Pseudo T-closed fields, approximations and NTP_2

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A local-global principle

For $i < n$, let $\mathcal{L}_i \supseteq \mathcal{L}_{\text{rg}}$ and T_i be an $\mathcal{L}_i$-theory of large fields.

Proposition

Let K be a field. The following are equivalent:

1. For every geometrically integral variety V over K, if, for any $L \models T_{i,K}$, $V_{\text{sm}}(L) \neq \varnothing$, then $V(K) \neq \varnothing$; — equiv. $\overline{V(K)}^{\text{z}} = V$;
2. Any regular extension $K \leq F$, which $\mathcal{L}_{\text{rg}}(K)$-embeds in some $L^\star \succcurlyeq L$, for every $L \models T_{i,K}$, is $\mathcal{L}_{\text{rg}}$-existentially closed.

When the above hold, we say that K is *pseudo T-closed*.

Example

- $T_0 = \text{ACF}$, $\text{P}T\text{C} = \text{PAC}$;
- $T_0 = \text{RCF}$, $\text{P}T\text{C} = \text{PRC}$;
- $T_{i<n} = \dots$, $\text{P}T\text{C} = \text{P}\mathcal{S}^\tau\text{CC}$;

A local-not-so-global principle

Let $\mathcal{L} = \bigcup_i \mathcal{L}_i$ and $T = \bigcup_i T_{i,\forall}$.

- Assume that every T_i eliminates quantifiers[1].

Proposition

Let $K \models T$. The following are equivalent:

1. K is pseudo T-closed;
2. Any regular extension $K \leq F \models T$ is $\mathcal{L}_{\mathrm{rg}}$-existentially closed.

Example

- $T_{i<n} = \mathrm{RCF}_<$, $PTC = n\mathrm{-PRC}$;
- $T_{i<n} = \mathrm{ACVF}$, $PTC = \mathrm{PAC}$ (+ n valuations).

[1] From now on, we will only consider quantifier free $\mathcal{L}_i$-formulas.

Strong pseudo T-closed fields

Assume that:

- there is a definable henselian V-topology τ_i in T_i;
- for all $L_i \models T_i$, any $\mathcal{L}_i(L_i)$-definable set X has non empty τ_i-interior in $\overline{X}^{z}$.

Proposition

Let $K \models T$. The following are equivalent:

1. Any regular extension $K \leq L \models T$ is $\mathcal{L}$-existentially closed.
2. For every geometrically integral variety V over K, and $p_i \in \mathcal{S}_i(K)$ generic in V, $\bigcup_i p_i$ is realised in some $K^{\star} \succcurlyeq K$;
3. For every $L_i \models T_{i,K}$, every geometrically integral variety V over K and every $\mathcal{L}_i(K)$-definable non-empty τ_i-open $U_i \subseteq V_{\mathrm{sm}}(L_i)$, we have $\bigcap_i U_i(K) \neq \varnothing$.

When the above hold, we say that K is *strongly pseudo T-closed*.

Examples

- Prestel: $T_{i<n} = \mathrm{RCF}_<$, SPTC = n–PRC = PTC.
- $T_{i<n} = p\mathrm{CF}_{\mathrm{Mac}}$, SP$T$C = n–PpC = PTC.
- Heinemann-Prestel, Schmid: Fix $K \models T$, let $K_i = K^s \cap L_i \models T_{i,K}$. Assume that no K_i embeds in some K_j, with $i \neq j$, and $K_i \neq K_i^s$. Then $K \models$ PTC if and only if $K \models$ SPTC.
- Kollar: T_0 = ACVF, SPTC = PAC + 1 valuation = PTC.
- Johnson: $T_{0<i\leq n}$ = ACVF, $\mathcal{L}$-existentially closed models of T are SPTC — and they are exactly the models of T_0 with n independent valuations, independent from τ_0.

Density

Fix $K \models \mathrm{P}T\mathrm{C}$, $L_i \models T_{i,K}$, $K_i = K^{\mathrm{s}} \cap L_i$ and τ a V-topology on K.

