Asymptotic imaginaries in $(\mathbb{F}_p((t))^{\mathrm{alg}}, \mathrm{v}, \phi_p)$

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What is your name?

Definition

A valuation on a field K is a group morphism $\mathrm{v} : K^\times \to \Gamma$, where $(\Gamma, +, <)$ is an ordered Abelian group such that:

- for all $x, y \in K$, $\mathrm{v}(x+y) \geq \min\{\mathrm{v}(x), \mathrm{v}(y)\}$.

We often denote $\mathrm{v}(0) = \infty > \Gamma$.

We write:

- $\mathcal{O} := \{x \in K \mid \mathrm{v}(x) \geq 0\}$;
- $\mathfrak{m} := \{x \in K \mid \mathrm{v}(x) > 0\}$;
- $k := \mathcal{O}/\mathfrak{m}$.

Example

- $k(t)$ with the degree of the root/pole at 0;
- its completion $k((t))$;
- more generally, the Hahn series field $k((\Gamma))$.

What is your quest?

Let M be an $\mathcal{L}$-structure and $X \subseteq Y \times Z$ be $\mathcal{L}$-definable.

- We wish to understand:

$$\{X_y \mid y \in Y\}$$

  where $X_y := \{z \in Z \mid (y, z) \in X\}$.

- In other words, we wish to describe Y/E where:

$$y_1 E y_2 \iff \forall z \, (y_1, z) \in X \leftrightarrow (y_2, z) \in X.$$

- In practice, we are looking for an $\mathcal{L}$-definable $f : Y \to W$ such that:

$$y_1 E y_2 \iff f(y_1) = f(y_2).$$

  Then f induces an embedding:

$$Y/E \to W.$$

- If such an f always exists, we say that T eliminates imaginaries.

What is the air speed velocity of an unladen swallow?

The structure $(\mathbb{F}_p((t))^{\mathrm{alg}}, \mathrm{v}, \phi_p)$ is a definable expansion of $\mathrm{ACVF}_{p,p}$.

- $S_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$, the moduli space of free rank n $\mathcal{O}$-submodules of K^n;
- $T_n := \bigsqcup_{s \in S_n} s/\mathfrak{m}s$, a family of n-dimensional k-vector spaces.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

ACVF eliminates imaginaries up to $\mathcal{G} := K \cup \bigcup_n (S_n \cup T_n)$.

Question

What can we say as p goes to infinity?

Let $\mathcal{F}$ be an non principal ultrafilter on the set of primes.

$$\prod_{p \to \mathcal{F}} (\mathbb{F}_p((t))^{\mathrm{alg}}, \mathrm{v}) \models \mathrm{ACVF}_{0,0}.$$

An African or European swallow?

Let $(K, \mathrm{v}, \sigma) := \prod_{p \to \mathcal{F}} (\mathbb{F}_p((t))^{\mathrm{alg}}, \mathrm{v}, \phi_p)$.

- Its residue field is $(k, \sigma_k) := \prod_{p \to \mathcal{F}} (\mathbb{F}_p^{\mathrm{alg}}, \phi_p)$;
- Its value group is $(\Gamma, \sigma_\Gamma) := \prod_{p \to \mathcal{F}} (\mathbb{Q}, p)$

Theorem (Durhan/Azgin, 2010)

$$(K, \mathrm{v}, \sigma) \equiv (k((t^\Gamma)), \mathrm{v}, \sigma)$$

where $\sigma(\sum_{\gamma \in \Gamma} c_\gamma t^\gamma) := \sum_{\gamma \in \Gamma} \sigma_k(c_\gamma) t^{\sigma_\Gamma(\gamma)}$.

- (Γ, σ_Γ) is an ω-increasing, torsion free, divisible $\mathbb{Z}[\sigma_\Gamma]$-module.

Theorem (Hrushovski)

$$(k, \sigma_k) \models \mathrm{ACFA}_0.$$

And that is how we know the Earth to be banana-shaped.

