On groups definable in geometric fields with generic derivations

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We study groups definable in existentially closed geometric fields with commuting derivations. Our main result is that such a group can be definably embedded in a group interpretable in the underlying geometric field. Compared to earlier work of the first two authors toguether with K. Peterzil, the novelty is that we also deal with infinite dimensional groups.

Introduction

This paper is about comparing groups definable in a given structure with groups definable in a reduct. These latter groups are usually better understood, providing in return many insight on the original groups.

The case of interest here is to compare groups definable in a field with a derivation (and potentially additional structure), to groups definable without the derivation. In the case of a differentially closed field[1], the second author proved in [Pil97] that a differential algebraic group differentially algebraically embeds in an algebraic group (answering questions of Kolchin). The methods were stability-theoretic and reminiscent of Weil's construction of an algebraic group out of a pregroup [Wei55].

Our main goal here is to consider a generalization of this result to an unstable context. We will be working with enriched geometric fields. Following [HP94, Definition 2.9], but allowing additional structure, we say that a (complete) theory T of fields with additional structures is *geometric* if:

- In models of T, model theoretic algebraic closure coincides with relative field theoretic algebraic closure (over $\operatorname{dcl}(\emptyset)$).

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[1] With no further structure.

- Models of T are perfect and T eliminates the $\exists^\infty$ quantifier.

As noted by multiple authors, [For11; JY23a], the first hypothesis implies the second. Such a theory T is also said to be *algebraically bounded*.

Examples include real closed fields, p-adically closed fields, more generally characteristic zero henselian valued fields, bounded pseudo algebraically closed fields (such as pseudofinite fields), open theories of topological fields [CP23], bounded perfect pseudo T-closed fields [MR23] and curve excluding fields [JY23b], among others.

So let us fix some geometric theory T of enriched fields in some language L. At the cost of Morleyizing, we can always assume that T eliminates quantifiers in a relational expansion of the ring language. Also fix some integer ℓ and $\Delta = \{\partial_1, \dots, \partial_\ell\}$. Consider the theory of models of T with ℓ commuting derivations in the language $L_\Delta = L \cup \Delta$. By [FT23, Theorem 4.1], this theory admits a model completion that we will denote T_Δ.

In this context, we prove the following.

Theorem (Theorem 3.6)**.** *Let $K \models T_\Delta$ and let Γ be a group which is L_Δ-definable in K. Then there is a group G which is L-interpretable in (the reduct to L of) K and an L_Δ-definable embedding $\Gamma \to G$.*

Moreover, if Γ is definable over some $A \subseteq K$, then the group G and the embedding can also be chosen over A.

In [PPP22] the first three authors proved the finite-dimensional case (see [PPP22, Defintion 2.2]) of our main result — actually first considering $K \models RCF$, then generalizing. Assuming that Γ is finite dimensional, one recovers a generically given L-definable group, and proceeds from there. In the case of possibly infinite-dimensional Γ one will obtain some kind of generically given L-definable group but living on infinite tuples, and there are additional technical complications. It turns out that general results from [HR19] are in a sense tailor-made to handle such situations, so we will appeal to them. Possibly, the methods of [KP02] giving another account of [Pil97] would also adapt to the present setting.

On the face of it, the main theorem for finite-dimensional definable groups follows from the theorem for arbitrary definable groups. However in [PPP22], the group G can be chosen to be L-definable and not only L-interpretable — no quotient is required.

Note also that, in [PPP22], the finite-dimensional case was also considered in other contexts, namely o-minimal expansions of real closed fields, which fall outside of the scope of the present paper. Also, in [PPP23], Buium's notion of an "algebraic D-group" is adapted from the context of algebraically closed fields with a derivation to models of T with a derivation and finite-dimensional L_Δ-definable groups are shown to be precisely groups of "sharp points" of D-groups.

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1 Geometric fields with generic derivations

Let us start by recalling some facts about geometric fields.

Definition 1.1. *Let $K \models T$ and let $X \subseteq K^n$ be definable (with parameters). The dimension* $\dim(X)$ *of X is the dimension (in the sense of algebraic geometry) of the Zariski closure of X in affine n-space.*

If X is definable over some $A \subseteq K$ and if K is sufficiently saturated, $\dim(X)$ is the maximal transcendence degree of some $c \in X(K)$ over A.

