Analytic difference fields

Elimination of field quantifiers and Ax-Kochen-Eršov principle

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Take Martin's talk, insert analytic everywhere.

Thank you

More seriously

Let K be a complete valued field.

- Let $(a_i)_{i\in\mathbb{N}}$ be a sequence in K, $\sum_i a_i$ converges if and only if $a_i \to 0$.
- Let $f = \sum_i a_i X^i \in \mathcal{O}[[X]]$ and $c \in \mathfrak{M}$, then $a_i c^i \to 0$ and hence f can be evaluated at c.
- Let $f = \sum_i a_i X^i \in \mathcal{O}\langle X\rangle = \{\sum_i a_i X^i : a_i \to 0\}$ and $c \in \mathcal{O}$ then $a_i c^i \to 0$ and hence f can be evaluated at c
- Similarly, any $f \in \mathcal{O}\langle\overline{X}\rangle[[\overline{Y}]]$ can be evaluated at any $\overline{c} \in \mathcal{O}^{|\overline{X}|} \times \mathfrak{M}^{|\overline{Y}|}$.

Fields with analytic structure

Let A be a Noetherian ring, I an ideal and suppose that A is complete and separated in its I-adic topology. Let $\mathcal{A}_{m,n} := A\langle\overline{X}\rangle[[\overline{Y}]]$ where $|\overline{X}| = m$ and $|\overline{Y}| = n$, $\mathcal{A} := \bigcup_{m,n} \mathcal{A}_{m,n}$.

Definition: Field with (separated) (A, I)-analytic structure

A field with (A, I)-analytic structure is a valued field K with ring morphisms $i_{m,n} : \mathcal{A}_{m,n} \to \mathcal{O}^{\mathcal{O}^m \times \mathfrak{M}^n}$ such that:

- $i_{0,0}(I) \subseteq \mathfrak{M}$;
- $i_{m,n}(X_i) : \mathcal{O}^m \times \mathfrak{M}^n \to \mathcal{O}$ is the i-th projection function;
- $i_{m,n}(Y_j) : \mathcal{O}^m \times \mathfrak{M}^n \to \mathcal{O}$ is the $(m + j)$-th projection function (followed by the inclusion $\mathfrak{M} \subseteq \mathcal{O}$);
- The $i_{m,n}$ are compatible with the obvious injections $\mathcal{A}_{m,n} \to \mathcal{A}_{m+k,n+l}$.

Example

Any complete field K has a natural $(\mathcal{O}, \mathfrak{M})$-analytic structure.

Valued fields language

Definition: Angular component maps

An angular component map on a valued field K is a group morphism $\mathrm{ac} : K^\star \to k^\star$ such that:

$$\mathrm{ac}|_{\mathcal{O}^\star} = \mathrm{res}|_{\mathcal{O}^\star}$$

Example

- On $K((X))$, $\sum_{i>n} a_i X_i \mapsto a_i$ where $a_i \neq 0$ is an angular component map.
- On $\mathbb{Q}_p$, $\sum_{i>n} a_i p^i \mapsto a_i$ where $a_i \neq 0$ is also an angular component map.
- Any $\aleph_1$-saturated valued field can be endowed with an angular component map.

Valued fields language

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We will be considering valued fields in the following language $\mathcal{L}$:

- sorts $\mathbf{K}$, Γ, $\mathbf{k}$;
- the ring language on $\mathbf{K}$ and $\mathbf{k}$;
- the language of ordered abelian groups on Γ;
- $\mathrm{v} : \mathbf{K} \to \Gamma$ and $\mathrm{ac} : \mathbf{K} \to \mathbf{k}$.

Remark

This is not the right language to consider mixed characteristic (or equicharacteristic p) in. Angular component add new definable sets. There exists another language known as the RV language that does not have this flaw.

Analytic language

Fix a ring A and an ideal I as previously.

Definition

Let $\mathcal{L}_\mathcal{A} := \mathcal{L} \cup \mathcal{A} \cup \{\mathcal{Q}\}$ where

- $f \in \mathcal{A}_{m,n}$ is a function symbol $\mathbf{K}^{m+n} \to \mathbf{K}$;
- $\mathcal{Q}$ is a function symbol $\mathbf{K}^2 \to \mathbf{K}$;

Any field K with analytic (A, I)-structure can be naturally endowed with an $\mathcal{L}_\mathcal{A}$-structure:

- the symbols $f \in \mathcal{A}_{m,n}$ are interpreted as $i_{m,n}(f)$ (extended by 0 outside of their domain);
- $\mathcal{Q}(x, y)$ is interpreted as x/y when y is not 0 and 0 otherwise;

Henselian valued fields

Definition

A valued field K is said to be Henselian if for all $P \in \mathcal{O}[X]$ and $a \in \mathcal{O}$ such that:

$$\mathrm{v}(P(a)) > 2\,\mathrm{v}(P'(a)),$$

then there exists $b \in \mathcal{O}$ such that:

$$P(b) = 0 \text{ and } \mathrm{v}(b - a) = \mathrm{v}(P(a)) - \mathrm{v}(P'(a)).$$

Example

- Any $K((X))$ is Henselian.
- $\mathbb{Q}_p$ is Henselian.

