THE TILTING EQUIVALENCE AS A BI-INTERPRETATION

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Abstract. We propose a model theoretic interpretation of the theorems about the equivalence between mixed characteristic perfectoid spaces and their tilts.

1. Introduction

A perfectoid field is a complete non-trivially normed field $(K, |\cdot|)$ of residue characteristic p for some prime number p for which the set of norms $|K^\times|$ is dense in $\mathbb{R}_+$ and the map $x \mapsto x^p$ is surjective as a self-map on the ring $\mathcal{O}(K)/p\mathcal{O}(K)$ where $\mathcal{O}(K) = \{x \in K : |x| \leq 1\}$ is the ring of integers of K. Given a perfectoid field K, the tilt $K^\flat$ is obtained, as a multiplicative monoid, as the projective limit of the system in which K maps to itself repeatedly via $x \mapsto x^p$:

$$K^\flat := \varprojlim(\cdots \longrightarrow K \xrightarrow{x\mapsto x^p} K \xrightarrow{x\mapsto x^p} K)$$

The tilt $K^\flat$ carries a natural structure of a complete normed field of characteristic p. (We recall the details of this construction in Section 4.) It can happen that two different perfectoid fields have isomorphic tilts. For a fixed complete perfect nontrivially normed field L of characteristic p the family of untilts, that is, characteristic zero perfectoid fields K with $K^\flat \cong L$, is parameterized by the Fargues-Fontaine curve [5]. That is, there is a natural correspondence between perfectoid fields and perfect non-trivially normed fields fields of positive characteristic given with a point on the Fargues-Fontaine curve.

This tilting correspondence has been used to transfer properties between mixed and positive characteristic. For example, in [12], Scholze extends the tilting operation to adic spaces, establishes an equivalence of categories of adic spaces over K and its tilt $K^\flat$, and uses this equivalence to transfer the truth of the weight monodromy conjecture for complete intersections in positive characteristic to mixed characteristic. For another celebrated use of perfectoids, consider André's proof of the direct summand conjecture in [1] in which he ports positive characteristic proof techniques to mixed characteristic.

Unlike the case of the Ax-Kochen-Ershov-type theorems in which theories of henselian fields of mixed characteristic and of positive characteristic converge as the residue characteristic grows, the tilt/untilt correspondence is not asymptotic; it allows for direct comparisons between mixed characteristic perfectoid fields and positive characteristic perfect fields with exactly the same residue fields. This is curious from the standpoint of mathematical logic as the correspondence could not possibly reflect an equality of theories, since, for instance, the characteristic of the field is captured by the theory. This article is motivated by the problem of answering the riddle of how the tilt/untilt correspondence might be explained through mathematical logic.

An answer is given by Jahnke and Kartas in [10]. Their main theorem is that if K is a perfectoid field, then for any non-principal ultrapower $K^{\mathfrak{U}}$, the tilt $K^\flat$ of K may be realized as an elementary structure of a residue field of $K^{\mathfrak{U}}$. (See [10, Theorem 6.2.3] for details.) Their proof gives an equivalence of categories between the category of finite étale algebras over K and the corresponding category over $K^\flat$ and the theorem itself allows them to replace parts of almost mathematics [6] with ordinary commutative algebra.

Our answer is that the tilt/untilt correspondence is a quantifier-free bi-interpretation in continuous logic, inspired by Fargues and Fontaine's account of the tilting equivalence [5]. We note in Section 5 that the existence of this bi-interpretation implies that various important features of perfectoid fields are preserved by the tilt/untilt correspondence. These consequences include the Fontaine-Wintenberger theorem that a perfectoid field and its tilt have canonically isomorphic Galois groups, identifications between adic spaces over a perfectoid field and adic space over its tilt, and an approximation lemma whereby the tilt of an algebraic variety over a perfectoid field may be approximated by sets defined by quantifier-free first-order formulas in the language of valued fields.

This present paper is long in the tooth. The main observations and constructions were already completed while the three authors were working together at Berkeley during the 2016/7 academic year. Our delay in preparing the work for publication resulted from our thwarted attempts to upgrade the approximation lemma to a strong enough form to allow for a transfer of further theorems in positive characteristic to mixed characteristic, the weight monodromy conjecture without a restriction to complete intersections being our target application. We doubt that our methods will suffice to achieve such an end, but, perhaps, a reader cleverer than ourselves could implement the strategy.

This paper is organized as follows. In Section 2 we recall the formalism of continuous logic for metric structures and work out the basics of interpretations in continuous logic. In Section 3 we introduce and develop the theories of metric valued fields and, more specifically, of perfectoid fields from the standpoint of continuous logic. Section 4 comprises the core this paper. There we construct bi-interpretations between three theories: perfectoid fields, truncated perfectoid rings, and perfect metric valued fields of positive characteristic equipped with a Fargues-Fontaine parameter. In Section 5 we express some consequences of the existence of the bi-interpretations produced in the earlier section.

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2. Some continuous logic

In this section we recall some of the basics of continuous logic with a special emphasis on the theory of interpretations.

2.1. **Continuous logic formalism.**

We work in the setting of bounded continuous logic as presented in [3]. Let us recall the basic notions.

Definition 2.1. A *language* is a set $\mathfrak{L}$ of function symbols, constants and relation symbols with the following additional data:

- for each relation symbol R, an arity k_R, a bounded interval $I_R \subseteq \mathbb{R}$ and a modulus of continuity $\Delta_R : (0,1] \to (0,1]$;
- for each function symbol f, an arity k_f and a modulus of continuity $\Delta_f : (0,1] \to (0,1]$;
- a real constant $D_{\mathfrak{L}}$.

An $\mathfrak{L}$-*structure* is then a complete metric space (M, d) of diameter at most $D_{\mathfrak{L}}$ with, in addition,

- for each relation symbol R, a function $R^M : M^{k_R} \to I_R$ which is Δ_R-uniformly continuous;
- for each k-ary function symbol f, a function $f^M : M^{k_f} \to M$ which is Δ_f-uniformly continuous;
- for each constant symbol c, an element c^M of M.

From now on, we assume that $D_{\mathfrak{L}} = 1$ and that $I_R = [0,1]$ for all R. The general case poses no extra difficulty.

Definition 2.2. *Terms* of $\mathfrak{L}$ are built from the variables and constant symbols by composition with function symbols.

Formulas of $\mathfrak{L}$ are formed from the relations and function symbols of the language $\mathfrak{L}$, using terms, connectives – that is, continuous functions $[0,1]^{\mathbb{N}} \to [0,1]$ – and quantifiers $\sup_x$ and $\inf_x$, for any variable x. We allow formulas with countably many (free) variables.

If x is a tuple of n variables, then we will use the notation M^x for the Cartesian power $M^{|x|}$. Note that we allow for the possibility that x is a countable tuple $x = (x_0, x_1, x_2, \ldots)$. We treat M^x itself as a complete metric space. To be concrete, we define a metric by $d_{M^x}(u, v) := \sup 2^{-i} d(u_i, v_i)$ where u_i is the i^{th} co-ordinate of u. When x is a finite tuple, this metric is equivalent to any of the other natural choices, *e.g.*, $\|\cdot\|_\infty$ or $\|\cdot\|_2$, that one might prefer. In the case where x is infinite, we need to be careful to ensure that the projection maps $M^x \to M$ are uniformly continuous. Given an $\mathfrak{L}$-structure M, a formula with free variables x is interpreted in the natural way, yielding a uniformly continuous function $M^x \to [0,1]$.

Definition 2.3. Fix a language $\mathfrak{L}$. A *partial type* in variables x is a finitely consistent collection of conditions of the form $\phi(x) = 0$. A *type* is a maximal partial type. A closed partial type, that is a partial type in the empty tuple of variables, is called a *theory*.

Definition 2.4. A *definable predicate* on M^x is the interpretation of a formula $\phi(x)$ with free variables x. A subset $X \subseteq M^x$ is *definable* if $d(x, X) = \inf_{y \in X} d(x, y)$ is a definable predicate. A *definable function* is a function $f : M^x \to M$ for which $d(f(x), y)$ is a definable predicate on $M^{x,y}$.

Remark 2.5. In some presentations of continuous logic, a distinction is made between the definable predicates realized as the interpretations of formulas and uniform limits of such predicates. Since we permit x to be a possibly countably infinite tuple of variables and allow all continuous functions on $[0,1]^x$ as connectives, there is no distinction between these two classes of predicates. Indeed, by restricting to a subsequence, any uniform limit of predicates may be realized as a limit of predicates which are uniformly Cauchy for any given modulus of Cauchy uniformity. Let us fix one such, $\epsilon : \mathbb{N} \to (0,1]$. The set $C_\epsilon := \{a \in [0,1]^{\mathbb{N}} : (\forall n,m)|a_n - a_m| \le \epsilon(\min\{n,m\})\}$ is a closed subset of the normal space $[0,1]^{\mathbb{N}}$. The function $f : C_\epsilon \to [0,1]$ given by $f(a) := \lim_{n\to\infty} a_n$ is continuous and bounded. Thus, by the Tietze-Urysohn extension theorem it extends to a continuous function $F : [0,1]^{\mathbb{N}} \to [0,1]$. Given an ϵ-Cauchy sequence of sequence $(P_n(y))_{n=0}^{\infty}$ of definable predicates, we may compute $\lim_{n\to\infty} P_n(y)$ as the connective F applied to the sequence $(P_n)_{n=0}^{\infty}$.

Remark 2.6. An $\mathfrak{L}$-structure M in the sense of ordinary first-order logic may be regarded as a structure with respect to continuous logic by giving M the discrete metric defined by $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$ and interpreting each relation symbol R in the variables x as the uniformly continuous predicate $R : M^x \to \mathbb{R}$ defined as $R(a) = 0$ if $M \models R(a)$ and $R(a) = 1$ if $M \models \neg R(a)$ where we may take 1 as the modulus of uniform continuity. In this way, for any first-order formula ϑ in the free variables x we have a predicate $\vartheta : M^x \to \mathbb{R}$ also defined by $\vartheta(a) = 0$ if $M \models \vartheta(a)$ and $\vartheta(a) = 1$ otherwise. Because we may apply continuous connectives in order to produce new definable predicates, there are many other definable predicates even for this discrete structure. A dense set of such predicates may be obtained as follows. Let $\vartheta_1, \dots, \vartheta_n$ be a finite sequence formulas in the free variable x (which may be a tuple). Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be a sequence of real numbers of the same length. Then $\sum_{i=1}^{n} \alpha_i \vartheta_i$ is a definable predicate from M^x to $\mathbb{R}$.

2.2. **Interpretations in continuous logic.** We need to extend the theory of interpretations to continuous logic. For an account in the context of (possibly infinitary) discrete logics see [8, Chapter 5].

Definition 2.7. An interpretation of the $\mathfrak{L}_2$-structure N in the $\mathfrak{L}_1$-structure M is given by an $\mathfrak{L}_1$-definable set $\widetilde{N}$ in some M^x and a surjective function $I : \widetilde{N} \to N$ so that for every $\mathfrak{L}_2$-definable predicate $Q : N^y \to \mathbb{R}$ the composite $Q \circ I^y : \widetilde{N}^y \to \mathbb{R}$ is a $\mathfrak{L}_1$-definable predicate, by which we mean that it extends to an $\mathfrak{L}_1$-definable predicate on $(M^x)^y$.

In Definition 2.7 we wrote "I^y" for the function on the Cartesian power $\widetilde{N}^y$ of $\widetilde{N}$ indexed by the variables y. From now on we will simply write I for the natural extension of I to Cartesian powers and restrictions to subsets. Using these natural extensions, we may compose interpretations. If M_i is an $\mathfrak{L}_i$ structure for $i = 1$, 2, or 3 and we have interpretations $I : \widetilde{M_2} \to M_2$ and $J : \widetilde{M_3} \to M_3$ of M_2 in M_1 and of M_3 in M_2 respectively, where $\widetilde{M_2} \subseteq M_1^x$ and $\widetilde{M_3} \subseteq M_2^y$, then $J \circ I$, by which we mean $J \circ I^y$, is an interpretation of M_3 in M_1.

Remark 2.8. We discussed in Remark 2.6 how a first-order structure may be regarded as a structure for continuous logic. The structures interpretable in the usual sense of first-order logic are also interpretable in this extended sense. However, there are other structures which may have non-discrete metrics which we can interpret in these discrete structures. For example, the construction of hyperimaginaries in the sense of first-order logic becomes a simple interpretation in continuous logic.

As with interpretations in the context of discrete structures, it is only necessary to check that the basic structure pulls back to definable predicates.

Proposition 2.9. *We are given two languages $\mathfrak{L}_1$ and $\mathfrak{L}_2$, an $\mathfrak{L}_1$-structure M, an $\mathfrak{L}_2$-structure N, an $\mathfrak{L}_1$-definable set $\widetilde{N}$ in M^x, and a surjective function $I : \widetilde{N} \to N$. If I respects the basic structure on N in the sense that each of*

- *$d_N(I(x_1), I(x_2))$ for d_N the distinguished metric on N,*
- *$d_N(I(y), c)$ for c an $\mathfrak{L}_2$-constant symbol,*
- *$d_N(I(y), f(I(z)))$ for f an $\mathfrak{L}_2$-function symbol in the variables z, and*
- *$P(I(z))$ for P an $\mathfrak{L}_2$-predicate in the variable z*

is a definable predicate on the appropriate $\widetilde{N}^z$, then I is an interpretation of N in M.

This proposition follows by induction on the construction of continuous logic formulas as in the classical case. We leave the details to the reader.

