PSEUDO T-CLOSED FIELDS

SAMARIA MONTENEGRO AND SILVAIN RIDEAU-KIKUCHI

Abstract. Pseudo algebraically closed, pseudo real closed, and pseudo p-adically closed fields are examples of unstable fields that share many similarities, but have mostly been studied separately. In this text, we propose a unified framework for studying them: the class of pseudo T-closed fields, where T is an enriched theory of fields. These fields verify a "local-global" principle for the existence of points on varieties with respect to models of T. This approach also enables a good description of some fields equipped with multiple V-topologies, particularly pseudo algebraically closed fields with a finite number of valuations. One important result is a (model theoretic) classification result for bounded pseudo T-closed fields, in particular we show that under specific hypotheses on T, these fields are NTP_2 of finite burden.

Keywords: Model theory, valued fields, ordered fields, NTP_2, PAC, PRC and PpC fields.
Mathematics Subject Classification: 03C98, 03C45, 12L12, 12J10, 12J15.

1. Introduction

One of the main examples (and motivation) for the development of unstable theories, especially simple theories, was that of pseudo algebraically closed fields (PAC fields), and especially those that have a small Galois group — *i.e.* those with finitely many extensions of any given degree d, usually called *bounded* fields. This class of fields was introduced by Ax in [Ax68] to describe the theory of finite fields. A PAC field is a field K such that any geometrically integral variety over K admits a K-rational point; in other words, they are existentially closed (in the language of rings) in any regular extension.

This notion has proven to be very fruitful, whether from the arithmetic point of view (see for example [FJ08]) and the model-theoretic point of view where it continues to be the source of many examples of moderate structures, see for example [Cha02]. It was eventually reinterpreted as a special case of a wide range of local-global principles for the existence of rational points with the emergence of the pseudo real closed fields (PRC) of Prestel [Pre81] (the existence of smooth points in any real closure implies

The first author was partially supported by Proyecto Semilla 821-C0-464, Vicerrectoría de Investigación, UCR.

The second author was partially supported by GeoMod AAPG2019 (ANR-DFG), Geometric and Combinatorial Configurations in Model Theory.

the existence of rational points) and pseudo p-adically closed fields (PpC) [Gro87; HJ88] (the analogous notion for p-adic valuations) and, eventually, the pseudo classically closed fields of Pop [Pop03] or the regularly T-closed fields of Schmid [Sch00].

Despite the great activity in model theory surrounding PAC fields, for quite some time, the model theoretic study of these fields was mostly confined to questions of completeness and decidability. In [Mon17b], the first author initiated the classification (in the sense of Shelah) of bounded PRC and PpC fields by showing, among other things, that bounded PRC and PpC fields are NTP_2 of finite burden. This was followed by further work on imaginaries [Mon17a] and [MR21] and definable groups [MOS20].

Notwithstanding an apparent similarity in techniques, PRC fields and PpC fields are studied separately in those works. The purpose of this text is to propose a common framework for the study of these fields. This has, of course, already been attempted before, and our approach is closely related to that of Schmid [Sch00]: given a theory T of (large) fields, we say that a field K is pseudo T-closed if every geometrically integral variety over K with a smooth point in every model of T containing K has a point in K (see Definition 3.4). This class generalizes several of the notions that appear in the literature (see Example 3.6).

What sets this work apart, however, is that we aim for a more model theoretic flavor both in our definitions and in our goals. In particular, we also aim to describe pseudo algebraically closed fields endowed with a finite number of valuations, a family of structure that has, to the best of our knowledge escaped scrutiny so far, despite its apparent similarities with the aforementioned examples.

One striking phenomenon that seems to arise in the study of all these "local-global" principles is the fact that an apparently arithmetic hypothesis, phrased uniquely in terms of existence of rational points in varieties has strong topological implications on the density of points for *a priori* unrelated topologies. In other words, existential closure as a topological field comes for free from existential closure as a field. This was first observed for PRC fields by Prestel [Pre81], and was later generalized to other classes. However, in all of these examples, that behavior is not completely surprising since the topologies considered can be defined in the field structure alone. Somewhat more surprisingly, this is also true of algebraically closed fields by the historic theorem of Robinson [Rob77] which states that algebraically closed fields are existentially closed as valued fields despite the valuation very much not being definable. This was generalized much later to PAC fields by Kollar [Kol07] and was also noticed by Johnson [Joh22] who showed that this happens as well when adding independent valuations to a Henselian valued field.

This stronger form of existential closure has been shown to have model theoretic consequences [GJ13; Hon23], and plays a central (albeit somewhat hidden) role in the first author's work. It is therefore natural, given a set I

of completions of T — which we think of as "localizations of T" and come with their topologies — to define the notion of I-pseudo T-closed fields (see Definition 4.4) which isolates this stronger notion of existential closure. Two obvious questions then arise:

- Are all pseudo T-closed fields I-pseudo T-closed?
- Are I-pseudo T-closed fields interesting from a model theoretic perspective?

The first half of this text gives a positive answer to the first question (under some finiteness hypothesis), showing that earlier instances of this phenomenon were not isolated incidents and that, indeed, I-pseudo T-closure is essentially just independence of the involved topologies on top of pseudo T-closure — *cf.* Theorem 5.11 for a precise statement. For example, PAC fields remain existentially closed when endowed with independent valuations.

The second half of the text aims at giving a partial answer to the second question by showing that, at least in the bounded perfect case, I-pseudo T-closed display tame model theoretic behavior, in line with what we could expect from generalized PAC fields. The presence of topologies is an obvious obstruction to the simplicity of these fields, but we show that they are essentially the only obstructions and that they are essentially only as complicated as their "localizations". For example, we show that their burden is the sum of the burden of the "localizations" (Theorem 7.9).

Paired with the positive answer to the first question, this provides a wealth of new examples of finite burden, strong and NTP_2 fields. One of the most natural finite burden example is the aforementioned case of bounded perfect PAC fields with several independent valuations. In the appendix, we also explain how to get rid of the independence assumption.

The organization of the text is as follows. In section 2, we give some preliminaries on burden, forking, large fields and V-topologies. In section 3, we define the class of pseudo T-closed fields for a given theory T of (enriched) large fields and give several equivalent definitions in terms of "local-global" principle and existential closure. We then proceed to prove an approximation theorem for rational points of varieties (Theorem 3.17) under some finiteness hypothesis.

In section 4, we define, given a set I of completions of T, the class of I-pseudo T-closed fields, and we give several equivalent definitions in terms of approximation properties and existential closure. We also prove that this is an elementary class under some mild hypotheses that will always be verified when the theories in I are theories of Henselian valued (or real closed) fields.

In section 6, we initiate the model theoretic study of perfect bounded I-pseudo T-closed fields. We show that the theory is model complete once the Galois group is fixed, and types are given by the isomorphism type of their algebraic closures (Proposition 6.7). We then deduce that these fields enjoy a form of quantifier elimination "up to density" (*cf.* Corollary 6.13).

This last property can also be naturally rephrased as an "open core" property that is typical of generic topological structure: any open definable set is quantifier free definable (Theorem 6.12). Note that our methods are, in fact, more general and also allow proving such "open core" properties for topological fields with additional non-topological generic structure.
In section 7, generalizing work of the first author, we prove that, although the presence of topologies is an obvious obstruction to the simplicity of these fields, it is essentially the only obstruction. For example, 3-amalgamation holds up to quantifier free obstructions (*cf.* Theorem 7.1). Along with the earlier density result, this allows us to compute burden in perfect bounded I-pseudo T-closed fields (Theorem 7.9) and characterize forking (Corollary 8.5).
We conclude with a short appendix describing how, although I-pseudo T-closure forces all the considered topologies to be independent, following an approach of Johnson [Joh19], we can also describe what happens for dependent valuations.
We thank the anonymous referee for his careful reading and insightful remarks on an earlier version of this text.

2. Preliminaries

In this section we will recall some notions (and fix some notation) that will be used throughout the paper.

2.1. **Model Theory.** Most of the structures in this paper, if not all, will be fields that we consider in (some expansion of) the ring language $\mathcal{L}_{\mathrm{rg}} := \{+,-,\cdot,0,1\}$.

Notation 2.1. Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. We write acl and dcl for the model theoretic algebraic and definable closure in M. If a and b are tuples in M and $A \subseteq M$, then we say that $a \equiv_A b$ ($a \equiv_A^{\mathrm{qf}} b$) if a and b have the same (quantifier-free) type over A. We will write $\mathcal{S}(A)$ for the space of $\mathcal{L}$-types with parameters in A.

Definition 2.2. If K is a field (considered as a structure in some expansion of $\mathcal{L}_{\mathrm{rg}}$), and X is a $\mathcal{L}(K)$-definable set, we will write $\dim(X)$ for the algebraic dimension of X:

$$\dim(X) := \max\{\mathrm{trdeg}(a/K) : a \in X(M) \text{ where } K \preccurlyeq M\}.$$

Definition 2.3. Let X be definable in some $\mathcal{L}$-structure M and κ a cardinal.

1. An *inp-pattern* of depth κ in X consists of definable subsets $\phi_l(M, a_{lj})$ of X, with $l < \kappa$ and $j < \omega$, such that:
   - for every $l < \kappa$, $(\phi_l(x, a_{lj}))_{j<\omega}$ is k_l-inconsistent, for some $k_l \geq 1$;
   - for every $f : \kappa \to \omega$, $(\phi_l(x, a_{lf(l)}))_{l<\kappa}$ is consistent.
2. The *burden* of X, denoted $\mathrm{bdn}(X)$ is greater or equal to κ, if there exists an inp-pattern of depth κ in X and

$$\mathrm{bdn}(X) = \sup\{\kappa : \mathrm{bdn}(X) \geq \kappa\}.$$

To keep track of finer estimates, burden can be computed in $\mathrm{Card}^\star$, *cf.* [Tou18, Definition 1.27].

(3) The burden of M, or equivalently the burden of $\mathrm{Th}(M)$, is the burden of the home sort (provided M has a unique dominant sort).

Remark 2.4. (1) By Lemma 2.2 of [Che14] if $\phi_l(M, a_{lj})$ is an inp-pattern, then you can always assume that the sequences $(a_{lj})_{j<\omega}$ are mutually indiscernible over any given $A \subseteq M$, meaning that $(a_{lj})_{j<\omega}$ is indiscernible over $A \cup \{a_{l'j} : l' \neq l, j < \omega\}$.

(2) By Lemma 7.1 of [Che14] if $\phi_{l,0}(M, a_{l,0,j}) \vee \phi_{l,1}(M, a_{l,1,j})$ is an inp-pattern, then $\phi_{l,f(l)}(M, a_{l,f(l),j})$ is an inp-pattern for some $f : \lambda \to \{0, 1\}$.

(3) If X, Y are definable, then $\mathrm{bdn}(X) + \mathrm{bdn}(Y) \leq \mathrm{bdn}(X \times Y)$.

(4) If X,Y are definable, then $\mathrm{bdn}(X \cup Y) = \max(\mathrm{bdn}(X), \mathrm{bdn}(Y))$.

(5) If $f : Y \to X$ is definable and surjective, then $\mathrm{bdn}(Y) \geq \mathrm{bdn}(X)$.

(6) If $f : Y \to X$ is definable and finite-to-one, then $\mathrm{bdn}(X) \geq \mathrm{bdn}(Y)$.

Definition 2.5. A theory T is called *strong* if there are no inp-patterns of infinite depth in its models. It is NTP_2 if there is a (cardinal) bound on the depth of inp-pattern in its models.

The burden of a real closed field (in the ring language, or equivalently the ordered ring language) is 1. In examples, we will also consider valued fields. Given a valued field (K, v), we denote $\mathcal{O}$ its valuation ring with maximal ideal $\mathfrak{m}$, $\mathrm{k} = \mathcal{O}/\mathfrak{m}$ its residue field, $\Gamma = v(K^\times)$ its valuation group and $\mathrm{RV} = K/(1 + \mathfrak{m})$ its leading terms. We refer the reader to [Tou18] for a very throughout introduction to their model theory and the proof of the following burden computations:

Fact 2.6 ([Tou18, Corollary 3.13 and Theorem 3.21])**.** *Let (K, v) be a valued field.*

(1) *If K eliminates quantifiers relatively to RV_1, is elementary equivalent to a maximally complete valued field and the $\mathrm{k}^\star/\mathrm{k}^{\star n}$ are finite, then*
$$\mathrm{bdn}(K) = \max\{\mathrm{bdn}(\mathrm{k}), \mathrm{bdn}(\Gamma)\}.$$

(2) *If K eliminates quantifiers relatively to $\bigcup_n \mathrm{RV}_n$, is elementary equivalent to a maximally complete valued field and K is finitely ramified — that is, for all $n \geq 1$, $(0, v(n))$ is finite — then*
$$\mathrm{bdn}(K) = \max\{\aleph_0 \cdot \mathrm{bdn}(\mathrm{k}), \mathrm{bdn}(\Gamma)\}.$$

These equalities hold resplendently for Γ-expansions of k-expansions.

In [Tou18], the results are stated under stronger assumptions, but as noted there, they hold at this level of generality — up to the fact that Touchard considers unramified mixed characteristic fields (and not finitely ramified ones), but the proof there also applies since the higher residue rings $\mathrm{R}_n = \mathcal{O}/n\mathfrak{m}$ are also interpretable in k in that case.

Definition 2.7. Let M be a (sufficiently saturated) $\mathcal{L}$-structure and $A \leq M$.

(1) We say that the formula $\phi(x,a)$ *divides over* A if there exists an indiscernible sequence $(a_j)_{j\in\omega}$ over A such that $a_0 = a$ and $\{\phi(x,a_j) : j \in \omega\}$ is inconsistent.
(2) We say that the formula $\phi(x,a)$ *forks over* A if there is an integer m and formulas $\psi_j(x,a_j)$ for $j < m$ such that $\phi(x,a) \vdash \bigvee_{j<m} \psi_j(x,a_j)$ and $\psi_j(x,a_j)$ divides over A for every $j < m$.
(3) A partial type p *forks* (respectively *divides*) *over* A if there is a formula in p which forks (respectively divides) over A.
(4) We say that A is *an extension base* if for all tuples a in M, $tp(a/A)$ does not fork over A.

By Theorem 1.2 of [CK12], in an NTP_2 theory, forking equals dividing over extension bases.

2.2. **Large fields.** In this text, by a variety V over a field K, we mean a separated integral scheme of finite type over K. We denote by $V(K)$ the set of its K-rational points. We also denote by $V_{sm}(K)$ the set of smooth K-rational points.

Remark 2.8 (*cf.* [Stacks, Tag 01V7])**.** A point $x \in V(K)$ is smooth if there exists an affine neighborhood $U \subseteq V$ of x and an étale morphism $U \to \mathbb{A}^d$ — that is U is isomorphic to $\mathrm{Spec}(S)$ where $S = K[x_1,\dots,x_{c+d}]/(f_1,\dots,f_c)$ and the matrix $(\frac{df_i}{dx_j})_{i\le c, j\le c}$ is invertible in S.

Definition 2.9. A field K is *large* if for every irreducible variety V, if $V_{\mathrm{sm}}(K) \neq \varnothing$, then $V(K)$ is Zariski dense in V — equivalently, by compatness, $K(V)$ can be $\mathcal{L}_{\mathrm{rg}}(K)$-embedded in some $K^\star \succcurlyeq K$.

