ON GROUPS DEFINABLE IN GEOMETRIC FIELDS WITH GENERIC DERIVATIONS

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Abstract. We study groups definable in existentially closed geometric fields with commuting derivations. Our main result is that such a group can be definably embedded in a group interpretable in the underlying geometric field. Compared to earlier work of the first two authors together with K. Peterzil, the novelty is that we also deal with infinite dimensional groups.

Introduction

This paper is about comparing groups definable in a given structure with groups definable in a reduct. These latter groups are usually better understood, providing in return many insights on the original groups.

The case of interest here is to compare groups definable in a field with a derivation (and potentially additional structure), to groups definable without the derivation. In the case of a differentially closed field[1], the second author proved in [Pil97] that a differential algebraic group differentially algebraically embeds in an algebraic group (answering questions of Kolchin). The methods were stability-theoretic and reminiscent of Weil's construction of an algebraic group out of a pregroup [Wei55].

Our main goal here is to consider a generalization of this result to an unstable context. We will be working with enriched geometric fields of characteristic 0. Following [HP94, Definition 2.9], but allowing additional structure, we say that a (complete) theory T of fields with additional structures is *a geometric theory of enriched fields* if:

- In models of T, model theoretic algebraic closure coincides with relative field theoretic algebraic closure (over $\mathrm{dcl}(\emptyset)$).
- Models of T are perfect and T eliminates the $\exists^\infty$ quantifier.

As noted by multiple authors, [For11; JY23a], the first hypothesis implies the second. Such a theory T is also said to be *algebraically bounded*.

Examples include real closed fields, p-adically closed fields, more generally characteristic zero henselian valued fields, bounded pseudo algebraically closed fields (such as pseudofinite fields), open theories of topological fields [CP23], bounded perfect pseudo T-closed fields [MR25] and curve excluding fields [JY23b], among others.

Recall that a geometric theory is a complete theory in an arbitrary language such that in models of T algebraic closure is a pregeometry and T eliminate the $\exists^\infty$ quantifier. So a geometric theory of enriched fields is a geometric theory, but the

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[1] With no further structure.

key point is that algebraic closure coincides with (relative) field-theoretic algebraic closure.

So let us fix some geometric theory T of enriched fields in some language L. At the cost of Morleyizing, we can always assume that T eliminates quantifiers in a relational expansion of the ring language. Also fix some integer ℓ and $\Delta = \{\partial_1, \ldots, \partial_\ell\}$. Consider the theory of models of T with ℓ commuting derivations in the language $L_\Delta = L \cup \Delta$. By [FT23, Theorem 4.1], this theory admits a model completion that we will denote T_Δ.

In this context, we prove the following.

Theorem (Theorem 3.6)**.** *Let $K \models T_\Delta$ and let Γ be a group which is L_Δ-definable in K. Then there is a group G which is L-interpretable in (the reduct to L of) K and an L_Δ-definable embedding $\Gamma \to G$.*

Moreover, if Γ is definable over some $A \subseteq K$, then the group G and the embedding can also be chosen over A.

In [PPP22] the first two authors and K. Peterzil proved the finite-dimensional case (see [PPP22, Definition 2.2]) of our main result — first considering $K \models RCF$, then generalizing. Assuming that Γ is finite dimensional, one recovers a generically given L-definable group, and proceeds from there. In the case of possibly infinite-dimensional Γ one will obtain some kind of generically given L-definable group but living on infinite tuples, and there are additional technical complications. It turns out that general results from [HR19] are in a sense tailor-made to handle such situations, so we will appeal to them. Possibly, the methods of [KP02] giving another account of [Pil97] would also adapt to the present setting.

On the face of it, the main theorem for finite-dimensional definable groups follows from the theorem for arbitrary definable groups. However in [PPP22], the group G can be chosen to be L-definable and not only L-interpretable — no quotient is required.

Note also that, in [PPP22], the finite-dimensional case was also considered in other contexts, namely *o*-minimal expansions of real closed fields, which fall outside the scope of the present paper. Also, in [PPP25], Buium's notion of an "algebraic D-group" is adapted from the context of algebraically closed fields with a derivation to models of T with a derivation and finite-dimensional L_Δ-definable groups are shown to be precisely groups of "sharp points" of D-groups.