Let $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathcal{H},0,0}$ be the $\mathcal{L}_{\mathcal{A}}$-theory of equicharacteristic zero Henselian fields with (A, I)-analytic structure and angular components.

Completions

Definition: Balls

Let K be a valued field, $c \in K$ and $\gamma \in \mathrm{v}(K)$.

- The open ball of radius γ around c is $\{x \in K : \mathrm{v}(x-c) > \gamma\}$.
- The closed ball of radius γ around c is $\{x \in K : \mathrm{v}(x-c) \geq \gamma\}$.

Definition: Spherically complete valued fields

A valued field K is said to be spherically complete for every decreasing sequence of balls $(b_i)_{i\in I}$ in K, there exists

$$c \in \bigcap_{i\in I} b_i$$

Proposition

Every spherically complete valued field is Henselian.

Analytic results I

Theorem: Cluckers-Lipshitz-Robinson...

$T^{ac}_{\mathcal{A},\mathcal{H},0,0}$ eliminates field quantifiers resplendently.

Definition: Resplendent elimination of field quantifiers

An $\mathcal{L}_{\mathcal{A}}$-theory T is said to eliminate field quantifiers resplendently if whenever $\mathcal{L}'$ is an enrichment of $\mathcal{L}_{\mathcal{A}}$ on $\mathbf{k}$ and Γ and $T' \supseteq T$ is an $\mathcal{L}'$-theory, then T' eliminates field quantifiers.

Remark

The key property behind this result is Weierstrass preparation that allows us to reduce questions about 1-types in $T^{ac}_{\mathcal{A},\mathcal{H},0,0}$ to purely algebraic considerations (and use field quantifier elimination result in Henselian fields).

Analytic results II

Let $\mathcal{T}$ be the set of all $\mathcal{L}_{\mathcal{A}}$-terms (from the sort **K**) without free variables.

Corollary: Analytic Ax-Kochen-Eršov

Let K and $L \models \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathcal{H},0,0}$ then:

$$K \equiv L \iff \begin{cases} \mathbf{k}(K) \equiv \mathbf{k}(L) & \text{as rings with constants added for the } \mathrm{ac}(t) \text{ where } t \in \mathcal{T}; \\ \Gamma(K) \equiv \Gamma(L) & \text{as ordered abelian groups with constants added for the } \mathrm{v}(t) \text{ where } t \in \mathcal{T}. \end{cases}$$

Corollary: Analytic NIP Ax-Kochen-Eršov

Let $K \models \mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\mathcal{H},0,0}$ then:

$$\mathrm{Th}(K) \text{ is NIP} \iff \mathrm{Th}(\mathbf{k}(K)), \text{ as a ring, is NIP}.$$

Remark

Resplendent versions of these statements hold.

And now for something completely different...

Isometries

Definition

Let K be a valued field. An isometry on K is an automorphism σ of valued fields such that for all $x \in K$, $\mathrm{v}(\sigma(x)) = \mathrm{v}(x)$.

- Any valued field automorphism induces an automorphism of the residue field $\sigma_{\mathbf{k}}(x)$.
- We will write $\mathrm{Fix}(K)$ for the fixed field.

Definition: Residually linearly closed

An valued field with an isometry K is said to be residually linearly closed if for every non zero tuple $\overline{a} \in \mathbf{k}$ and $b \in \mathbf{k}$, the equation $\sum_i a_i \sigma_{\mathbf{k}}^i(x) = b$ has a solution.

Definition: Enough Constants

A valued field with an isometry K is said to have enough constants if for every $x \in K$ there exists $y \in \mathrm{Fix}(K)$ such that $\mathrm{v}(x) = \mathrm{v}(y)$.

A new notion of Henselianity

We will write $\overline{\sigma}(a)$ for the tuple $a, \dots, \sigma^n(a)$.

Definition: σ-Henselianity

A valued field with an isometry K is said to be σ-Henselian if for any $P \in \mathcal{O}[X_0, \dots, X_n]$ and $a \in \mathcal{O}$ such that:

$$v(P(\overline{\sigma}(a))) > 2\min_i\{\mathrm{v}(\frac{\partial P}{\partial X_i}(\overline{\sigma}(a)))\},$$

then there exists $b \in \mathcal{O}$ such that:

$$P(\overline{\sigma}(b)) = 0 \text{ and } \mathrm{v}(b - a) = \mathrm{v}(P(\overline{\sigma}(a))) - \min_i\{\mathrm{v}(\frac{\partial P}{\partial X_i}(\overline{\sigma}(a)))\}.$$

Proposition

Every residually linearly closed spherically complete field with an isometry is σ-Henselian.

Yet another language

Definition

Let $\mathcal{L}_\sigma := \mathcal{L} \cup \{\sigma, \sigma_{\mathbf{k}}\}$ where σ is a function symbol $\mathbf{K} \to \mathbf{K}$ and $\sigma_{\mathbf{k}}$ is a function symbol $\mathbf{k} \to \mathbf{k}$.