As in the classical case, it makes sense to regard interpretations syntactically so that one has a notion an interpretation of one theory in another. That is, if we are given formulas whose interpretations give each of the predicates described in Proposition 2.9, for example, predicates $\varphi_{\widetilde{N}}(x)$ giving the distance to $\widetilde{N}$, $\varphi_{d_N}(x_1, x_2)$ giving the pullback of the metric of N to $\widetilde{N}$, *etc.*, so that for each $\mathfrak{L}_1$-structure M which a model of some given theory T_1, we obtain an $\mathfrak{L}_2$-structure N modeling a theory T_2 by taking the universe of N to be the zero set $\{a \in M^x : \varphi_{\widetilde{N}}(a) = 0\}$ modulo the equivalence relation $\{(a, b) \in \widetilde{N}^2 : \varphi_{d_N}(a, b) = 0\}$ with the metric given by φ_{d_N}, and the remaining structure described by the formulas corresponding to the bullet points of Proposition 2.9, then we would say that we have an interpretation of T_2 in T_1. We will abuse notation writing I for the interpretation given by this choice of formulas and then for any model M of T_1 we will write $I(M)$ for the model of T_2 given by these formulas.

To make the syntactic approach to interpretations more precise, we should talk about interpretable sets in continuous logic.

Definition 2.10. Let M be an $\mathfrak{L}$-structure, $X \subseteq M^x$ a definable set, and $\partial : X^2 \to \mathbb{R}_{\geq 0}$ a definable predicate giving a pseudometric on X. Define an equivalence relation E on X by $xEy :\Longleftrightarrow \partial(x, y) = 0$. The interpretable set $\mathfrak{X}^{\text{full}}$ has as its universe X/E, the set of E-equivalence classes, metric $\overline{\partial}$ induced by ∂, and for each $\mathfrak{L}$-definable predicate $P : X^y \to \mathbb{R}$ which is uniformly continuous with respect to ∂ (note that such a P is necessarily invariant with respect to E) an induced predicate $\overline{P} : (X/E)^y \to \mathbb{R}$.

Using the notion of interpretable sets in continuous logic, we have another characterization of interpretable structures.

Proposition 2.11. *If $I : \widetilde{N} \to N$ is an interpretation of the $\mathfrak{L}_2$-structure N in the $\mathfrak{L}_1$-structure M, then there is an interpretable set $\mathfrak{X}^{full}$ with structural metric $\overline{\partial}$ and an isometry $\overline{I} : \mathfrak{X}^{full} \to N$ so that for each $\mathfrak{L}_2$-definable predicate $P : N^y \to \mathbb{R}$, the pullback $P^{\overline{I}} := P \circ \overline{I} : (\mathfrak{X}^{full})^y \to \mathbb{R}$ is definable in $\mathfrak{X}^{full}$. Hence, every structure interpretable in M is isomorphic to a reduct of some interpretable set in M.*

The syntactic presentation of an interpretation gives rise through the proof of Proposition 2.9 to a translation $P \mapsto P^I$ from the interpreted language $\mathfrak{L}_2$ to the language $\mathfrak{L}_1$ having the property that for any $\mathfrak{L}_2$-predicate P we have $P^I = P \circ I$.

On the face of it, two interpretations given by equivalent (modulo T_1) formulas are the same in that the predicates are the same. However, there may be cases in which it pays to be more careful about the choice of the formulas. For example, if we can take the formulas to be simple enough, then an interpretation would restrict to give interpretations for substructures.

Proposition 2.12. *With the notation as in Proposition 2.9, if*

- *the predicates giving the distance to $\widetilde{N}$, the pullback of the distance on d_N, and the pullback of the distinguished predicates on N are quantifier-free definable,*
- *for each constant symbol c in $\mathfrak{L}_2$ there is a uniform limit of closed $\mathfrak{L}_1$-terms $\widetilde{c}$ so that $I(\widetilde{c}^M) = c^N$, and*
- *for each function symbol f with variable y in $\mathfrak{L}_2$ there is a uniform limit of $\mathfrak{L}_1$-terms $\widetilde{f}$ so that $I \circ \widetilde{f} = f \circ I$,*

then the restriction of I to any substructure of M in an interpretation of a substructure of N.

We will call the interpretations of Proposition 2.12 *quantifier-free interpretations.* Note that requirement that the basic function symbols and constant symbols be represented by terms is stronger than merely asking that the pullback of the distance to the graphs of the interpretations of the function symbols and to the interpretations of the constant symbols be quantifier-free definable predicates. The translations associated to quantifier-free interpretations preserve the quantifier complexity of predicates.

Definition 2.13. As with the classical theory, we say that two interpretations I and J of N in M are *homotopic* if the function $J^{-1} \circ I$ from the definable set $\widetilde{N}_I$, where $\widetilde{N}_I$ is the domain of I, to the interpretable set $\widetilde{N}_J/\sim$ where $\widetilde{N}_J$ is the domain of J and $\sim$ is the definable equivalence relation induced by $x \sim y :\Longleftrightarrow J(x) = J(y)$ is definable. In the special case that we have a pair of interpretations I of N in M and J of M in N, we say that this is a *bi-interpretation* if the composite $I \circ J$ is homotopic to the identity interpretation of M in itself and $J \circ I$ is homotopic to the identity interpretation of N in itself.

2.2.1. *Preservation of definability by interpretations.* An important consequence of a pair of interpretations forming a bi-interpretation is that the image of a definable set under a definable predicate is itself definable in the interpreted model. In fact, we only need half of the condition on being a bi-interpretation for this fact to hold.

Proposition 2.14. *Suppose that I is an interpretation of N in M and that J is an interpretation of M in N. Suppose moreover that $J \circ I$ is homotopic to the identity interpretation of M in M. If P is an $\mathfrak{L}_2$-predicate which is invariant under J, then the function $\overline{P}$ on M defined by $\overline{P}(x) := P(J^{-1}(x))$ is an $\mathfrak{L}_1$-definable predicate.*

Proof. Let $\alpha : (J \circ I)(M) \to M$ be the definable isomorphism witnessing that $J \circ I$ is homotopic to the identity self-interpretation of M. By [3, Proposition 9.23], α is a uniformly continuous function. Since I is an interpretation, $P^I := P \circ I$ is an $\mathfrak{L}_1$-definable predicate. Unwinding the definition of $\overline{P}$ we see the $\overline{P} = P^I \circ \alpha$ is also an $\mathfrak{L}_1$-definable predicate. □

As a corollary of Proposition 2.14 we have the following consequence on the images of definable sets.

Corollary 2.15. *Under the hypotheses of Proposition 2.14, if $X \subseteq \widetilde{M}^y$ is an $\mathfrak{L}_2$-definable set and the function $J : \widetilde{M} \to M$ is an isometry, then $J(X) \subseteq M^y$ is an $\mathfrak{L}_1$-definable set.*

Proof. Let $P(y) = d_N(y, X)$ be the $\mathfrak{L}_2$-definable predicate expressing the distance to X. Since J is an isometry, it is in particular injective so that P is invariant under J. By Proposition 2.14, $\overline{P} = P \circ J^{-1}$ is $\mathfrak{L}_1$-definable. Since J is an isometry, we have $d_M(z, J(X)) = d_N(J^{-1}(z), X) = P(J^{-1}(z)) = \overline{P}(z)$. That is, the distance to $J(X)$ is an $\mathfrak{L}_1$-definable predicate, showing that $J(X)$ is $\mathfrak{L}_1$-definable. □

Remark 2.16. Corollary 2.15 could be achieved under the weaker hypothesis that the pseudometric $d_M^J(x, y) := d_M(J(x), J(y))$ on $\widetilde{M}$ given by pulling back the structural metric of M is a metric equivalent to the restriction of the structural metric d_{N^y} of N^y.

2.2.2. *Preservation of existential closedness by interpretations.* A quantifier-free interpretation of one theory in another will transform existentially closed extensions to existentially closed extensions. Before we prove this proposition, let us recall what we mean by an existentially closed extension in continuous logic.

Definition 2.17. The $\mathfrak{L}$-structure M_1 is existentially closed in M_2, written $M_1 \preceq_\exists M_2$, if M_1 is a substructure of M_2 and for every pair of variables x and y, quantifier-free $\mathfrak{L}$ predicate $P(x, y)$ in the variables (x, y) taking values in $[0, \infty)$, and point $a \in M_1^y$, if $\inf_x P^{M_2}(x, a) = 0$, then $\inf_x P^{M_1}(x, a) = 0$.

We say that M_1 is existentially closed relative to the theory T if for every extension $M_2 \models T$ of M_1, we have $M_1 \preceq_\exists M_2$.

Proposition 2.18. *Let I be a quantifier-free interpretation of the $\mathfrak{L}_2$ theory T_2 in the $\mathfrak{L}_1$ theory T_1. If $M_1 \preceq_\exists M_2$ is an existentially closed extension of models of T_1, then $I(M_1) \preceq_\exists I(M_2)$.*

Proof. Let us write $\widetilde{N}$ for the definable set in the variables x giving the universe of the interpretation I. Let $Q(y, z)$ be a quantifier-free $\mathfrak{L}_2$-predicate and $a \in I(M_1)^z$ a point so that $\inf_y Q^{I(M_2)}(y, a) = 0$. Let $\widetilde{a} \in \widetilde{N}(M_1)$ with $I(\widetilde{a}) = a$. Then $\inf_{\widetilde{N}^y} (Q^I)^{M_2}(y, \widetilde{a}) = 0$ and Q^I is a quantifier-free predicate as I is quantifier-free. Note that we are using the fact [3, Theorem 9.17] that quantification over a definable set gives a definable predicate. Since $M_1 \preceq_\exists M_2$, we have $\inf_{\widetilde{N}^y} (Q^I)^{M_1}(y, \widetilde{a}) = 0$ as well. Unwinding the meaning of this equality, we have $\inf_{I(M_1)^y} Q^{I(M_1)}(y, a) = 0$, as required. □

It follows from Proposition 2.18 that a quantifier-free bi-interpretation between theories will take existentially closed models to existentially closed models. Let us express this result using a weaker condition than bi-interpretation.

Proposition 2.19. *Let I be a quantifier-free interpretation of the $\mathfrak{L}_2$ theory T_2 in the $\mathfrak{L}_1$ theory T_1 having the property that for every extension $N_1 \subseteq N_2$ of models of T_2 there is an extension $M_1 \subseteq M_2$ of models of T_1 and isomorphisms $\rho_i : I(M_i) \cong N_i$ for $i = 1$ and 2 fitting into the following commuting square.*

$$\begin{array}{ccc} I(M_2) & \xrightarrow{\rho_2} & N_2 \\ \uparrow & & \uparrow \\ I(M_1) & \xrightarrow{\rho_1} & N_1 \end{array}$$

If M is an existentially closed model with respect to T_1, then $I(M)$ is existentially closed with respect to T_2.

Proof. Let M be an existentially closed model with respect to T_1. Since I interprets T_2 in T_1, $I(M) \vDash T_2$. Let $I(M) \subseteq N'$ be an extension of models of T_2. By our hypothesis, possibly after applying an isomorphism, we may find an extension $M \subseteq M'$ of models of T_1 so that the extension $I(M) \subseteq N'$ is the extension $I(M) \subseteq I(M')$ where the inclusion is the extension is obtained from the extension $M \subseteq M'$ via the functoriality of I. Since M is existentially closed in the class of models of T_1, $M \preceq_\exists M'$. By Proposition 2.18, $I(M) \preceq_\exists I(M') = N'$. Hence, $I(M)$ is existentially closed with respect to T_2 □

Corollary 2.20. *If I is half of a quantifier-free bi-interpretation of the the theory T_2 in the theory T_1 and M is existentially closed with respect to T_1, then $I(M)$ is existentially closed with respect to T_2.*

Proof. Let J be the other half of the bi-interpretation. Then any extension $N_1 \subseteq N_2$ of models of T_2 is isomorphic to $I(J(N_1)) \subseteq I(J(N_2))$. Hence, Proposition 2.19 applies. □

Sometimes we may check existential closedness relative to a weaker theory by considering only existential closedness relative to a stronger one. The following lemma will be used in Section 5.2.

Lemma 2.21. *If $T \subseteq T'$ are $\mathfrak{L}$ theories having the property that for any model $M \vDash T$ of T there is an extension $M \subseteq N \vDash T'$ to a model of T', then a model $M \vDash T'$ is existentially closed with respect to T if and only if it is existentially closed relative to T'.*

Proof. This is proven just as it would be in the first-order case. □

2.3. **Types in continuous logic.** We modify the notion of type spaces from [3, Section 3] slightly. Recall Definition 2.10 of an interpretable set and its induced predicates.

Definition 2.22. Let M be an $\mathfrak{L}$-structure and X be an interpretable set in M.

We write S_X for the set of types in X, by which we mean complete types in the variable x ranging over X. We give S_X the weakest topology so that for each φ the function $S_X \to \mathbb{R}$ defined by $p \mapsto \varphi^p := \inf_r \phi(x) \dot{-} r \in p$ is continuous. If $A \subseteq M$ is a subset of M, then $S_X(A)$ is the type space S_X relative to the $\mathfrak{L}_A$-structure M_A where the new constant symbols $a \in A$ are interpreted by $a^{M_A} := a$.

Since each formula $\varphi(x)$ is constrained to take values in a compact interval, I_φ, evaluation at all of the formulas in the variable x realizes S_X as a closed, and hence, compact subset of the product $\prod_\varphi I_\varphi$ of compact intervals over the set of all such formulas. Thus, S_X is itself a compact Hausdorff space.

Since, as discussed in Remark 2.5, we allow for a process of taking uniform limits as a connective, using [3, Proposition 3.4], we see that the space $C(S_X, \mathbb{R})$ of continuous real-valued functions on the type space S_X may be identified with the set of $\mathfrak{L}$ formulas on X via evaluation. This observation yields the following proposition about maps on type spaces induced by interpretations.