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1. Geometric fields with generic derivations

Let us start by recalling some facts about geometric fields.

Definition 1.1. *Let $K \models T$ and let $X \subseteq K^n$ be definable (with parameters). The dimension* $\dim(X)$ *of X is the dimension (in the sense of algebraic geometry) of the Zariski closure of X in affine n-space.*

If X is definable over some subfield $A \subseteq K$ and if K is sufficiently saturated, $\dim(X)$ is the maximal transcendence degree of some $c \in X(K)$ over A.

In a geometric theory, dimension is definable in the sense that for all definable $X \subseteq Y \times Z$, for every $n \geq 0$, $\{z \in Z : \dim(X_z) = n\}$ is definable, where $X_z = \{y \in Y : (y, z) \in X\}$ is the fiber of X at z.

Let us now consider the properties of models of T_Δ. Let $K, M \models T_\Delta$ and let $A \subseteq K$. We say that an injective ring embedding $f : A \to M$ is an L-elementary embedding if it preserves L-types (between the reducts of K and M to L) and that it is differential if it preserves the derivations. If we want to specify that we work in the reduct to L, we will indicate it with a subscript $_L$, like in the notation dcl_L for the L-definable closure. Notation with a subscript $_{\Delta\mathrm{rg}}$ will refer to the differential field structure and notions with a subscript $_\Delta$, or no subscript, will refer to the full L_Δ-structure.

Lemma 1.2. *Let $K, M \models T_\Delta$, let $A \leq_{\Delta\mathrm{rg}} K$ be a differential subfield and let $f : A \to M$ be a differential L-elementary embedding. Then there exists an L_Δ-elementary extension $M' \succcurlyeq M$ and a differential L-elementary embedding $g : K \to M'$ extending f such that $g(K)$ is algebraically independent of M over $g(A)$.*

Proof. Let $A_0 = A^{\mathrm{alg}} \cap K$ be the relative field algebraic closure of A in K. Since f is L-elementary, and L contains the ring language, we can extend f to $\overline{f} : A_0 \to M$ as an L-elementary embedding. Moreover, since Δ extends uniquely to A^{alg}, the embedding $\overline{f}$ is also a differential field embedding.

Fix some tuple $a \in K^m$ and let V be its (geometrically integral) locus over A_0. For any $X \in \mathrm{tp}_L(a/A_0)$ (namely X is the solution set of a formula in the type) contained in V, let W be the Zariski closure of X over K. Then $W \subseteq V$, it contains a and, by invariance, it is defined over $\mathrm{dcl}_L(A_0) = A_0$ in the language L. So $W = V$. By compactness, $tp_L(a/A_0) \cup \{\neg\phi(x) : \phi$ defining a proper Zariski closed subset of V, defined over $K\}$ is consistent. It follows that we can extend $\overline{f}$ to an L-elementary $g_L : K \to M' \succcurlyeq_L M$ such that $g_L(K)$ is algebraically independent of M over $g_L(A_0)$.

Since $A_0 \leq K$ is a regular extension, it follows that the compositum $g_L(K)M$ is isomorphic to (the fraction field of) $K \otimes_{A_0} M$ which can therefore be made into a substructure of a model of T extending the L-structure on both K and M. The derivations on K and M also extend uniquely to $K \otimes_{A_0} M$ and the resulting derivations commute. Since $M \leq K \otimes_{A_0} M$ is L_Δ-existentially closed, $K \otimes_{A_0} M$ can be embedded into some L_Δ-elementary extension $M_1 \succcurlyeq M$, concluding the proof. ∎

Let Θ denote the commutative monoid generated by Δ. Its elements are of the form $\theta = \partial_1^{e_1} \dots \partial_\ell^{e_\ell}$ for all integers $e_1, \dots e_\ell \geq 0$. For such a θ, we define $|\theta| = \sum_{i \leq \ell} e_i$. We order the elements of Θ by lexicographic order on $|\theta|$ and the components of θ.

For any tuple a in a differential field (K, Δ) and for any integer $n \geq 0$, we write $\nabla_n(a) = (\theta a_i)_{i, |\theta| \leq n}$. We also write $\nabla_\omega(a) = (\theta a)_{i,\theta}$.

