Imaginaries in pseudo-p-adically closed fields

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Bounded pseudo-p-adically closed fields

Definition

- A valuation is *p-adic* if the residue field is $\mathbb{F}_p$ and p has minimal positive valutation.
- A field extension $K \leq L$ is *totally p-adic* if every p-adic valuation of K can be extended to a p-adic valuation of L.
- A field K is *pseudo-p-adically closed* if it is existentially closed (as a ring) in every regular totally p-adic extension.
- A field is *bounded* if it has finitely many extensions of any given degree.

Proposition (Montenegro)

Let K be a bounded pseudo-p-adically closed field. There are finitely many p-adic valuations on K and they are definable in the ring language.

The geometric language

Let (K, v) be a valued field.

Definition (Geometric sorts)

- We define $\mathbf{S}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ and $\mathbf{T}_n := \mathrm{GL}_n(K)/\mathrm{GL}_{n,n}(\mathcal{O})$.
- The *geometric language* $\mathcal{L}^{\mathcal{G}}$ has sorts $\mathbf{F}$, $\mathbf{S}_n$ and $\mathbf{T}_n$ for all $n \geq 1$. It also contains the ring language on $\mathbf{F}$, the canonical projections $s_n : \mathrm{GL}_n(\mathbf{F}) \to \mathbf{S}_n$ and $t_n : \mathrm{GL}_n(\mathbf{F}) \to \mathbf{T}_n$.

Remark

$\mathbf{S}_1 \simeq \Gamma$ and s_1 can be identified with the valuation.

Theorem

- Algebraically closed valued fields eliminate imaginaries in $\mathcal{L}^{\mathcal{G}}$ (HHM).
- $\mathbb{Q}_p$ eliminates imaginaries in $\mathcal{L}^{\mathcal{G}}$ (HMR).

An orthogonality result

- Let K be a bounded pseudo-p-adically closed fields with n p-adic valuations $(\mathrm{v}_i)_{i\leq n}$.
- Let $\mathcal{L}_i$ denote n copies of $\mathcal{L}^{\mathcal{G}}$, with sorts $\mathcal{G}_i$, sharing the sort $\mathbf{F}$.
- Let $K_0 \preccurlyeq K$, $\mathcal{L} = \bigcup_i \mathcal{L}_i \cup K_0$ and $T = \mathrm{Th}_{\mathcal{L}}(K)$.
- Let $M \vDash T$, $\overline{M}_i$ be the algebraic closure of M with an extension of v_i and M_i be the p-adic closure of M inside $\overline{M}_i$

Remark

Let $U_i \neq \varnothing$ be v_i-open, then $\bigcap_i U_i \neq \varnothing$.

Proposition

Let $K_0 \subseteq A \subseteq \mathbf{F}(M)$ and $s_i, t_i \in \mathbf{S}_{i,n}(M)$. If

$$\forall i,\ s_i \equiv^{M_i}_{\mathcal{L}_i(A)} t_i$$

then

$$(s_i)_{i\leq n} \equiv^{M}_{\mathcal{L}(A)} (t_i)_{i\leq n}.$$

A local density result

Let $A \subseteq M^{\mathrm{eq}}$ containing $\bigcup_i \mathcal{G}_i(\mathrm{acl}^{\mathrm{eq}}_M(A))$.

Proposition

Let $c \in \mathbf{F}(M)$. Then, for all i, there exists an $\mathcal{G}_i(A)$-invariant $\mathcal{L}_i(\overline{M}_i)$-type p_i such that $\mathrm{tp}^M_{\mathcal{L}}(c/A) \cup \bigcup_i p_i$ is consistent.

Proposition

Let $c \in \mathbf{F}(M)$ and $d \in \mathcal{G}_i(\mathrm{acl}^{\mathrm{eq}}_M(Ac))$ be some tuples. Assume $\mathrm{tp}^{\overline{M}_i}_{\mathcal{L}_i}(c/\overline{M}_i)$ is $\mathcal{G}_i(A)$-invariant, then so is $\mathrm{tp}^{\overline{M}_i}_{\mathcal{L}_i}(d/\overline{M}_i)$.