In a geometric theory, dimension is definable in the sense that for all definable $X \subseteq Y \times Z$, for every $n \geq 0$, $\{z \in Z : \dim(X_z) = n\}$ is definable, where $X_z = \{y \in Y : (y, z) \in X\}$ is the fiber of X at z.

Let us now consider the properties of models of T_Δ. Let $K, M \models T_\Delta$ and let $A \subseteq K$. We say that an injective ring embedding $f : A \to M$ is an L-elementary embedding if it preserves L-types (between the reducts of K and M to L) and that it is differential if it preserves the derivations. If we want to specify that we work in the reduct to L, we will indicate it with a subscript $_L$, like in the notation dcl_L for the L-definable closure. Notation with a subscript $_{\Delta\mathrm{rg}}$ will refer to the differential field structure and notions with a subscript $_\Delta$, or no subscript, will refer to the full L_Δ-structure.

Lemma 1.2. *Let $K, M \models T_\Delta$, let $A \leq_{\Delta\mathrm{rg}} K$ be a differential subfield and let $f : A \to M$ be a differential L-elementary embedding. Then there exists an L_Δ-elementary extension $M' \succcurlyeq M$ and a differential L-elementary embedding $g : K \to M'$ extending f such that $g(K)$ is algebraically independent from M over $g(A)$.*

Proof. Let $A_0 = A^{\mathrm{alg}} \cap K$ be the relative field algebraic closure of A in K. Since f is L-elementary, we can extend f to $\overline{f} : A_0 \to M$ as an L-elementary embedding. Moreover, since Δ extends uniquely to A^{alg}, $\overline{f}$ is also a differential field embedding.

Fix some tuple $a \in K^m$ and let V be its (geometrically integral) locus over A_0. For any $X \in \mathrm{tp}(a/A_0)$ contained in V, let W be the Zariski closure of X over K. Then $W \subseteq V$, it contains a and, by invariance, it is defined over $\mathrm{dcl}_L(A_0) = A_0$. So $W = V$. By compactness, it follows that we can extend $\overline{f}$ to an L-elementary $g_L : K \to M' \succcurlyeq_L M$ such that $g_L(K)$ is algebraically independent from M over $g_L(A_0)$.

Since $A_0 \leq K$ is a regular extension, it follows that the compositum $g_L(K)M$ is isomorphic to (the fraction field of) $K \otimes_{A_0} M$ which can therefore be made into a substructure of a model of T extending the L-structure on both K and M. The derivations on K and M also extend uniquely to $K \otimes_{A_0} M$ and the resulting derivations commute. Since $M \leq K \otimes_{A_0} M$ is L_Δ-existentially closed, $K \otimes_{A_0} M$ can be embedded into some L_Δ-elementary extension $M' \succcurlyeq M$, concluding the proof. $\square$

Let Θ denote the commutative monoid generated by Δ. Its elements are of the form $\theta = \partial_1^{e_1} \ldots \partial_n^{e_n}$ for all integers $e_1, \ldots e_n \geq 0$. For such a θ, we define $|\theta| = \sum_{i \leq n} e_i$. We order the elements of Θ by lexicographic order on, first $|\theta|$ and the the components of θ.

For any tuple a in a differential field (K, Δ) and for any integer $n \geq 0$, we write $\nabla_n(a) = (\theta(a_i))_{i, |\theta| \leq n}$. We also write $\nabla_\omega(a) = (\theta(a))_{i,\theta}$.

From Lemma 1.2, we immediately recover a strong form of quantifier elimination[2]. Let $K \models T_\Delta$ and let $A \subseteq K$.

[2] This is implicit in [FT23], but it is only stated explicitly for a single derivation.

Corollary 1.3. *If X is L_Δ-definable over A in K, there exists an integer n and a set Y which is L-definable over A such that $x \in X$ if and only if $\nabla_n(x) \in Y$.*

Proof. Let $K, M \models T_\Delta$ be sufficiently saturated, let $A \leq_{\Delta\mathrm{rg}} K$ be a small differential subfield and $f : A \to M$ be a differential L-elementary embedding. Let $B \leq M$ be small and contain A. By Lemma 1.2, we may extend f to a differential L-elementary embedding $B \to N$ — growing B, we may first assume that it is an L_Δ-elementary substructure of M. In other words, differential L-elementary isomorphisms between small differential subfields have the back-and-forth and hence are L_Δ-elementary.