From Lemma 1.2, we immediately recover a strong form of quantifier elimination[2]. Let $K \models T_\Delta$ and let $A \subseteq K$.

Corollary 1.3. (1) *Differential L-elementary maps between differential subfields of models of T_Δ are L_Δ-elementary.*
(2) *If X is L_Δ-definable over A in K, there exists an integer n and a set Y which is L-definable over A such that $x \in X$ if and only if $\nabla_n(x) \in Y$.*

Proof. Let $K, M \models T_\Delta$ be sufficiently saturated, let $A \leq_{\Delta\mathrm{rg}} K$ be a small differential subfield and $f : A \to M$ be a differential L-elementary embedding. Let $B \leq K$ be

[2] This is implicit in [FT23], but it is only stated explicitly for a single derivation.

small and contain A. By Lemma 1.2, we may extend f to a differential L-elementary embedding $B \to M$ — enlarging B, we may assume that it is an L_Δ-elementary substructure of K. In other words, differential L-elementary isomorphisms between small differential subfields have the back-and-forth property and hence are L_Δ-elementary. The first statement is proved.

Now, it follows that, for every tuple a and b in K, if $\mathrm{tp}_L(\nabla_\omega(a)) = \mathrm{tp}_L(\nabla_\omega(b))$ then $\mathrm{tp}(a) = \mathrm{tp}(b)$. The second statement follows by compactness. □

We can also immediately characterize algebraic and definable closure in models of T_Δ.

Corollary 1.4. (1) *The L_Δ-algebraic closure $\mathrm{acl}_\Delta(A)$ of A is the relative field algebraic closure in K of the differential field generated by A.*
(2) *The L_Δ-definable closure $\mathrm{dcl}_\Delta(A)$ of A is the L-definable closure of the differential field generated by A.*
(3) *Let $f : X \to Y$ be L_Δ-definable over A. There exists $F : Z \to W$ which is L-definable over A and an integer $n \geq 0$ such that $\nabla_n(X) \subseteq Z$ and, for all $x \in X$, $f(x) = F(\nabla_n(x))$.*

Proof. Let $A = A^{\mathrm{alg}} \cap K \leq_{\Delta\mathrm{rg}} K$ be a relatively algebraically closed differential subfield. By Lemma 1.2, there exists a differential L-elementary embedding $g : K \to K' \succcurlyeq K$ extending the identity on A and such that $g(K)$ and K are algebraically independent over A. Since $A \leq K$ is regular, K and $g(K)$ are linearly disjoint over A. Moreover, by Corollary 1.3, g is L_Δ-elementary and thus $g(K) \preccurlyeq K'$. It follows that $\mathrm{acl}_\Delta(A) \subseteq K \cap g(K) = A$, concluding the proof of the first item.

Now, let $A = \mathrm{dcl}_L(A) \leq_{\Delta\mathrm{rg}} K$ be a dcl_L-closed differential subfield of K. By the first item, we have $\mathrm{dcl}_\Delta(A) \subseteq \mathrm{acl}_\Delta(A) \subseteq A^{\mathrm{alg}}$. Consider $a \in A^{\mathrm{alg}} \cap K \setminus A$. Then, since $a \notin \mathrm{dcl}_L(A)$, it has at least one other L-conjugate $a' \in M$ over A. Since $a, a' \in A^{\mathrm{alg}}$, any L-elementary embedding sending a to a' is also a differential embedding and hence, by Corollary 1.3, it is L_Δ-elementary. So $a \notin \mathrm{dcl}_\Delta(A)$ and $\mathrm{dcl}_\Delta(A) \subseteq A$, proving the second item.