Let $\mathrm{RV} = K^\times/1+\mathfrak{m}$, $\mathcal{L}_{\mathrm{RV}}$ be the language with sorts K and RV.

Definition

Let VFA_0 be the common theory of non principal ultraproducts:

$$\prod_{p \to \mathcal{F}} (\mathbb{F}_p((t))^{\mathrm{alg}}, \mathrm{v}, \phi_p)$$

in $\mathcal{L} := \mathcal{L}_{\mathrm{RV}} \cup \{\sigma, \sigma_{\mathrm{RV}}\}$.

Theorem (Pal, 2012)

VFA_0 eliminates field quantifiers.

Theorem (Hils-RK.)

VFA_0 eliminates imaginaries up to $\mathcal{G}$.

One...

Let T be an enrichment of ACVF, $M \models T$ be sufficiently large and X be (strict pro-)$\mathcal{L}(M)$-definable. We define:

- $\mathfrak{A} := \{\mathrm{aut}(M) \mid \sigma(X) = X\}$;
- $E := \{e \in \mathcal{G}(M) \mid \mathfrak{A} \cdot e \text{ is finite}\}$.

Let tp_0 denote the type in the underlying ACVF-reduct.

Proposition (RK., 2015)

Assume that k and Γ eliminate $\exists^\infty$ in their full induced structure. Then, there exists $a \in X$ in some elementary extension, with $\mathrm{tp}_0(a/M)$ $\mathcal{L}(E)$-definable.

In VFA_0:

- Γ is stably embedded o-minimal;
- k is a stably embedded pure model of ACFA_0.

Two...

Let $M \models \mathrm{VFA}_0$, $E \subseteq \mathcal{G}(A)$ and $a \in N \succcurlyeq M$ a tuple.

- Let $\nabla(a) := (\sigma^n(a))_{n>0}$.

Theorem (Field quantifier elimination)

Let $\rho(a) \subseteq \mathrm{dcl}_0(K\nabla(a))$ with $\mathrm{rv}(K(\nabla(a))) \subseteq \mathrm{dcl}_0(K\rho(a))$. Then

$$\mathrm{tp}_0(\nabla(a)/K) \cup \mathrm{tp}(\rho(a)/K) \vdash \mathrm{tp}(a/K).$$

Assume $\mathrm{tp}_0(\nabla(a)/K)$ is $\mathcal{L}(E)$-definable.

- Let $\mathrm{Lin}_E := \bigsqcup_{s \in S(E)} s/\mathfrak{m}s$ and $\mathrm{D}_E := E \cup \mathrm{RV} \cup \mathrm{Lin}_E$.

Proposition

There exists a tuple $\rho(a) \in \mathrm{D}_E(\mathrm{dcl}_0(K\nabla(a)))$, whose germ is $\mathrm{aut}(K/E)$-invariant, such that $\mathrm{rv}(K(\nabla(a))) \subseteq \mathrm{dcl}_0(\mathrm{D}_E(K)\rho(a))$.

Corollary

The type $\mathrm{tp}(a/K)$ is $\mathrm{aut}(K/\mathrm{D}_E(K))$-invariant.

Five!

- Lin_E is a family of k-vector spaces and for every $s \in E$, an isomorphism $\sigma_s : s/\mathfrak{m}s \to \sigma(s)/\mathfrak{m}\sigma(s)$.

Proposition (adapted from Hrushovski, 2012)

Lin_E eliminates imaginaries.

- RV is a short exact sequence of $\mathbb{Z}[\sigma]$-modules with an enriched kernel.

Proposition

- The structure RV eliminates imaginaries relative to $\mathrm{RV} \cup \Gamma$.
- The structure D_E eliminates imaginaries relative to $\mathrm{D}_E \cup \Gamma$.

Theorem

VFA_0 eliminates imaginaries up to $\mathcal{G}$.

Fetchez la vache!