It follows that, for every tuple a and b in K, if $\mathrm{tp}_L(\nabla_\omega(a)) = \mathrm{tp}_L(\nabla_\omega(b))$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$. The corollary follows by compactness. □

We can also immediately characterize algebraic and definable closure in models of T_Δ.

Corollary 1.4. *1. The L_Δ-algebraic closure $\mathrm{acl}_\Delta(A)$ of A is the relative field algebraic closure in K of the differential field generated by A.*

2. The L_Δ-definable closure $\mathrm{dcl}_\Delta(A)$ of A is the L-definable closure of the differential field generated by A.

3. Let $f : X \to Y$ be L_Δ-definable over A. There exists $F : Z \to W$ which is L-definable over A and an integer $n \geq 0$ such that $\nabla_n(X) \subseteq Z$ and , for all $x \in X$, $f(x) = F(\nabla_n(x))$.

Proof. Let $A = A^{\mathrm{alg}} \cap K \leq_{\Delta\mathrm{rg}} K$ be a relatively algebraically closed differential subfield. By Lemma 1.2, there exists an differential L-elementary embedding $g : K \to K' \succcurlyeq K$ extending the identity on A and such that $g(K)$ and K are algebraically independent over A. Since $A \leq K$ is regular, K and $g(K)$ are linearly disjoint over A. Moreover, by Corollary 1.3, g is L_Δ-elementary and thus $g(K) \preccurlyeq K'$. It follows that $\mathrm{acl}_\Delta(A) \subseteq K \cap g(K) = A$, concluding the proof of the first item.

Now, let $A = \mathrm{dcl}_L(A) \leq_{\Delta\mathrm{rg}} K$ be a dcl_L-closed differential subfield of K. By the first item, we have $\mathrm{dcl}_\Delta(A) \subseteq \mathrm{acl}_\Delta(A) \subseteq A^{\mathrm{alg}}$. Consider $a \in A^{\mathrm{alg}} \cap K \setminus A$. Then, since $a \notin \mathrm{dcl}_L(A)$, it has at least one other L-conjugate $a' \in M$ over A. Since $a, a' \in A^{\mathrm{alg}}$, any L-elementary embedding sending a to a' is also a differential embedding and hence, by Corollary 1.3, it is L_Δ-elementary. So $a \notin \mathrm{dcl}_\Delta(A)$ and $\mathrm{dcl}_\Delta(A) \subseteq A$, proving the second item.

Finally, let $f : X \to Y$ be L_Δ-definable over A. For every $x \in X$, by the previous item, we have $f(x) \in \mathrm{dcl}_L(A\nabla_\omega(x))$. By compactness, it follows that there are finitely many maps F_i which are L-definable over A such that for every $x \in X$, there exists an i such that $f(x) = F_i(\nabla_\omega(x))$. Let $X_i = \{x \in X : f(x) = F_i(\nabla_\omega(x))\}$ and, by Corollary 1.3, let Z_i be L-definable over A such that, for some sufficiently large n, $x \in X_i$ if and only if $\nabla_n(x) \in Z_i$. We may assume that the Z_i are disjoint. We define F on $Z = \bigcup_i Z_i$ by $F(z) = F_i(z)$ if $z \in Z_i$. Then, for any $x \in X$, we have $f(x) = F(\nabla_n(x))$. □

Let us conclude this section with a purely differential statement which is implicit in the proof of the existence of the Kolchin polynomial.

Lemma 1.5. *Let (K, Δ) be a differential field (with finitely many commuting derivations). Let $K \leq_{\Delta\mathrm{rg}} K\langle a_1, \ldots, a_m\rangle_\Delta$ be a finitely generated differential field extension. Then there exists an $n \neq 0$ such that the extension $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental.*

Proof. We order the set of $\theta(a_i)$, for all $\theta \in \Theta$ and $i \leq n$ by lexical order first on θ and then on i — this is a well founded ordering. Let E be the set of θa_i such that $\theta a_i \in K(\theta' a_j : \theta' a_j < \theta a_i)^{\mathrm{alg}}$. For any $\theta a_i \in E$ and any $\eta \in \Theta \setminus \{1\}$, we have $\eta\theta a_i \in K(\theta' a_j : \theta' a_j \leq \theta a_i)$.