Finally, let $f : X \to Y$ be L_Δ-definable over A. For every $x \in X$, by the previous item, we have $f(x) \in \mathrm{dcl}_L(A\nabla_\omega(x))$. By compactness, it follows that there are finitely many maps F_i which are L-definable over A such that for every $x \in X$, there exists an i such that $f(x) = F_i(\nabla_\omega(x))$. Let $X_i = \{x \in X : f(x) = F_i(\nabla_\omega(x))\}$ and, by Corollary 1.3, let Z_i be L-definable over A such that, for some sufficiently large n, $x \in X_i$ if and only if $\nabla_n(x) \in Z_i$. We may assume that the Z_i are disjoint. We define F on $Z = \bigcup_i Z_i$ by $F(z) = F_i(z)$ if $z \in Z_i$. Then, for any $x \in X$, we have $f(x) = F(\nabla_n(x))$. □

Let us conclude this section with a purely differential statement which is implicit in the proof of the existence of the Kolchin polynomial.

Lemma 1.5. *Let (K, Δ) be a differential field (with finitely many commuting derivations). Let $K \leq_{\Delta\mathrm{rg}} K\langle a_1, \ldots, a_m\rangle_\Delta$ be a finitely generated differential field extension. Let $a = (a_1, .., a_m)$. Then there exists an $n \neq 0$ such that the extension $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental.*

Proof. We order the set of θa_i, for all $\theta \in \Theta$ and $i \leq n$ by lexical order first on θ and then on i — this is a well order isomorphic to ω. Let E be the set of θa_i such that $\theta a_i \in K(\theta' a_j : \theta' a_j < \theta a_i)^{\mathrm{alg}}$. For any $\theta a_i \in E$ and any $\eta \in \Theta \setminus \{1\}$, we have $\eta\theta a_i \in K(\theta' a_j : \theta' a_j \leq \theta a_i)$.

Let E_0 be the set of $\theta a_i \in E$ which are not proper derivatives of any element of E. Identifying $\{\theta a_i : \theta \in \Theta \text{ and } i \leq m\}$ naturally with a subset of $\omega^\ell \times \{1, \dots, m\}$, if E_0 is infinite, it contains a strictly increasing sequence for the product order — indeed starting with any sequence of pairwise distinct elements in E, one can iteratively extract subsequences to make each projection increasing. So there is some $i \leq m$ and some $\theta, \eta \in \Theta$ such that θa_i and $\eta\theta a_i$ are in E_0. This contradicts the previous paragraph so E_0 is finite.

Also, we have $E = \Theta E_0$. Let $n = \max_{\theta \in E_0} |\theta|$. Then $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental. Indeed, for any θa_i with $|\theta| > n$, either $\theta a_i \in E$ in which case $\theta a_i \in K(\theta' a_j : \theta' a_j < \theta a_i)$ or $\theta a_i \notin E$ in which case θa_i is transcendental over $K(\theta' a_j : \theta' a_j < \theta a_i)$. □

2. Generic points

Let $K \models T_\Delta$ be sufficiently saturated and homogeneous. We also fix an elementary extension $M \succcurlyeq K$ which is $|K|^+$-saturated in which to realize (partial) types over K. Let $L_{\Delta\mathrm{rg}}$ denote the language of differential rings.

Recall that the $L_{\Delta\mathrm{rg}}$-theory $\mathrm{DCF}_{\Delta,0}$ of differentially closed fields of characteristic 0 with respect to the commuting derivations in Δ, is ω-stable and has quantifier elimination. We let $\mathcal{U}$ be a big model of $DCF_{\Delta,0}$ containing M. So complete quantifier free types in $L_{\Delta\mathrm{rg}}$ over subsets of $\mathcal{U}$ correspond to complete types in the sense of $\mathrm{DCF}_{\Delta,0}$, so, as such, have an ordinal valued Morley rank. For a tuple a from M we will define the Morley rank of the quantifier-free $L_{\Delta\mathrm{rg}}$ -type of a over K to be its Morley rank in $\mathrm{DCF}_{\Delta,0}$.

Let X be L_Δ-definable in K. Let S be the set of complete quantifier-free types over K (in the sense of $\mathrm{DCF}_{\Delta,0}$) which are finitely satisfiable in K by elements of X, equivalently realized in M by an element of X. Then by compactness (in T_Δ), the set S is a closed subset of the space of quantifier-free complete types over K, in the sense of $\mathrm{DCF}_{\Delta,0}$. So S corresponds to a partial (quantifier-free) type Σ over K in the sense of $\mathrm{DCF}_{\Delta,0}$. Let α be the Morley rank of Σ. By properties of Morley rank, we have:

Lemma 2.1. *Σ extends to finitely many complete quantifier-free types over K of Morley rank α, say $p_1, .., p_k$.*

We expect that $p_1, .., p_k$ are also the types in the space S of maximal Cantor-Bendixon rank, but we will not need to know this.