We know that algebraically closed fields, real closed fields, p-adically closed fields and, more generally, Henselian fields are large; as well as pseudo algebraically closed fields, pseudo real closed fields and pseudo p-adically closed fields, and further generalizations of these notions.

Finally, given two extensions M and K of a field F, we write $M \mathop{\downarrow}^{\ell}_F K$ if the field M is linearly disjoint from K over F and $L \mathop{\downarrow}^{\mathrm{a}}_F K$ if M and K are algebraically independent over F. We also denote by K^{a} the algebraic closure of K and by K^{s} its separable closure. Recall that a variety V over K is geometrically integral (equivalently absolutely irreducible defined over K) if the extension $K \le K(V)$ is regular — that is $K(V) \mathop{\downarrow}^{\ell}_K K^{\mathrm{a}}$.

2.3. **V-topological fields.**

Definition 2.10. Let (K,τ) be a (non-discrete) topological field.
(1) A subset $S \subseteq K$ is said to be *bounded* if for every for every open neighbourhood of the identity $U \subset K$, there exists $x \in K^\times$ such that $xS = \{x\cdot y : y \in S\} \subseteq U$ — equivalently, $\{x \in K : xS \subseteq U\}$ is a (non-trivial) neighborhood of zero.
(2) A subset $S \subseteq K$ is said to be *bounded away from* 0 if there exists a neighborhood U of zero such that $S \cap U = \varnothing$.

(3) The topology τ is said to be a *V-topology* if, whenever $S^{-1} = \{x^{-1} : x \in S\}$ is bounded away from 0, S is bounded.
(4) The topology τ is said to be *Henselian* if every integer $n \geq 1$ there is a neighborhood U of zero such that every polynomial in $X^{n+1} + X^n + U[X]^{n-1}$ has a zero in K — where $U[X]^{n-1}$ denotes the set of polynomials of degree at most $n-1$ with coefficients in U.

By a theorem of Dürbaum-Kowalsky, and independently Fleischer, a topology is a V-topology if and only if it is induced by either an order or a valuation. We refer the reader to [PZ78] for further detail on V-topologies. In this text, the most useful consequence of Henselianity (and an equivalent characterization) is the inverse function theorem:

Fact 2.11 ([HHJ20, Proposition 2.8]). *Let (K, τ) be a Henselian V-topological field and $f : V \to W$ be an étale morphism of varieties. Then any $x \in V(K)$ admits a neighborhood U such that $f|_U$ is a homeomorphism onto its open image.*

Definition 2.12. A V-topology on a field K is said to be definable if it admits a uniformly definable basis — equivalently if there exists one definable bounded open set.

Remark 2.13. In fact, by (the proof of) [EP05, Lemma B.2], if τ is a definable V-topology on K, then there exists an open bounded definable neighborhood U of 0 such that $K = U \cup (U \setminus \{0\})^{-1}$.

For more details on definable V-topologies, we refer the reader to [HHJ20, Section 3].

3. The class of pseudo T-closed fields

Let $\mathcal{L}$ enrich the language of rings, T be an $\mathcal{L}$-theory of large fields and K be an $\mathcal{L}$-structure that is a field — we will refer to those as $\mathcal{L}$-fields from now on. We write T_K for the theory of models of T containing K.

Definition 3.1.
- We say that a field extension $K \leq F$ is *totally* T if, for every $M \models T_K$, F can be $\mathcal{L}_{\text{rg}}(K)$-embedded in some $M^\star \succcurlyeq M$.
- We say that a variety V over K is *totally* T if, for every $M \models T_K$, $V_{\text{sm}}(M) \neq \varnothing$.

Observe that if V is a geometrically irreducible variety over K, then V_{sm} is Zariski open in V ([Stacks, Tag 056V]). In particular $V_{\text{sm}}(K(V)) \neq \varnothing$. It follows that, since models of T are large, V is totally T if and only if $K \leq K(V)$ is totally T.

Remark 3.2. Observe that if T is RCF in $\mathcal{L}_{\text{rg}}$, we have that $K \leq F$ is totally T if and only if it is a totally real extension, meaning that every order on K extends to some order on F. Similarly, if T is pCF in $\mathcal{L}_{\text{rg}}$, then $K \leq F$ is totally T if every p-adic valuation on K extends to some p-adic valuation on F.

Proposition 3.3. *The following are equivalent:*

(i) for every geometrically integral (affine) totally T variety V over K, we have $V(K) \neq \varnothing$;

(ii) for every geometrically integral (affine) totally T variety V over K, $V(K)$ is Zariski dense in V;

(iii) Any regular totally T extension $K \leq F$ is $\mathcal{L}_{\mathrm{rg}}$-existentially closed.

Proof. We obviously have that (ii) implies (i).

(i)⇒(iii) Let F be as in (iii) and let $\phi(x)$ be some quantifier free $\mathcal{L}_{\mathrm{rg}}$-formula such that $F \vDash \exists x \phi(x)$. Adding existential quantifiers, we may assume that ϕ is of the form $\bigwedge_{i<n} P_i(x) = 0$ where $P_i \in K[x]$. Let $a \vDash \phi$ in F and V be the locus of a over K. Since $K \leq K(a) \leq F$ is regular, V is geometrically integral and, by ([Stacks, Tag 056V]), $V_{\mathrm{sm}}(F) \neq \varnothing$. Moreover, by hypothesis on F, for every $M \vDash T_K$, there exists $M^\star \succcurlyeq M$ such that $F \leq M^\star$. So $V_{\mathrm{sm}}(M^\star) \neq \varnothing$ and hence $V_{\mathrm{sm}}(M) \neq \varnothing$. By (i), we find $c \in V(K)$. In particular, $K \vDash \bigwedge_i P_i(c) = 0$.

(iii)⇒(ii) Let V be as in (ii). Any $M \vDash T_K$ is large and hence, if $V_{\mathrm{sm}}(M) \neq \varnothing$, then some $M^\star \succcurlyeq M$ contains a K-generic point a — that is a is not in any proper sub-variety of V over K. In other terms, $F := K(V)$ is $\mathcal{L}_{\mathrm{rg}}(K)$-embeddable in $M^\star$. By (iii), F is $\mathcal{L}_{\mathrm{rg}}(K)$-embeddable in $K^\star \succcurlyeq K$, equivalently, $V(K)$ is Zariski dense in V. □

Definition 3.4. The $\mathcal{L}$-field K is said to be *pseudo T-closed* (PTC) if the statements of Proposition 3.3 hold.

Remark 3.5. (1) It follows from Proposition 3.3.(ii), that PTC fields are large.

(2) The class of PTC fields is inductive in $\mathcal{L}$, *cf.* Proposition 4.9.

Example 3.6. For different choices of T, the following are examples of PTC fields :

- If T = ACF in $\mathcal{L}_{\mathrm{rg}}$, then P$T$C fields are exactly PAC fields.
- If T = RCF in $\mathcal{L}_{\mathrm{rg}}$, then P$T$C fields are exactly PRC fields.
- If $T = \mathrm{RCF}_<$ in $\mathcal{L}_{\mathrm{rg}} \cup \{<\}$, then P$T$C fields are exactly $1-$PRC fields, *i.e.* PRC fields whose only order is $<$. The equivalence of those two classes is not obvious from the definition and relies crucially on Proposition 3.12 to show the only order on a PTC field is $<$.
- Let $\mathcal{L} = \mathcal{L}_{\mathrm{rg}} \cup \{<_1, \dots, <_n\}$ and $T = \bigvee_i \mathrm{RCF}_{<_i}$, whose models are the $\mathcal{L}$-structures M with $M|_{\mathcal{L}_i} \vDash \mathrm{RCF}_{<_i}$, for some i. Then PTC fields where the $<_i$ define distinct orders are exactly $n-$PRC fields. This equivalence relies again on Proposition 3.12.
- If $T = p$CF in $\mathcal{L}_{\mathrm{rg}}$, then P$T$C-fields are exactly P$p$C fields.
- For $i < n$, let $\mathcal{L}_i$ be copies of Macintyre's language sharing the ring language and $p\mathrm{CF}_{v_i}$ the $\mathcal{L}_i$-theory of p-adically closed fields. Then PTC fields whose $\mathcal{L}_i$-structure induces distinct p-adic valuations are exactly $n-$PpC fields. Once again the equivalence relies on Proposition 3.12.

- Fehm's pseudo $\mathcal{S}$-closed fields [Feh13] also fit in this framework.

Lemma 3.7. *Let F, K be $\mathcal{L}$-fields, $F \leq M$ be regular and totally T, and $f : F \to K$ be an $\mathcal{L}$-embedding. Then $K \leq K \otimes_F M$ is regular and totally T.*

Proof. We may assume that M is finite type over F. Then $M = F(V)$, where V is a geometrically integral totally T variety over F. Let $E \vDash T_K$. Then $E \vDash T_F$ and hence $V_{\mathrm{sm}}(E) \neq \varnothing$. Since E is large, we find an $\mathcal{L}_{\mathrm{rg}}(E)$-embedding $g : E \otimes_F M \simeq E(V) \to E^\star \succcurlyeq E$ — in particular, we have an $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $K(V) \simeq K \otimes_F M \to E^\star$. □

Convention 3.8. Replacing T by the theory whose models are either models of T or algebraically closed does not change the notion of pseudo T-closed fields: algebraically closed fields always have rational smooth points in any variety. So, from now on, we will assume that any algebraically closed field is a model of T.

We will now study the topological properties of PTC fields. We fix K a PTC field.

Notation 3.9.

- For every $M \vDash T_K$, let $K_M \coloneqq M \cap K^{\mathrm{s}}$.
- For every V-topology τ on K, let $C_\tau \coloneqq \{K_M$ that can be endowed with a Henselian V-topology that induces τ on $K\}$.
- We say that $F \in C_\tau$ is minimal if any $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $E \to F$, with $E \in C_\tau$, is surjective.
- For every V-topology τ on K, let $K_\tau = \widehat{(K, \tau)} \cap K^{\mathrm{s}}$, where $\widehat{(K, \tau)}$ is the completion of (K, τ).

Note that, by convention, $K^{\mathrm{a}} \vDash T$ and hence $K^{\mathrm{s}} \in C_\tau$, for every τ. Also, by uniqueness of the Henselian V-topology on a non-separably closed field — cf. [PZ78, Theorem 7.9] — if τ and τ' are distinct V-topologies on K, $C_\tau \cap C_{\tau'} = \{K^{\mathrm{s}}\}$.

Lemma 3.10. *For every τ, C_τ admits minimal elements.*

Proof. By Zorn's lemma, it suffices to show that any decreasing chain $(K_i)_{i\in I}$ in C_τ admits a lower bound. Let $M_i \vDash T_K$ such that $K_i = M_i \cap K^{\mathrm{s}}$. The theory $T_K \cup \{\forall x\ P(x) \neq 0 : P \in K[x]$ such that, for some $i \in I$, $M_i \vDash \forall x\ P(x) \neq 0\}$ is consistent since any finite subset is realized in an M_i and let M be a model. For all $i \in I$ and $P \in K[x]$, if $K_M \vDash \exists x\ P(x) = 0$, then we also have $M_i \vDash \exists x\ P(x) = 0$, so K_M can be embedded in M_i over K, *i.e.* it is a lower bound. □

As in [Sch00, Proposition 4.5], we will see that minimal elements are unique (up to isomorphism) under the following hypothesis:

Hypothesis 3.11. Until the end of Section 3, we assume :

(H) For every $M \vDash T_K$, K_M can be endowed with at least one Henselian V-topology.

Proposition 3.12. *Let τ be a* V*-topology on K and $F \in C_\tau$ be minimal. Then F is $\mathcal{L}_{\mathrm{rg}}(K)$-homeomorphic to K_τ, in particular, K is τ-dense in F.*

Proof. We start with two intermediary results:

Claim 3.12.1. *There exists a τ-continuous $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $\sigma : K_\tau \to F$.*

Proof. There is a continuous $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $\sigma : \widehat{(K,\tau)} \to \widehat{(F,\tau)}$, by universality of the completion. Since $K \subseteq K_\tau$ is separable and F, being topologically Henselian, is separably closed in $\widehat{(F,\tau)}$, *cf.* [PZ78, Corollary 7.6], we have $\sigma(K_\tau) \leq \widehat{(F,\tau)} \cap K^{\mathrm{s}} = F$. □

Claim 3.12.2. *There exists $M \vDash T_K$ and an $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $\rho : K_M \to \widehat{(K,\tau)}$.*

Proof. If no K_M embeds in $\widehat{(K,\tau)}$, then, by compactness, there exists a (non-trivial) separable $P \in K[x]$ with a zero in each K_M but no zero in $\widehat{(K,\tau)}$. Then there exists a neighborhood $U \subseteq K$ of 0 such that $P(\widehat{(K,\tau)}) \cap U = \varnothing$. In particular, $P(K) \cap U = \varnothing$. It follows that $P(K)^{-1}$ is bounded and hence so is $1 - P(0)P(K)^{-1}$. Let $c \in K^\star$ be such that $c(1 - P(0)P(K)^{-1}) \subseteq U$. Let $Q(x,y) = P(x)(1 - c^{-1}P(y)) - P(0) \in K[x,y]$. By [HP84, Proposition 1.1], the zero locus V of Q is geometrically integral. For every $M \vDash T$, one can check that $(0,a) \in V_{\mathrm{sm}}(M)$, where $a \in M$ is a zero of P. So there exists $(b,a) \in V(K)$, *i.e.* $P(b) = c(1 - P(0)P(a)^{-1}) \in U$, a contradiction. □

Note that, since $K \leq K_M$ is separable, $\rho(K_M) \leq K_\tau$. If $K_M \in C_{\tau'}$, for some $\tau' \neq \tau$, then τ' extends to a Henselian V-topology on K_τ, distinct from τ. Moreover, K_τ is, by definition, separably closed in its τ-completion. It follows by [PZ78, Improved Theorem 7.9], that $K_\tau = K^{\mathrm{s}}$, and hence F $\mathcal{L}_{\mathrm{rg}}(K)$-embeds in K_τ. So we may assume that $K_M \in C_\tau$.
Now, the $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $\sigma \circ \rho : K_M \to K_\tau \to F$ must be surjective by minimality of F and hence σ is a τ-continuous $\mathcal{L}_{\mathrm{rg}}(K)$-isomorphism. In particular, K is τ-dense in F. It follows that the identity on K extends to a (unique) τ-continuous $\mathcal{L}_{\mathrm{rg}}(K)$-morphism $F \to K_\tau$ which is the inverse of σ; and σ is indeed an $\mathcal{L}_{\mathrm{rg}}(K)$-homeomorphism. □

Notation 3.13. Fix $(\tau_i)_{0 \leq i < n}$ distinct V-topologies on K and $K_i = K_{\tau_i} \in C_{\tau_i}$ minimal.

Since, by Stone's approximation theorem (see [PZ78, Theorem 4.1]), distinct V-topologies are independent, we deduce the following approximation result on the affine line:

Corollary 3.14. *For every non-empty τ_i-open $O_i \leq K_i$, $\bigcap_i O_i \cap K \neq \varnothing$.*

Our goal now is to extend this approximation result to arbitrary geometrically integral totally T varieties. We don't know, however, if we can do so without the following finiteness hypothesis:

Hypothesis 3.15. Until the end of Section 3, we assume :

($\mathbf{F}_1$) There are only finitely many non-separably closed K_τ, among all possible τ.