Let E_0 be the set of $\theta a_i \in E$ which are not proper derivatives of any element of E, then E_0 is finite and $E = \Theta E_0$. Let $n = \max_{\theta \in E_0} |\theta|$. Then $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental. Indeed, for any θa_i with $|\theta| > n$, either $\theta a_i \in E$ in which case $\theta a_i \in K(\theta' a_j : \theta' a_j < \theta a_i)$ or $\theta a_i \notin E$ in which case θa_i is transcendental over $K(\theta' a_j : \theta' a_j < \theta a_i)$. □

2 Generic points

Let $K \models T_\Delta$ be sufficiently saturated and homogeneous. We also fix an elementary extension $M \succcurlyeq K$ which is $|K|^+$-saturated in which to realize (partial) types over K.

Let $L_{\Delta\mathrm{rg}}$ denote the language of differential rings. In what follows we will be using "Morley rank" in K. We define it in the following manner. The space S of quantifier free $L_{\Delta\mathrm{rg}}$-types over A which are finitely satisfiable in K has a Cantor-Bendixon rank — also called the Morley rank — which is ordinal, since DCF is ω-stable and the set of types over K which are finitely satisfiable in K is closed.

Let X be L_Δ-definable in K, then the set of quantifier free $L_{\Delta\mathrm{rg}}$-types over K consistent with X is closed and by definition of Morley rank, we get:

Lemma 2.1. *The set of quantifier free $L_{\Delta\mathrm{rg}}$-types over K of maximal Morley rank which are consistent with X is finite.*

We denote those types $p_1, \ldots, p_k$.

Definition 2.2. *We say that $a \in X$ — in M — is generic (over K) if it realizes one of the p_i — in other words, a is generic in X if a has maximum Morley rank in X over K.*

Lemma 2.3. *Let $f : X \to X$ be definable over K and injective. Let a be generic in X, then $f(a)$ is also generic.*

Proof. By hypothesis, $\mathrm{dcl}(Ka) = \mathrm{dcl}(Kf(a))$. In follows from Corollary 1.4.1, that the field algebraic closure of the differential fields generated by a and $f(a)$ over K are identical. It follows that a and $f(a)$ have the same Morley rank over K. So a is generic in X if and only if $f(a)$ is. □

Let $\Sigma_X(x_\omega)$ be the common L-type over K of $\nabla_\omega(a)$, where $a \in X$ is generic over K. By quantifier elimination (Corollary 1.3), $\Sigma_X(\nabla_\omega(x))$ is the partial type of generics in X.

Lemma 2.4. *Let $A = \mathrm{dcl}_\Delta(A) \subseteq K$ be such that X is definable over A. Then the partial type Σ_X is L-definable over A — that is, for every formula $\phi(x_\omega, y)$, the set of $a \in K^y$ such that $\Sigma_X(x_\omega) \models \phi(x_\omega, a)$ is L-definable over A.*

Proof. For every $n \geq 0$, let $W_{i,n}$ be the Zariski locus of $\nabla_n(a)$ over K for any (equivalently all) $a \models p_i$. Then $a \models p_i$ if and only if, for every n, $\nabla_n(a) \in W_{i,n}$ and $\dim(\nabla_n(a)/K) = \dim(W_{i,n})$.

Now, fix a formula $\phi(x_\omega, y)$. Let $n \geq 0$ be sufficiently large so that all the variables of x_ω that actually appear in ϕ are in $x_n = \{x_\theta : |\theta| \leq n\}$. We will write ϕ as $\phi(x_n, y)$. Also assume that n is sufficiently large so that $x \in X$ if and only if $\nabla_n(x) \in Y$, for some Y which is L-definable over A. Finally, assume that n is sufficiently large so that, by Lemma 1.5, for every i and every $a \models p_i$, the extension $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental.