Definition 2.2. *We say that $a \in X$ — in M — is generic (over K) if it realizes one of the p_i — in other words, a is generic in X if a has maximum Morley rank in X over K.*

Lemma 2.3. *Let $f : X \to X$ be L_Δ-definable over K and injective. Let a be generic in X, then $f(a)$ is also generic.*

Proof. By hypothesis, $\mathrm{dcl}_\Delta(Ka) = \mathrm{dcl}_\Delta(Kf(a))$. It follows from Corollary 1.4.1, that the field algebraic closure of the differential fields generated by a and $f(a)$ over K are identical, and hence that a and $f(a)$ have the same Morley rank over K. So a is generic in X if and only if $f(a)$ is. □

Let $\Sigma_X(x_\omega)$ be the common L-type over K of $\nabla_\omega(a)$, where $a \in X$ is generic over K. By quantifier elimination (Corollary 1.3), $\Sigma_X(\nabla_\omega(x))$ is the partial type of generics in X.

Lemma 2.4. *Let $A = \mathrm{dcl}_\Delta(A) \subseteq K$ be such that X is definable over A. Then the partial type Σ_X is L-definable over A — that is, for every formula $\phi(x_\omega, y)$, the set of $a \in K^y$ such that $\Sigma_X(x_\omega) \models \phi(x_\omega, a)$ is L-definable over A.*

Proof. We write $\dim(a/K)$ for the transcendence degree of $K(a)$ over K.

For every $n \geq 0$, let $W_{i,n}$ be the Zariski locus of $\nabla_n(a)$ over K for any (equivalently all) $a \models p_i$. Then $a \models p_i$ if and only if, for every n, $\nabla_n(a) \in W_{i,n}$ and $\dim(\nabla_n(a)/K) = \dim(W_{i,n})$.

Assume that n is sufficiently large so that $x \in X$ if and only if $\nabla_n(x) \in Y$, for some Y which is L-definable over A (by Corollary 1.3 (ii)). Also assume that n is sufficiently large so that, by Lemma 1.5, for every i and every $a \models p_i$, the extension $K(\nabla_n(a)) \leq K(\nabla_\omega(a))$ is purely transcendental. For future reference, let us fix some $N_0 \in \mathbb{N}$ such that the above conditions hold for all $n \geq N_0$.

Claim 2.5. *Let $n \geq N_0$. Fix an i and let $a_n \in W_{i,n}$ be such that* $\dim(a_n/K) = \dim(W_{i,n})$. *Then there exists $b \models p_i$ such that* $\mathrm{tp}_L(\nabla_n(b)/K) = \mathrm{tp}_L(a_n/K)$.

Let $b \models p_i$ — a priori we can choose b in M but for now we ignore the L-structure induced by M on $K(\nabla_n(b))$. We have a field isomorphism $f_n : K(a_n) \to K(\nabla_n(b))$ sending a_n to $\nabla_n(b)$. By saturation, we can find $c = (c_i)_{i<\omega} \in M$ transcendental and algebraically independent over $K(a_n)$. As, by choice, $K(\nabla_\omega(b))$ is purely transcendental over $K(\nabla_n(b))$, the isomorphism f_n extends to a ring isomorphism $f : K(a_n, c) \to K(\nabla_\omega(b))$.

Now $K(a_n)(c)$ has its L-structure (as a substructure of M), and the isomorphism f induces a new L-structure on the differential field $K(\nabla_\omega(b))$ which is compatible with the ring structure and extends the L-structure on K. Let us write K_1 for the differential field $K(\nabla_\omega(b))$ with this new L-structure. By construction, f is an L-embedding, and hence the quantifier free L-type of $\nabla_n(b)$ over K in K_1 equals the quantifier free L-type of a_n over K. Also note that K_1 is the expansion of a substructure of a model of T by ℓ commuting derivations containing the model K of T_Δ. As M is a saturated model of T_Δ (the model completion of models of T with ℓ commuting derivations), there is an embedding h of the L_Δ-structure K_1 into M over K. Let $u = h(b)$. Then a_n and $\nabla_n(u) = h(\nabla_n(b))$ have the same quantifier free L-type over K. As T has quantifier elimination they have the same L-type over K in the model M of T. This proves Claim 2.5.