Proposition 3.16. *Let C be a totally T smooth projective geometrically integral curve over K. For every non-empty τ_i-open sets $O_i \subseteq C(K_i)$, $\bigcap_{i<n} O_i \cap C(K)$ is infinite.*

This statement (and its proof) is inspired by similar statements in two important and orthogonal subcases: Henselian fields enriched with further valuations [Joh22, Theorem 4.1] (see also [HHJ20, Proposition 4.2]) and PAC fields with a valuation [Kol07, Theorem 2].

Proof. We proceed by induction on n. If $n = 0$, since $C(K)$ is Zariski dense in C, it is, in particular, infinite. Let us now assume $n > 0$. By induction, $\bigcap_{i>0} O_i \cap C(K)$ contains a finite set P of size d larger than the genus of C. By Riemann-Roch, there exists a non-constant rational function f on C over K, whose set of poles is contained in P and whose poles are all simple. This gives rise to a morphism $f : C \to \mathbb{P}^1$ over K with $f^{-1}(1:0) \subseteq P \subseteq \bigcap_{0<i<m} O_i \cap C(K)$ and which is étale above $(1:0)$.
Let τ be any V-topology on K distinct from τ_0. If $\tau = \tau_i$, let $O_\tau = O_i$ and otherwise, let $O_\tau = V(K_\tau)$. By the inverse function theorem, Fact 2.11, for every $p \in P \subseteq O_\tau$ there exists a τ-open neighborhood $W_{p,\tau} \subseteq O_\tau$ of p such that $f|_{W_{p,\tau}}$ is a homeomorphism unto its open image. We may assume that the $W_{p,\tau}$ do not intersect when p varies. Let $U_\tau := \bigcap_{p\in P} f(W_{p,\tau})$. It is open (and hence Zariski dense) in $\mathbb{P}^1(K_\tau)$. For every $y \in U_\tau$, $f^{-1}(y)$ consists of d distinct elements of O_τ.
Let $e \geq 1$ be maximal such that$\{t \in \mathbb{P}(K_0) : \exists^{\neq} y_0 \dots y_{e-1} \in C(K_0)\ f(y_j) = t$ and $y_0 \in O_0\}$ is infinite. Let D be the normalization of the irreducible component of the e-fold product of C over $\mathbb{P}^1$ containing $y_{<e}$. Then $K \leq K(D) = K(y_{<e})$ is regular and D is geometrically integral. Let $g : D \to C$ be the projection on the first coordinate and $h = f \circ g$. Let $W_{\tau_0} := \{x \in O_0 : \exists y \in D(K_0) : g(y) = x$ and g is smooth a $y\}$ and $U_{\tau_0} = f(W_{\tau_0})$. Recall that, τ_0 being Henselian, f and g are local homeomorphisms at smooth points, so both sets have non-empty τ_0-interior. Shrinking them, we may assume that they are τ_0-open and that U_{τ_0} is a ball.
Let $J := \{\tau : K_\tau \neq K_\tau^{\mathrm{s}}\} \cup \{\tau_i : i < n\}$, which is finite by **($\mathbf{F}_1$)**. By Corollary 3.14, there exists $a \in \bigcap_{\tau\in J} U_\tau \cap K$. Composing f with a degree one morphism, we may assume that $a = (1:0)$. For every $s \in \mathbb{A}^1$, let $B_s \subseteq D \times D$ be defined by $(s^2 u_1 u_2 - v_1 v_2) \circ (h \times h) = 0$, where $([u_1 : v_1], [u_2, v_2])$ are coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$. By [Kol07, Lemma 15], for all but finitely many s, B_s is geometrically integral. Applying Corollary 3.14 again, we find $s \in K^\times$ such that B_s is geometrically integral and $(1:s) \in \bigcap_{\tau\in J} U_\tau$. Then, for every $\tau \in J$ and $y \in D(K_\tau)$ with $h(y) = (1:s)$, (y, y) is a smooth point of $B_s(K_\tau)$ — with $(h \times h)(y, y) \in U_\tau \times U_\tau$. Note that for every $M \vDash T_K$, if K_M is not separably closed, it contains some non-separably closed K_τ and hence there

is a smooth M-point on B_s. If K_M is separably closed, B_s also has a smooth M-point, so B_s is totally T.

By induction, we can find infinitely many $(y_1, y_2) \in B_s(K)$ with $h(y_k) \in \bigcap_{0<i} U_{\tau_i} \setminus \{(0:1)\}$. Considering the affine coordinates v/u on $\mathbb{P}^1$, we see that one of the $h(y_k)$ is τ_0-closer to 0 than s and hence $h(y_k) \in U_{\tau_0}$. In other words, we have found infinitely many $t \in \bigcap_{i<n} U_{\tau_i}(K)$ and $(x_{t,i})_{i<e} \in C(K)$ with $f(x_{t,i}) = t$. By construction there exist $x_t \in O_0$ with $f(x_t) = t$. By maximality of e, x_t is distinct from the $x_{t,i}$ for at most finitely many t. So we may assume that $x_{t,0} = x_t \in O_0(K)$. Moreover, for every $i > 0$, since $f(x_t) = t \in U_{\tau_i}$, we have $x_t \in f^{-1}(U_{\tau_i}) \subseteq O_i$ and hence $x_t \in \bigcap_i O_i(K)$. □

Theorem 3.17. *Let V be a geometrically integral K-variety. Let $O_i \subseteq V_{\mathrm{sm}}(K_i)$ be non-empty τ_i-open sets, for every $i < n$. Then $\bigcap_{i<n} O_i \cap V(K) \neq \varnothing$.*

Proof. For every $i < n$, fix an $a_i \in O_i \subseteq V(K^{\mathrm{s}})$. By [JR98, Lemma 10.1] there exists a smooth geometrically integral affine K-curve $C \subseteq V$ containing the a_i. Let $U_i = O_i \cap C$ which is a τ_i-open non-empty subset of $C(K_i)$. Let $\overline{C}$ be a smooth projective model of C. By Proposition 3.16, there exists $y \in \bigcap_i U_i \cap \overline{C}(K) \subseteq \bigcap_i O_i \cap V(K)$. □

Remark 3.18. Schmid proved a similar result, [Sch00, Theorem 4.9], without any finiteness hypothesis, but requiring that no minimal K_M is separably closed — a case that we certainly do not want to omit if we want to say anything about valued pseudo algebraically closed fields.

On the other hand, the proof we give here relies heavily on hypothesis $(\mathbf{F}_1)$. Without it, we would have to find points in infinitely many (uniformly defined) open sets for independent topologies. This is reminiscent of known approximation theorems (*eg.* [ADF20]), but it is not obvious that they apply here.

4. The class of I-pseudo T-closed fields

Let us fix the following notation for the rest of the text. As before, let $\mathcal{L}$ enrich the language of rings, T be an $\mathcal{L}$-theory of large fields.

Let I be a (potentially infinite) set of theories which eliminate quantifiers in (disjoint) relational expansions of the language of rings. We will often denote by T_i the element $i \in I$ and $\mathcal{L}_i$ its language. From now on, all $\mathcal{L}_i$-formulas will be assumed to be quantifier free. Let $\mathcal{L}_I = \bigcup_i \mathcal{L}_i$ and $T_I = \bigcup_i T_{i,\forall}$.

For every $K \models T_{i,\forall}$, we denote by $T_{i,K}$ the theory of models of T_i containing K. We will also assume that any $M_i \models T_{i,K}$ can be expanded to a model of T_K. When I is finite, this can always be assumed by replacing T by $T \vee \bigvee_i T_i$ whose models are the $\mathcal{L} \cup \mathcal{L}_I$-structures M such that $M|_{\mathcal{L}} \models T$ or $M|_{\mathcal{L}_i} \models T_i$ for some i.

For each $i \in I$, we will denote by acl_i and dcl_i the algebraic and definable closure in models of T_i. We will also write $\mathcal{S}^i(K)$ for the space of (quantifier free) $\mathcal{L}_i$-types with parameters in K. We will use the notation $\mathrm{tp}_i(a/A)$ for the type of a over A in the language $\mathcal{L}_i$.

Let us fix some $\mathcal{L}$-field $K \vDash T_I$ and $M_i \vDash T_{i,K}$, for all $i \in I$.

Lemma 4.1. *Let V be an irreducible variety over K and let us assume that $\mathrm{dcl}_i(K) \subseteq K^{\mathrm{a}}$. Then, for any $\mathcal{L}_i(K)$-definable $X \subseteq V$, $X(M_i)$ is Zariski dense in V if and only if it is K-Zariski dense in V (*i.e.* is not contained in any proper sub-variety of V over K).*

Proof. Note first that this statement is clear for $K = M_i$ — *i.e.* $X(M_i)$ is Zariski dense in V if and only if it is M_i-Zariski dense in V. Let us now assume that $X(M_i)$ is K-Zariski dense in V. We may assume M_i is sufficiently saturated and homogeneous. Let $W \subseteq V$ be the Zariski closure of $X(M_i)$. Then W is $\mathrm{aut}(M_i/K)$-invariant and hence it is defined over $\mathrm{dcl}_i(K) \subseteq K^{\mathrm{a}}$. Let $(W_j)_{j \leq n}$ be the K-conjugates of W. Then $X(M_i) \subseteq \bigcup_i W_i \subseteq V$. So $V = \bigcup_i W_i = W$. □

Remark 4.2. Since finite sets in fields can be coded using symmetric polynomials, we have that $\mathrm{acl}_i(K) \subseteq \mathrm{dcl}_i(K)^{\mathrm{a}}$. It follows that $\mathrm{acl}_i(K) \subseteq K^{\mathrm{a}}$ if and only if $\mathrm{dcl}_i(K) \subseteq K^{\mathrm{a}}$.

Proposition 4.3. *The following are equivalent:*

(i) any regular totally T extension $K \leq F \vDash T_I$ is $\mathcal{L}_I$-existentially closed;

(ii) for every geometrically integral (affine) totally T variety V over K, and, for all $i \in I$, every $p_i \in \mathcal{S}^i(K)$ K-generic in V, $\bigcup_i p_i$ is realised in some $K^\star \succcurlyeq K$;

(iii) for every geometrically integral (affine) totally T variety V over K, and, for all $i \in I_0 \subseteq I$ finite, every $\mathcal{L}_i(K)$-definable K-Zariski dense $X_i(M_i) \subseteq V$, $\bigcap_i X_i(K) \neq \varnothing$ — equivalently, is Zariski dense in V;

(iv) for all $i \in I$, $\mathrm{dcl}_i(K) \subseteq K^{\mathrm{a}}$ and for every geometrically integral (affine) totally T variety V over K, and, for all $i \in I' \subseteq I$ finite, every $\mathcal{L}_i(K)$-definable Zariski dense $X_i(M_i) \subseteq V$, $\bigcap_i X_i(K) \neq \varnothing$ — equivalently, is Zariski dense in V;

Proof.(i)⇒(ii) Let p_i and V be as in (ii) and let $a_i \vDash p_i$ (in some $M_i^\star \succcurlyeq M_i$). Note that $K \leq K(a_i)$ is isomorphic to $K(V)$. These isomorphisms allow us to make $K(V)$ into a model of T_I where all the a_i coincide. Since $K \leq K(V)$ is regular and totally T, we find an $\mathcal{L}_I(K)$-embedding $f : K(V) \to K^\star \succcurlyeq K$. The common image of the a_i in $K^\star$ is a realisation of $\bigcup_i p_i$.

(ii)⇒(iii) Let X_i and V be as in (iii). If $i \notin I_0$, let $X_i(M_i) = V(M_i)$. Note that since M_i can be made into a model of T_K and V is totally T, $V(M_i)$ is Zariski dense in V. For all $i \in I$, by compactness, we find $a_i \in X_i(M_i^\star)$, where $M_i^\star \succcurlyeq M_i$ which is K-generic in V. Applying (ii) to $p_i = \mathrm{tp}_i(a_i/K)$, we find $a \vDash \bigcup_i p_i$ in some $K^\star \succcurlyeq K$; in particular, $a \in \bigcap_{i \in I_0} X_i \smallsetminus W$ where $W \subseteq V$ is any K-subvariety.

(iii)⇔(iv) Let us first assume (iii) and prove that $\mathrm{dcl}_i(K) \subseteq K^{\mathrm{a}}$. By contradiction, consider some $a \in \mathrm{acl}_i(K) \smallsetminus K^{\mathrm{a}}$. Let X_i be $\mathcal{L}_i(K)$-definable such that $X_i(M_i)$ is minimal finite containing a. Then $X_i(M_i)$ is K-Zariski dense in $\mathbb{A}^1$. So $X_i(K) \neq \varnothing$, contradicting the minimality of X_i. The equivalence now follows from Lemma 4.1.

(iii)⇒(i) Let F be as in (i) and X be a quantifier free $\mathcal{L}(K)$-definable set with $X(F) \neq \varnothing$. We may assume that $X = \bigcap_{i \in I_0} X_i$, where $I_0 \subseteq I$ finite and X_i is $\mathcal{L}_i(K)$-definable. Fix some $a \in X(F)$. Let V be the locus of a over K. Since $K \leq F$ is regular and totally T, V is geometrically integral and totally T. By construction, $X_i(M_i)$ is K-Zariski dense in V. By (iv), $X(K) = \bigcap_i X_i(K) \neq \varnothing$. □

Definition 4.4. The $\mathcal{L}$-field K is said to be *I-pseudo T-closed* (PTC$_I$) if the statements of Proposition 4.3 hold.

Conditions (i) and (ii) have a similar flavor to the interpolative fusion of [KTW21], although the main difference here is that K is not required to be a model of the T_i but only of $T_{i,\forall}$.

Example 4.5. The following are examples of PTC$_I$ fields for different choices of T_i and T:

- By [Pre81, Theorem 1.7], n-PRC fields are PTC$_I$, for $T_i = \mathrm{RCF}_{<_i}$ and $T = \bigvee_i T_i$, as in Example 3.6.
- By [MR21, Theorem 2.17], n-PpC fields are PTC$_I$, for $T_i = p\mathrm{CF}_{v_i}$ and $T = \bigvee_i T_i$, as in Example 3.6.
- More generally, it follows from [Sch00, Theorem 4.9] that, if K is PRC and $(<_i)_{i \in I}$ are distinct orders on K, then it is PTC$_I$, for T = RCF and $T_i = \mathrm{RCF}_{<_i}$. Similarly, for PpC fields with named p-adic valuations.
- It follows from [Kol07, Theorem 2], that a PAC field with one valuation is PTC$_I$, for T = ACF and T_0 = ACVF.
- In the present paper we generalize this result by showing that a PAC field with n distinct valuations is PTC$_I$, for T = ACF and $(T_i)_{i<n}$ copies of ACVF. This is a consequence Theorem 5.11.
- By [Joh22, Theorem 4.1], if K is real closed (respectively p-adically closed or algebraically closed), and the $(v_i)_{0<i\leq n}$ are distinct valuations, then it is PTC$_I$, for $T_0 = \mathrm{RCF}_{<_0}$ (respectively $p\mathrm{CF}_{v_0}$ or ACVF), T_i = ACVF, for $0 < i < n$ and $T = \bigvee_i T_i$.
- As a consequence of Theorem 5.11, we also generalize this result by showing that if $I = \{\mathrm{Th}(K_i) : i < n\}$ where K_i is either real closed or a characteristic zero Henselian valued field — in an adequate language to eliminate quantifiers — and $T = \bigvee_i T_i$, then every PTC field where the T_i induce distinct topologies is PTC$_I$. Note that if all T_i but T_0 are equal to ACVF, then any model of T_0 is PTC.