Claim 2.5. *Fix an i and let $a_n \in W_{i,n}$ be such that $\dim(a_n/K) = \dim(W_{i,n})$. Then there exists $b \models p_i$ such that $\mathrm{tp}_L(\nabla_n(b)/K) = \mathrm{tp}_L(a_n/K)$.*

Let $b \models p_i$ — a priori we can choose b in M but for now we ignore the L-structure induced by M on $K(\nabla_n(b))$. We have a field isomorphism $f_n : K(a_n) \to K(\nabla_n(b))$ sending a_n to $\nabla_n(b)$. By saturation, we can find $c = (c_i)_{i<\omega} \in M$ transcendental and algebraically independent over $K(a_n)$. As, by choice, $K(\nabla_\omega(b))$ is purely transcendental over $K(\nabla_n(b))$, the isomorphism f_n extends to a ring isomorphism $f : K(a_n, c) \to K(\nabla_\omega(b))$.

Now $K(a_n)(c)$ has its L-structure (as a substructure of M), and the isomorphism f induces a new L-structure on the differential field $K(\nabla_\omega(b))$ which is compatible with the ring structure and extends the L-structure on K. Let us write K_1 for the differential field $K(\nabla_\omega(b))$ with this new L-structure. By construction, f is an L-embedding, and hence the quantifier free L-type of $\nabla_n(b)$ over K in K_1 equals the quantifier free L-type of a_n over K. Also note that K_1 is the expansion of a substructure of a model of T by ℓ commuting derivations containing the model K of T_Δ. As M is a saturated model of T_Δ (the model completion of models of T with ℓ commuting derivations), there is an embedding h of the L_Δ-structure K_1 into M over K. Let $c = h(b)$. Then a_n and $\nabla_n(c) = h(\nabla_n(b))$ have the same quantifier free L-type over K. As T has quantifier elimination they have the same L-type over K in the model M of T. This proves Claim 2.5.

Now, for any $d \in K^y$, if $\dim(x_n \in Y \cap W_{i,n} \wedge \phi(x_n, d)) = \dim(W_{i,n})$, by definition, there exists an $a_n \in Y(M)$ such that $\dim(a_n/K) = \dim(W_{i,n})$. Then, by Claim 2.5, we find $b \models p_i$ such that $\nabla_n(b) \in Y$ — and hence $b \in X$. Conversely, if p_i is consistent with p_i, then $\dim(x_n \in Y \cap W_{i,n} \wedge \phi(x_n, d)) = \dim(W_{i,n})$.

Then $\phi(x_n, d) \in \Sigma_X$ if and only if for any $b \in X$ generic over K, we have $M \models \phi(\nabla_n(b), d)$; equivalently,

$$\dim(x_n \in Y \cap W_{i,n} \wedge \neg\phi(x_n, b)) < \dim(W_{i,n}).$$

Since dimension is definable in T, this concludes the proof that Σ_X is L-definable. There remains to show that it is L-definable over $\mathrm{dcl}_\Delta(A)$. However, note that, for any n, the finite set of (codes of the) $W_{i,n}$ is L_Δ-definable over A. By elimination of imaginaries in algebraically closed fields, it is quantifier free definable in the ring language over $\mathrm{dcl}_\Delta(A)$. Hence Σ_X is L-definable over $\mathrm{dcl}_\Delta(A)$. □

3 Groups

For now, let T be any theory, let $A \subseteq K \models T$. Assume that K is $|A|^+$-saturated. As previously, we also fix an elementary extension $M \succcurlyeq K$ which is $|K|^+$-saturated in which we realize partial types over K.

A $*$-definable set over A is the set of realizations in K of a partial type over A in at most $|A|$-many variables. A $*$-interpretable set over A is a set which is $*$-definable over A in the expansion K^{eq} of K by all quotients of $\emptyset$-definable sets by $\emptyset$-definable equivalence relation. These are also known as pro-interpretable sets. Finally, a $*$-definable group (resp. $*$-interpretable) over A is a $*$-definable (resp. $*$-interpretable) G over A together with a $*$-definable group law over A.

Given a $*$-interpretable set X, a (global) partial type Σ concentrating on X is a partial type in the same variables as X, which is over K (not necessarily over a small set of parameters from K) and such that $\Sigma(x) \models x \in X$. We assume that partial types are closed under finite conjunctions and consequences. For example a complete type $p(x)$ over K implying X is such a global partial type concentrating on X.