Proposition 2.23. *If $I : \widetilde{N} \to N$ is an interpretation of the $\mathfrak{L}_2$-structure N in the $\mathfrak{L}_1$-structure, then I induces a continuous map $I_* : S_{\widetilde{N}} \to S_N$.*

Proof. For $p \in S_{\widetilde{N}}$, we define $I_*(p)$ by specifying $\varphi^{I_*(p)} := (\varphi^I)^p$ for each φ an $\mathfrak{L}_2$-formula in the variable y ranging over N. Continuity of I_* is an immediate consequence of the fact that precomposition of I with a definable predicate is itself a definable predicate: the basic open subsets of S_N take the form $[r < \varphi < s] := \{p \in S_N : r < \varphi^p < s\}$ for φ an $\mathfrak{L}_2$-formula in the variable y and $r < s$ real numbers. The preimage of this set under I_* is $[r < \varphi^I < s]$ which is open in $S_{\widetilde{N}}$. □

We would like to upgrade Proposition 2.23 to the assertion that if I is half of a bi-interpretation, then I_* is a homeomorphism. Such an assertion is not literally true, but it becomes so once we replace the domains of the interpretations with their associated interpretable sets.

Proposition 2.24. *Let $I : \widetilde{N} \to N$ and $J : \widetilde{M} \to M$ be a pair of interpretations forming a bi-interpretation. Let $\mathfrak{N}$ be the M interpretable set obtained as the quotient of $\widetilde{N}$ by the equivalence relation coming from the pseudometric d_N^I and let $\overline{I} : \mathfrak{N} \to N$ be the induced map. Then $\overline{I}_* : S_{\mathfrak{N}} \to S_N$ is a homeomorphism.*

Proof. We already know that $\overline{I}_*$ is continuous. Let us check that it is open. Let $P : \mathfrak{N} \to \mathbb{R}$ be a definable predicate and let $r < s$ be real numbers. Since $\overline{I}$ is an isometry, P is $\overline{I}$ invariant. Thus, by Proposition 2.14, $P^{\overline{I}^{-1}} := P \circ \overline{I}^{-1} : N \to \mathbb{R}$ is a definable predicate. From the definition of $\overline{I}_*$, we see that $\overline{I}_*([r < P < s]) = [r < P^{\overline{I}} < s]$, implying that $\overline{I}_*$ is an open mapping, and, hence, a homeomorphism. □

Under an interpretation which is an isometry, the pullback of a definable set is a definable set. It follows that in Proposition 2.24 for any definable set $X \subseteq N^y$, there is a definable set $\widetilde{X} := \overline{I}^{-1} X$ for which $\overline{I}$ induces a homeomorphism $S_{\widetilde{X}} \to S_X$. We may go further. If $\partial : X \times X \to \mathbb{R}_{\geq 0}$ is an N definable pseudometric on X, then $\partial^{\overline{I}}$ is an M definable pseudometric on $\widetilde{X}$ and $\overline{I}$ induces a map between the interpretable structures $\overline{\widetilde{X}}$ and $\overline{X}$ obtained by quotienting by the induced equivalence relations. The induced map on the type spaces $S_{\overline{\widetilde{X}}} \to S_{\overline{X}}$ is a homeomorphism.

3. Metric valued fields

Definition 3.1. Let $\mathfrak{L}_{\mathrm{rg}}$ denote the bounded continuous ring language with binary function symbols $+$ and $\cdot$, unary function symbol $-$ and constants $0, 1$.

The distance takes values in $[0,1]$ and all three symbols come with the modulus of continuity $x \mapsto 2x$.

Definition 3.2. Let (K, v) be a valued field which admits a rank one coarsening $v_0 : K \to (\mathbb{R}, +)$. We consider the $\mathfrak{L}_{\mathrm{rg}}$-structure $(\mathcal{O}(K), |x-y|, +, \cdot, -, 0, 1)$ where $|x| = e^{-v_0(x)}$.

Lemma 3.3. *The class* MVF *of all such structures is elementary. It is axiomatized by:*

- *the (universal) theory of ultrametric domains;*
- *the axiom* $\sup_{x,y} \inf_z \min(xz - y, yz - x) = 0$.

Proof. Any structure as above is a model of that theory. Conversely, any model R of that theory is a valuation ring and the open unit ball for $|\cdot|$ is an ideal, *i.e.* $|\cdot|$ is a (necessarily rank one) coarsening of the valuation associated to R. □

Remark 3.4. Let (K,v) be a valued field. For every $\gamma \in vK$, let $\gamma\mathcal{O}$ denote $\{x : v(x) \geq \gamma\}$. Fix some $\gamma \in vK^\times$ and consider the ring $\mathcal{O}_\gamma := \varprojlim_n \mathcal{O}/\gamma^n\mathcal{O}$ which is naturally isomorphic to the residue field for the coarsening of v by the smallest convex subgroup containing γ.

Then, after an choice of normalization, $\mathcal{O}_\gamma$ is a model of MVF and every model of MVF is obtained this way. So MVF is the theory of a continuously interpretable structure in the discrete theory of valued fields.

Definition 3.5. Let ACMVF be the $\mathfrak{L}_{\mathrm{rg}}$-theory of algebraically non-discrete metric valued fields which consists of MVF and the axioms:

- $\inf_x \max\{|x| \dot{-} 1/2, 1/2 \dot{-} |x|\} = 0$,
- $\sup_x \inf_y |P(x,y)| = 0$, for any finite tuple of variables x, any variable y and any polynomial $P \in \mathbb{Z}[x,y]$ monic (non-constant) in the variable y.

Fix p a prime. We write $\mathrm{ACMVF}_{0,p}$ for the $\mathfrak{L}_{\mathrm{rg}}$ theory of algebraically closed non-discrete metric valued fields of characteristic zero with residue field of characteristic zero and ACMVF_p for the $\mathfrak{L}_{\mathrm{rg}}$ theory of algebraically closed non-discrete metric valued fields of characteristic p. For fixed $\alpha \in (0,1)$ we let $\mathrm{PERF}_{|p|=\alpha}$ be the theory of (valuation rings of) mixed characteristic $(0,p)$ perfectoid fields which consists of MVF and the axioms:

- $|p| = \alpha$,
- $\inf_x \max\{|x| \dot{-} \alpha^{1/p}, \alpha^{1/p} \dot{-} |x|\} = 0$,
- $\sup_x \inf_y \inf_z |x - y^p - pz| = 0$.

Note that our axioms for mixed characteristic perfectoid fields depend on a choice of normalization.

The theory $\mathrm{PERF}_{|p|=0}$ is the theory of (valuation rings of) characteristic p perfectoid fields which consists of MVF and the axioms:

- $|p| = 0$,
- $\inf_x \max\{|x| \dot{-} 1/2, 1/2 \dot{-} |x|\} = 0$,
- $\sup_x \inf_y |x - y^p| = 0$.

Note that all models of $\mathrm{PERF}_{|p|=0}$ are perfect.

Let $\mathrm{D}(x,y)$ be the predicate $\inf_{z\in\mathcal{O}} |y - xz|$. Note that for every $x, y \in \mathcal{O}$,

$$|x - y\mathcal{O}^\times| = \begin{cases} 0 & \text{if } v(x) = v(y) \\ \max(|x|,|y|) & \text{otherwise} \end{cases} = \max(D(x,y), D(y,x)).$$

This is a pseudo metric whose associated imaginary is $v\mathcal{O}$ — with some non discrete norm. The induced topology is discrete at every point except for $v(0)$. A basis of neighborhoods of $v(0)$ is given by upward closed sets.

Also, this shows that $\mathcal{O}^\times$ is definable, and since $|x - \mathfrak{m}| = 1 \dot{-} d(x, \mathcal{O}^\times)$, so is $\mathfrak{m}$. The pseudo norm $|x - y - \mathfrak{m}|$ is thus also a predicate and the associated imaginary is the residue field Kv with the discrete norm. Similarly, for any non-zero $\varpi \in \mathcal{O}$, the set $\varpi\mathcal{O}$ is definable and the pseudo-norm $|x - y - \varpi\mathcal{O}|$ gives rise to the imaginary $\mathcal{O}/(\varpi)$ with the norm induced by $|\cdot|$ — the associated topology is trivial.

Definition 3.6. Let $\mathfrak{L}_{\mathrm{D}}$ denote $\mathfrak{L}_{\mathrm{rg}} \cup \{\mathrm{D}\}$.

Proposition 3.7. *The theory* ACMVF *eliminates quantifiers in* $\mathfrak{L}_{\mathrm{D}}$ *and is the model completion of* MVF.

Let $\mathfrak{L}_{\mathrm{div}}$ denote the discrete language of rings with a divisibility predicate interpreted in valued fields as $v(x) \le v(y)$. Recall the (discrete) $\mathfrak{L}_{\mathrm{div}}$-theory ACVF of non-trivially valued algebraically closed fields eliminates quantifiers.

Proof. First note that in any $\aleph_0$-saturated $K \vDash$ ACMVF, we have $|\mathcal{O}| = [0,1]$. Indeed, if $a \in \mathcal{O}$ is such that $|a| \in (0,1)$, then $\mathbb{Q} \cdot |a| \subseteq |\mathcal{O}|$ is dense in $[0,1]$.

Claim 3.8. *Let $M \vDash$ ACMVF be κ-saturated (with induced valuation v). There exists a κ-saturated $N \vDash$ ACVF containing (M,v) such that $\mathcal{O}(M)$ is a section of the map $\mathcal{O}(N) \to \mathcal{O}(N)/I_M$ where $I_M := \{x \in \mathcal{O}(N) : v(x) > v(M)\}$.*

Proof. Let K be the fraction field of $\mathcal{O}(M)$, let Γ be a κ-saturated divisible ordered abelian group and let $N := K((\Gamma))$ with the valuation $w(\sum_{i \ge i_0} a_i t^i) = (v(a_{i_0}), i_0) \in v(K) \times \Gamma$ if $a_{i_0} \neq 0$. One can check that $w(N)$ is a κ-saturated divisible ordered abelian group. The residue field of (N,w) is isomorphic to the residue field of (K,v) which is κ-saturated. Since N is maximally complete, N is κ-saturated. □

Let now $M, N \vDash$ ACMVF, let $A \le M$ and $f : A \to N$ be an $\mathfrak{L}_{\mathrm{D}}$-embedding. Assume that N is $|M|^+$-saturated. Let $N_1 \supseteq N$ be as in the claim. If $|\cdot|$ is discrete on A, let $c \in M$ be such that $|c| \in (0,1)$ and $d \in N$ such that $|d| = |c|$ — note that f then extends to an $\mathfrak{L}_{\mathrm{div}}$-embedding sending $v(c)$ to $v(d)$. Since $D(x,y) = 0$ defines the usual divisibility predicate of ACVF, by quantifier elimination, f extends to an $\mathfrak{L}_{\mathrm{div}}$-embedding $g : M \to N_1$ — sending $v(c)$ to $v(d)$ if A is discrete. Since $g(M) \cap I_N = g(M \cap I_M) = \{0\}$, this embedding induces an $\mathfrak{L}_{\mathrm{D}}$-embedding $h : M \to N$ extending f. Quantifier elimination (and the rest of the statement) follows. □

Remark 3.9. A similar reduction shows quantifier elimination for ACMVF in the three sorted language where we add the value group and residue field.

Let us conclude this section by noting the relationship between the discrete structure and the continuous structure of a perfectoid field. This relies on a deep result of Jahnke and Kartas [10].

Fix $\alpha < 1$ and let $M \vDash \mathrm{PERF}_{|p|=\alpha}$ with associated valuation v. Let us assume that v is henselian and $v(M)$ has bounded regular rank. An ordered group has bounded regular rank if all definable convex subgroups (in elementary extensions) are $\varnothing$-definable — these groups are also known as groups with finite spines. Note that if v is a rank one valuation, *i.e.* coincides with $-\log|\cdot|$, then both hypotheses are verified.

Lemma 3.10. *Let $\Gamma \preccurlyeq \Gamma'$ be an elementary extension of bounded regular rank groups (in the discrete language of ordered groups). Let Δ be the convex hull of Γ in Γ'. Then Γ'/Δ is divisible.*

Proof. Otherwise, if $\gamma \notin \Delta + n\Gamma$, then the largest convex subgroup H such that $\gamma \notin H + n\Gamma$ is definable in Γ' (see [4, Lemma 2.1]) but it cannot be $\varnothing$-definable as $\Gamma \le \Delta \le H < \Gamma'$. □

Let $\mathfrak{U}$ be a non-principal ultrafilter. Let $M^{\mathfrak{U}}$ denote the discrete ultrapower of $\mathfrak{L}_{\mathrm{div}}$-structure and let $M_0^{\mathfrak{U}}$ denote the continuous ultrapower of $\mathfrak{L}_{\mathrm{rg}}$-structures. If $\varpi \in \mathcal{O}(M) \smallsetminus \{0\}$ is topologically nilpotent, then $M_0^{\mathfrak{U}}$ is isomorphic to the residue field of $\mathcal{O}(M^{\mathfrak{U}})[1/\varpi]$, whose valuation we denote v_0. Also, $v(M_0^{\mathfrak{U}})$ is the convex hull of $v(M)$ inside $v(M^{\mathfrak{U}})$.

By [10, Theorem 4.3.1], $(M^{\mathfrak{U}}, v_0)$ is henselian with perfect residue field. So we can find a lift $f : M_0^{\mathfrak{U}} \to M^{\mathfrak{U}}$. It is also an embedding of valued fields.