Lemma 4.6. *Fix an $i \in I$ and let $K, M \vDash T_{i,\forall}$, $F \leq M$ be regular with $\mathrm{acl}_i(F) \cap M \subseteq F$ and $f : F \to K$ be an $\mathcal{L}_i$-embedding. Then $K \otimes_F M$ can be made into a model of $T_{i,\forall}$, extending the $\mathcal{L}_i$-structure on both K and M.*

Proof. By quantifier elimination, we can extend f to some $g : M \to N \models T_i$. Since $\mathrm{acl}_i(F) \cap M \subseteq F$, any $\mathcal{L}_i(F)$ definable set containing a tuple in $M \smallsetminus F$ is infinite and thus, in (a sufficiently saturated elementary extension of N), it contains a realization in $N \smallsetminus K$. By compactness, we may assume that $K \downarrow^{\mathrm{a}}_F g(M)$. Since $F \leq M$ is regular, we have $K \downarrow^{\ell}_F M$. Then $K \otimes_F M \simeq Kg(M) \models T_{i,\forall}$. □

Corollary 4.7. *Let $K, M \models T_I$, $F \leq M$ be regular and totally T, with $\mathrm{acl}_i(F) \cap M \subseteq F$, for all $i \in I$, and $f : F \to K$ be an $\mathcal{L}_I$-embedding. Then $K \leq K \otimes_F M$ is a regular totally T extension that can be made into a model of T_I extending the $\mathcal{L}_I$-structure on K and M.*

Proof. This follows from Lemmas 3.7 and 4.6. □

We will also need the following orthogonal case of free amalgamation:

Lemma 4.8. *Fix an $i \in I$ and let $K, M \models T_{i,\forall}$, $F \leq M$ be algebraic and $f : F \to K$ be an $\mathcal{L}_i$-embedding with $M \otimes_F K$ integral. Then $K \otimes_F M$ can be made into a model of $T_{i,\forall}$, extending the $\mathcal{L}_i$-structure on both K and M.*

Proof. By quantifier elimination we can extend f to some $g : M \to N \models T_i$. Since $F \leq M$ is algebraic and $K \otimes_F M$ is integral, we have $K \otimes_F M \simeq Kg(M) \models T_{i,\forall}$. □

Let us now prove that, in the cases which we will later consider, $\mathrm{P}T\mathrm{C}_I$ is elementary:

Proposition 4.9. *Assume that, for all $i \in I$:*

- **($\mathbf{A}_i$)** *for every $F \models T_{i,\forall}$, $\mathrm{acl}_i(F) \subseteq F^{\mathrm{a}}$;*
- **($\mathbf{Z}_i$)** *maximal Zariski dimension is definable in family: for every $\mathcal{L}_i$ definable sets $X \subseteq \mathbb{A}^{n+m}$, the set $\{s \in \mathbb{A}^n : \dim(X_s) = m\}$ is $\mathcal{L}_i$-definable, where X_s is the fiber of X in s.*

Then the class of $\mathrm{P}T\mathrm{C}_I$ fields is elementary — in fact, inductive — in $\mathcal{L} \cup \mathcal{L}_I$.

Remark 4.10. Instead of assuming Hypothesis $(\mathbf{A}_i)$, we can allow the following generalization. For all $i \in I$, let $T_i' \supseteq T_{i,\forall}$ be such that for every $F \models T_i'$, $\mathrm{dcl}_i(F) \subseteq F^{\mathrm{a}}$. Then the class of $\mathrm{P}T\mathrm{C}_I$ models of $T_I' := \bigcup_i T_i'$ is elementary.
For example, one could take T_i' to be the class of dcl_i-closed models of $T_{i,\forall}$.

Proof of Proposition 4.9. Let us first show that if $F \leq K \models T_I$ is $\mathcal{L}_I$-existentially closed and K is $\mathrm{P}T\mathrm{C}_I$, then so is F. Let $F \leq M \models T_I$ be regular and totally T. By Hypothesis $(\mathbf{A}_i)$, for all i, $\mathrm{acl}_i(F) \cap M \subseteq F^{\mathrm{a}} \cap M = F$ and hence, by Corollary 4.7, $K \leq K \otimes_F M \models T_I$ is regular and totally T and the $\mathcal{L}_I$-structure on $K \otimes_F M$ extends both that of M and K. Since K is $\mathrm{P}T\mathrm{C}_I$, it follows that this extension is $\mathcal{L}_I$-existentially closed and hence so is $F \leq M$. Now, we let J be a set of indices and let K_j be $\mathrm{P}T\mathrm{C}_I$, for all $j \in J$, and $\mathfrak{U}$ be an ultrafilter on J. We wish to show that $K := \prod_{j \to \mathfrak{U}} K_j$ is $\mathrm{P}T\mathrm{C}_I$. Let V be a smooth geometrically integral totally T variety over K such that for all $M \models T_K$, $V(M) \neq \emptyset$. By definability of irreducibility in ACF, we can find smooth geometrically irreducible varieties V_j over K_j such that $V = \prod_{j\to\mathfrak{U}} V_j$. Let $Y := \{j : \text{for all } M_j \vDash T_{K_j}, V_j(M_j) \neq \varnothing\}$. If $Y \notin \mathfrak{U}$, then for every $j \notin Y$, let $M_j \vDash T_{K_j}$ such that $V_j(M_j) = \varnothing$. Then $M := \prod_{j\to\mathfrak{U}} M_j \vDash T_K$, but $V(M) = \varnothing$, a contradiction. So, we may assume that $Y = J$. For $i \in I_0 \subseteq I$ finite, let now $M_i \vDash T_{i,K}$ and $X_i(M_i) \subseteq V$ $\mathcal{L}_i(K)$-definable and Zariski dense — in other words $\dim(X_i) = \dim(V)$ or equivalently, any coordinate projection which is dominant from V is dominant from X_i. By Hypothesis $(\mathbf{Z}_i)$, we find $M_{ji} \vDash T_{i,K_j}$ and $X_{ji}(M_{ji}) \subseteq V_j$ $\mathcal{L}_i(K)$-definable and Zariski dense with $\prod_{j\to\mathfrak{U}} X_{ji} = X_i$. Since K_j is $\mathrm{P}T\mathrm{C}_I$, $\bigcap_i X_{ji}(K_j) \neq \varnothing$. Hence, $\bigcap_i X_i(K) \neq \varnothing$. It follows that the class $\mathrm{P}T\mathrm{C}_I$ is elementary in $\mathcal{L} \cup \mathcal{L}_I$. Note that, since it is in fact closed under existentially closed substructures, it is, in fact, inductive this can be seen, *e.g.* by working in the enrichment of $\mathcal{L}\cup\mathcal{L}_I$ by all existential formulas which yields a class closed under all substructures, *i.e.* a universal class. □

5. V-Topological theories

We now fix an $i \in I$ and $M_i \vDash T_i$. We now wish to consider that $\mathcal{L}_i$-definable sets are essentially open. This is closely related to the notion of "t-theory" in [Dri78, Chapter III].

Hypothesis 5.1. In this section, we will assume that:

- $(\mathbf{H}_i)$ T_i admits a definable (non-discrete) Henselian V-topology τ_i and, for every $M_i \vDash T_i$, any $\mathcal{L}_i(M_i)$-definable set X_i has non-empty τ_i-interior in (the M_i-points of) its Zariski closure.

Example 5.2. (1) Hypothesis $(\mathbf{H}_i)$ holds if T_i is the theory of real closed fields. Indeed, the order topology is a definable Henselian V-topology and any definable set is a disjoint union of sets of the form $V \cap U$ where V is Zariski closed and U is open.

(2) Similarly, Hypothesis $(\mathbf{H}_i)$ also holds if T_i is a theory of (RV_1-enriched) Henselian valued fields which eliminates quantifiers relatively to RV_1 — or, more generally, relatively to $\bigcup_n \mathrm{RV}_n$ in characteristic zero.

Remark 5.3. (1) Whenever $(\mathbf{H}_i)$ holds, Hypothesis $(\mathbf{Z}_i)$ of Proposition 4.9 also holds since a definable set is Zariski dense in $\mathbb{A}^n$ if and only if it has non-empty τ_i-interior.

(2) As we will see in Lemma 5.4, Hypothesis $(\mathbf{A}_i)$ of Proposition 4.9 also holds provided $\mathrm{dcl}_i(\varnothing) \subseteq F^{\mathrm{a}}$ where F is the field generated by the constants — constants that we may add for this exact purpose.

As it turns out, we can almost control the parameters in Hypothesis $(\mathbf{H}_i)$:

Lemma 5.4. *Let $F \leq M_i$, the following are equivalent:*

(i) $\mathrm{dcl}_i(\varnothing) \subseteq F^{\mathrm{a}}$*;*

(ii) $\mathrm{acl}_i(F) \subseteq F^{\mathrm{a}}$*;*

(iii) any $\mathcal{L}_i(F)$-definable set X has non-empty τ_i-interior in its F-Zariski closure.

Proof. It is clear that (ii) implies (i).

(i)⇒(ii) Let $E = \mathrm{dcl}_i(\varnothing)$. We prove, by induction on $\dim(V)$, that for any variety V over E and $X \subseteq V \times \mathbb{A}^1$ $\mathcal{L}_i$-definable with $X_a := \{y \in \mathbb{A}^1 : (a,y) \in X\}$ finite for every $a \in X(M_i)$, we have $X_a(M_i) \subseteq E(a)^{\mathrm{a}}$. We may assume that V is irreducible. By finiteness of X_a, X is not τ_i-open in $V \times \mathbb{A}^1$ and hence is not Zariski dense. So there exists $P = \sum_j p_j(x) y^j \in E[x,y]$ with $P(V \times \mathbb{A}^1) \neq 0$ but $P(X) = 0$. Let V_j be the zero locus of p_j in V and $J := \{j : V_j \subset V\} \neq \varnothing$. By induction, for any $j \in J$ and $a \in V_j$, $X_a(M_i) \subseteq E(a)^{\mathrm{a}}$ and by construction, for any $a \in V \smallsetminus \bigcup_j V_j$, $P(a,y) \neq 0$ and $P(a, X_a) = 0$.

(ii)⇒(iii) By Lemma 4.1, $X(M_i)$ is Zariski dense in its F-Zariski closure V. It now follows from $(\mathbf{H}_i)$ that X has non-empty interior in V.

(iii)⇒(ii) Let $X \subseteq \mathbb{A}^1$ be a minimal finite $\mathcal{L}_i(F)$-definable set. Let V its F-Zariski closure. Then either $\dim(V) = 0$, in which case $X \subseteq F^{\mathrm{a}}$ as required, or $\dim(V) = 1$ and $V = \mathbb{A}^1$. By (iii), the finite set X is τ_i-open in $\mathbb{A}_1$, contradicting its non-discreteness. □

Lemma 5.5. *Let $F \leq M_i$ be such that, for any non-empty open $\mathcal{L}_i(F)$-definable subset of $\mathbb{A}^1(M_i)$ has an F-point. Then the topology generated by the $X(F)$, where $X \subseteq M_i$ is open and $\mathcal{L}_i(F)$-definable, is a V-topology. If, moreover, $F^{\mathrm{s}} \cap M_i \subseteq F$, then it is Henselian.*

We will also denote this topology on F by τ_i even if it might not be the induced topology, per se.

Proof. Let S be $\mathcal{L}_i(F)$-definable and bounded (in M_i) and let U be $\mathcal{L}_i(F)$-definable neighborhood of zero. Then $\{x \in M_i : xS \subseteq U\}$ is a $\mathcal{L}_i(F)$-definable neighborhood of 0. So, by hypothesis, its τ_i interior contains a point distinct from 0; in other terms, $S(F)$ is bounded in F. So, if $U \subseteq M_i$ is an $\mathcal{L}_i(F)$-definable neighborhood of 0, $(M_i \smallsetminus U)^{-1}$ is bounded in M_i and hence in F, proving that the topology is a V-topology.
The second statement follows from the fact that Henselianity states the existence of roots to certain separable polynomials. □

Lemma 5.6. *Let $F \leq M_i \vDash T_i$ be such that $F^{\mathrm{s}} \cap M_i \subseteq F$ and any non-empty τ_i-open $\mathcal{L}_i(F)$-definable subset of $\mathbb{A}^1(M_i)$ has an F-point. Let V be a geometrically irreducible variety over F and $X(M_i) \subseteq V(M_i)$ be τ_i-open and $\mathcal{L}_i(F)$-definable. The following are equivalent:*

(i) $X(M_i)$ *is Zariski dense in* V*;*

(ii) $X(M_i) \cap V_{\mathrm{sm}} \neq \varnothing$*;*

(iii) $X(F) \cap V_{\mathrm{sm}} \neq \varnothing$.

Proof. Note that (i) obviously implies (ii), and so does (iii). Also, shrinking V, we may assume (Remark 2.8) that it is affine smooth and that there exists an étale map $f : V \to \mathbb{A}^n$ over F.

(ii)⇒(iii) By Fact 2.11, $f(X) \subseteq \mathbb{A}^n$ has non-empty interior, so there exists $y \in f(X) \cap \mathbb{A}^n(F)$ and hence $x \in X(M_i)$ with $f(x) = y$. Then $x \in F(f(x))^{\mathrm{s}} \cap M_i = F$.

(ii)⇒(i) By Fact 2.11 we can find a non-empty $\mathcal{L}_i(M_i)$-definable τ_i-open $U \subseteq X(M_i)$ such that $f|_U$ is a homeomorphism onto its open image. Since open subset of $\mathbb{A}^n$ are Zariski dense, (i) follows. □

Lemma 5.7. *Let $F \leq M_i$, with $\mathrm{dcl}_i(\varnothing) \subseteq F^{\mathrm{a}}$, let V be a geometrically irreducible variety over F and $X \subseteq V$ be $\mathcal{L}_i(F)$-definable. The following are equivalent:*

(i) $X(M_i)$ is Zariski dense in V;
(ii) $X(M_i)$ is F-Zariski dense in V;
(iii) the τ_i-interior of $X(M_i)$ in V contains a smooth point.

Proof. It is clear that (i) implies (ii) and, by Lemma 5.6, (iii) implies (i). So let us prove that (ii) implies (iii) and assume that $X(M_i)$ is F-Zariski dense in V and let U be its τ_i-interior in V. Since $X \smallsetminus U$ has empty τ_i-interior in V, by Lemma 5.4, it is not F-Zariski dense in V. It follows that U is F-Zariski dense in V. In particular $U \cap V_{\mathrm{sm}} \neq \varnothing$. □

Remark 5.8. Note that Hypothesis $(\mathbf{Z}_i)$ of Proposition 4.9 follows from $(\mathbf{H}_i)$ (see Hypothesis 5.1). Indeed, by Lemma 5.7, a definable set $X \subseteq \mathbb{A}^n$ is Zariski dense if and only if it has non empty τ_i-interior.
Also, if $F \vDash T_{i,\forall}$ is such that $\mathrm{dcl}_i(\varnothing) \subseteq F^{\mathrm{a}}$, then, by Lemma 5.4, $T_{i,F,\forall}$ satisfies the conditions of Remark 4.10. Therefore, for topological theories, the conditions for $\mathrm{P}T\mathrm{C}_I$ to be elementary given in Proposition 4.9 and Remark 4.10 are easy to ensure.