Definition 3.1. *Let G be a $*$-interpretable group and Σ a global partial type concentrating on G. We say that Σ is (left) translation invariant if for every $g \in G(K)$ and $a \models \Sigma$, we have $g \cdot a \models \Sigma$ — equivalently, if $\Sigma \models X$ and $g \in G(K)$, then $\Sigma \models g \cdot X$.*

If $\Sigma(x)$ is a global partial type, we write $\Sigma|_A$ for its restriction to formulas with parameters in A. Also, as in Lemma 2.4, we say that Σ is definable over A if for every formula $\phi(x, y)$, the set of tuples $a \in K^y$ such that $\Sigma(x) \models \phi(x, a)$ is L-definable over A. Note that, in [HR19], a global partial type concentrating on G is called a definable generic if it is both definable and translation invariant.

Definition 3.2. *Let Σ be a global partial type (concentrating on some $*$-interpretable set) which is definable over A and let $F(x, y)$ be a map $*$-definable over A. We say that (Σ, F) is a pregroup[3] over A if:*

1. *If $a \models \Sigma|_A$ and $b \models \Sigma$, then $F(a, b)$ is defined and $F(a, \Sigma) = \Sigma$.*
2. *If $a \models \Sigma|_A$ and $b \models \Sigma|_{Aa}$, then a and b are interdefinable over $A \cup \{F(a, b)\}$.*
3. *If $a \models \Sigma|_A$, if $b \models \Sigma|_{Aa}$ and if $c \models \Sigma|_{Aab}$, then $F(a, F(b, c)) = F(F(a, b), c)$.*

These are the main results we will use on these notions:

Proposition 3.3 ([HR19, Prop. 3.15])**.** *Let (Σ, F) be a pregroup over A. Then there exists a $*$-interpretable group G over A and an injective map f which is $*$-interpretable over A and such that:*

[3] Note that we diverge from the terminology in [HR19] where such a pair is called a group chunk.

1. *for any* $a \models \Sigma|_A$*, we have* $f(a) \in G$*;*

2. *for any* $a \models \Sigma|_A$ *and* $b \models \Sigma|_{Aa}$*,* $f(F(a,b)) = f(a) \cdot f(b)$*;*

3. *the global partial type* $f(\Sigma)$ *is translation invariant in* G*.*

Proposition 3.4 ([HR19, Prop. 3.4])**.** *Let* G *be a* $*$*-interpretable group over* A *and let* Σ *be a translation invariant global partial type concentrating on* G *definable over* A*. Then* G *is* $*$*-interpretably over* A *isomorphic to a projective limit of groups interpretable over* A*.*

Proposition 3.5 ([HR19, Prop. 3.16])**.** *Let* G *and* H *be* $*$*-interpretable groups over* A*, let* Σ *be a translation invariant partial type concentrating on* G *definable over* A *and let* f *be a* $*$*-interpretable map over* A *such that for every* $a \models \Sigma|_A$*,* $f(a) \in H$*. Assume moreover, that for all* $a \models \Sigma|_A$ *and* $b \models \Sigma|_{Aa}$*,* $f(a \cdot b) = f(a) \cdot f(b)$*. Then there exists a unique* $*$*-interpretable (over* A*) group morphism* $g : G \to H$ *agreeing with* f *of realizations of* Σ *— which is injective if* f *is.*

In other words, there is an equivalence of categories between pregroups, $*$-interpretable groups with a translation invariant partial type and projective limits of interpretable groups with a translation invariant partial type.

Now, let T be a geometric theory of enriched characteristic zero fields.