Proposition 3.11. *The* $\mathfrak{L}_{\mathrm{div}}$*-embedding* $f:(M_0^{\mathfrak{U}},v)\to(M^{\mathfrak{U}},v)$ *is elementary.*

Proof. This follows immediately from [10, Theorem 5.1.4]. Note that, looking at the proof of [10, Theorem 5.1.2], the assumption that $v(M_0^{\mathfrak{U}})\preccurlyeq v(M^{\mathfrak{U}})$ can be lifted given Lemma 3.10. □

Corollary 3.12. *Let* $M,N\vDash \mathrm{PERF}_{|p|=\alpha}$ *whose valuation is henselian and whose value group has bounded regular rank.*

(1) *If* M *and* N *are* $\mathfrak{L}_{\mathrm{rg}}$*-elementarily equivalent then they are* $\mathfrak{L}_{\mathrm{div}}$*-elementarily equivalent.*
(2) *Let* $\varpi_M\in\mathcal{O}(M)$ *(resp.* $\varpi_N\in\mathcal{O}(N)$*) be such that* $0<|\varpi_M|=|\varpi_N|<1$*. If* (M,ϖ_M) *and* (N,ϖ_N) *are* $\mathfrak{L}_{\mathrm{div}}$*-elementarily equivalent, they are also* $\mathfrak{L}_{\mathrm{rg}}$*-elementarily equivalent.*

Proof. Let us first assume that M and N are equivalent as $\mathfrak{L}_{\mathrm{rg}}$-structure. Then for some non-principal ultrafilter $\mathfrak{U}$, by the Keisler-Shelah theorem, the $\mathfrak{L}_{\mathrm{rg}}$-ultrapowers $M_0^{\mathfrak{U}}$ and $N_0^{\mathfrak{U}}$ are isomorphic. By Proposition 3.11, the $\mathfrak{L}_{\mathrm{div}}$-ultrapowers $M^{\mathfrak{U}}$ and $N^{\mathfrak{U}}$ are elementarily equivalent, and hence so are the $\mathfrak{L}_{\mathrm{div}}$-structures M and N.

The second item follows from the fact that, as noticed above, the $\mathfrak{L}_{\mathrm{rg}}$-structure is continuously interpretable in the $\mathfrak{L}_{\mathrm{div}}$-structure (with parameter ϖ). □

Remark 3.13. (1) Under the hypotheses of Lemma 3.10, we can prove that $\Delta\leq\Gamma'$ is elementary. Corollary 3.12 would hold for any class of value groups with this property. However, we do not know if this holds beyond bounded regular rank.
(2) It is not clear either if assuming that the valuation is henselian is necessary in Corollary 3.12.

4. Bi-interpretations

If R is a ring, let $R^\flat=\varprojlim_{x\mapsto x^p}R$. A priori it is only a multiplicative monoid, unless R has characteristic p, in which case it is a ring.

Lemma 4.1. *Let* $M\vDash\mathrm{PERF}_{|p|=\alpha}$ *for some* $\alpha<1$*. Let* $\varpi\in\mathcal{O}(M)\smallsetminus\{0\}$ *be topologically nilpotent divisor of* p*. The natural map* $f:\mathcal{O}(M)^\flat\to(\mathcal{O}(M)/(\varpi))^\flat$ *is a bijection. The inverse map is given by* $g:x\mapsto(\lim_j\tilde{x}_{i+j}^{p^j})_i$*, where* $\tilde{x}_i$ *reduces to* x_i *mod* ϖ*.*

In particular, $\mathcal{O}(M)^\flat$ can be made into a ring.

Proof. The crucial fact that is the following: if $a,b\in\mathcal{O}(M)$ are congruent modulo ϖ^i, then $a^p\equiv b^p\mod\varpi^{i+1}$. It follows that if $a\equiv b\mod\varpi$, then $a^{p^i}\equiv b^{p^i}$ mod ϖ^{i+1}.

Now consider some $x=(x_i)_i\in(\mathcal{O}(M)/(\varpi))^\flat$, since $x_{i+j}^{p^j}=x_i$, the sequence $\tilde{x}_{i+j}^{p^j}$ is Cauchy and its limit $y_i\in\mathcal{O}(M)$ does not depend on the choice of $\tilde{x}_i$. Note also that $(\lim_j\tilde{x}_{i+j+1}^{p^j})^p=\lim_j\tilde{x}_{i+j+1}^{p^{j+1}}=\lim_j\tilde{x}_{i+j}^{p^j}$ so $(y_i)_{i\in\mathbb{N}}$ is indeed an elements of $\mathcal{O}(M)^\flat$. Since $y_i\equiv x_i\mod\varpi$, we have $f\circ g=\mathrm{id}$. Finally, for every $x\in\mathcal{O}(M)^\flat$, we have $\lim_j x_{i+j}^{p^j}=x_i$ and hence $g\circ f=\mathrm{id}$. □

Let $M\vDash\mathrm{PERF}_{|p|=\alpha}$, where $0<\alpha<1$. The projection $\mathrm{res}_p:\mathcal{O}(M)\to\mathcal{O}(M)/(p)$ is an interpretation of the discrete ring $\mathcal{O}(M)/(p)$ in M. Our goal now is to axiomatize

$\mathcal{O}(M)/(p)$. Let TVR (truncated valuation ring) be the first other theory in the language $\mathfrak{L}_{\mathrm{div}}$ of (discrete) rings with a divisibility predicate $x\,|\,y$ axiomatized by:

- The theory of rings;
- for all x, y, we have $x\,|\,y$ if and only if x divides y;
- for all x and y, either $x\,|\,y$ or $y\,|\,x$;
- for all x and y, if $x \neq 0$ and $x = yx$ then y is a unit.

Let $R \vDash \mathrm{TVR}$, we can consider the map $v : a \mapsto (a)$ into the the set Γ of principal ideals of R ordered by reverse inclusion — so $v(x) \leq v(y)$ if and only if $x\,|\,y$. It is a linearly ordered commutative monoid with respect to multiplication of principal ideals (that we denote $+$). Its minimal element is the neutral element $0 = v(1)$. The maximal element is $\infty = v(0)$, we have $\gamma + \infty = \infty$ for all $\gamma \in \Gamma$ and for every $x \in R$, if $v(x) = \infty$ then $x = 0$. Moreover, for every $\gamma, \delta \in R$, if $\gamma \leq \delta \neq \infty$ then there exists a unique $\epsilon \in \Gamma$ such that $\gamma + \epsilon = \delta$ — we also write $\epsilon = \delta - \gamma$. Indeed, if $\delta = (y) \subseteq (x) = \gamma$, there exists $z \in R$ such that $y = xz$; and if, for some $z_1, z_2 \in R$, we have $xz_1 = xz_2 \neq 0$, then we may assume $z_2 = z_1 a$ for some a and thus $xz_1 = xz_1 a$. So $a \in R^\times$ and $v(z_1) = v(z_2)$. Finally, for every $x, y \in R$, we have $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \max\{v(x), v(y)\}$. We say that v is a truncated valuation on R.

Fix some $\alpha \in (0,1)$. If $\varpi \in R \setminus \{0\}$ be topologically nilpotent, then v induces a unique ultrametric norm $|\cdot| : \Gamma \to (0,1)$ such that $|\varpi| = \alpha$.

Lemma 4.2. *Then norm $|\cdot|$ is continuously quantifier-free ϖ-definable in R.*

Proof. We present the proof under the further assumption that $p^{-n}v(\varpi)$ exists for all $n \geq 0$, as this will simplify the notation and is the case we will use afterwards. Fix some $n \geq 1$. For every $x \in R$ and $1 \leq m < p^{2n}$, if $m/(p)^n \cdot v(\varpi) \leq v(x) < (m+1)/(p)^n \cdot v(\varpi)$, then $\alpha^{(m+1)/(p)^n} \leq |x| \leq \alpha^{m/(p)^n}$. In that case, set $t_n(x) = \alpha^{m/(p)^n}$ — if $v(x) \geq p^n \cdot v(\varpi)$, set $t_n(x) = 0$. Then the t_n are quantifier-free ϖ-definable as they take finitely many values and the fibers are quantifier-free definable. Moreover, they uniformly converge to $|\cdot|$. □

Let the theory TPERF_p be the $\mathfrak{L}_{\mathrm{div}}$-theory containing TVR and :

- $p = 0$;
- $\phi : x \mapsto x^p$ is surjective;
- there exists $\varpi \neq 0$ such that $\varpi^p = 0$ and for all x, if $x^p = 0$, then $\varpi\,|\,x$ — we say that ϖ is a pseudo uniformizer.

Remark 4.3. If $M \vDash \mathrm{PERF}_{|p|=\alpha}$, with $0 < \alpha < 1$ and $\varpi \in \mathcal{O}(M) \setminus \{0\}$ is a topologically nilpotent divisor of p, then $\mathcal{O}(M)/(\varpi) \vDash \mathrm{TPERF}_p$.

Let $\mathfrak{L}_{\mathrm{div},\varpi} = \mathfrak{L}_{\mathrm{div}} \cup \{\varpi_i : i \geq 1\}$ and let $\mathrm{TPERF}_{p,\varpi}$ be the $\mathfrak{L}_{\mathrm{div},\varpi}$-theory

$$\mathrm{TPERF}_p \cup \{\varpi_1 \text{ is a pseudo uniformizer}, \varpi_{i+1}^p = \varpi_i : i \geq 1\}.$$

Proposition 4.4. *Let $R \vDash \mathrm{TPERF}_{p,\varpi}$.*

(1) *Let $v^\flat : R^\flat \to \Gamma^\flat = \varprojlim_{x \mapsto px} \Gamma$ the map induced by v. Then $(R^\flat, v^\flat)$ is a characteristic p perfect valued ring.*
(2) *The map*

$$\begin{array}{rccc} \iota : & \Gamma & \to & \Gamma^\flat \\ & \gamma \neq \infty & \mapsto & (p^{-i}\gamma)_i \\ & \infty & \mapsto & \infty \end{array}$$

is strictly increasing. The image of $\Gamma \smallsetminus \{\infty\}$ *by* ι *is the initial segment* $< pv(\varpi_1)$ *of* $\Gamma^\flat$. *Let* $\varpi^\flat \in R^\flat$ *is the element* $(0, \varpi_1, \varpi_2, \ldots)$. *Using this identification, then* $v^\flat(\varpi^\flat) = pv(\varpi_1)$.

(3) *The projection on the first co-ordinate* $\natural : R^\flat \to R$ *induces an isomorphism* $R^\flat/(\varpi^\flat) \cong R$.

(4) $R^\flat$ *is a* $\varpi^\flat$*-adically complete and separated valuation ring.*

Proof. (1) It is routine to check that $v^\flat$ is a truncated valuation. To show that it is a valuation, it suffices to check that if $\gamma, \delta \in \Gamma^\flat \smallsetminus \{\infty\}$, then $\gamma + \delta \neq \infty$ — in that case, $\Gamma^\flat \smallsetminus \{\infty\}$ is the positive part of an ordered abelian group and hence $v^\flat$ is a valuation. Note first that since $\gamma = (\gamma_i)_{i \geq 0} \neq \infty$, we have $(p\gamma)_{i+1} = \gamma_i$ and hence $p\gamma \neq \infty$. As we may assume that $\delta \leq \gamma$, it follows that $\gamma + \delta \leq 2\gamma \leq p\gamma < \infty$. So $(R^\flat, v^\flat)$ is a valued ring and it is perfect of characteristic p by construction.

(2) For every $\gamma \leq \delta \in \Gamma$, if $p\gamma = p\delta \neq \infty$, then $p(\delta - \gamma) = 0$ and hence $\delta = \gamma$. It follows that ι is well-defined. It is strictly increasing. The set $\iota(\Gamma \smallsetminus \{\infty\})$ is the set of $\gamma = (\gamma_i)_{i \geq 0} \in \Gamma^\flat$ such that $\gamma_0 \neq \infty$, it is an initial segment of $\Gamma^\flat$. Moreover, if $\gamma < pv(\varpi_1)$, then $\gamma_1 < v(\varpi_1)$ and hence $\gamma_0 \neq \infty$.

With this identification, we have $p^{-1} v^\flat(\varpi^\flat) = v^\flat(\varpi^\flat)_1 = v(\varpi_1) < \infty$.

(3) We have $(\varpi^\flat)^\natural = 0$. Conversely, if $x = (x_i)_{i \geq 0} \in R^\flat \smallsetminus \{0\}$ is such that $x_0 = 0$, then, for some $i > 0$, $x_i \neq 0$. Since $x_i^{p^i} = x_0 = 0$, it follows that $v(\varpi_1) \leq p^{i-1} v(x_i)$. So $v^\flat(\varpi^\flat) = pv(\varpi_1) \leq p^i v(x_i) = v^\flat(x)$ and the kernel of $\natural$ is indeed (ϖ).

Now, the isomorphism $R^\flat/(\varpi) \cong R^\flat/(\varpi^{p^n})$ given by $x \mapsto x^{p^n}$ induces an isomorphism $R^\flat \cong \varprojlim_{x \mapsto x^p} R^\flat/(\varpi) \cong \varprojlim R^\flat/(\varpi^n)$. Since the left to right map can be checked to be the natural inclusion into the ϖ-adic completion, it follows that $R^\flat$ is ϖ-adically complete.