We will now show the remarkable fact that, under some finiteness hypothesis, $\mathcal{L}_I$-existential closure comes essentially from independence of the τ_i topologies.

Hypothesis 5.9. We will consider the following hypothesis:

($\mathbf{F}_2$) The set I is finite and for every $M \vDash T_K$, there exists $i \in I$ and an $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $M_i \to M^\star \succcurlyeq M$.

In particular, the minimal fields of the form $M \cap K^{\mathrm{s}}$, where $M \vDash T_K$ are among the $K_i = M_i \cap K^{\mathrm{s}}$, where $M_i \vDash T_{i,K}$. Note that it also follows that every extension $K \leq M \vDash T_I$ is totally T.
It follows that a field is then I-pseudo T-closed if and only if it is I-pseudo $\bigvee_{i\in I} T_i$-closed, so the mention of T is now redundant.

Definition 5.10. If I is finite, we will say that K is pseudo I-closed (PIC) if it is I-pseudo $\bigvee_{i\in I} T_i$-closed.

Theorem 5.11. *Assume $(\boldsymbol{H}_i)$ for all $i \in I$ (see 5.1) and $(\boldsymbol{F}_2)$. The following are equivalent:*

(i) $K \vDash \mathrm{P}I\mathrm{C}$;
(ii) the following hold:
  (a) $K \vDash \mathrm{P}T\mathrm{C}$;
  (b) for all $i \in I$, $\mathrm{dcl}_i(\varnothing) \subseteq K^{\mathrm{a}}$;
  (c) the τ_i are distinct topologies on K as $i \in I$ varies;

(d) for all $i \in I$, any non-empty open $\mathcal{L}_i(K)$-definable $X \subseteq \mathbb{A}^1$ contains a K-point.

Proof. Assuming (i), (a) obviously follows. Condition (b) follows from Proposition 4.3.(iv) and conditions (c) and (d) follow from Proposition 4.3.(iii) applied to $\mathbb{A}^1$. Conversely, let us assume (ii). By Proposition 4.3 and Lemmas 5.6 and 5.7, it suffices to prove that, given V a geometrically integral totally T variety over K and $X_i \subseteq V$ $\mathcal{L}_i(K)$-definable τ_i-open with $X_i(K_i) \cap V_{\mathrm{sm}} \neq \emptyset$, we have $\bigcap_i X_i(K) \neq \emptyset$.
Note that, by Hypothesis $(\mathbf{F}_2)$, any $K_M = M \cap K^{\mathrm{s}}$, where $M \vDash T_K$, is an algebraic extension of some K_i, which, by Lemma 5.5 admits a Henselian V-topology extending τ_i. So **(H)** holds and if K_M is minimal in some C_τ and not separably closed, then $\tau = \tau_i$ and $K_M \simeq K_i$ — in particular, $(\mathbf{F}_1)$ holds. Since the τ_i are all distinct, it also follows that every K_i is minimal in C_{τ_i}. We can, therefore, conclude with Theorem 3.17. ☐

Remark 5.12. If $(\mathbf{F}_2)$ does not hold but, instead, **(H)** holds and no $M \vDash T_K$ contains K^{s}, then, by [Sch00, Theorem 4.9], Theorem 5.11 also holds.

6. Bounded PIC fields

From now on, we assume that I is finite and that $T = \bigvee_{i\in I} T_i$. In particular, any extension $K \leq M \vDash T_I$ is totally T, and we will therefore prefer the terminology pseudo I-closed (see Definition 5.10).
We fix some $I_\tau \subseteq I$, and we assume that, for every $i \in I_\tau$, $(\mathbf{H}_i)$ holds (see Hypothesis 5.1). By Remark 5.3, hypotheses $(\mathbf{Z}_i)$ and $(\mathbf{A}_i)$ of Proposition 4.9 holds for every $i \in \tau$ (provided we add constants). We also assume that $(\mathbf{Z}_i)$ holds for every $i \notin I_\tau$, and we fix T_i' as in Remark 4.10.

Definition 6.1. A field K is called *bounded* if for any integer n, K has finitely many extensions of degree n.

Notation 6.2. Fix $\mathfrak{d} : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. Let $\mathcal{L}_\mathfrak{d} \coloneqq \mathcal{L}_I \cup \{c_j : j > 0 \text{ and } |c_j| = \mathfrak{d}(j)\}$. We denote by $T_\mathfrak{d}$ the $\mathcal{L}_\mathfrak{d}$-theory of fields such that $P_i \coloneqq X^{\mathfrak{d}(i)} + \sum_{j<\mathfrak{d}(i)} c_{i,j} X^j$ is irreducible and every separable polynomial of degree i is split modulo P_i. Let PIC$_\mathfrak{d}$ be the (elementary) class of PIC models of $T_\mathfrak{d} \cup \bigcup_{i\notin I_\tau} T_i'$.
From now on, we will always work in the language $\mathcal{L}_\mathfrak{d}$, so if K is PIC$_\mathfrak{d}$ and $A \leq K$, then $\mathrm{acl}(A), \mathrm{dcl}(A), \mathrm{acl}_i(A), \mathrm{dcl}_i(A)$ always contain the constants.

Remark 6.3. Any model of $T_\mathfrak{d}$ is bounded. Conversely, any bounded model of T can be made into a model of $T_\mathfrak{d}$ for some well-chosen $\mathfrak{d}$.

Lemma 6.4. *Let $K \vDash T$ and $F \leq K$. The following are equivalent:*
(1) $K \vDash T_\mathfrak{d}$ *and* $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ *is surjective;*
(2) $F \vDash T_\mathfrak{d}$ *and* $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ *is a homeomorphism;*
(3) $K \vDash T_\mathfrak{d}$ *and* $F \vDash T_\mathfrak{d}$.

This is rather standard, but we could not find a reference, so we sketch the proof.

Proof. Let us first assume (1) and prove (3). The polynomial P_n is clearly irreducible over F. Let $Q \in F[X]$ be an irreducible separable polynomial of degree n and a one of its roots. Since Q is split in $K[X]/P_n$, P_n admits a root $e \in F^{\mathrm{s}}$ of P_n such that $a \in K[e]$. Then $\mathrm{Gal}(F[e]) = \mathrm{res}(\mathrm{Gal}(K[e])) \leq \mathrm{res}(\mathrm{Gal}(K[a])) = \mathrm{Gal}(F[a])$ where the equalities follow from surjectivity; and hence $F[a] \leq F[e]$.
If (3) holds, then every separable extension of K is included by one generated by an element of F^{s} and hence $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ is injective. Also, any separable extension of F, is contained in one generated by a root of a P_n which is also irreducible over K. It follows that F^{s} is linearly disjoint from K over F and hence $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ is surjective, so (2) holds. Conversely, if (2) holds, then, by Galois correspondence, the P_n are irreducible over K and $K^{\mathrm{s}} = KF^{\mathrm{s}}$ so $K \vDash T_{\mathfrak{d}}$ and (1) holds. □

Lemma 6.5. *Let $K, M \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$, with K perfect, and $F \leq K$ with $\mathrm{acl}_i(F) \cap K \subseteq F$, for all $i \in I$. Then any $\mathcal{L}_{\mathfrak{d}}$-embedding $f : F \to M$ can be extended to an $\mathcal{L}_{\mathfrak{d}}$-embedding $g : K \to M^\star \succcurlyeq M$ with $\mathrm{acl}_i(g(K)) \cap L^\star \subseteq g(K)$, for all $i \in I$.*

Proof. Note that F is perfect and hence $F \leq K$ is regular. So g exists by Corollary 4.7 and the fact that $M \vDash \mathrm{P}I\mathrm{C}$. There remains to show that $\mathrm{acl}_i(g(K)) \cap M^\star \subseteq g(K)$. Since $\mathrm{acl}_i(K) \subseteq K^{\mathrm{a}}$, it suffices to show that $g(K)^{\mathrm{a}} \cap M^\star \subseteq g(K)$. By Lemma 6.4, since $K \vDash T_{\mathfrak{d}}$ and $F \leq K$ is regular, $F \vDash T_{\mathfrak{d}}$ and hence, by Lemma 6.4 again, $\mathrm{Gal}(M^\star) \to \mathrm{Gal}(g(F))$ is a homeomorphism. It factorizes through the homeomorphism $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ and hence $\mathrm{res} : \mathrm{Gal}(M^\star) \to \mathrm{Gal}(g(K))$ is surjective; *i.e.* $g(K)^{\mathrm{a}} \cap M^\star = g(K)$. □

Definition 6.6. For every $K \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ and $A \subseteq K$, let $\mathrm{acl}_I(A) \subseteq K$ be the smallest subset $C \subseteq K$ containing A such that $\mathrm{acl}_i(C) \subseteq C$, for all $i \in I$.

Proposition 6.7. *Let $K, M \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ be perfect, $F \leq K$ and $f : F \to M$ be an $\mathcal{L}_{\mathfrak{d}}$-embedding. The following are equivalent:*
(1) *f extends to an $\mathcal{L}_{\mathfrak{d}}$-embedding $g : \mathrm{acl}_I(F) \cap K \to M$;*
(2) *f is $\mathcal{L}_{\mathfrak{d}}$-elementary (from K to M).*

Proof.
(1) ⇒ (2) Let $g : \mathrm{acl}_I(F) \cap K \to M$ extend f. By a standard back and forth argument using Lemma 6.5, we build $K \preccurlyeq K^\star$, $M \preccurlyeq M^\star$ and an $\mathcal{L}_{\mathfrak{d}}$-isomorphism $h : K^\star \to M^\star$ extending g. Elementarity of f follows immediately.
(2) ⇒ (1) This is immediate. □

Corollary 6.8. *Any $K \leq M$, with both fields $\mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ perfect, is $\mathcal{L}_{\mathfrak{d}}$-elementary.*

Proof. By Lemma 6.4, the morphism $\mathrm{res} : \mathrm{Gal}(K) \to \mathrm{Gal}(F)$ is surjective. Since $\mathrm{acl}_i(K) \subseteq K^{\mathrm{a}}$, it follows that $\mathrm{acl}_I(K) \cap M \subseteq K$ and hence, by Proposition 6.7 the inclusion is elementary. □

For the rest of the section we fix a perfect $K \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ and $M_i \vDash T_{i,K}$.

Proposition 6.9. *Let $F \le K$. Then* $\operatorname{acl}(F) = \operatorname{acl}_I(F) \cap K$.

Proof. Let $E = \operatorname{acl}_I(F) \cap K \subseteq \operatorname{acl}(F)$. By Corollary 4.7, we find an $\mathcal{L}_\mathfrak{d}(E)$-embedding $f : K \to K^\star \succcurlyeq K$ with $f(K) \downarrow^\ell_E K$. By Corollary 6.8, f is elementary. It follows that $\operatorname{acl}(F) \subseteq K \cap f(K) = E$. □

Notation 6.10. For every $p \in \mathcal{S}(F)$, where $F \le K$, let p_i denote the underlying (quantifier free) $\mathcal{L}_i$-type.

Proposition 6.11. *Let $F = \operatorname{acl}(F) \le K$, V an irreducible variety over F, $p \in \mathcal{S}(F)$ F-generic in V and, for $i \in I_\tau$, let $O_i \subseteq V(M_i)$ be $\mathcal{L}_i(K)$-definable τ_i-open sets consistent with p_i (modulo $T_{i,K}$). Then p is consistent with $\bigcap_{i \in I_\tau} O_i(K)$.*

Proof. Let $a \models p$ (in some $K^\star \succcurlyeq K$). Let ac enumerate $\operatorname{acl}(Fa)$. By Proposition 6.7, it suffices to find $de \equiv^{\mathrm{qf}}_F ac$ with $d \in \bigcap_{i \in I_\tau} O_i$. By compactness, we may assume c is finite. Let $q_i = \operatorname{tp}_i(ac/F)$, for every $i \in I$, and W be the geometrically integral algebraic locus of ac. For every $i \in I_\tau$, we also denote by O_i the corresponding open subset of W.
Let us fix some $i \in I_\tau$. For every $\mathcal{L}_i(F)$-definable X, by $(\mathbf{H}_i)$, $q_i \vdash X$ if and only if $q_i \vdash \mathring{X} \cap W_{\mathrm{sm}}$, where $\mathring{X}$ denotes the τ_i-interior of $X \cap W$ in W. In particular, $\mathring{X} \cap W_{\mathrm{sm}} \cap O_i \ne \varnothing$ and, by Lemma 5.6, $X \cap O_i$ is consistent with the K-generic of W. So, by compactness, we find $a_i c_i \models q_i$ K-generic in W with $a_i \in O_i$. If $i \notin I_\tau$, by Lemma 4.6 we can also find $a_i c_i \models q_i$ K-generic in W. Let $r_i := \operatorname{tp}_i(a_i c_i/K) \supseteq q_i$. Note that since $\operatorname{tp}(ac/F)$ is finitely satisfiable in K, $W_{\mathrm{sm}}(K) \ne \varnothing$. By Proposition 4.3, there exists $de \models \bigcup_i r_i$ in some $K^\star \succcurlyeq K$. By construction, for every $i \in I_\tau$, we have $d \in O_i$ and, for every $i \in I$, we have $de \models p_i$, concluding the proof. □

Let τ be the topology generated by the τ_i, for $i \in I_\tau$.

Theorem 6.12. *Let $F = \operatorname{acl}(F) \le K$, and X be $\mathcal{L}_\mathfrak{d}(F)$-definable and τ-closed (respectively τ-open) in its Zariski closure V. Then X is quantifier free $\bigcup_{i \in I_\tau} \mathcal{L}_i(F)$-definable.*

Proof. Let assume that X is τ-closed. Let $x \in X(K)$ and $y \in V(K)$ be such that $\bigcup_{i \in I_\tau} \operatorname{tp}_i(x/F) = \bigcup_{i \in I_\tau} \operatorname{tp}_i(y/F)$. By compactness, it suffices to show that $y \in X$. For every $i \in I_\tau$, $O_i \subseteq V(M_i)$ τ_i-open containing y. Let W be the locus of x (and hence y) over F. By Proposition 6.11, $\operatorname{tp}(x/F)$ is consistent with $W \cap \bigcap_i O_i$, In particular, there exists $z \in X \cap \bigcap_i O_i$. So y is in the τ-closure of X, which is X itself. □

Considering the τ-closure of a definable set, we have generalized the "density theorem" of [Mon17b, Theorem 3.17]:

Corollary 6.13. *Let $F = \operatorname{acl}(F) \le K$, and X be $\mathcal{L}_\mathfrak{d}(F)$-definable with Zariski closure V. Then there exists an integer m, $\mathcal{L}_i(F)$-definable τ_i-open sets $Y_{ij} \subseteq V$, for $i \in I_\tau$ and $j < m$, and a quantifier free $\mathcal{L}_\mathfrak{d}(F)$-definable set*

Y*, with* $\dim(Y) < \dim(X)$*, such that*

$$X \subseteq \bigcup_j \bigcap_i Y_{ij} \cup Y \text{ is } \tau\text{-dense.}$$

Proof. Applying Theorem 6.12 to the τ-closure of X, we find $\mathcal{L}_i(F)$-definable sets Y_{ij} such that $X \subseteq \bigcup_j \bigcap_i Y_{ij}$ is τ-dense. Further dividing, we may assume that Y_i is either τ_i-open or has empty τ_i-interior (and hence has lower dimension by $(\mathbf{H}_i)$). □

Remark 6.14. The previous result generalizes the common pattern, that, when adding a "generic" topology to some structure, the "open core" will consist only of sets definable in the topology on its own. In fact, this result provides a broad setting in which to consider these questions, which does not depend on the initial structure and allows for more than one topology.

