gamma(E)})^{\mathrm{eq}}(E)).$$

*In particular for $A = \mathrm{acl}(A) \subseteq M^{\mathrm{eq}}$,*

$$(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}}(A)).$$

*Proof.* With hypothesis (a), this is [HR21, Proposition 5.3.1]. With hypothesis (b), it is a direct consequence of Proposition 6.2 with $R = \mathbb{Z}$. Note that $\Gamma$ is torsion free, as it is an ordered abelian group. □

Finally, let us relate $\mathrm{Lin}_A^{\mathrm{eq}}$ to the linear imaginaries $\mathrm{k}^{\mathrm{leq}}$:

**Lemma 6.4.** *Let $A = \mathrm{dcl}(A) \subseteq M$. Then $\mathrm{Lin}_A^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\mathrm{k}^{\mathrm{leq}}(A))$.*

*Proof.* Recall that $\mathrm{Lin}_A$ is a stably embedded collection of k-vector spaces — see Remark 3.19. Take $e \in \mathrm{Lin}_A^{\mathrm{eq}}(A)$. Then $e$ is the code of a definable set $X_a \subseteq \prod_i \mathrm{red}(R_i)$ where $a \in \prod_j \mathrm{red}(R'_j)$ and $R_i$ and $R'_j$ are $A$-definable $\mathfrak{m}$-avoiding module. Then $R = \prod R_i \times \prod_j R'_j$ is an $A$-definable $\mathfrak{m}$-avoiding module, and, adding zero coordinates, we may assume that we have $X_a \subseteq \mathrm{red}(R)$ and $a \in \mathrm{red}(R)$. Let $aEa'$ be the equivalence relation defined by $X_a = X'_a$ and $c$ be the type of $R$. Then $e \in \mathrm{dcl}(\mathrm{Lin}_{c,V/E})$. □

We can now prove our main results:

**Theorem 6.5.** *Further assume that $M$ is a* k-$\Gamma$*-expansion of* $\mathrm{Hen}_{0,0}$ *and that* either one *of the following conditions holds:*

*(a) for every $n \in \mathbb{Z}_{\geq 0}^{\geq 2}$ one has $[\Gamma : n\Gamma] < \infty$ and the pre-image in* RV *of any coset of $n\Gamma$ contains a point which is algebraic over $\varnothing$;*

*(b) or, the multiplicative group $\mathrm{k}^\times$ is divisible.*

*Then $M$ weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

*Proof.* Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(A)$. By Proposition 6.1, we have

$$e \in \mathrm{dcl}(\mathcal{G}(A) \cup (\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A)).$$

By Corollary 6.3, we have

$$(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}}(A) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A), \mathrm{Lin}_A^{\mathrm{eq}}(A)) \subseteq \mathrm{dcl}(\Gamma^{\mathrm{eq}}(A), \mathrm{k}^{\mathrm{leq}}(A)),$$

where the last inclusion follows from Lemma 6.4. □

**Theorem 6.6.** *Further assume that $M$ admits $\mathfrak{L}$-definable angular components. Then $M$ weakly eliminates imaginaries down to $K \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$.*

*Proof.* Let $e \in M^{\mathrm{eq}}$ and $A = \mathrm{acl}(A)$. By Proposition 6.1, $e \in \mathrm{dcl}(\mathcal{G}(A), (\mathrm{RV} \cup \mathrm{Lin}_A^{\mathrm{eq}})(A))$. Since RV is $\varnothing$-definably isomorphic to $\mathrm{k}^\times \times \Gamma$, then $(\mathrm{RV} \cup \mathrm{Lin}_A)^{\mathrm{eq}} \subseteq (\Gamma \cup \mathrm{Lin}_A)^{\mathrm{eq}}$. The statement now follows from orthogonality of $\Gamma$ and $\mathrm{Lin}_A$ and Lemma 6.4. □

As an illustration, we conclude this paper with the complete classification of (almost) k-internal sets, when the value group is dense.

**Corollary 6.7.** *Let $M$ be as in Theorem 6.5 or Theorem 6.6 and assume that $\Gamma(M)$ is dense. Let $A \subseteq M^{\mathrm{eq}}$ and $X$ be $A$-definable. The following statements are equivalent:*

1. *$X$ is* k*-internal;*
2. *$X$ is almost* k*-internal;*
3. *$X$ is orthogonal to $\Gamma$;*
4. *$X \subseteq \mathrm{dcl}(\mathrm{acl}(A), \mathrm{Lin}_A)$.*

*Proof.* The fourth statement is a particular case of the first statement. The second statement is a particular case of the first, and it implies the third since k and $\Gamma$ are orthogonal. There remains to prove that if $X$ is orthogonal to $\Gamma$ then it is a subset of $\mathrm{acl}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}}$. By Theorems 6.5 and 6.6, any element $a \in X$ is weakly coded in by some tuple $\eta \in \mathrm{K} \cup \mathrm{k}^{\mathrm{leq}} \cup \Gamma^{\mathrm{eq}}$. Then $\eta$ also lies on a $A$-definable set orthogonal to $\Gamma$. Since $X$ is orthogonal to $\Gamma$, if $\eta \in \mathrm{K} \cup \Gamma^{\mathrm{eq}}$, then $\eta \in \mathrm{acl}(A)$. If $\eta \in \mathrm{Gr}$, then, by Proposition 3.21, $\eta \in \mathrm{acl}(A)$. Finally, if $\eta \in \mathrm{red}(R_s)^{\mathrm{eq}}$, for some $s \in \mathrm{Gr}$, then $s \in \mathrm{acl}(A)$ and hence $\eta \in \mathrm{Lin}_A^{\mathrm{eq}}$. It follows that $e$ is weakly coded in $\mathrm{acl}(A) \cup \mathrm{Lin}_A^{\mathrm{eq}}$, as required. □

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