Any valued field with an isometry can be naturally endowed with an $\mathcal{L}_\sigma$-structure.

Definition

Let $T^{ac}_{\sigma-\mathcal{H},0,0}$ be the $\mathcal{L}_\sigma$-theory of equicharacteristic zero σ-Henselian non trivially valued fields with an isometry and enough constants.

Remark

Models of $T^{ac}_{\sigma-\mathcal{H},0,0}$ are residually linearly closed.

Valued difference results

Theorem: Azgin-van den Dries, 2010

$T^{ac}_{\sigma-\mathcal{H},0,0}$ eliminates field quantifiers resplendently.

Corollary: Valued difference Ax-Kochen-Eršov

Let K and $L \vDash T^{ac}_{\sigma-\mathcal{H},0,0}$ then:

$$K \equiv L \iff \begin{cases} \mathbf{k}(K) \equiv \mathbf{k}(L) & \text{as difference rings;} \\ \Gamma(K) \equiv \Gamma(L) & \text{as ordered abelian groups.} \end{cases}$$

Corollary: Valued difference NIP Ax-Kochen-Eršov

Let $K \vDash T^{ac}_{\sigma-\mathcal{H},0,0}$ then:

$$\mathrm{Th}(K) \text{ is NIP} \iff \mathrm{Th}(\mathbf{k}(K)), \text{as a difference ring, is NIP}.$$

And now...

In stereo.

Analytic valued fields with an isometry

Definition

Let K be a field with (A, I)-analytic structure. For all m, n choose an automorphism $\begin{array}{ccc} \mathcal{A}_{m,n} & \to & \mathcal{A}_{m,n} \\ f & \mapsto & f^{\sigma} \end{array}$. An isometry of K (as an analytic field) is an isometry of K such that for all $f \in \mathcal{A}$ and $\bar{x} \in K$,

$$\sigma(f(\bar{x})) = f^{\sigma}(\sigma(\bar{x}))$$

also holds.

If all the automorphisms are the identity then we are just asking that σ is an automorphism of K as a field with (A, I)-analytic structure.

Definition

Let $\mathcal{L}_{\mathcal{A},\sigma} := \mathcal{L}_{\mathcal{A}} \cup \mathcal{L}_{\sigma}$.

Some differentiation

Definition: Linear approximation

Let K be a valued field with an isometry, $f : K^n \to K$, $\overline{d} \in K^n$, $a \in K$ and $\gamma \in \mathrm{v}(K)$. We say that $\overline{d}$ is a linear approximation of f (at prolongations) around a with radius γ if for all $b \in K$ such that $\mathrm{v}(b - a) > \gamma$ and $\varepsilon \in K$ such that $\mathrm{v}(\varepsilon) > \gamma$:

$$\mathrm{v}(f(\overline{\sigma}(b+\varepsilon)) - f(\overline{\sigma}(b)) - \sum_i d_i \sigma^i(\varepsilon)) > \min_i\{\mathrm{v}(d_i)\} + \mathrm{v}(\varepsilon)$$

Remark

- Not very compatible with sum, product, composition...
- If f is continuous differentiable around a then it is linearly approximated by its derivatives.
- Very much akin to the Jacobian property of Cluckers-Lipshitz.

σ-Henselianity, take 2

Definition

An analytic field with an isometry K is said to be σ-Henselian if for every $\mathcal{L}_{\mathcal{A},\sigma}$-term $t : \mathbf{K}^n \to \mathbf{K}$, $a \in K$, $\overline{d} \in K^n$ and $\gamma \in \mathrm{v}(K)$ such that $\overline{d}$ linearly approximates t around a with radius γ and:

$$\mathrm{v}(t(\overline{\sigma}(a))) > \min_i\{\mathrm{v}(d_i)\} + \gamma,$$

then there exists $b \in K$ such that:

$$t(\overline{\sigma}(b)) = 0 \text{ and } \mathrm{v}(b - a) = \mathrm{v}(t(\overline{\sigma}(a))) - \min_i\{\mathrm{v}(d_i)\}.$$

Remark

As any $P \in \mathcal{O}[X_0, \ldots, X_n]$ is linearly approximated by its (formal) derivatives at any $a \in \mathcal{O}$ with radius $\min_i\{\mathrm{v}(\frac{\partial P}{\partial X_i}(\overline{\sigma}(a)))\}$, for difference polynomial, this new form of σ-Henselianity is actually equivalent to the previous one.

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Let $\mathrm{T}^{\mathrm{ac}}_{\mathcal{A},\sigma-\mathcal{H},0,0}$ be the theory of equicharacteristic zero σ-Henselian non trivially valued fields with (A, I)-analytic structure, an isometry and enough constants.

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Proposition

Every residually linearly closed spherically complete field with (A, I)-analytic structure and an isometry is σ-Henselian.

Analytic difference results

Theorem: R., 2013

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Thank you

(For real this time)