Generic derivations. To illustrate that point, let us consider the following example: let $T_1 = \mathrm{DCF}_0$ the (Morleyized) theory of characteristic zero differentially closed fields with a derivation, $T_2 = \mathrm{ACVF}$, $I := \{T_1, T_2\}$. Then existentially closed valued fields with a derivation are PIC. Hence, definable closed sets in such fields are definable using only the valued field structure, recovering a special case of [CP23, Theorem 3.1.11].
The proof given here would extend immediately to bounded perfect PAC fields with independent valuations and generic commutative derivations (which are PIC for $T_1 = \mathrm{DCF}_n$ and $T_2, \ldots T_m = \mathrm{ACVF}$), if we knew that this class is indeed closed under elementary extensions. Proposition 4.9 does not apply in that context, but a geometric axiomatization seems plausible, along the lines of [Tre05].
There is another issue if we wish to consider fields that are not PAC, which is our reliance on relational languages. If $T_2 = \mathrm{RCF}_<$ and $T_1 = \mathrm{DCF}_0$ (Morleyized), then existentially closed ordered differential fields might not be PIC. Indeed, the order on a subfield of a differential field might not extend to the whole field. However, the setup presented here can be adapted to allow for non-relational $\mathcal{L}_i$; the main issue being, again, the elementarity of the class.

7. Amalgamation and Burden in PIC$_\mathfrak{d}$

Recall that we now assume that I is finite and that $T = \bigvee_{i \in I} T_i$. In particular, any extension $K \leq M \vDash T_I$ is totally T. We now work with $I_\tau = I$ and therefore assume that $(\mathbf{H}_i)$ holds for every $i \in I$ (see Hypothesis 5.1) and τ will denote the topology generated by the τ_i for all $i \in I$. Finally, we fix $K \vDash \mathrm{P}I\mathrm{C}_\mathfrak{d}$ perfect sufficiently saturated and homogeneous (see Notation 6.2) and $M_i \vDash T_{i,K}$.
We start by proving that obstructions to 3-amalgamation only arise from the quantifier free structure.

Theorem 7.1. *Let* $F \leq K$ *and* $a_1, a_2, c_1, c_2, c \in K$*. Suppose that* $\mathrm{dcl}_i(\varnothing) \subseteq F^{\mathrm{a}}$*, for all* i*,* $F = \mathrm{acl}(F)$*,* a_j *enumerates* $A_j := \mathrm{acl}(Fa_j)$ *and* c_j *enumerates*

$C_j := \operatorname{acl}(Fc_j)$, *for* $j = 1,2$. *Assume also that* $A_1 \cap A_2 = F$, $c \downarrow^{\mathrm{a}}_F a_1a_2$, $c_1 \equiv_F c_2$ *and* $c \equiv^{\mathrm{qf}}_{A_j} c_j$, *for* $j = 1,2$. *Then*

$$\operatorname{tp}(c_1/A_1) \cup \operatorname{tp}(c_2/A_2) \cup \operatorname{qftp}(c/A_1A_2)$$

is consistent.

This is a minor variation on [Mon17b, Theorem 3.21].

Proof. First note that, since A_1A_2 is a regular extension of A_1 and of A_2, by [Cha02, Lemma 2.1], we have that $A_1^{\mathrm{a}} \cap A_2^{\mathrm{a}} = (A_1 \cap A_2)^{\mathrm{a}} = F^{\mathrm{a}}$. Note also that for any $F \le E \le K$, by Lemma 5.4, $\operatorname{acl}(E) = E^{\mathrm{a}} \cap K$.
Let $C := F(c)$ and $A := \operatorname{acl}(A_1A_2)$. Since $c \equiv^{\mathrm{qf}}_{a_j} c_j$, we have an $\mathcal{L}_I(A_j)$-isomorphism $f_j : A_jC_j \to A_jC$ sending c_j to c. This map extends to an $\mathcal{L}_{\mathrm{rg}}(A_j)$-embedding $B_j := \operatorname{acl}(A_jC_j) \to (A_jC)^{\mathrm{a}}$. We endow $D_j := f_j(B_j)$ with the $\mathcal{L}_I$-structure making f_j into an $\mathcal{L}_I(A_j)$-isomorphism. This structure extends the $\mathcal{L}_I$-structure on A_jC.
Note that both $A_2 \le A$ and $A_2 \le D_2 \simeq B_2$ are regular and that, since $a_1 \downarrow^{\mathrm{a}}_{a_2} c$, we have $A \downarrow^{\mathrm{a}}_{A_2} D_2$. It follows that $A \le D_2A$, $A_2 \le D_2A$ and $F \le D_2A$ are regular. So $D_2A \downarrow^{\ell}_F F^{\mathrm{a}}$ and hence $D_2A \downarrow^{\ell}_{CA_1} F^{\mathrm{a}}CA_1$. Moreover, since $C^{\mathrm{a}} \downarrow^{\ell}_{F^{\mathrm{a}}} A_1^{\mathrm{a}}A_2^{\mathrm{a}}$, by [Cha02, lemma 2.5.(2)] we have:

$$(CA_1)^{\mathrm{a}} \cap (CA_2)^{\mathrm{a}}(A_1A_2)^{\mathrm{a}} = (C^{\mathrm{a}}(A_1^{\mathrm{a}} \cap A_2^{\mathrm{a}}))^{\mathrm{a}}A_1^{\mathrm{a}} = C^{\mathrm{a}}A_1^{\mathrm{a}}.$$

By Lemma 6.4, $\operatorname{Gal}(C) \simeq \operatorname{Gal}(F)$ and hence, $C^{\mathrm{a}} = F^{\mathrm{a}}C$. Similarly, $A_1^{\mathrm{a}} = F^{\mathrm{a}}A_1$ and thus $C^{\mathrm{a}}A_1^{\mathrm{a}} = F^{\mathrm{a}}CA_1$. So, $(CA_1)^{\mathrm{a}} \cap D_2A = F^{\mathrm{a}}CA_1 \cap D_2A = CA_1$; *i.e.* $CA_1 \le D_2A$ is regular. By symmetry, $CA_2 \le D_1A$ is also regular.
Since $D_j \downarrow^{\ell}_{CA_j} CA \le D_kA$, for $k \ne j$, by Lemma 4.6, D_jA can be made into a model of T whose $\mathcal{L}_I$-structure extends that of D_j and CA. Since $D_1 \downarrow^{\ell}_{CA_1} D_2A$, we also have $D_1A \downarrow^{\ell}_{CA} D_2A$ and, by Lemma 4.8, D_1D_2A can be made into a model of T whose $\mathcal{L}_I$-structure extends that of CA, D_1 and D_2. Also, since $CA_1 \le D_2A$ is regular and $D_1 \le (CA_1)^{\mathrm{a}}$, it follows that $D_1 \le D_1D_2A$, and thus $F \le D_1D_2A$, are regular. So $D_1D_2A \downarrow^{\ell}_A F^{\mathrm{a}}A = A^{\mathrm{a}}$ and $A \le D_1D_2A$ is regular.
By Corollary 4.7, there exists an $\mathcal{L}_I(A)$-embedding $g : D_1D_2A \to K^\star \succcurlyeq K$. Let $c^\star = g(c)$. We have $c^\star \equiv^{\mathrm{qf}}_{\mathcal{L}_I(a_1a_2)} c$. Moreover, by Proposition 6.7, $g \circ f_j$ is a $\mathcal{L}_I(A_j)$-elementary isomorphism sending c_j to $c^\star$. □

7.1. **Burden.** We will now study the burden of perfect bounded PIC fields in terms of the burden of each theory T_i.

Notation 7.2. We denote by bdn_i the burden computed in models of $T_{i,K}$.

Lemma 7.3. *Let* X *be an* $\mathcal{L}_i(M_i)$*-definable set of dimension* d. *Then*

$$\operatorname{bdn}_i(X) = \operatorname{bdn}_i(\mathbb{A}^d).$$

Proof. By Remark 2.13, there exists an $\mathcal{L}_i(M_i)$-definable bounded τ_i-open neighborhood O_i of 0 such that $M_i \subseteq O_i \cup (O_i \setminus \{0\})^{-1}$. Then $\mathbb{A}^d(M_i) = \bigcup_{\varepsilon\in 2^d} \prod_{j<d} O_i^{\varepsilon_j}$ where $O_i^1 = O_i$ and $O_i^0 = (O_i \setminus \{0\})^{-1}$. Since $\operatorname{bdn}_i(\prod_{j<d} O_i) =$

$\prod_{j<d} O_i^{\varepsilon_j}$ for every ε, it follows that $\mathrm{bdn}_i(\prod_{j<d} O_i) = \mathrm{bdn}_i(\mathbb{A}^d)$. If $U \subseteq \mathbb{A}^d$ is an $\mathcal{L}_i(M_i)$-definable open, it contains a definable subset of the form $\prod_{j<d} U_j$ where $U_j \subseteq M_i$ is in (affine) $\mathcal{L}_i(M_i)$-definable bijection with O_i. It follows that

$$\mathrm{bdn}_i(\mathbb{A}^d) = \mathrm{bdn}_i(\prod_{j<d} O_i) = \mathrm{bdn}_i(\prod_{j<d} U_i) \leq \mathrm{bdn}_i(U) \leq \mathrm{bdn}_i(\mathbb{A}^d).$$

Now, since $\dim(X) = d$, there exists an $\mathcal{L}_i(M_i)$-definable finite to one map $f : X \to \mathbb{A}^d$ whose image is Zariski dense and hence has non-empty interior. So, we have $\mathrm{bdn}_i(X) = \mathrm{bdn}_i(f(X)) = \mathrm{bdn}_i(\mathbb{A}^d)$. □

Proposition 7.4. *Let $A = \mathrm{acl}(A) \subseteq K$, X be quantifier free $\mathcal{L}_I(A)$-definable and non-empty, $\phi(x,y)$ be an $\mathcal{L}_{\mathfrak{d}}$-formula and $(a_j)_{j\in\omega} \in K^y$ an indiscernible sequence over A. Suppose that $\phi(K,a_0)$ is τ-dense in X. Then $\bigwedge_j \phi(x,a_j)$ is consistent.*

Proof. By compactness, extend $(a_j)_{j\in\omega}$ to an A-indiscernible sequence indexed by some sufficiently large cardinal κ. For every finite $J \subseteq \kappa$, let a_J enumerate $\mathrm{acl}(Aa_j : j \in J)$ — in a way compatible with inclusions: if $f : J_1 \to J_2$ is an increasing bijection, then for any $J \subseteq J_1$, a_J and $a_{f(J)}$ appear in the same place in a_{J_1} and a_{J_2} respectively.

Claim 7.4.1. *There exists $c \in X$ such that $c \mathop{\downarrow}^{\mathrm{a}}_A \bigcup_{j<\omega} a_j$ and, for all finite $J \subseteq \omega$, $\mathrm{tp}(a_J/Ac)$ only depends on $|J|$.*

Proof. Note that, by invariance, the Zariski closure of X is over A and hence we can find $c \in X$ with $c \mathop{\downarrow}^{\mathrm{a}}_A \bigcup_j a_j$. By iterated applications of Erdös-Rado and at the cost of reducing κ (exactly as when extracting an indiscernible sequence), we find $J_0 \subseteq \kappa$, with order type ω, such that for all finite $J \subseteq J_0$, $\mathrm{tp}(a_J/Ac)$ only depends on $|J|$. By homogeneity (and changing c), we may assume that $J_0 = \omega$. □

Note that if $J_1, J_2 \subseteq \omega$ are disjoint of the same cardinality, if $e \in a_{J_1} \cap a_{J_2}$, then there is some n, m with $a_{J_1,n} = e = a_{J_2,m}$. Spreading J_1 and J_2 apart, we can find J_3 such that $J_1 \cup J_2$, $J_1 \cup J_3$ and $J_2 \cup J_3$ are ordered in the same way. It follows that $a_{J_1,n} = a_{J_3,m} = a_{J_2,n}$. So e appears in the same place in all a_J, with J of the given cardinality, and $\mathrm{tp}(a_J/Ace)$ only depends on $|J|$. So we may assume that for disjoint $J_1, J_2 \subseteq \omega$, $a_{J_1} \cap a_{J_2} = A$.

Claim 7.4.2. *There exists $c \in \phi(K,a_0)$ such that $c \mathop{\downarrow}^{\mathrm{a}}_A \bigcup_{j<\omega} a_j$ and, for all finite $J \subseteq \omega$, $\mathrm{qftp}(a_J/Ac)$ only depends on $|J|$.*

Proof. Let $\Sigma_i(x)$ be the (closure under consequence of the) partial type expressing that $\mathrm{tp}_i(a_J/Ax)$ only depends on $|J|$. By compactness, it suffices, given $\psi_i(x) \in \Sigma_i(x)$ for every i and $U \subseteq V$ Zariski open over $A(a_j)_{j<\omega}$, to find $e \in U$ realizing $\phi(x,a_0)$ and the ψ_i.
Let c be such as in Claim 7.4.1 and let V be its algebraic locus over A, which is also its algebraic locus over $Aa_{j<\omega}$. Then c is in the τ_i-interior of $\psi_i(K)$ in V and, since $c \in \psi_i(K) \cap X$, $\psi_i(K)$ has non-empty τ_i-interior in X. The

existence of e as above now follows from the τ-density of $\phi(K, a_0)$ in X and the fact U is τ-open (in V and hence in X). □

Note that if E is an $\mathcal{L}_i(Ac)$-definable finite equivalence relation, for some finite $J_1, J_2 \subseteq \omega$ we have $a_{J_1} E a_{J_2}$ and hence this holds for all J_1, J_2. In other words, $\mathrm{tp}_i(a_J/\mathrm{acl}_i(Ac))$ only depends on $|J|$ and hence, since $\mathrm{acl}(Ac) \subseteq A(c)^{\mathrm{a}} \subseteq \mathrm{acl}_i(Ac)$, for all i, $\mathrm{qftp}(a_J/\mathrm{acl}(Ac))$ only depends on $|J|$. Let d enumerate $\mathrm{acl}(Ac)$.