(4) Let $x = (x_i)_{i \geq 0}, y = (y_i)_{i \geq 0} \in R^\flat$ be such that $v^\flat(x) \leq v^\flat(y)$. Taking p-th roots, we may assume that $v^\flat(y) < v^\flat(\varpi^\flat)$. Then, for all $i \geq 0$, we have $v(x_i) \leq v(y_i) < \infty$ so we find some $a_{i,0}$ such that $y_i = x_i a_{i,0}$. Choose some $a_i \in R^\flat$ whose first co-ordinate is $a_{i,0}$. Then $y^{p^{-i}} \equiv x^{p^{-i}} a_i \mod \varpi^\flat$. It follows that $v^\flat(a_{i+1}^{p^{i+1}} - a_i^{p^i}) \geq p^i v^\flat(\varpi^\flat) - v^\flat(x) \geq p^{i-1} v^\flat(\varpi^\flat)$. So a_i is Cauchy and let $a = \lim a_i$. Then $y = \lim_i x a_i^{p^i} = xa$. □

Let $\alpha \in (0,1)$ and $R \models \mathrm{TPERF}_{p,\varpi}$. Let $|\cdot|_\alpha^\flat : R^\flat \to [0,1]$ be the coarsening of $v^\flat$ normalized so that $|\varpi^\flat|_\alpha^\flat = \alpha$. Then $R_\alpha^\flat = (R^\flat, |\cdot|_\alpha^\flat, \varpi^\flat)$ is a model of the $\mathfrak{L}_{\mathrm{D}} \cup \{\varpi\}$-theory $\mathrm{PERF}_{|p|=0,|\varpi|=\alpha}$.

Let $N \models \mathrm{PERF}_{|p|=0,|\varpi|=\alpha}$. The $\mathfrak{L}_{\mathrm{div},\varpi}$-structure $N^\natural = (\mathcal{O}(N)/(\varpi), \mathrm{res}_\varpi(\varpi_i) : i \geq 1)$ is a model of $\mathrm{TPERF}_{p,\varpi}$.

Theorem 4.5. *Fix* $\alpha \in (0,1)$. *Let* $R \models \mathrm{TPERF}_{p,\varpi}$ *and* $N \models \mathrm{PERF}_{|p|=0,|\varpi|=\alpha}$.

(1) *The set* $\Omega(R) := \{x \in R^{\mathbb{N}} : \forall i \in \mathbb{N},\ x_{i+1}^p = x_i\}$ *is quantifier-free (continuously) definable in* R.

(2) *The map* $\mathrm{id}_\Omega : \Omega(R) \to R^\flat$ *is a quantifier-free interpretation of* $R_\alpha^\flat$ *in* R.

(3) *The map* $\mathrm{res}_\varpi : \mathcal{O}(N) \to \mathcal{O}(N)/(\varpi)$ *is a quantifier-free interpretation of* $N^\natural$ *in* N.

(4) *The maps* id_Ω *and* res_ϖ *form a quantifier-free bi-interpretation between* $\mathrm{TPERF}_{p,\varpi}$ *and* $\mathrm{PERF}_{|p|=0,|\varpi|=\alpha}$.

Proof. (1) The predicate $d_n(x) = \inf_y \sup_{i\leq n} |x_i - y^{p^{n-i}}|2^{-i}$ on $R^{\mathbb{N}}$ uniformly converges to $|x - \Omega|$ — indeed, the error is at most 2^{-n-1}. So Ω is continuously quantifier-free definable.

(2) Let $|\cdot|_\alpha : R \to [0,1]$ be the norm such that $|\varpi_1|_\alpha = \alpha^{1/(p)}$ and for every $x, y \in R$ let

$$D_\alpha(x,y) = \begin{cases} 0 & \text{if } x \,|\, y \\ |y|_\alpha & \text{otherwise.} \end{cases}$$

This is a (continuous) quantifier-free predicate in R. Let $D^\flat$ denote the interpretation of D in $R^\flat_\alpha$. Then for every $x, y \in R^\flat$, we have $D^\flat(x,y) = \lim_i D_\alpha(x_i, y_i)^{p^i}$ which is quantifier-free definable. Since $|x|^\flat_\alpha = D^\flat(0,x)$, the first item is proved.

(3) For every $x, y \in \mathcal{O}(N)$, we have $\mathrm{res}_\varpi(x) \,|\, \mathrm{res}_\varpi(y)$ if and only if

$$\min\{\mathrm{D}(\varpi, x), \mathrm{D}(x,y)\} = 0,$$

and otherwise, we have $\min\{\mathrm{D}(x,y), \mathrm{D}(\varpi,x)\} = |x|$. So $|$ is induced by a quantifier-free definable predicate and the second item follows.

(4) The double interpretation of R in itself has domain $\Omega(R)$ and the projection on the first co-ordinate $\natural : \Omega(R) \to R$, which is a term, induces an isomorphism of $R^\flat/(\varpi)$ with R.

The domain of the double interpretation of N in itself is $X = \{x \in \mathcal{O}^{\mathbb{N}} : x_{i+1}^p \equiv x_i \mod \varpi\}$. By Lemma 4.1, the term $x \mapsto \lim_i x_i^{p^i}$ induces an isomorphism $(\mathcal{O}(N)/(\varpi))^\flat \to \mathcal{O}(N)^\flat \to \mathcal{O}(N)$. ☐

Recall that if R is a ring, then $\mathrm{W}_n(R)$ is a ring with underlying set R^n where addition, subtraction and multiplication are given by polynomials over $\mathbb{Z}$. The projection $\mathrm{W}_{n+1}(R) \to \mathrm{W}_n(R)$ is a ring morphism — in other words the n-th co-ordinate of addition, subtraction and multiplication are computed using only the first n co-ordinates — and the limit of this projective system is denoted $\mathrm{W}(R)$. For every n, $\mathrm{w}_n(x) = \sum_{i<n} x_i^{p^{n-i}} p^i$ defines a ring morphism $\mathrm{W}_n(R) \to R$. We write $[\cdot] : R \to \mathrm{W}(R)$ for the map sending a to $(a, 0, 0, \ldots)$; it is a multiplicative section of the projection $\mathrm{W}(R) \to R$ on the first co-ordinate. The W_n — and hence W — are functorial and for any ring morphism $f : R_1 \to R_2$, $\mathrm{W}_n(f) : x \mapsto (f(x_i))_{i<n}$ is a ring morphism. By construction, $\mathrm{W}(R)$ is p-adically complete and separated. Note that if R has characteristic p and the Frobenius morphism is bijective on R, then, for any $a \in \mathrm{W}(R)$, we have $a = \sum_i [a_i]^{p^{-i}} p^i$.

Let now $N \vDash \mathrm{PERF}_{|p|=0}$. We fix $\varpi \in \mathcal{O}(N)$ such that $0 < |\xi| < 1$ and $b \in \mathrm{W}(\mathcal{O}(N))^\times$. Note that, $b \in \mathrm{W}(\mathcal{O}(N))^\times$ if and only if $b_0 \in \mathcal{O}(N)^\times$. Let $\xi = [\varpi] - pb \in \mathrm{W}(\mathcal{O}(N))$.

Lemma 4.6. *For any $x \in \mathrm{W}(\mathcal{O}(N))$, there exists $y \in \mathcal{O}(N)$ and $a \in \mathrm{W}(\mathcal{O}(N))^\times$ such that $x \equiv [y]a \mod \xi$.*

Proof. Let us first assume that $v(x_0) < v(\varpi)$. Let $x'' = (x - [x_0] - [x_1]^{1/(p)}p)p^{-2}$ and $a = \varpi x_0^{-1}$. As $p \equiv [\varpi]b^{-1}$, we have $x \equiv [x_0]([1 + ax_1] + p[a]x'')$. Note that $v(a) > 0$, so $1 + ax_1 \in \mathcal{O}(N)^\times$ and hence $x[x_0]^{-1} = [1 + ax_1] + p[a]x'' \in W(\mathcal{O}(N))^\times$.

Let us now assume that $v(x_0) \geq v(\varpi)$. Let $y_0 = x_0\varpi^{-1}$ and $x' = (x - [x_0])p^{-1}$. Then $x \equiv [\varpi]([y_0] + b^{-1}x') \equiv p([y_0]b + x')$. It follows by induction that if the lemma fails for x, then, for all n, there exists $y_n, z_n \in \mathrm{W}(\mathcal{O}(m))$ such that $x = \xi y_n + p^n z_n$ — in other words, x is in the p-adic closure $\bigcap_n (\xi) + (p^n)$ of ξ. However, note that

$\xi(y_{n+1}-y_n) = p^n(z_n - z_{n+1})$. Since $\xi_0 = \varpi \neq 0$, it follows that $y_{n+1} - y_n \in (p^n)$. So the sequence y_n is p-adically Cauchy. Let $y = \lim_n y_n$. We have $x = \lim_n \xi y_n + p^n z_n = \xi y \in (\xi)$ and hence $x \equiv [0] \mod \xi$. □

For every $x \in \mathrm{W}(\mathcal{O}(N))$ let y and a as in the lemma. We define $|x|_\xi = |y|$ and $v_\xi(x) = v(y)$.

Proposition 4.7. (1) *The map v_ξ induces a valuation on $A_\xi = \mathrm{W}(\mathcal{O}(N))/(\xi)$ and A_ξ is its valuation ring.*

(2) *The map $|\cdot|_\xi$ is a pseudo-norm on $\mathrm{W}(\mathcal{O}(N))$ with kernel (ξ). The associated norm on A_ξ is the (norm associated to the) rank one coarsening of v_ξ.*

(3) *the map $\mathrm{res}_\xi \circ [\cdot] : \mathcal{O}(N) \to A_\xi$ induces an isomorphism $\mathcal{O}(N)/(\varpi) \to A_\xi/(p)$.*

(4) *We have $(A_\xi, |\cdot|_\xi) \models \mathrm{PERF}_{|p|=|\varpi|}$.*

(5) *Let D_ξ be the interpretation of D in $(A_\xi, |\cdot|_\xi)$. The map*

$$\begin{array}{ccc} \mathrm{W}(\mathcal{O}(N))^2 & \to & [0,1] \\ (x,y) & \mapsto & D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(y)) \end{array}$$

is quantifier-free (ϖ, b)-definable in the $\mathfrak{L}_{\mathrm{D}}$-structure N.

(6) *We have an isomorphism*

$$\begin{array}{ccc} (A_\xi)^\flat & \to & \mathcal{O}(N) \\ (x_i)_i & \mapsto & \lim_i x_{i,0}^{p^i} \\ (\mathrm{res}_\xi([x^{p^{-i}}]))_i & \leftarrow & x. \end{array}$$

Proof. (1) First of all, by construction, $v_\xi(x) = \infty$ if and only if $x \in (\xi)$ and $v_\xi(1) \neq \infty$. Moreover, if $x, y \in \mathcal{O}(N)$ and $a, b \in \mathrm{W}(\mathcal{O}(N))^\times$, then $[x]a[y]b = [xy]ab$ and hence v_ξ is multiplicative. Let us now assume that $v_\xi(x) \le v_\xi(y)$ and let $z = yx^{-1}$. Then $[x]a + [y]b = [x](a + [z]b) \equiv [x][s]c \mod \xi$ for some $s \in \mathcal{O}(N)$ and $c \in \mathrm{W}(\mathcal{O}(N))^\times$. So $v_\xi([x]a + [y]b) \ge v([x]a)$ and v_ξ does induce a valuation on A_ξ. Also $[y]b = [x][z]b \in [x]aW(\mathcal{O}(N))$. So A_ξ is indeed its valuation ring.

(2) The second statement follows immediately from the definition and the fact that $|\cdot|$ is the norm associated to v on N.

(3) The kernel of the map $\mathcal{O}(N) \to A_\xi/(p)$ induced by $\mathrm{res}_\xi \circ [\cdot]$ is the set of $x \in \mathcal{O}(N)$ such that $v(x) \ge v_\xi(p)$. Since $p \equiv [\varpi]b^{-1} \mod \xi$, we have $v_\xi(p) = v(\varpi)$ and the kernel is indeed (ϖ).

(4) So far, we have shown that $(A_\xi, |\cdot|) \models \mathrm{MVF}$ and that $|p|_\xi = |\varpi|$. Since $|A_\xi|_\xi = |\mathcal{O}(N)|$, it is dense in $[0,1]$ and since $A_\xi/(p) \cong \mathcal{O}(N)/(\varpi)$ and $\mathcal{O}(N)$ is perfect, the Frobenius is surjective. This concludes the proof.

(5) For any $x \in \mathrm{W}(\mathcal{O}(N))$ such that $x_0 = \varpi z$, we write $x' = p^{-1}(x - [x_0]) + b[z]$. Then $\mathrm{res}_\xi(px') = \mathrm{res}_\xi(x - [x_0] + pb[z]) = \mathrm{res}_\xi(x - [z]([\varpi] - pb)) = \mathrm{res}_\xi(x)$.

Let $x, y \in \mathrm{W}(\mathcal{O}(N))$. If $x, y \notin (\varpi)$, then $v_\xi(x) = v(x_0)$, $v_\xi(y) = v(y_0)$ and hence $D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(y)) = \mathrm{D}(x_0, y_0)$. If $x_0 \notin (\varpi)$ but $y \in (\varpi)$, then $D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(py')) = 0 = \mathrm{D}(x_0, y_0)$ as $v_\xi(x) = v(x_0) < v(\varpi) = v_\xi(p) \le v_\xi(py')$. If $x_0 \in (\varpi)$ but $y_0 \notin \varpi$, then $D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(y)) = D_\xi(\mathrm{res}_\xi(px'), \mathrm{res}_\xi(y)) = |y|_\xi = |y_0|$ as $v_\xi(x) \ge v_\xi(p) > v_\xi(y)$. Finally if $x, y \in (\varpi)$, then

$$D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(y)) = D_\xi(\mathrm{res}_\xi(px'), \mathrm{res}_\xi(py')) = |p|_\xi D_\xi(\mathrm{res}_\xi(x'), \mathrm{res}_\xi(y')).$$

It follows that the quantifier-free predicates t_n defined by $t_0(x, y) = 0$ and

$$t_{n+1}(x) = \begin{cases} \mathrm{D}(x_0, y_0) & \text{if } x_0 \notin (\varpi) \\ |y_0| & \text{if } x_0 \in (\varpi) \text{ and } y_0 \notin \varpi \\ |\varpi| t_n(x', y') & \text{if } x_0, y_0 \in (\varpi) \end{cases}$$

uniformly converges to $D_\xi(\mathrm{res}_\xi(x), \mathrm{res}_\xi(y))$.