- Assume (K_i, τ_i) is not discrete.
- Let $C_\tau := \{K_i : \tau_i \text{ induces } \tau\} \cap \{K^{\mathrm{s}}\}$.
- Say that $F \in C_\tau$ is minimal if any $\mathcal{L}_{\mathrm{rg}}(K)$-embedding $E \to F$, with $E \in C_\tau$, is surjective.
- Let $K_\tau = \widehat{(K, \tau)} \cap K^{\mathrm{s}}$.

Proposition

K_τ is $\mathcal{L}_{\mathrm{rg}}(K)$-homeomorphic to any minimal $F \in C_\tau$.

- In particular, K is τ-dense in F.

Density, proof

Proposition

K_τ is $\mathcal{L}_{\mathrm{rg}}(K)$-homeomorphic to any minimal $F \in C_\tau$.

- If $P \in K[x]$ is separable and has a root in $L_i \vDash T_{i,K}$, for all i, then P has a root in $\widehat{(K,\tau)}$.
- Some $E \in C_\tau$ can be $\mathcal{L}_{\mathrm{rg}}(K)$-embedded in K_τ.
- K_τ continuously $\mathcal{L}_{\mathrm{rg}}(K)$-embeds in any $E \in C_\tau$.

Approximation

Let $\tau_{j<m}$ be distinct V-topologies on K and K_j be minimal in C_{τ_j}.

Corollary

For every non-empty τ_j-open $U_j \leq K_j$, $\bigcap_j U_j(K) \neq \varnothing$.

Proposition

Let C be a smooth geometrically integral projective curve over K, with $C(K) \neq \varnothing$. For every non-empty τ_j-open sets $U_j \subseteq C(K_j)$, $\bigcap_{j<m} U_j(K)$ is infinite.

Corollary

Let V be a geometrically integral variety over K, with $V(K) \neq \varnothing$. For every non-empty τ_j-open sets $U_j \subseteq V_{\mathrm{sm}}(K_j)$, $\bigcap_{j<m} U_j(K) \neq \varnothing$.

Approximation, proof I

Proposition

Let C be a smooth geometrically integral projective curve over K, with $C(K) \neq \varnothing$. For every non-empty τ_j-open sets $U_j \subseteq C(K_j)$, $\bigcap_{j<m} U_j(K)$ is infinite.

- There exists $f : C \to \mathbb{P}^1$ over K with $f^{-1}(0:1) \subseteq \bigcap_{j>0} U_j(K)$ which is smooth above $(0:1)$.
- Let e be maximal such that

$$\{t \in \mathbb{P}(K_0) : \exists y_1 \dots y_e \in C(K_0)\, f(y_j) = t \wedge y_1 \in U_0\}$$

  is infinite, D be some irreducible component of the e-fold product of C over $\mathbb{P}^1$, $g : D \to C$ and $h = f \circ g$.
- By the $\mathbb{P}^1$ case, we may assume that τ_0-locally around $(0:1)$, $f|_{U_0}$ is surjective and h is surjective and smooth.

Approximation, proof II

$$\begin{tikzcd} D \arrow[r, "g"] \arrow[rr, bend right, "h"'] & C \arrow[r, "f"] & \mathbb{P}^1 \end{tikzcd}$$

- Let $B_s \subseteq D \times D$ be given by $(ss_1s_2 - t_1t_2) \circ (h \times h) = 0$.

Lemma (Kollar)

For all but finitely many s, B_s is geometrically integral.

- By the $\mathbb{P}^1$ case, we find $(1 : s) \in \mathbb{P}^1(K)$ arbitrarily close to $(1 : 0)$ for each τ_j, such that B_{s^2} is geometrically integral.
- By induction, we find infinitely many $(y_1, y_2) \in B_{s^2}(K)$ with $h(y_\ell) \in \bigcap_{j>0} f(U_j)$.
- For one ℓ, $h(y_\ell) \in f(U_0)$.
- By maximality of e, we may assume that $g(y_\ell) \in U_0$.
- So $g(y_\ell) \in \bigcap_{j<m} U_j$.

Automatic $\mathcal{L}$-existential closedness

- Assume that K_i is dense in some $L_i \models T_{i,K}$.