Claim 7.4.3. *For all $n > 0$, there exists $d_n \equiv^{\mathrm{qf}}_{a_{\{0,\ldots,2^n-1\}}} d$ such that $K \models \bigwedge_{j<2^n} \phi(d_n, a_j)$.*

Proof. We proceed by induction on n and let us assume we have found d_n — note that $d_0 = d$ works. Let $a^1 = a_{\{0,\ldots,2^n-1\}}$ and $a^2 = a_{\{2^n,\ldots,2^{n+1}-1\}}$, $d^1 = d_n$ and d^2 be such that $d^2 a^2 \equiv_A d^1 a^1$, in particular, $d^2 \models \bigwedge_{2^n \leq j < 2^{n+1}} \phi(x, a_j)$. Then $a^1 \cap a^2 = A$, $d \mathop{\downarrow}^{\mathrm{a}}_A a^1 a^2$, $d^1 \equiv_A d^2$, $d^1 a^1 \equiv^{\mathrm{qf}}_A d a^1$ and $d^2 a^2 \equiv_A d^1 a^1 \equiv^{\mathrm{qf}}_A d a^1 \equiv^{\mathrm{qf}}_A d a^2$. By Theorem 7.1, we find d_{n+1} such that $d_{n+1} a^j \equiv_A d^j a^j$ — and hence $d_{n+1} \in \bigcap_{j<2^{n+1}} \phi(K, a_j)$ — and $d_{n+1} \equiv^{\mathrm{qf}}_{a^1 a^2} d$. □

So $\bigwedge_j \phi(x, a_j)$ is indeed consistent. □

Proposition 7.5. *Let X be $\mathcal{L}_{\mathfrak{d}}(K)$-definable of dimension d. If for every $i \in I$, $\mathrm{bdn}_i(\mathbb{A}^d) \leq \kappa_i$ for some cardinal κ_i, then*

$$\mathrm{bdn}(X) \leq \sum_{i \in I} \kappa_i.$$

Moreover, if $\mathrm{bdn}_i(\mathbb{A}^d) < \kappa$, where κ is an infinite cardinal, then

$$\mathrm{bdn}(X) < \sum_{i \in I} \kappa.$$

Proof. We proceed by induction on d and assume K is sufficiently saturated. Let $\phi_l(x, a_{lj})_{l<\kappa, j<\omega}$ be an inp-pattern of depth κ in X — where $\kappa > \sum_i \mathrm{bdn}_i(\mathbb{A}^d)$ in the first case. Let $A = \mathrm{acl}(A) \leq K$ be such that the ϕ_l and X are all over A. We may assume that the $(a_{lj})_{j<\omega}$ are mutually indiscernible over A. Let V be the Zariski closure of X.

By [Che14, Lemma 7.1] (see Remark 2.4(2)), Theorem 6.12 and indiscernability, we find (uniformly) $\mathcal{L}_i(a_{lj})$-definable $Y_{lji} \subseteq V$ such that $\phi_l(K, a_{lj})$ is τ-dense in $Y_{lj} := \bigcap_i Y_{lji}$. If any $Y_{l_0 j_0 i_0}$ has dimension smaller than d, then, by induction, $\mathrm{bdn}(Y_{l_0 j_0 i_0}) \leq \sum_{i \in I} \lambda_i$, where $\lambda_i + 1 = \kappa_i$. However, the $\phi_l(x, a_{lj})_{l \neq l_0, j<\omega}$ is an inp-pattern of depth λ in $Y_{l_0 j_0 i_0}$, where $\lambda + 1 = \kappa$. So $\kappa = \lambda + 1 \leq \mathrm{bdn}(Y_{l_0 j_0 i_0}) + 1 \leq \sum_{i \in I} \lambda_i + 1 \leq \sum_{i \in I} \kappa_i < \kappa$, a contradiction. In the second case, by induction, we would have $\kappa \leq \mathrm{bdn}(Y_{l_0 j_0 i_0}) < \kappa$, which is also a contradiction.

So we may assume that all the Y_{lji} have dimension d. In particular, they have non-empty τ_i-interior in V. Since the border of Y_{lji} in V has lower dimension, by [Che14, Lemma 7.1] and induction, we may further assume that the Y_{lji} are τ_i-open in V.

Since $\phi_l(x, a_{lj})$ is an inp-pattern, for all $f : \kappa \to \omega$ we have that $\bigwedge_l Y_{l,f(l),i}$ is consistent. It follows that there are at most κ_i many l (respectively strictly less than κ in the second case) such that $\bigwedge_l Y_{lji}$ is inconsistent. In both cases, it follows that we can find an l such that, for all i, $\bigwedge_j Y_{lji}$ is consistent.
Since the Y_{lji} are τ_i-open (in V), by compactness, $\bigwedge_j Y_{lji}$ contains some $\mathcal{L}_i(K)$-definable τ_i-open $U_i \subseteq V$. Let $U = \bigcap_i U_i$. Then, for every j, $\phi(K, a_{lj})$ is τ-dense in U. Let $B \supseteq A$ be such that U is quantifier free $\mathcal{L}_I(B)$-definable. We may assume that a_{lj} is B-indiscernible. By Proposition 7.4, $\bigwedge_j \phi(K, a_{lj})$ is consistent — contradicting that $\phi_l(x, a_{lj})_{l<\kappa, j<\omega}$ is an inp-pattern. □

We also have the following qualitative corollary:

Corollary 7.6. *Suppose that, for all $i \in I$, T_i is NTP_2 (respectively strong), then any perfect $K \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ also is.*

The bound in Proposition 7.5 is actually tight (although the argument is surprisingly more involved than we expected at first).

Lemma 7.7. *If, for some $i \in I$, $\mathrm{bdn}_i(\mathbb{A}^d) \geq \kappa$ for some cardinal κ, then there exists an inp-pattern $(\phi_l(x, a_{lj}))_{l<\kappa, j<\omega}$ of depth κ in any τ_i-open $\mathcal{L}_i(K)$-definable set $X \subseteq \mathbb{A}^d$ such that $a_{lj} \in K$ and $\phi_l(x, a_{lj})$ defines a τ_i-open set.*

Proof. Note first that, at the cost of reducing X, we may assume that $X = \prod_{k<d} X_k$ is a product of open (bounded) subsets of $\mathbb{A}^1$. By Lemma 7.3, we find $M_i \vDash T_{i,K}$ and $(\phi_l(x, a_{lj}))_{l<\kappa, j<\omega}$ an inp-pattern of depth κ in $X(M_i)$. Writing $\phi(M_i, a_{lj})$ as the union of its interior and a lower dimension set, by Remark 2.4.(2), we may assume that either $\phi(M_i, a_{lj})$ is τ_i-open or $\dim(\phi(M_i, a_{lj})) < d$. In the latter case, it follows by Remark 2.4.(6), that $\mathrm{bdn}(\mathbb{A}^{d-1}) \geq \lambda$, where $\lambda + 1 = \kappa$. By induction, we find an inp-pattern of depth λ in $\prod_{k<d-1} X_k$ with $\psi_l(M_i, c_{lj})$ consisting of τ_i-open sets. Together with a uniformly definable family of disjoint τ_i-open subsets of X_{d-1}, they yield an inp-pattern of depth κ in X.
So we may assume that the $\phi_l(M_i, a_{lj})$ are τ_i-open and there only remains to prove that we can choose the a_{lj} in K. Let c_{lj} be generic in some $\mathbb{A}^d$ such that $a_{lj} \in \mathrm{acl}(c_{lj})$. We may assume that a_{lj} enumerates $\mathrm{acl}_i(c_{lj})$, that $(c_{lj} a_{lj})_j$ are mutually indiscernible sequences and that it is middle part of a sequence indexed by $\omega + \omega + \omega^\star$. Let A_l consist of the union of K, the ω initial segment and the $\omega^\star$ final segment of the sequence. For each j, let E_{lj} be the set of $\mathcal{L}_i(A_l c_{lj})$-conjugates of a_{lj}.

Claim 7.7.1. *For every $j_0 < \omega$, E_{lj_0} is a complete type over $A_l(c_{lj})_j(a_{lj})_{j \neq j_0}$.*

Proof. Assume there exists $E \subset E_{lj}$ non-empty defined over $A_l(c_{lj})_j(a_{lj})_{j \neq j_0}$. Then, by indiscernability, moving the parameters to the initial and final segment, such a set also exists with parameters in A. □

Let $X_{lj} = \bigcup_{a \in E_{lj}} \phi_l(x, a)$. If the X_{lj} are consistent, then, for every k and $j < k$ there exists $e_{lj} \in E_{lj}$ such that $\bigwedge_{j<k} \phi_l(x, e_{lj})$ is consistent. But, by Claim 7.7.1, $\prod_{j<k} E_{lj}$ is a complete type. This would imply that, $\bigwedge_{j<k} \phi_l(x, a_{lj})$

is consistent, contradicting that the $\phi_l(x, a_{lj})$ form an inp-pattern. It follows that the X_{lj} form an inp-pattern (of depth κ in X).
So, at the cost of allowing $\phi_l(x, y)$ to be an $\mathcal{L}_i(A)$-formula for some $A \subseteq M_i$ containing K, we may assume the a_{lj} to be generic in some $\mathbb{A}^m$ (over A). Let C be a transcendence basis of A over K and $b \in A \subseteq \mathrm{acl}_i(C)$ be such that $\phi_l(x, y) = \psi(x, y, b)$, where ψ is an $\mathcal{L}_i(K(C))$-formula. As above, possibly enlarging C, we may assume that the set E of $\mathcal{L}_i(K(C))$-conjugates of b is a complete type over $K(C, c_{lj})$. Then, for every $e \in E$, the $\psi(x, a_{lj}, e)$ are inconsistent. It follows that the $\theta(x, a_{lj}) = \bigvee_{e\in E} \psi(x, a_{lj}, e)$ are inconsistent and hence form an inp-pattern (of depth κ in X). So we may assume that $A = K(C)$ is generated over K by a (finite) set of algebraically independent elements. Then $C(a_{lj})_{l<\kappa, j<\omega}$ is a tuple of algebraically independent elements over K, in other words any finite sub-tuple is K-generic in the affine space of the right dimension. It follows, *e.g.* by Proposition 4.3.(ii), that its $\mathcal{L}_i(K)$-type can be realized in K. □

Proposition 7.8. *Let X be $\mathcal{L}_{\mathfrak{d}}(K)$-definable of dimension d. If, for every $i \in I$, $\mathrm{bdn}_i(\mathbb{A}^d) \geq \kappa_i$ for some cardinal κ_i, then*

$$\mathrm{bdn}(X) \geq \sum_{i\in I} \kappa_i.$$

Proof. By Remark 2.4.(6), we may assume that $X \subseteq \mathbb{A}^d$. By Corollary 6.13, we may further assume that X is τ-dense in $\bigcap_i X_i$ where X_i is $\mathcal{L}_i(K)$-definable and τ_i-open. By Lemma 7.7, assuming that K is sufficiently saturated, for every $i \in I$, we find an inp-pattern $\phi_{i,l}(x, a_{i,l,j})_{l<\kappa_i, j<\omega}$ of τ_i-open sets in X_i,with $a_{i,l,j} \in K$.
Note that for any finite $Y_i \subseteq \kappa_i$ and $f_i : Y_i \to \omega$, $\bigcap_{l\in Y_i} \phi_{i,l}(M_i, a_{i,l,f(i,l)})$ is τ_i-open and hence, by τ density, $X \cap \bigcap_{i,l\in Y_i} \phi_{i,l}(M_i, a_{i,l,f(i,l)}) \neq \varnothing$. It follows that together, the $\phi_{i,l}(x, a_{i,l,j})$ form an inp-pattern of depth $\sum_i \kappa_i$ in X. □

We have thus proved the following equality:

Theorem 7.9. *For every perfect $K \models \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ and $\mathcal{L}_{\mathfrak{d}}(K)$-definable set X of dimension d, we have*

$$\mathrm{bdn}(X) = \sum_i \mathrm{bdn}_i(\mathbb{A}^d).$$

The equality holds in Card*, *cf.* [Tou18, Definition 1.29].

Corollary 7.10. *Let* $\mathrm{K} \models \mathrm{PAC}$ *be perfect and bounded and v_i be n independent non-trivial valuations on K. Then*

$$\mathrm{bdn}(K, v_1, \dots, v_n) = n.$$

A similar corollary holds for PRC and PpC fields, but we refer the reader to the more general statement in Corollary A.3.

8. Forking and dividing

In this section we will study the relationship between forking (and dividing) in the theories T_i and in the theory $\mathrm{P}I\mathrm{C}_{\mathfrak{d}}$.
Let us recall our conventions and notation. We assume that I is finite and that $T = \bigvee_{i\in I} T_i$. We also assume that $(\mathbf{H}_i)$ holds for every $i \in I$ (see Hypothesis 5.1). We denote by τ the topology generated by the τ_i, and we fix $K \vDash \mathrm{P}I\mathrm{C}_{\mathfrak{d}}$ perfect sufficiently saturated and homogeneous. We fix some $M_i \vDash T_{i,K}$.

Proposition 8.1. *Let $A = \mathrm{acl}(A) \le K$ and X be a $\mathcal{L}_{\mathfrak{d}}(K)$-definable subset of K such that $X \subseteq \bigcap_i X_i$ is τ-dense, where X_i is $\mathcal{L}_i(K)$-definable and τ_i-open. Then X does not divide over A if and only if for all i, X_i does not divide over A (in M_i).*

Proof. ($\Leftarrow$) We assume that, for every i, $X_i := \theta_i(M_i, b)$ does not divide over A in M_i, and we write X as $\psi(K, b)$. Let κ be a sufficiently large cardinal and $(b_j)_{j<\kappa}$ be an indiscernible sequence over A with $b_0 = b$. Let $X_{ij} = \theta_i(M_i, b_j)$. By indiscernability $X_j = \psi(K, b_j) \subseteq \bigcap_i X_{ij}$ is τ-dense. Since X_i does not divide over A, we have $\bigcap_j X_{ij} \neq \varnothing$ and, by compactness, it contains some non-empty $\mathcal{L}_i(K)$-definable τ_i-open U_i. Enlarging A and taking an appropriate subsequence of the b_j, we may assume that U_i is $\mathcal{L}_i(A)$-definable.
Note that since $U := \bigcap_i U_i$ is τ-open, X is τ-dense in U. It follows, by Proposition 7.4, that $\bigcap_j (X_j \cap U) \neq \varnothing$, and hence X does not divide over A.
($\Rightarrow$) We now assume that, for some fixed i, $X_i := \theta_i(M_i, b)$ divides over A and let $(b_j)_{j<\omega}$ be an A-indiscernible sequence such that $b_0 = b$ and $\bigwedge_j \theta(x, b_j)$ is inconsistent. We may assume that the b_j are algebraically independent over A. For all j, $A(B_j) \simeq A(b) \le K$. Since $A \le A(b_j)$ is regular, we have $A(b_{j+1}) \mathop{\smile\!\!\!\!\!\mid}^{l}_{A} A(b_0 \dots b_j)$ and, by Lemma 4.6 and induction, we see that $A(b_j : j < \omega) \vDash T_{i,\forall}$. By Corollary 4.7, $K \le K(b_j : j < \omega) \simeq K \otimes_A A(b_j : j < \omega)$ is $\mathcal{L}_I$-existentially closed, so we may assume that the b_j are in K. It follows that $X_i \cap K$ divides over A in K, and hence, so does $X \subseteq X_i \cap K$. $\square$

Corollary 8.2. *Let a be a tuple of K and $A \subseteq B \subseteq K$ such that $A = \mathrm{acl}(A)$ and $B = \mathrm{acl}(B)$. Then $\mathrm{tp}(a/B)$ does not divide over A if and only if, for all $i \in \{1, \dots, n\}$, $\mathrm{tp}_i(a/B)$ does not divide over A (in M_i).*

Proof. This follows from Proposition 8.1 and Corollary 6.13. $\square$

Theorem 8.3. *Let $A = \mathrm{acl}(A) \le K$. If A is an extension base of T_i, for all $i \in I$, then A is an extension base of K.*

Proof. By transitivity of forking, it suffices to show that no $\mathcal{L}_{\mathfrak{d}}(A)$-definable set $X \subseteq K$ forks over A. So, if $X \subseteq \bigcup_{j<m} Y_j$, where the Y_j are $\mathcal{L}_{\mathfrak{d}}(K)$-definable, we have to show that one of the Y_j does not divide over A. To do so, we may assume:

(i) $X \subseteq \bigcap_i X_i$ is τ-dense, where the X_i are $\mathcal{L}_i(A)$-definable.
(ii) $X \smallsetminus \bigcup_j Y_j$ is finite.

