Prolongations in valued fields

Silvain Rideau

UC Berkeley

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where $(k, \sigma_k) \models \mathrm{ACVA}_0$, Γ is divible and σ_{Γ} is ω-increasing, or:

$$\prod_{p \to \mathfrak{U}} (\overline{\mathbb{F}_p(t)}, \mathrm{v}_t, \phi_p)$$

where $\mathfrak{U}$ is a non-principal ultrafilter on the set of primes.

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  or, when V is irreducible, with the complete type in $\mathcal{L}_{\mathrm{rg}} = \{+, -, \cdot, 0, 1\}$ over K:

$$p_\omega := \{P(\mathcal{D}_n(x)) = 0 : P \in I\} \cup \{P(\mathcal{D}_n(x)) \neq 0 : P \notin I\}.$$

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VFA_0 The map $p \mapsto p_\omega$ is not injective. Indeed the corresponding map on the residue field is not.

Definable types

- Let $K = \overline{K}$, $p(x)$ a complete type in $\mathcal{L}_{\text{rg}}$ over K, $I = \{P : "P(x) = 0" \in p\}$ and $V = V(I)$. For all varieties W over K, we have:

$$"x \in W" \in p \text{ if and only if } V \subseteq W.$$

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Definition

A complete type $p(x)$ over M is said to be definable if for every formula $\phi(x,y)$, there exists a formula $\theta(y)$ (with parameters in M) such that, for all $a \in M^y$:

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- All complete types in algebraically (respectively differentially) closed fields are definable.
- The generic type of a ball in an algebraically closed valued field is definable.

Prolongations of definable types

- If $p(x)$ is definable, then for all $\mathcal{L}$-formula $\phi(x, y)$, there exists an $\mathcal{L}_{\mathcal{D}}$-formula $\theta(y)$ such that for all $a \in K^y$, $\phi(x, a) \in p_\omega$ if and only if $\theta(a)$ holds.

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Proposition (R.-Simon, Hils-Kamensky-R., Hils-R.)

In $\mathrm{VDF}_{\mathcal{EC}}$, $\mathrm{SCVH}_{p,e}$ and VFA_0, if p is definable then p_ω is definable.

Density of definable types

- Ideally, each definable X set should contain a ”canonical” definable type: a definable type which is defined over the same parameters as X and does not depend on the choice of parameters.

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In $\mathrm{VDF}_{\mathcal{EC}}$ and $\mathrm{SCVH}_{p,e}$, if X is a definable set, there exists a definable type $p(x)$ containing "$x \in X$". Moreover, for any A such that X is definable over A, p is definable over A.

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- In fact, we build p_ω first and then get p.
- A similar result holds for VFA_0 but because of the lack of elimination of quantifiers, we have to consider quantifier free types.

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- Algebraically closed valued fields do not eliminate imaginaries. By a result of Haskell-Hrushovski-Macpherson, they do once you add points for elements of $\mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$ and $\mathrm{GL}_n(K)/\ker(\rho)$ where $\rho : \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{m})$ is the reduction map. We then say that we are working in the geometric language.

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Theorem (R.,Hils-Kamensky-R.)

$\mathrm{VDF}_{\mathcal{EC}}$ and $\mathrm{SCVH}_{p,e}$ eliminate imaginaries in the geometric language.