Theorem

The following are equivalent:

- $K \models \mathrm{P}T\mathrm{C}$ and the τ_i are pairwise distinct.
- $K \models \mathrm{SP}T\mathrm{C}$.

Example

- $T_{i<n} = \mathrm{ACVF}$, $\mathrm{SP}T\mathrm{C} = \mathrm{PAC} + n$ independent valuations.
- $T_{i<n} = \mathrm{RCF}_<$ and $T_{n\leq i<n+m} = \mathrm{ACVF}$, $\mathrm{SP}T\mathrm{C} = n{-}\mathrm{PRC} + m$ independent valuations.

Types in bounded perfect SPTC fields

Fix $\mathfrak{d} : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. Let $\mathcal{L}_{\mathfrak{d}} := \mathcal{L} \cup \{c_i : |c_i| = \mathfrak{d}(i)\}$ and

$$\begin{aligned}\mathrm{SP}T\mathrm{C}^{\mathrm{perf}}_{\mathfrak{d}} := {} & \text{perfect } \mathrm{SP}T\mathrm{C} \\ & \cup \{P_i := X^{\mathfrak{d}(i)} + \textstyle\sum_{j<\mathfrak{d}(i)} c_{i,j} X^j \text{ irreducible}\} \\ & \cup \{\text{separable polynomials of degree } i \text{ split modulo } P_i\}.\end{aligned}$$

Fix $K, L \vDash \mathrm{SP}T\mathrm{C}^{\mathrm{perf}}_{\mathfrak{d}}$, $F \leq K$ and $f : F \to L$ an $\mathcal{L}_{\mathfrak{d}}$-embedding.

Embedding lemma

If $F = F^{\mathrm{a}} \cap K$, then f can be extended to an $\mathcal{L}_{\mathfrak{d}}$-embedding $g : K \to L^\star \succcurlyeq L$ with $g(K)^{\mathrm{a}} \cap L^\star = g(K)$.

Corollary

The following are equivalent:

1. f extends to an $\mathcal{L}_{\mathfrak{d}}$-embedding $g : F^{\mathrm{a}} \cap K \to L$;
2. $f : F \subseteq K \to L$ is $\mathcal{L}_{\mathfrak{d}}$-elementary.

Local density

Fix $K \models \mathrm{SPTC}_{\mathfrak{d}}^{\mathrm{perf}}$, $L_i \models T_{i,K}$ and $F \leq K$.

- For every $p \in \mathcal{S}(F)$, let p_i denote its restriction to $\mathcal{L}_i(F)$.

Proposition

Assume that $F = F^{\mathrm{a}} \cap K$. Let $p \in \mathcal{S}_n(F)$ and $U_i \subseteq L_i^n$ be $\mathcal{L}_i(K)$-definable τ_i-open sets. The following are equivalent:

1. for every i, p_i is consistent with U_i;
2. p is consistent with $\bigcap_i U_i$.

Let τ be the coarsest topology refining all the τ_i.

Corollary

The τ-closure of any $\mathcal{L}(F)$-definable X is quantifier free $\mathcal{L}(F)$-definable.

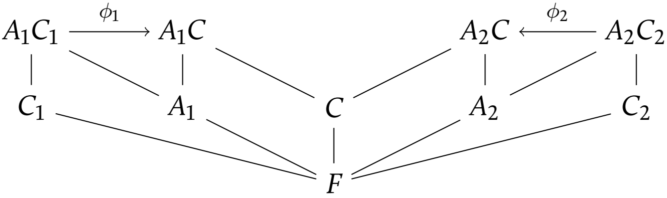
- X is τ-dense in $\bigcup_j \bigcap_i X_{j,i}$ where $X_{j,i}$ is $\mathcal{L}_i(F)$-definable.