Theorem 3.6. *Let* $K \models T_\Delta$*, let* $A = \mathrm{dcl}_\Delta(A) \subseteq K$ *and let* Γ *be a group* L_Δ*-definable in* K *over* A*. Then there is a group* G *which is* L*-interpretable over* A *and a group embedding* $\Gamma \to G$ *which is* L_Δ*-definable in* K *over* A*.*

Proof. Let $\Sigma(x_\omega)$ be the global partial L-type such that $\Sigma(\nabla_\omega(x))$ is the partial type of generics in Γ. By Lemma 2.4, it is L-definable over A. Then Σ is L-definable over A. By Corollary 1.3.3, there exists a map F which is $*$-definable in L over A such that for every $a, b \in \Gamma$, we have $\nabla_\omega(a \cdot b) = F(\nabla_\omega(a), \nabla_\omega(b))$. Likewise there are functions G_1 and G_2 which are $*$-definable in L over A such that for any $a, b \in \Gamma$, we have $\nabla_\omega(a \cdot b^{-1}) = G_1(\nabla_\omega(a), \nabla_\omega(b))$ and $\nabla_\omega(a^{-1} \cdot b) = G_2(\nabla_\omega(a), \nabla_\omega(b))$.

Claim 3.7. (Σ, F) *is a pregroup.*

Proof. First fix $a \in \Gamma(K)$ and $b \in \Gamma(M)$. By Lemma 2.3, $a \cdot b$ is generic in Γ over K if and only if b is. Namely $\nabla_\omega(b)$ realizes Σ if and only if $\nabla_\omega(a \cdot b) = F(\nabla_\omega(a), \nabla_\omega(b))$ also does. As $\Sigma(x_\omega)$ is the L-type of all tuples $\nabla_\omega(b)$ for $b \in \Gamma$ generic over K, it follows that for any $b_\omega \models \Sigma$, we have $F(\nabla_\omega(a), b_\omega) \models \Sigma$ and moreover, that every $c_\omega \models \Sigma$ is of the form $F(\nabla_\omega(a), b_\omega)$ for some $b_\omega \models \Sigma$. So we have $F(\nabla_\omega(a), \Sigma) = \Sigma$.

By definability of Σ, the set of tuples $a_\omega \in K$ such that $F(a_\omega, \Sigma) = \Sigma$ is $*$-definable over A. As it includes $\nabla_\omega(a)$ for any $a \in \Gamma$, it also includes all all realizations of $\Sigma|_A$. This yields condition 1 in Definition 3.2.

Condition 2 and 3 hold for similar reasons. For example, let us consider condition 2. Again, fix $a \in \Gamma(K)$ and $b \in \Gamma(M)$ and let $c = a \cdot b$. Then $\nabla_\omega(c) = F(\nabla_\omega(a), \nabla_\omega(b))$.

We also have $\nabla_\omega(a) = G_1(\nabla_\omega(c), \nabla_\omega(b))$ and $\nabla_\omega(b) = G_2(\nabla_\omega(a), \nabla_\omega(c))$. So, by definition of Σ, for every $b_\omega \models \Sigma$, we have $\nabla_\omega(a) = G_1(F(\nabla_\omega(a), b_\omega), b_\omega)$ and $b_\omega = G_2(\nabla_\omega(a), F(\nabla_\omega(a), b_\omega))$. Again, by definability of Σ over A, the set of tuples a_ω such that $a_\omega = G_1(F(a_\omega, b_\omega), b_\omega) = G_2(b_\omega, F(a_\omega, b_\omega))$ is $*$-definable in L over A and it contains $\nabla_\omega(a)$ for all $a \in \Gamma(K)$. In particular, it contains all realizations of $\Sigma|_A$. This yields condition 2 in Definition 3.2. $\square$

Let us now come back to the proof of the theorem. By Propositions 3.3 and 3.4, we obtain a projective limit $G = \varprojlim_i G_i$ of groups which are L-interpretable over A, as well as a $*$-definable in L over A map f from realizations of $\Sigma|_A$ to G (in M) such that for every $a \models \Sigma|_A$ and $b \models \Sigma|_{Aa}$, $f(F(a,b)) = f(a) \cdot f(b)$.

Note that $\Sigma(\nabla_\omega(x))$ is the global definable over A translation invariant L_Δ-type of generics of Γ over K. We can therefore apply Proposition 3.5 to the map $f \circ \nabla_\omega$ from realizations of $\Sigma(\nabla_\omega(x))|_A$ to G to obtain a group embedding $g : \Gamma \to \varprojlim_i G_i$ which is $*$-definable over A. As G is a projective limit, the composition of g with the projection on some G_i is already injective and this completes the proof. $\square$

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