(6) Consider the isomorphisms

$$A_\xi^\flat \cong (\mathcal{O}(N)/(\varpi))^\flat \cong \mathcal{O}(N)^\flat \cong \mathcal{O}(N),$$

where the first one is induced by res_p, the second is described in Lemma 4.1 and the last one is induced by projection to the first co-ordinate, giving the formula above. The inverse can be checked to be given by $x \mapsto (x^{p^{-i}})_i \mapsto (\mathrm{res}_\varpi(x^{p^{-i}}))_i \mapsto (\mathrm{res}_\xi([x^{p^{-i}}]))_i$. □

Let $M \vDash \mathrm{PERF}_{|p|=\alpha}$, where $0 < \alpha < 1$, and let $^\sharp : \mathcal{O}(M)^\flat \to \mathcal{O}(M)$ denote the projection on the first co-ordinate. Recall that, by Lemma 4.1, $\mathcal{O}(M)^\flat \cong (\mathcal{O}(M)/(p))^\flat$ is a model of $\mathrm{PERF}_{|p|=0}$ that we denote $M^\flat$ whose valuation and norm we denote $v^\flat$ and $|\cdot|^\flat$ respectively. The identification of $v(\mathcal{O}(M)/(p))$ inside $v^\flat(\mathcal{O}(M)^\flat)$ induces an isomorphism $v(\mathcal{O}(M)) = v^\flat(\mathcal{O}(M)^\flat)$. The normalization of $|\cdot|^\flat$ is chosen so that, if $\varpi \in \mathcal{O}(M)^\flat$ has valuation $v(p)$, then $|\varpi|^\flat = \alpha$.

Fact 4.8. (1) *Let A be a p-adically complete ring, let R be a characteristic p ring with bijective Frobenius morphism, and $f : R \to A/(p)$ be a ring morphism. Then there exists a (unique) ring morphism $g : \mathrm{W}(R) \to A$ such that*

$$\begin{array}{ccc} \mathrm{W}(R) & \xrightarrow{\ g\ } & A \\ \downarrow & & \downarrow \\ R & \xrightarrow[f]{} & A/(p) \end{array}$$

commutes. If f lifts to a multiplicative morphism $\overline{f} : R \to A$, then for every $x \in \mathrm{W}(R)$, we have

$$g(x) = \sum_i \overline{f}(x_i^{p^{-i}}) p^i.$$

(2) *For every $x \in Val(M)^\flat$, we have $v^\flat(x) = v(x^\sharp)$ and $|x|^\flat = |x^\sharp|$.*

(3) *The map $^\sharp : \mathcal{O}(M)^\flat \to \mathcal{O}(M)$ induces an isomorphism $\mathcal{O}(M)^\flat/(\varpi) \to \mathcal{O}(M)/(p)$. Also, for every $x \in$*

(4) *The map $\theta : \mathrm{W}(\mathcal{O}(M)^\flat) \to \mathcal{O}(M)$ defined by*

$$\theta : \sum_i [x_i] p^i \mapsto \sum_i x_i^\sharp p^i$$

is a surjective ring homomorphism with kernel $([\varpi] - pb)$, for any $\varpi \in \mathcal{O}(M)^\flat$ and $b \in \mathrm{W}(\mathcal{O}(M)^\flat)$ such that $v(\varpi) = v(p)$ and $\theta(b) = \varpi^\sharp p^{-1}$ — in particular, $v(b_0) = 0$.

Proof. (1) For every $n \in \mathbb{Z}_{>0}$ and $x \in R$, we have $\mathrm{w}_n(px) = \sum_i x_i^{p^{-i}} p^n \in p^n A$. It follows that w_n induces a well defined ring morphism $\mathrm{w}_n : \mathrm{W}_n(A/(p)) \to A/(p)^n$. Then

$$g_n := \mathrm{w}_n \circ \mathrm{W}_n(f) \circ \mathrm{W}_n(\phi^{-n}) : \mathrm{W}_n(R) \to \mathrm{W}_n(R) \to \mathrm{W}_n(A/(p)) \to A/(p)^n$$

is a ring homomorphism and for every $x \in \mathrm{W}_n(R)$, we have $g_n(x) = \sum_i \overline{x_i^{p^{-n}}}^{p^{n-i}} p^i$, where $\overline{f(x_i^{p^{-n}})} \in A/(p)^n$ is any lift of $f(x_i^{p^{-n}}) \in A$. Since $\overline{f(x_i^{p^{n+1}})}^p$ is a lift of $f(x_i^{p^{-n}}) \in A$, it follows that the g_n are compatible with the projections and hence yield a ring morphism $g : \mathrm{W}(R) \to \varprojlim_n A/(p)^n \cong A$. The formula for $g(x)$ when $A \to A/(p)$ admits a multiplicative lifting follows from the formulas for g_n.

(2) Recall that if $x \in (\mathcal{O}(M)/(p))^\flat$ is such that $x_i \neq 0$, then $v^\flat(x) = p^i v(x_i)$. It follows that if $x \in \mathcal{O}(M)^\flat$, $v^\flat(x) = p^i v(x_i) = v(x_0) = v(x^\sharp)$, for $i \gg 1$. As $|\varpi|^\flat = \alpha = |p|$ when $v^\flat(\varpi) = v(p)$, the normalizations match and we have $|x|^\flat = |x^\sharp|$.

(3) The map induced by $\sharp$ coincides with the isomorphism

$$\mathcal{O}(M)^\flat/(\varpi) \cong (\mathcal{O}(M)/(p))^\flat/(\varpi) \cong \mathcal{O}(M)/(p)$$

— the last isomorphisms is described in Proposition 4.4.(3).

(4) Applying the first statement to the ring morphism $\sharp : \mathcal{O}(M)^\flat \to \mathcal{O}(M) \to \mathcal{O}(M)/(p)$ yields a ring morphism $\theta : \mathrm{W}(\mathcal{O}(M)^\flat) \to \mathcal{O}(M)$ which is exactly given by $x = \sum_i [x_i] p^i \mapsto \sum_i (x_i)^\sharp p^i$.

By construction, $[\varpi] + pb \in \ker(\theta)$ and $v(b_0) = v(\theta(b)) = v(\pi) - v(p) = 0$. There remains to prove that $\overline{\theta} : \mathrm{W}(\mathcal{O}(M)^\flat)/([\varpi] + pb \in \ker(\theta)) \to \mathcal{O}(M)$ is an isomorphism. Note that the map $\mathrm{W}(\mathcal{O}(M)^\flat)/([\varpi] + pb, p) \cong \mathcal{O}(M)^\flat/(\varpi) \to \mathcal{O}(M)/(p)$, induced by $\overline{\theta}$ modulo p, is the map induced by $\sharp$ which is an isomorphism.

Since p is not a zero divisor in $\mathcal{O}(M)$, it now follows by induction that $\overline{\theta}$ is an isomorphism modulo p^n for all $n > 0$; and hence, since $\mathrm{W}(\mathcal{O}(M)^\flat)$ is p-adically complete, $\overline{\theta}$ is an isomorphism. □

Lemma 4.9. *Let $M \models \mathrm{PERF}_{|p|=\alpha}$, where $0 \leq \alpha < 1$. The set $\Omega(M) := \{x \in \mathcal{O}(M)^{\mathbb{N}} : \forall i \in \mathbb{N},\ x_{i+1}^p = x_i\}$ is quantifier-free definable.*

Proof. The definable predicates $d_n(x) := \inf_{y \in \mathcal{O}/(p)} \sup_{i \leq n} |x_i - y^{p^{n-i}}| 2^{-i}$ converge uniformly to $d(x, \Omega)$. Indeed for any $y \in \mathcal{O}(M)$, there exists $z \in \Omega(M)$ such that $y \equiv z \mod p$ in which case $y^{p^{n-i}} \equiv z^{p^{n-i}} \mod p^{n-i}$. It follows that $|x - \Omega| \leq d_n(x) \leq |x - \Omega| + 2(\min\{|p|, 2^{-1}\})^n$. □

If $a_i \in \Omega(M)$, we write $a^\flat = (a_i)_i \in \mathcal{O}(M)^\flat$.

Remark 4.10. The norm on Ω and the pullback of the norm from $\mathcal{O}(M)^\flat$ are equivalent. Indeed the map $x^\flat \to x_n$ induces an isomorphism $\mathcal{O}(M)^\flat/(\varpi^{p^n}) \cong \mathcal{O}(M)/(p)$ so, $x^\flat \equiv y^\flat \mod \varpi^{p^n}$ if and only if $x_n \equiv y_n \mod p$ — in which case $x_{n-i} \equiv y_{n-i} \mod p^{i+1}$.

Fix some $a \in (0,1)$. Let $\mathrm{PERF}_{|p|=0,|\xi|=\alpha}$ be the $\mathfrak{L}_{\mathrm{D}} \cup \{\varpi_i, b_{i,j} : i,j \geq 0\}$-theory

$$\mathrm{PERF}_{|p|=0} \cup \{|\varpi_0| = \alpha, b_{0,0} \in \mathcal{O}^\times, \varpi \in \Omega, b_j \in \Omega : j \geq 0\}.$$

It might seem odd to name p-th roots in characteristic p, but we have to if we aim for quantifier-free interpretations as the inverse Frobenius is not a term in the language. To alleviate notation, we confuse ϖ with ϖ_0 and b with $\sum_j [b_{j,0}] p^j$. As before, we $\xi = [\varpi] - pb$.

Let $\mathrm{PERF}_{|p|=\alpha,\xi}$ be the $\mathfrak{L}_{\mathrm{D}} \cup \{\varpi_i, b_{i,j} : i,j \geq 0\}$-theory

$$\mathrm{PERF}_{|p|=0} \cup \{\ker(\theta) = ([\varpi^\flat] - p\sum_{j\geq 0}[b_j^\flat]p^j)\}.$$

If $M \vDash \mathrm{PERF}_{|p|=\alpha,\xi}$, then $M^\flat = (\mathcal{O}(M)^\flat, (\varpi_i^\flat)^{p^{-i}}, (b_j^\flat)^{p^{-i}} : i,j \geq 0)$ is a model of $\mathrm{PERF}_{|p|=0,|\xi|=\alpha}$ by Fact 4.8.(4).

Theorem 4.11. *Fix $\alpha \in (0,1)$. Let $M \vDash \mathrm{PERF}_{|p|=\alpha,\xi}$ and $N \vDash \mathrm{PERF}_{|p|=0,|\xi|=\alpha}$.*

(1) *The map $\mathrm{id}_\Omega : \Omega(M) \to \mathcal{O}(M)^\flat$ is a quantifier-free interpretation of $M^\flat$ in M.*
(2) *Let $f : \mathcal{O}(N) \to (A_\xi)^\flat$ be the isomorphism of Proposition 4.7.(6). Then $N_\xi = (A_\xi, |\cdot|_\xi, f(\varpi), \mathrm{W}(f(b)))$ is a model of $\mathrm{PERF}_{|p|=\alpha,\xi}$.*
(3) *The map $\mathrm{res}_\xi : \mathrm{W}(\mathcal{O}(N)) \to A_\xi$ is a quantifier-free interpretation of N_ξ in N.*
(4) *The maps id_Ω and res_ξ are a quantifier-free bi-interpretation between the theories $\mathrm{PERF}_{|p|=\alpha,\xi}$ and $\mathrm{PERF}_{|p|=0,|\xi|=\alpha}$.*

Proof. (1) Note that the map $^\sharp : \Omega \to \mathcal{O}$ given by the first co-ordinate is a term and addition on Ω which is given by $\lim_i (x_i + y_i)^{p^i}$ is a uniform limit of terms. Also, if $D^\flat$ is the interpretation of D in $M_\xi^\flat$, then $D^\flat(x,y) = \mathrm{D}(x^\sharp, y^\sharp)$ is quantifier free definable. The second statement follows.

(2) The following diagram

$$\begin{array}{ccc} \mathrm{W}(\mathcal{O}(N)) & \xrightarrow{\mathrm{res}_\xi} & A_\xi \\ {\scriptstyle \mathrm{W}(f)}\downarrow & \nearrow_{\theta} & \\ W(A_\xi^\flat) & & \end{array}$$

commutes: if $x = (x_j)_j \in \mathrm{W}(\mathcal{O}(N))$, then $\mathrm{W}(f)(x) = (\mathrm{res}_\xi([x_j^{p^{-i}}]))_{i,j}$ and hence $\theta(\mathrm{W}(f)(x)) = \sum_j \mathrm{res}_\xi([x_j^{p^{-j}}])p^j = \mathrm{res}_\xi(x)$. So the kernel of θ is generated by $\mathrm{W}(f)(\xi) = [f(\varpi_0)] + p\mathrm{W}(f)(b)$.

(3) By Proposition 4.7.(5), the interpretation of D in N_ξ is induced by a quantifier-free definable predicate in N — and so is $|\cdot|$ since $|x| = \mathrm{D}(x,0)$. Since the map f is induced by $x \mapsto (x^{p^{-i}})_i$ it is a term on $\varpi, b_j \in \Omega$.