Now we want to finish the proof of Lemma 2.4. Fix an L-formula $\phi(x_\omega, y)$. We have to prove that the set of $d \in K$ such that $\phi(x_\omega, d) \in \Sigma_X(x_\omega)$ is L-definable over A.

Let $n \geq N_0$ be sufficiently large so that all variables of x_ω that actually appear in ϕ are in $x_n = \{x_\theta : |\theta| \leq n\}$. We will write ϕ as $\phi(x_n, y)$. Now $\phi(x_n, d) \in \Sigma_X(x_\omega)$ if and only if for all $i = 1, .., k$ and $a \in X$ realizing p_i, we have that $\phi(\nabla_n(a), d)$. Using Claim 2.5 this is equivalent to $\dim(\text{“}x_n \in Y \cap W_{i,n}\text{”} \wedge \neg\phi(x_n, d)) < \dim(W_{i,n})$, which by definability of dimension in T is an L-definable condition on d.

It remains to be seen that Σ_X is L-definable over $A = \mathrm{dcl}_\Delta(A)$. However, note that, for any n, the finite set of (codes of the) $W_{i,n}$ is L_Δ-definable over A. By elimination of imaginaries in algebraically closed fields, it is quantifier free definable in the ring language over A. Hence, Σ_X is L-definable over $\mathrm{dcl}_\Delta(A)$. □

3. Groups

For now, let T be any theory, and let $A \subseteq K \models T$. Assume that K is $|A|^+$-saturated. As previously, we also fix an elementary extension $M \succcurlyeq K$ which is $|K|^+$-saturated in which we realize partial types over K.

We will use the language of pro-definable sets as in [HR19, Section 2]. A pro-definable set X over A is a (small) projective filtered system $(X_i)_{i\in I}$ of definable sets and definable maps over A. We think of X as $\varprojlim_{i\in I} X_i$. For simplicity and in terms of the application, there is no harm in assuming that the index set I is countable. As in our applications (to definable groups in T_Δ) we do not necessarily eliminate imaginaries we distinguish between pro-definable and pro-interpretable (so pro-interpretable means pro-definable in T^{eq}). A pro-definable map over A between pro-definable sets X and $Y = \varprojlim_{j\in J} Y_j$ is a compatible system of definable maps $f_j : X_{i_j} \to Y_j$ over A. Finally, a pro-definable group over A is a pro-definable set G together with a pro-definable group law both over A. Note that pro-definable sets can alternatively be described as $*$-definable sets, as in [Hru90, Section 3]: solutions in M of a partial type over a small set of parameters in (possibly infinitely) many variables.

Given a pro-interpretable set $X = \varprojlim_{i\in I} X_i$, a (global) partial type Σ concentrating on X is a compatible collection of partial types p_i over K (not necessarily over a small set of parameters from K) such that $p_i(x_i) \models x_i \in X_i$. We can see partial types concentrating on X as satisfiable collections of pro-definable subsets of X. We assume that partial types are closed under finite conjunctions and consequences (these are called "filters" in [HR19]). For example a complete type $p(x)$ over K implying X is such a global partial type concentrating on X. When we talk about a realization of a global partial type (in maybe infinitely many variables) we mean a realization in M, unless we say otherwise. If Σ is a global partial type concentrating on a pro-definable set X and $f : X \to Y$ is pro-definable, the "pushforward" $f(\Sigma)$ is the global partial type whose set of realizations in M is precisely $\{f(a) : a \text{ realizes } \Sigma\}$; in other words, for any pro-definable set Z, we have $f(\Sigma)(y) \models y \in Z$ if and only if $\Sigma(x) \models f(x) \in Z$.