(iii) $Y_j \subseteq \bigcap_i Y_{ji}$ is τ-dense, where $Y_{ji} \subseteq X_i$ is τ_i-open and $\mathcal{L}_i(K)$-definable.
(iv) For every i and j_1, j_2, $Y_{j_1 i} = Y_{j_2 i}$ or $Y_{j_1 i} \cap Y_{j_2 i} = \varnothing$.
(v) $X_i \smallsetminus \bigcup_j Y_{ji}$ is finite and does not contain any element of A.

Indeed, by Corollary 6.13, X is τ-dense in $\bigcup_l \bigcap_i X_{il} \cup Z$ where the X_{il} are $\mathcal{L}_i(A)$-definable and Z is finite and $\mathcal{L}_\mathfrak{d}(A)$-definable. If Z is non-empty, it does not fork over A, and we are done. Otherwise, it suffices to show that any of the $X \cap \bigcap_i X_{il}$ does not fork over A. Similarly, by Corollary 6.13, Y_j is τ-dense in $\bigcup_l \bigcap_i Y_{jil} \cup Z_j$ where the Y_{jil} are $\mathcal{L}_i(K)$-definable τ_i-open and Z_j is finite. Replacing the Y_j by the $Y_j \cap \bigcap_i (Y_{jil} \cap X_i)$ and setting the finite sets aside, we can ensure (ii) and (iii).

Replacing the Y_{ji} by the τ_i-interior of the atoms of Boolean algebra that they generate (for fixed i), we can further assume (iii). Now, for some finite set Z, we have $X \subseteq \bigcup_j Y_{ji} \cup Z \subseteq X_i$ and since X is τ_i-dense in X_i, $Z_i := X_i \smallsetminus (\bigcup_j Y_{ji} \cup Z)$ has empty τ_i-interior and is therefore finite. Now, if $(Z_i \cup Z) \cap A \neq \varnothing$, we can remove those points from X and the X_i without losing any of the other properties.

There remains to show that one of the Y_j does not divide over A. Note first that, for every $f : I \to m$, by τ-density, there exists an $a \in X \cap \bigcap_i Y_{f(i)i}$. Let j be such that $a \in Y_j \subseteq \bigcap_i Y_{ji}$. Then, by (iv), for all i we have $Y_{f(i)i} = Y_{ji}$.

Now, by hypothesis, X_i does not fork over A and since any finite set that does not divide over $A = \mathrm{acl}(A)$ must contain a point from A, there is an $f(i) < m$ such that $Y_{f(i)i}$ does not divide over A. As noted above, we may assume that $f(i) = j$ is constant. It now follows from Proposition 8.1, that Y_j does not divide over A. □

Definition 8.4. Let a be a tuple of K and $A, B \subseteq K$. We write $a \mathop{\smile\!\!\!\!\!\!\downarrow}_A B$ if $\mathrm{tp}(a/AB)$ does not fork over A (in K) and $a \mathop{\smile\!\!\!\!\!\!\downarrow}^i_A B$ if $\mathrm{tp}_i(a/AB)$ does not fork over A (in M_i).

Corollary 8.5. *Suppose that T_i is NTP$_2$ for all $i \in I$ and that forking equals dividing in T_i. Then forking equals dividing in K and if a, A, B are in K, then $a \mathop{\smile\!\!\!\!\!\!\downarrow}^i_A B$ for all $i \in I$ if and only if $a \mathop{\smile\!\!\!\!\!\!\downarrow}_A B$.*

Proof. Since K is NTP$_2$ (Corollary 7.6) to show that forking equals dividing in K it's enough to show that any set in K is an extension base, but this is trivial by Theorem 8.3.

The last equivalence is clear from Proposition 8.1 and the fact that forking equals dividing in K. □

Appendix A. Dependent topologies

Following the approach of [Joh19] we can also understand dependent valuations and orders by reduction to the case of independent topologies.

We now fix a finite tree I — that is a filtered partial order such that for all $i \in I$, the set $\{j \in I : i \leq j\}$ is totally ordered. Let r be its maximal element — its root. To every leaf i — that is every minimal element — we associate:

- An integer p_i which is either prime or zero.

- A theory complete T_i of fields of characteristic p_i which eliminates quantifiers in a relational expansion of the ring language such that $(\mathbf{H}_i)$ holds (see hypothesis 5.1).

To every non-leaf $i \in I$, we associate:

- An integer p_i which is either prime or zero. We assume that for all daughters j of i, the p_j are equal. Moreover, they are equal to p_i if it is non-zero. We also assume that p_r is equal to its daughter p_j.
- A theory complete $T_{i,\Gamma}$ of (enriched) ordered Abelian group. We assume that it is the theory of the trivial group if and only if $i = r$. If p_i is non-zero, we assume that models of $T_{i,\Gamma}$ are p_i-divisible. If p_i is zero and daughter p_j are non-zero, we require $T_{i,\Gamma}$ to come with a constant c_i for a positive element. We then assume that$[-c_i, c_i]$ is either p_j-divisible or finite — in the latter case, we say that i is finitely ramified.

For every non-leaf $i \in I$, we define the theories $T_{i,\mathrm{k}}$ and T_i, by induction:

- The theory $T_{i,\mathrm{k}}$ (the residual theory of i) is the theory of pseudo I_i-closed fields without finite extensions of degree divisible by p_i (if it is non-zero), where $I_i = \{T_j : j \text{ daughter of } i\}$.
- The theory T_i is a Morleyization of the theory of characteristic p_i algebraically maximal complete fields with residue field a model of $T_{i,\mathrm{k}}$ and value group a model of $T_{i,\Gamma}$. If p_i is zero and the daughter p_j are non-zero, then we also require that $c_i = \mathrm{val}(p_j)$.

Note that, by [HH19, Corollary A.3], if i is not finitely ramified, the reduct of T_i to the valued field language eliminates quantifiers relative to RV_1; and it eliminates quantifiers relative to $\bigcup_n \mathrm{RV}_n$ if i is finitely ramified, by [Bas91, Theorem B]. Moreover, each of completion admits maximally complete models. In particular, $(\mathbf{H}_i)$ holds in every T_i.

Definition A.1. We say that a field is pseudo I-closed if it is pseudo I_r-closed.

This is coherent with our earlier notation, provided we identify a set of independent valuations to the tree of height one with trivial valuation at the root, and the independent valuations as leaves.
We also define:

$$\kappa_i = \begin{cases} \mathrm{bdn}(T_i) & \text{if } i \text{ is a leaf,} \\ \max(\aleph_0 \cdot \sum_{j \in I_j} \kappa_j, \mathrm{bdn}(T_{i,\Gamma})) & \text{if } i \text{ is finitely ramified,} \\ \max(\sum_{j \in I_j} \kappa_j, \mathrm{bdn}(T_{i,\Gamma})) & \text{otherwise.} \end{cases}$$

Once again, for a finer estimate, κ_i can be computed in $\mathrm{Card}^\star$, *cf.* [Tou18, Definition 1.29].

Theorem A.2. *Let K be a bounded perfect pseudo I-closed field. Then*

$$\mathrm{bdn}(K) = \kappa_r.$$

Proof. We proceed by induction on i. By Theorem 7.9, $\mathrm{bdn}(T_{i,\mathrm{k}}) = \sum_{j \in I_i} \kappa_i$ (by Theorem 7.9) and, by Fact 2.6, $\mathrm{bdn}(T_i) = \kappa_i$. □

Corollary A.3. *Let K be a bounded perfect* PAC*, m-*PRC *or m-*P*p*C *field (setting $m = 0$ if K is* PAC*) and $v_1, \dots, v_l$ be valuations on K. Let n be the maximal number of non-dependent valuations among the v_i, that are also not dependent from the m orders (respectively p-adic valuations) if K is* PRC *(respectively* P*p*C*). Then*

$$\mathrm{bdn}(K, v_1, \dots, v_l) = m + n.$$

Proof. We assume K is PAC. Let I be the tree of valuations generated by the v_i, in other words the nodes of I are the valuation rings $\prod_{i \in J} \mathcal{O}_i$, where $J \subseteq I$ and $\mathcal{O}_i$ is the valuation ring of v_i (and the trivial valuation at the root). For any node $i \in I$ we also write v_i for the associated valuation. Since K is PAC, the residue field of any of the v_i is algebraically closed and its value group is divisible — this follows, for example, from Proposition 3.12. We annotate I with a copy of ACVF on each leaf and with the theory of divisible ordered Abelian groups on each non-leave node (except for the root), as well as the relevant characteristics and constants.
It follows from Theorem 5.11 that $K \vDash \mathrm{P}I\mathrm{C}$ and this structure is bi-interpretable with $(K, v_1, \dots, v_n)$. By induction on i we see that κ_i is the number of leaves below the node i. The corollary follows.
The PRC and PpC cases follow in a similar manner, noting that orders (respectively p-adic valuations) can only be leaves in the tree. □

References

[ADF20] S. Anscombe, P. Dittmann, and A. Fehm. "Approximation theorems for spaces of localities". *Math. Z.* 296.3-4 (2020), pp. 1471–1499.

[Ax68] J. Ax. "The elementary theory of finite fields". *Ann. of Math. (2)* 88 (1968), pp. 239–271.

[Bas91] . A. Basarab. "Relative elimination of quantifiers for Henselian valued fields". *Ann. Pure Appl. Logic* 53.1 (1991), pp. 51–74.

[Cha02] Z. Chatzidakis. "Properties of forking in ω-free pseudo-algebraically closed fields". *J. Symb. Log.* 67.3 (2002), pp. 957–996.

[Che14] A. Chernikov. "Theories without the tree property of the second kind". *Ann. Pure Appl. Logic* 165.2 (2014), pp. 695–723.

[CK12] A. Chernikov and I. Kaplan. "Forking and dividing in NTP_2 theories". *J. Symb. Log.* 77.1 (2012), pp. 1–20.

[CP23] P. Cubides Kovacsics and F. Point. "Topological fields with a generic derivation". English. *Ann. Pure Appl. Logic* 174.3 (2023). Id/No 103211, p. 38.

[Dri78] L. van den Dries. "Model theory of fields: decidability, and bounds for polynomial ideals". PhD thesis. Rijksuniversiteit Utrecht, 1978.

[EP05] A. J. Engler and A. Prestel. *Valued Fields*. Springer Monographs in Mathematics. Springer-Verlag, 2005.

[Feh13] A. Fehm. "Elementary geometric local-global principles for fields". *Ann. Pure Appl. Logic* 164.10 (2013), pp. 989–1008.

[FJ08] M. D. Fried and M. Jarden. *Field arithmetic*. Third. Vol. 11. Ergeb. Math. Grenzgeb. (3). Revised by Jarden. Springer-Verlag, Berlin, 2008, pp. xxiv+792.

[GJ13] W.-D. Geyer and M. Jarden. "Model completeness of valued PAC fields". *Trav. Math.* XXIII (2013), pp. 173–214.

[Gro87] C. Grob. "Die Entscheidbarkeit der Theorie der maximalen pseudo p-adisch abgeschlossenen Körper". *Konstanzer Dissertationen* 202 (1987).

[HH19] Y. Halevi and A. Hasson. "Eliminating field quantifiers in strongly dependent Henselian fields". *Proc. Amer. Math. Soc.* 147.5 (2019), pp. 2213–2230.

[HHJ20] Y. Halevi, A. Hasson, and F. Jahnke. "Definable V-topologies, Henselianity and NIP". *J. Math. Log.* 20.2 (2020), pp. 2050008, 33.

[HJ88] D. Haran and M. Jarden. "The absolute Galois group of a pseudo p-adically closed field". *J. Reine Angew. Math.* 383 (1988), pp. 147–206.

[HP84] B. Heinemann and A. Prestel. "Fields regularly closed with respect to finitely many valuations and orderings". *Quadratic and Hermitian forms (Hamilton, Ont., 1983).* Vol. 4. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1984, pp. 297–336.

[Hon23] J. Hong. "Quantifier elimination on some pseudo-algebraically closed valued fields". *Ann. Pure Appl. Logic* 174.1 (2023), Paper No. 103170, 23.

[JR98] M. Jarden and A. Razon. "Rumely's local global principle for algebraic P$\mathcal{S}$C fields over rings". *Trans. Amer. Math. Soc.* 350.1 (1998), pp. 55–85.

[Joh19] W. Johnson. "Finite burden in multivalued algebraically closed fields". arXiv:1905.04991. 2019.

[Joh22] W. Johnson. "Forking and dividing in fields with several orderings and valuations". *J. Math. Log.* 22.1 (2022), Paper No. 2150025, 43.

[Kol07] J. Kollár. "Algebraic varieties over PAC fields". *Israel J. Math.* 161 (2007), pp. 89–101.

[KTW21] A. Kruckman, C.-M. Tran, and E. Walsberg. "Interpolative fusions". *J. Math. Log.* 21.2 (2021), Paper No. 2150010, 38.

[Mon17a] S. Montenegro. "Imaginaries in bounded pseudo real closed fields". *Ann. Pure Appl. Logic* 168.10 (2017), pp. 1866–1877.

[Mon17b] S. Montenegro. "Pseudo real closed fields, pseudo p-adically closed fields and NTP_2". *Ann. Pure Appl. Logic* 168.1 (2017), pp. 191–232.

[MOS20] S. Montenegro, A. Onshuus, and P. Simon. "Stabilizers, NTP_2 groups with f-generics, and PRC fields". *J. Inst. Math. Jussieu* 19.3 (2020), pp. 821–853.

[MR21] S. Montenegro and S. Rideau-Kikuchi. "Imaginaries, invariant types and pseudo p-adically closed fields". *Trans. Amer. Math. Soc.* 374.2 (2021), pp. 803–828.

[Pop03] F. Pop. "Classically projective groups and pseudo classically closed fields". *Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999).* Vol. 33. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2003, pp. 251–283.

[Pre81] A. Prestel. "Pseudo real closed fields". *Set theory and model theory (Bonn, 1979).* Vol. 872. Lecture Notes in Math. Springer, Berlin-New York, 1981, pp. 127–156.

[PZ78] A. Prestel and M. Ziegler. "Model-theoretic methods in the theory of topological fields". *J. Reine Angew. Math.* 299-300 (1978), pp. 318–341.

[Rob77] A. Robinson. *Complete Theories.* Second. Stud. Logic Found. Math. 46. North-Holland Publishing Co., Amsterdam, 1977, pp. x+129.

[Sch00] J. Schmid. "Regularly T-closed fields". *Hilbert's tenth problem: relations with arithmetic and algebraic geometry (Ghent, 1999).* Vol. 270. Contemp. Math. Amer. Math. Soc., Providence, RI, 2000, pp. 187–212.

[Stacks] Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`. 2018.

[Tou18] P. Touchard. "Burden in Henselian Valued Fields". arXiv:1811.08756. 2018.

[Tre05] M. Tressl. "The uniform companion for large differential fields of characteristic 0". *Trans. Amer. Math. Soc.* 357.10 (2005), pp. 3933–3951.

Samaria Montenegro, Escuela de Matemática-CIMPA, Universidad de Costa Rica, San José, Costa Rica

Silvain Rideau-Kikuchi, DMA, École normale supérieure, Université PSL, CNRS, 75005 Paris, France