(4) The double interpretation of M in itself is $\mathrm{W}(\mathcal{O}(M)^\flat)/(\xi)$. The map θ, which is induced by a term as seen in its definition Fact 4.8.(4), induces an isomorphism with $\mathcal{O}(M)$.

Conversely, the double interpretation of N in itself is isomorphic to $\mathcal{O}(N)$ via the map of Proposition 4.7.(6) which is induced by a term. □

Let $\mathrm{TPERF}_{p,\xi}$ be the $\mathfrak{L}_{\mathrm{div},\varpi} \cup \{b_{i,j} : i,j \geq 0\}$-theory

$$\mathrm{TPERF}_{p,\varpi} \cup \{b_{0,0} \in R^\times, b_{i+1,j}^p = b_{i,j} : i,j \geq 0\}.$$

If $M \vDash \mathrm{PERF}_{|p|=\alpha,\xi}$, then $M^\natural = (\mathcal{O}(M)/(p), \mathrm{res}_p(\varpi_i), \mathrm{res}_p(b_{i,j}) : i,j \geq 0)$ is a model of $\mathrm{TPERF}_{p,\xi}$.

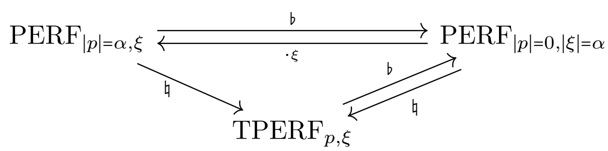
Let $\alpha \in (0,1)$ and $R \vDash \mathrm{TPERF}_{p,\xi}$. Then $R_\alpha^\flat = (R^\flat, |\cdot|_\alpha, (\varpi^\flat)^{p^{-i}}, (b_j^\flat)^{p^{-i}} : i,j \geq 0)$ is a model of $\mathrm{PERF}_{|p|=0,|\xi|=\alpha}$, where $|\cdot|_\alpha$ is normalized so that $\varpi^\flat = (0, \varpi_1, \varpi_2, \dots)$ has norm α.

Corollary 4.12. *Fix $\alpha \in (0,1)$. Let $M \vDash \mathrm{PERF}_{|p|=\alpha,\xi}$ and let $R \vDash \mathrm{TPERF}_{p,\xi}$.*

(1) *The map* $\mathrm{res}_p : \mathcal{O}(M) \to \mathcal{O}(M)/(p)$ *is a quantifier-free interpretation of* $M^\natural$ *in* M.
(2) *It forms a quantifier-free bi-interpretation with the composition* $\mathrm{res}_\xi \circ \mathrm{id}_\Omega$.

Proof. (1) As in Theorem 4.5.(3), this follows from the fact that the predicate $\mathrm{res}_p(x) \,|\, \mathrm{res}_p(y)$ is quantifier-free definable.
(2) The isomorphism $W(R^\flat)/(\xi, p) \cong R^\flat/(\varpi) \cong R$ is induced by the projection on the first co-ordinate. Conversely, the map $\sharp : (\mathcal{O}(M)/(p))^\flat \cong \mathcal{O}(M)^\flat \to \mathcal{O}(M)$ is induced by the uniform limit of terms $\lim_i x_i^{p^i}$ and so the isomorphism $\theta : \mathrm{W}((\mathcal{O}(M)/(p))^\flat)/(\xi) \cong \mathrm{W}(\mathcal{O}(M)^\flat)/(\xi) \cong \mathcal{O}(M)$ is also induced by a uniform limit of terms. □

In conclusion, we have the following quantifier-free bi-interpretations.



Remark 4.13. Note that the parameters here are not canonical, but the interpretation above also induce bi-interpretations (not necessarily quantifier-free) between $\mathrm{PERF}_{|p|=\alpha}$, $\mathrm{TPERF}_{p,(\xi)}$ and $\mathrm{PERF}_{|p|=0,|(\xi)|=\alpha}$ where, in the two last theories, we name the ideal generated by ξ.

Starting with a model N of $\mathrm{PERF}_{|p|=0}$ we thus get a uniform family of interpretations parametrized by the set $Y^1 = \{([\varpi] + pb) : 0 < |\varpi| < 1 \text{ and } b_0 \in \mathcal{O}(N)^\times\}$. Note that this is a type-definable subset of the interpretable set $\mathrm{W}(\mathcal{O}(N))/\mathrm{W}(\mathcal{O}(N))^\times$ — equipped with the norm $|x - y\mathrm{W}(\mathcal{O}(N))^\times| = |x_0 - y_0\mathcal{O}(N)^\times|$.

If we fix some $\varpi \in \mathcal{O}(N)$ with $0 < |\varpi| < 1$, we can consider $X^1 = \{\xi \in Y^1 : v(\varpi) \leq v(\xi_0) < pv(\varpi)\}$. This is a ϖ-interpretable set whose points parametrize all untilts up to the the identification induced by the Frobenius automorphism on N. This set corresponds to the degree one points on the Fargues-Fontaine curve.

Remark 4.14. If we consider Ben Yaacov's theory of metric valued fields [2], then the tilt is also interpretable. However, it is unclear if we can interpret untilts, or even axiomatize perfectoid fields, as the valuation ring is not definable.

5. Consequences

That the tilt-untilt correspondence is mediated by a quantifier-free bi-interpretation gives fundamental properties of this correspondence as formal consequences.

5.1. **Tilting elementary equivalence.** The tilt being a bi-interpretation, it preserves elementary equivalence as continuous $\mathfrak{L}_{\mathrm{rg}}$-structures. However, Corollary 3.12 allows us to recover results on the discrete $\mathfrak{L}_{\mathrm{div}}$-structure.

Proposition 5.1. *Fix* $\alpha \in (0,1)$ *and* $M, N \vDash \mathrm{PERF}_{|p|=\alpha}$ *have henselian valuation and bounded regular rank valuation group — for example the valuation is rank one. Let* $(\xi_M) \subseteq \mathrm{W}(\mathcal{O}(M^\flat))$ *(resp.* $(\xi_N) \subseteq \mathrm{W}(\mathcal{O}(N^\flat))$*) denote the kernel of* θ*. Then* M *and* N *are* $\mathfrak{L}_{\mathrm{div}}$*-elementarily equivalent if and only of* $(M^\flat, (\xi_M))$ *and* $(N^\flat, (\xi_N))$ *also are.*

Proof. If M and N are $\mathfrak{L}_{\text{div}}$-elementarily equivalent, then they are $\mathfrak{L}_{\text{rg}}$-elementarily equivalent (see Corollary 3.12.2). As tilting is a bi-interpretation (see Remark 4.13), it follows that $(M^\flat, (\xi_M))$ and $(N^\flat, (\xi_N))$ are $\mathfrak{L}_{\text{rg}}$-elementarily equivalent. By Corollary 3.12.1, they also are $\mathfrak{L}_{\text{div}}$-elementarily equivalent — note that the ideal (ξ_M) is principal, so choosing generators with the same type, we see that they remain elementarily equivalent when the ideal is named.

Conversely, if $(M^\flat, (\xi_M))$ and $(N^\flat, (\xi_N))$ are $\mathfrak{L}_{\text{div}}$-elementarily equivalent, then they are $\mathfrak{L}_{\text{rg}}$-elementarily equivalent — note that $v(\xi_{M,0})$ (resp. $v(\xi_{N,0})$) is well defined, allowing us to interpret the $\mathfrak{L}_{\text{rg}}$-structure in the $\mathfrak{L}_{\text{div}}$-structure. As before, by bi-interpretation, M and N are $\mathfrak{L}_{\text{rg}}$-elementarily equivalent and hence they are $\mathfrak{L}_{\text{div}}$-elementarily equivalent. □

5.2. **Fontaine-Wintenberger isomorphism.** What goes by the name of the Fontaine-Wintenberger theorem is (an elaboration of) the assertion that for K a perfectoid field of mixed characteristic, there is a canonical isomorphism between the absolute Galois group of K and the absolute Galois group of its tilt $K^\flat$. Let us note here that this theorem follows formally from quantifier-free bi-interpretation (Theorem 4.11).

Let us start by noting that tilting and untilting transform algebraically closed fields to algebraically closed fields.

Proposition 5.2. *If $M \models \text{ACMVF}_{0,p}$, then $M^\flat \models \text{ACMVF}$. Likewise, if $N \models \text{ACMVF}_p$ and we choose ξ and α so that $(N, \xi) \models \text{PERF}_{|p|=0,|\xi|=\alpha}$, then $N_\xi \models$ ACMVF.*

Proof. By Corollary 2.20, $M^\flat$ and N_ξ are existentially closed relative to $\text{PERF}_{|p|=0}$ and $\text{PERF}_{|p|=\alpha}$ respectively. By [2, Theorem 2.4], the models of ACMVF are precisely the existentially closed structures relative to MVF. As every model of MVF has a perfectoid extension, it follows from Lemma 2.21, that $M^\flat$ and N_ξ are existentially closed relative to MVF. Hence, $M^\flat \models$ ACMVF and $N_\xi \models$ ACMVF. That is, they are both algebraically closed. □

The Fontaine-Wintenberger theorem now follows using the usual formal arguments. For the sake of completeness, we reproduce those here.

As is standard, for a field K we write K^{alg} for its algebraic closure and if $K \subseteq L$ is a subfield of a complete field L, then we write $\widehat{K}$ for the completion of K realized as a subfield of L.

Lemma 5.3. *If $M \models \text{PERF}_{|p|=\alpha}$ with $\alpha > 0$, then $(\widehat{M^{\text{alg}}})^\flat = (\widehat{M^\flat)^{\text{alg}}}$. If $N \models \text{PERF}_{|p|=0,|\xi|=\alpha}$, then $\widehat{N^{\text{alg}}}_\xi = (\widehat{N_\xi)^{\text{alg}}}$.*

Proof. By Proposition 5.2, $(\widehat{M^\flat)^{\text{alg}}}$ is (or may be realized as) a subfield of $(\widehat{M^{\text{alg}}})^\flat$. Since tilting is part of a bi-interpretation between $\text{PERF}_{|p|=\alpha,\xi}$ and $\text{PERF}_{|p|=0,|\xi|=\alpha}$, there is some substructure $M' \subseteq \widehat{M^{\text{alg}}}$ with $(M')^\flat = (\widehat{M^\flat)^{\text{alg}}}$ and $M' = ((\widehat{M^\flat)^{\text{alg}}})_\xi$. Applying Proposition 5.2 again, $M' \models$ ACMVF. Hence, as $\widehat{M^{\text{alg}}}$ is the smallest with respect to inclusion complete algebraically closed subfield of itself containing M, $M' = \widehat{M^{\text{alg}}}$, from which we conclude that $(\widehat{M^{\text{alg}}})^\flat = (\widehat{M^\flat)^{\text{alg}}}$.

The argument for the untilt is essentially the same. □

Remark 5.4. Kedlaya and Temkin show in [11] that if k is a field positive characteristic and $K := \widehat{k((t))^{\text{alg}}}$ is the completion of the algebraic closure of the field of Laurent series over k, then there are non-surjective continuous field endomorphisms of K. Shahoseini and Pasandideh observe with [13, Example 29] that it follows from this theorem of Kedlaya and Temkin that that there is a complete algebraically closed subfield L of $\mathbb{C}_p^\flat = K$, where $\mathbb{C}_p$ is the completion of an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers for which L is not the tilt of any substructure of $\mathbb{C}_p$.

Such a field cannot contain any parameter ξ used to realize $\mathbb{C}_p = (\mathbb{C}_p^\flat)_\xi$. For example, we can choose $\xi = [t] - p$. In that case, any complete algebraically complete subfield of K containing t must indeed be equal to K.

Corollary 5.5. *If L/K is a finite Galois extension of models of $\mathrm{PERF}_{|p|=\alpha}$, then $L^\flat/K^\flat$ is a finite Galois extension and tilting induces an isomorphism $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L^\flat/K^\flat)$. Likewise, if L/K is a finite Galois extension of models of $\mathrm{PERF}_{|p|=0}$, then L_ξ/K_ξ is a finite Galois extension and untilting gives an isomorphism*

$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L_\xi/K_\xi).$$

Proof. By Lemma 5.3, $K^\flat \subseteq L^\flat \subseteq (K^{\mathrm{alg}})^\flat = \widehat{(K^\flat)^{\mathrm{alg}}}$ so that $L^\flat$ is the completion of an algebraic extension of $K^\flat$. Let N be the completion of the normal closure of $L^\flat$ in $\widehat{(K^\flat)^{\mathrm{alg}}}$. For any $\sigma \in \mathrm{Aut}(N/K)$, let $\sigma_\xi \in \mathrm{Aut}(N_\xi/K_\xi)$ be induced by the interpretation. Because the extension $(L^\flat)_\xi/(K^\flat)_\xi$ is (isomorphic to) the Galois extension L/K, we have $\sigma_\xi((L^\flat)_\xi) = (L^\flat)_\xi$. Therefore, we have $\sigma(L^\flat) = L^\flat$ and we see that $L^\flat/K^\flat$ is already a normal extension, and, thus, Galois as $K^\flat$ is perfect. The bi-interpretation then induces $\mathrm{Gal}(L/K) = \mathrm{Aut}(L/K) \cong \mathrm{Aut}(L^\flat/K^\flat) = \mathrm{Gal}(L^\flat/K^\flat)$.

The same argument applies to the case of untilting. □

The Fontaine-Wintenberger theorem follows from Corollary 5.5 from the presentation of the absolute Galois group of a field as the projective limit of the Galois groups of its finite Galois extensions.