Corollary

Let $c \in \mathbf{F}(M)$ be some tuple. Then, for all i, there exists a $\mathcal{G}_i(A)$-invariant $\mathcal{L}_i(M_i)$-type p_i such that $\mathrm{tp}^M_{\mathcal{L}}(c/A) \cup \bigcup_i p_i$ is consistent.

A criterion using amalgamation

- Let $(\mathcal{L}_i)_{i\in I}$ be languages, with sorts $\mathcal{R}_i$, sharing a dominant sort $\mathcal{D}$, and let $\mathcal{L} \supseteq \bigcup_i \mathcal{L}_i$.
- Let T_i be $\mathcal{L}_i$-theories and $T \supseteq \bigcup_i T_{i,\forall}$ be an $\mathcal{L}$-theory.
- Let $M \vDash T$ and $M \subseteq M_i \vDash T_i$ be sufficiently saturated and homogeneous.
- For all $C \leq M^{\text{eq}}$ and all tuples $a, b \in \mathcal{D}(M)$, write $a \downarrow_C b$ if there are $\mathcal{R}_i(A)$-invariant $\mathcal{L}_i(M_i)$-types p_i with $a \vDash \bigcup_i p_i|_{\mathcal{R}_i(C)b}$.

Proposition

Assume:

1. for all $A = \text{acl}^{\text{eq}}_M(A) \subseteq M^{\text{eq}}$ and tuple $c \in \mathcal{D}(M)$, there exists $d \equiv^M_{\mathcal{L}(A)} c$ with $d \downarrow_A M$;
2. For all $A = \text{acl}_M(A) \subseteq M$ and $a, b, c \in \mathcal{D}(M)$ tuples, if $b \downarrow_A a$, $c \downarrow_A ab$, $a \equiv^M_{\mathcal{L}(A)} b$ and $ac \equiv^{M_i}_{\mathcal{L}_i(\mathcal{R}_i(A))} bc$, for all i, then there exists d such that $db \equiv^M_{\mathcal{L}(A)} da \equiv^M_{\mathcal{L}(A)} ca$.

Then T weakly eliminates imaginaries.

Amalgamation

In pseudo-p-adically closed fields, Condition 2 follows from a more general result:

Proposition

Let $M \models T$, $A \subseteq M$ and $a_1, a_2, c_1, c_2, c \in \mathbf{F}(M)$ be tuples. Assume $\overline{\mathbf{F}(A)}^{\mathrm{a}} \cap M \subseteq A$, $\overline{\mathbf{F}(A)(a_1)}^{\mathrm{a}} \cap \overline{\mathbf{F}(A)(a_2)}^{\mathrm{a}} = \overline{\mathbf{F}(A)}^{\mathrm{a}}$, $c \downarrow_A a_1 a_2$, $c_1 \equiv^M_{\mathcal{L}(A)} c_2$, $c \equiv^{M_i}_{\mathcal{L}_i(Aa_1)} c_1$ and $c \equiv_{\mathcal{L}_i(Aa_2)} c_2$, for all i. Then

$$\mathrm{tp}^M_{\mathcal{L}}(c_1/Aa_1) \cup \mathrm{tp}^M_{\mathcal{L}}(c_2/Aa_2) \cup \bigcup_i \mathrm{tp}^{M_i}_{\mathcal{L}}(c/Aa_1a_2)$$

is satisfiable.

Remark

- If $A \subseteq \mathbf{F}$, this is an earlier result of Montenegro.
- The general result follows from the older version and the description of the structure on the geometric sorts given by the orthogonality result.

Elimination of imaginaries

Theorem

The theory T eliminates imaginaries.

Remark

- Coding finite sets is not completely obvious.
- Since the valuations v_i are discrete, $\mathbf{T}_{i,n}$ is coded in $\mathbf{S}_{i,n+1}$.
- Let $\mathcal{O} = \bigcap_i \mathcal{O}_i$. We have a bijection

$$\prod_i \mathbf{S}_{i,n} = \prod_i \mathrm{GL}_n(\mathbf{F})/\mathrm{GL}_n(\mathcal{O}_i) \simeq \mathrm{GL}_n(\mathbf{F})/\mathrm{GL}_n(\mathcal{O}).$$