If $\Sigma(x)$ is a global partial type, we write $\Sigma|_A$ for its restriction to formulas with parameters in A. Also, as in Lemma 2.4, we say that Σ is definable over A if for every formula $\phi(x, y)$, the set of tuples $a \in K^y$ such that $\Sigma(x) \models \phi(x, a)$ is L-definable over A.

Definition 3.1. *Let G be a pro-interpretable group and Σ a global partial type concentrating on G. We say that Σ is (left) translation invariant if for every $g \in G(K)$ and $a \models \Sigma$, we have $g \cdot a \models \Sigma$ — equivalently, if $\Sigma \models X$ and $g \in G(K)$, then $\Sigma \models g \cdot X$.*

So Σ is translation invariant in G if and only if $G = \{g \in G : g \cdot \Sigma = \Sigma\} = \mathrm{Stab}_G(\Sigma)$. Note that a global definable type concentrating on G which is translation invariant is a generic definable filter in the terminology of [HR19, Definition 3.1]. Conversely, if G admits a generic definable filter then it admits a definable and translation invariant global partial type by [HR19, Remark 3.3].

Definition 3.2. *Let Σ be a global partial type (concentrating on some pro-interpretable set) which is definable over A and let $F(x, y)$ be a map pro-definable over A. We say that (Σ, F) is an abstract group chunk, or pregroup, over A if the following hold.*

(1) *If* $a \models \Sigma|_A$ *and* $b \models \Sigma$*, then* $F(a,b)$ *is defined and* $F(a,\Sigma) = \Sigma$.
(2) *If* $a \models \Sigma|_A$ *and* $b \models \Sigma|_{Aa}$*, then* a *and* b *are interdefinable over* $A \cup \{F(a,b)\}$.
(3) *If* $a \models \Sigma|_A$*, if* $b \models \Sigma|_{Aa}$ *and if* $c \models \Sigma|_{Aab}$*, then* $F(a, F(b,c)) = F(F(a,b),c)$.

These are the main results we will use on these notions.

Proposition 3.3 ([HR19, Prop. 3.15])**.** *Let* (Σ, F) *be a pregroup over* A*. Then there exists a pro-interpretable group* G *over* A *and an injective map* $f : \Sigma|_A \to G$ *which is pro-interpretable over* A *and such that:*

(1) *for any* $a \models \Sigma|_A$ *and* $b \models \Sigma|_{Aa}$, $f(F(a,b)) = f(a) \cdot f(b)$*;*
(2) *the global partial type* $f(\Sigma)$ *is translation invariant in* G.

Proposition 3.4 ([HR19, Prop. 3.4])**.** *Let* G *be a pro-interpretable group over* A *and let* Σ *be a translation invariant global partial type concentrating on* G *definable over* A*. Then* G *is pro-interpretably over* A *isomorphic to a projective limit of groups interpretable over* A.

Proposition 3.5 ([HR19, Prop. 3.16])**.** *Let* G *and* H *be pro-interpretable groups over* A*, let* Σ *be a translation invariant partial type concentrating on* G *definable over* A *and let* f *be a pro-definable map over* A *such that for every* $a \models \Sigma|_A$, $f(a) \in H$*. Assume moreover, that for all* $a \models \Sigma|_A$ *and* $b \models \Sigma|_{Aa}$, $f(a \cdot b) = f(a) \cdot f(b)$*. Then there exists a unique pro-interpretable (over* A*) group morphism* $g : G \to H$ *agreeing with* f *on realizations of* Σ.

In other words, there is an equivalence of categories between pregroups, pro-interpretable groups with a translation invariant definable partial type, and projective limits of interpretable groups with a translation invariant definable partial type. In particular, in Proposition 3.5, the map g is injective if and only if f is.

Now, let T be a geometric theory of enriched characteristic zero fields.

Theorem 3.6. *Let* $K \models T_\Delta$*, let* $A = \mathrm{dcl}_\Delta(A) \subseteq K$ *and let* Γ *be a group* L_Δ*-definable in* K *over* A*. Then there is a group* G *which is* L*-interpretable over* A *and a group embedding* $\Gamma \to G$ *which is* L_Δ*-definable in* K *over* A.