Corollary 5.6. *For $K \vDash \mathrm{PERF}_{|p|=\alpha}$ with $\alpha > 0$, tilting induces an isomorphism of absolute Galois groups*

$$\mathrm{Gal}(K^{alg}/K) \cong \mathrm{Gal}((K^\flat)^{alg}/K^\flat).$$

Likewise, for $K \vDash \mathrm{PERF}_{|p|=0}$, untilting induces an isomorphism of absolute Galois groups

$$\mathrm{Gal}(K^{alg}/K) \cong \mathrm{Gal}((K_\xi)^{alg}/K_\xi).$$

5.3. **Approximation lemma.** A key step in the proof of the weight monodromy conjecture for complete intersections in [12] is an approximation lemma whereby the tilt of a hypersurface over a perfectoid field of mixed characteristic is approximated by hypersurfaces over the tilt in the sense that for any $\epsilon > 0$ it is possible to find a hypersurface contained in the ϵ-tubular neighborhood of the tilt of the original hypersurface. Our original motivation in pursuing this project was to extend the approximation lemma to all algebraic varieties, though we were not able to achieve that aim. Instead, we have a weaker approximation theorem whereby tilts of (quantifier-free) $\mathfrak{L}_{\mathrm{D}}$-definable sets may be approximated by (quantifier-free) $\mathfrak{L}_{\mathrm{div}}$-definable sets.

Proposition 5.7. *Let $M \vDash \mathrm{PERF}_{|p|=\alpha}$ be a perfectoid field of mixed characteristic, let y be a tuple of variables, and let $X \subseteq \Omega(M)^y$ be an $\mathfrak{L}_{\mathrm{D}}(M)$-definable subset. Define $X^\flat$ to be $\{x^\flat \in (\mathcal{O}(M)^\flat)^y : x \in X\}$. Then $X^\flat$ is $\mathfrak{L}_{\mathrm{D}}(M^\flat)$-definable.*

Moreover, for every $\gamma \in v(\mathcal{O}(M)) \setminus \{\infty\}$*, the set* $X^\flat_\gamma = \{x \in \mathcal{O}(M^\flat)^y : v(x - X^\flat) \geq \gamma\}$ *is* $\mathfrak{L}_{\mathrm{div}}(M^\flat)$*-definable.*

Proof. The tilting interpretation is part of a bi-interpretation. By Remark 4.10, the pullback of the norm on $\mathcal{O}^\flat$ is equivalent to the norm on Ω, so, by Corollary 2.15, the set $X^\flat$ is definable in $K^\flat$.

The second part of the statement follows form the fact that $X^\flat_\gamma$ is the preimage of a definable set in $\mathcal{O}/\gamma\mathcal{O}$ whose structure in the $\mathfrak{L}_{\mathrm{div}}(M)$-induced structure. □

We can upgrade the statement of Proposition 5.7 so that if X is itself quantifier-free definable, then the first-order formulas used in the approximation to $X^\flat$ may be taken to be quantifier-free.

Proposition 5.8. *With the hypotheses and notation of Proposition 5.7, if* X *is quantifier-free definable, then* $X^\flat$ *is quantifier free* $\mathfrak{L}_{\mathrm{D}}(M^\flat)$*-definable and* $X^\flat_\gamma$ *is* $\mathfrak{L}_{\mathrm{div}}(M^\flat)$*-definable.*

Given Remark 2.5, the distance to $X^\flat$ is therefore a uniform limit of linear combinations of $\mathfrak{L}_{\mathrm{div}}(M^\flat)$-formulas.

Proof. Since X is quantifier-free definable, we have that $X(K) = X(\widehat{K^{\mathrm{alg}}}) \cap K^y$. By Proposition 5.2, $(\widehat{K^{\mathrm{alg}}})^\flat$ is also algebraically closed. The proposition now follows from quantifier elimination in, respectively, ACMVF and ACVF. □

Remark 5.9. If $X \subseteq \mathcal{O}(M)^y$ is a zero set of a (quantifier free) $\mathfrak{L}_{\mathrm{D}}(M)$-definable and for some $\gamma \in v(\mathcal{O}(M)) \setminus \{\infty\}$, we have $X = X_\gamma = \{x \in \mathcal{O}(M)^y : v(y - X) \geq \gamma\}$, then for some $\delta \in v(\mathcal{O}(M)) \setminus \{\infty\}$, we have $X^\flat = X^\flat_\delta$. So, by the above, $X^\flat$ is itself (quantifier free) $\mathfrak{L}_{\mathrm{div}}(M^\flat)$-definable.

Let us now consider an affine variety V over $M \vDash \mathrm{ACMVF}$ — to be precise, we consider an irreducible affine scheme of finite type over $\mathcal{O}(M)$ with an $\mathcal{O}(M)$-point.

Lemma 5.10. *The set* $V(\mathcal{O}(M))$ *is quantifier-free* $\mathfrak{L}_{\mathrm{D}}(A)$*-definable.*

Proof. Let $(f_i)_{i \leq n} \in A[x]$ be equations defining V. Let $\alpha(x) = \min_i(v(f_i(x)))$ and let $d_V(x)$ be the $\mathfrak{L}_{\mathrm{div}}(M)$-definable function $x \mapsto \sup_{y \in V(\mathcal{O}(M))} v(x-y)$ — this map is well defined as $v(M)$ is definably complete. We show that, for some $n \in \mathbb{Z}_{>0}$ and $\gamma \in v(\mathcal{O}(M))$, we have $\alpha(x) \leq \gamma + n d_V(x)$. Our argument is extracted from the proof of [9, Lemma 6.6]. Our lemma follows by applying to this result to the rank one coarsening of M and [3, Proposition 2.19].

If the inequality fails, then in some (discrete) ultrapower $M^\star$ of (M, v), we find x with coordinates in $\mathcal{O}(M^\star)$ such that, if $\Delta \leq v(M^\star)$ denotes the convex group generated by $v(M)$ and $d_V(x)$, we have $\alpha(x) > \Delta$. If w is the coarsening associated to Δ and $\mathrm{res}_w : \mathcal{O} \to k_w$ denotes the reduction modulo the maximal ideal of w, then $\mathrm{res}_w(x) \in V(k_w)$. Then there exists $a \in V(\mathcal{O}(M^\star))$ such that $\mathrm{res}_w(a) = \mathrm{res}_w(x)$ (see [7, Theorem 3.2.4]). But then $v(x-a) > \Delta$ and hence $d_X(x) > \Delta$, contradicting the definition of Δ. □

Considering the irreducible components of the set defined by $x^{p^i} \in V$, we see that this set is also quantifier free $\mathfrak{L}_{\mathrm{D}}(M)$-definable. Taking a uniform limit, we see that $\{x \in \Omega : x_0 \in V\}$ is quantifier free $\mathfrak{L}_{\mathrm{D}}(M)$-definable. By Proposition 5.8, the set $V^\flat = \{x \in \mathcal{O}^\flat : x^\sharp \in V\}$ is quantifier free $\mathfrak{L}_{\mathrm{D}}(M^\flat)$-definable. If we choose some $\gamma \in v(\mathcal{O}(M)) \setminus \{\infty\}$, then $V^\flat_\gamma$ is $\mathfrak{L}_{\mathrm{div}}(M^\flat)$-definable.

Ideally we would like $V_\gamma^\flat$ to be the γ-neighborhood of some variety W_γ over $M^\flat$ of dimension equal to that of V. In [12], this is achieved through a "syntactic trick": one may trace through the translation given by tilting to see that for a hypersurface X defined by $f = 0$ where f is a polynomial, $X^\flat$ is defined by $g = 0$ for some g such that $g^\sharp$ is a good approximation of f. For a complete intersection, the approximating variety W is found by approximating the tilts of a minimal set of defining equations. The formulas we obtain need not have this form.

5.4. **Type spaces.** As we have seen with Propositions 2.23 and 2.24, interpretations induce continuous maps between type spaces and for an isometric bi-interpretation this induced map is a homeomorphism. Specializing to the case of the tilt/untilt correspondence, we may understand the type spaces themselves as adic spaces, albeit considered with a finer topology than usual, so that corresponding homeomorphisms between type spaces may be seen as a kind of equivalence of adic spaces over a perfectoid field and its tilt.

Let $M \vDash \mathrm{PERF}_{|p|=\alpha}$ be perfectoid and let $A \leq \mathcal{O}(M)$ be a perfectoid substructure. For every type-A-definable set $X \subseteq \Omega^n$, let $X^\flat \subseteq (\mathcal{O}^\flat)^n$ denote the corresponding type-$A^\flat$-definable set.

Theorem 5.11. *The spaces* $\mathrm{S}_X(A)$ *are* $\mathrm{S}_{X^\flat}(A^\flat)$ *are homeomorphic. Moreover, the homeomorphism is functorial and induces an equivalence of categories:*

$$\left\{ \begin{array}{c} \textit{type-A-definable subsets} \\ \textit{of (cartesian powers of) } \Omega(M) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{type-}A^\flat\textit{-definable subsets} \\ \textit{of (cartesian powers of) } \mathcal{O}(M^\flat) \end{array} \right\}.$$

Proof. This is a special case of Proposition 2.24. □

Let us now assume $M \vDash \mathrm{ACMVF}$. Then we are considering quantifier-free type spaces that are closely related to adic spaces.

In what follows, we write T for a (potentially infinite countable) tuple of variables and $A\langle T\rangle$ for the Tate algebra of convergent power series over A in those variables. We allow valuations to have kernels. Perhaps, the reader would prefer to call these "pseudovaluations".

Fact 5.12. *Let w be a valuation on $A\langle T\rangle$ extending v. The following are equivalent:*

(1) *for every $\gamma \in w(A\langle T\rangle)$, the set $\{f \in A\langle T\rangle : w(f) \geq \gamma\}$ contains $aA\langle T\rangle$ for some $a \in A$.*
(2) *for every $f \in A\langle T\rangle$, if $wf < \infty$, then there exists $\gamma \in v(A)$ such that $wf \leq \gamma$.*

Proof. Indeed for any $f \in A\langle T\rangle$ and any $a \in A$, we have $w(f) \leq v(a)$ of and only if $aA\langle T\rangle \subseteq \{g \in A\langle T\rangle : w(g) \geq \gamma\}$. □

A valuation satisfying the equivalent conditions of Lemma 5.12 is said to be continuous. Let $\mathrm{Cont}_v(A\langle T\rangle)$ be the set of continuous valuations on $A\langle T\rangle$ extending v, up to equivalence. We endow $\mathrm{Cont}_v(A\langle T\rangle)$ with the topology generated by the sets of the form $\{wf \geq wg \neq \infty\}$ for all $f, g \in A\langle T\rangle$. If v has finite rank on A, this space is exactly the fiber of the adic space $\mathrm{Spa}(A\langle T\rangle, A\langle T\rangle)$ over the point v of $\mathrm{Spa}(A, A)$.

Proposition 5.13. *There is a continuous bijection*

$$S_{\mathcal{O}^{|T|}}(A) \to \mathrm{Cont}_v(A\langle T\rangle).$$

The topology on $S_V(A)$ is the constructible topology associated to $\mathrm{Cont}_v(A\langle T\rangle)$.

Proof. Fix any $p \in S_{\mathcal{O}^{|T|}}(A)$, then we define a valuation v_p on $A\langle T\rangle$ by $v_p(f) \leq v_p(g)$ whenever $p(T) \models D(f,g) = 0$. Since $vA^\times$ is cofinal in the valuation group of any elementary extension of M, v_p is continuous. Injectivity follows from quantifier elimination (Proposition 3.7). Conversely, if v is a continuous valuation on $A\langle T\rangle$, then tensoring by $\mathcal{O}(M)$, modding out by the kernel and taking the valuation ring, we obtain an extension R of M. As M is existentially closed in R, the type of T over A is an then element of $S_{\mathcal{O}^{|T|}}(A)$ whose associated valuation is v itself.

For continuity, we need to prove that the pre-image of

$$\{w \in \mathrm{Cont}_v(A\langle T\rangle) : wf \geq wg \neq \infty\}$$

is open. Note that, for all non-zero $a \in A$, the set

$$\{w \in \mathrm{Cont}_v(A\langle T\rangle) : wf \geq wg \leq va\}$$

is clopen, as it factorizes through some quotient modulo $c \in A$. Since

$$\{w \in \mathrm{Cont}_v(A\langle T\rangle) : wf \geq wg \neq \infty\} = \bigcup_{a \in A^\times} \{w \in \mathrm{Cont}_v(A\langle T\rangle) : wf \geq wg \leq va\},$$

this set is open.

To prove the last statement, we have to prove that the open quasi-compact subsets of $\mathrm{Cont}_v(A\langle T\rangle)$ — i.e. the sets $\{wf_i \geq wg \neq \infty : i < n\}$ where $\mathfrak{m} \subseteq (f_i : i < n)$ — is clopen. But then some linear combination of the f_i is a non-zero constant and hence $\{wf_i \geq wg \neq \infty : i < n\} = \{wf_i \geq wg \leq va : i < n\}$ for some $a \in A^\times$, which is indeed clopen. □

The continuous bijection above is compatible with the projection to $\mathrm{Spec}(A\langle T\rangle)$. Thus, for every ideal $I \subseteq A\langle T\rangle$, it induces a continuous bijection

$$S_V(A) \to \mathrm{Cont}_v(A\langle T\rangle/I),$$

where $V \subseteq \mathcal{O}^{|T|}$ is the zero set of all $f \in I$. It also naturally induces a continuous bijection on affinoid subspaces.

Note that this continuous bijection identifies the space $S_{\Omega^n}(A)$ of Theorem 5.11 with the constructible space associated to $\mathrm{Cont}_v(A\langle T^{p^{-\infty}}\rangle)$.

Remark 5.14. Recovering the right topology on type spaces seems to be a matter of working in positive continuous logic; but the exact setup for this is not entirely obvious to the authors.

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