Proof. Let $\Sigma(x_\omega)$ be the global partial L-type such that $\Sigma(\nabla_\omega(x))$ is the partial type of generics in Γ. By Lemma 2.4, it is L-definable over A. By Corollary 1.4.3, there exists a map F which is pro-definable in L over A such that for every $a, b \in \Gamma$, we have $\nabla_\omega(a \cdot b) = F(\nabla_\omega(a), \nabla_\omega(b))$. Likewise, there are functions G_1 and G_2 which are pro-definable in L over A such that for any $a, b \in \Gamma$, we have $\nabla_\omega(a \cdot b^{-1}) = G_1(\nabla_\omega(a), \nabla_\omega(b))$ and $\nabla_\omega(a^{-1} \cdot b) = G_2(\nabla_\omega(a), \nabla_\omega(b))$.

Claim 3.7. (Σ, F) *is a pregroup.*

Proof. First fix $a \in \Gamma(K)$ and $b \in \Gamma(M)$. By Lemma 2.3, $a \cdot b$ is generic in Γ over K if and only if b is. Namely, $\nabla_\omega(b)$ realizes Σ if and only if $\nabla_\omega(a \cdot b) = F(\nabla_\omega(a), \nabla_\omega(b))$ also does. As $\Sigma(x_\omega)$ is the L-type of all tuples $\nabla_\omega(b)$ for $b \in \Gamma$ generic over K, it follows that for any $b_\omega \models \Sigma$, we have $F(\nabla_\omega(a), b_\omega) \models \Sigma$ and moreover, that every $c_\omega \models \Sigma$ is of the form $F(\nabla_\omega(a), b_\omega)$ for some $b_\omega \models \Sigma$. So we have $F(\nabla_\omega(a), \Sigma) = \Sigma$.

By definability of Σ, the set of tuples $a_\omega \in K$ such that $F(a_\omega, \Sigma) = \Sigma$ is pro-definable over A. As it includes $\nabla_\omega(a)$ for any $a \in \Gamma$, it also includes all realizations of $\Sigma|_A$. This yields condition 1 in Definition 3.2.

Condition 2 and 3 hold for similar reasons. For example, let us consider condition 2. Again, fix $a \in \Gamma(K)$ and $b \in \Gamma(M)$ and let $c = a \cdot b$. Then $\nabla_\omega(c) = F(\nabla_\omega(a), \nabla_\omega(b))$.

We also have $\nabla_\omega(a) = G_1(\nabla_\omega(c), \nabla_\omega(b))$ and $\nabla_\omega(b) = G_2(\nabla_\omega(a), \nabla_\omega(c))$. So, by definition of Σ, for every $b_\omega \models \Sigma$, we have $\nabla_\omega(a) = G_1(F(\nabla_\omega(a), b_\omega), b_\omega)$ and $b_\omega = G_2(\nabla_\omega(a), F(\nabla_\omega(a), b_\omega))$. Again, by definability of Σ over A, the set of tuples a_ω such that $a_\omega = G_1(F(a_\omega, b_\omega), b_\omega) = G_2(b_\omega, F(a_\omega, b_\omega))$ is pro-definable in L over A, and it contains $\nabla_\omega(a)$ for all $a \in \Gamma(K)$. In particular, it contains all realizations of $\Sigma|_A$. This yields condition 2 in Definition 3.2. □

Let us now come back to the proof of the theorem. By Propositions 3.3 and 3.4, we obtain a projective limit $G = \varprojlim_i G_i$ of groups which are L-interpretable over A, as well as an injective map f from realizations of $\Sigma|_A$ to G (in M) which is pro-definable in L over A and such that for every $a \models \Sigma|_A$ and $b \models \Sigma|_{Aa}$, we have $f(F(a,b)) = f(a) \cdot f(b)$.

Note that $\Sigma(\nabla_\omega(x))$ is the global definable over A translation invariant L_Δ-type of generics of Γ over K. We can therefore apply Proposition 3.5 to the map $f \circ \nabla_\omega$ from realizations of $\Sigma(\nabla_\omega(x))|_A$ to G to obtain a group embedding $g : \Gamma \to G = \varprojlim_i G_i$ which is pro-definable over A. By compactness, the composition of g with the projection on some G_i is already injective and this completes the proof. □

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