Proposition

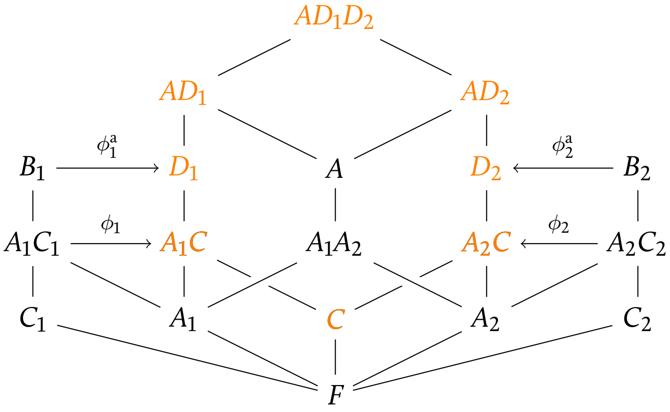
Let $a_1, a_2, c_1, c_2, c \in K$, with a_j enumerating $A_j := \mathrm{acl}(Fa_j)$ and c_j enumerating $C_j := \mathrm{acl}(Fc_j)$, for $j = 1, 2$. Assume:

- $A_1 \cap A_2 = F$;
- $\mathrm{tp}(c_1/F) = \mathrm{tp}(c_2/F)$;
- $c \downarrow^{\mathrm{a}}_{F} a_1 a_2$;
- $\mathrm{qftp}(c/A_j) = \mathrm{qftp}(c_j/A_j)$, for $j = 1, 2$.

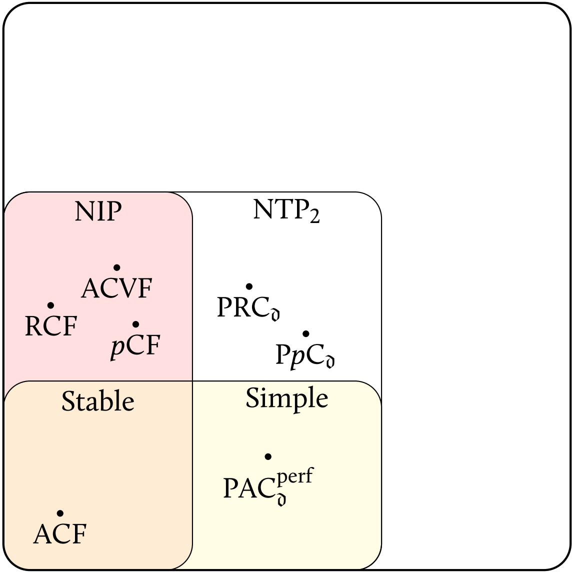
Then $\mathrm{tp}(c_1/A_1) \cup \mathrm{tp}(c_2/A_2) \cup \mathrm{qftp}(c/A_1A_2)$ is consistent.



Amalgamation, proof



The universe

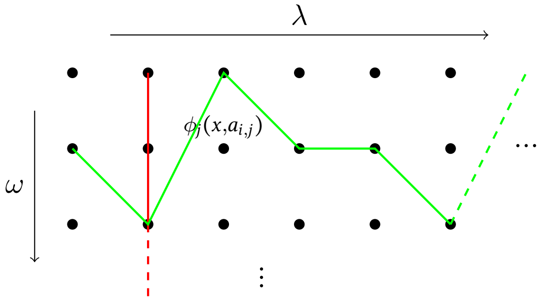


Burden

Definition

Let λ be a cardinal. An inp-pattern of depth λ consists of tuples $(a_{i,j})_{i<\omega, j<\lambda}$, $\mathcal{L}$-formulas $(\phi_j(x,y))_{j<\lambda}$ and intergers $(k_j)_{j<\lambda}$ such that:

- $\{\phi_j(x, a_{i,j}) : i < \omega\}$ is k_j-inconsistent, for every $j < \lambda$;
- $\{\phi_j(x, a_{f(j),j}) : i < \lambda\}$ is consistent, for every $f : \lambda \to \omega$.



Burden

Definition

Let λ be a cardinal. An inp-pattern of depth λ consists of tuples $(a_{i,j})_{i<\omega,j<\lambda}$, $\mathcal{L}$-formulas $(\phi_j(x,y))_{j<\lambda}$ and intergers $(k_j)_{j<\lambda}$ such that:

- $\{\phi_j(x,a_{i,j}) : i < \omega\}$ is k_j-inconsistent, for every $j < \lambda$;
- $\{\phi_j(x,a_{f(j),j}) : i < \lambda\}$ is consistent, for every $f : \lambda \to \omega$.

- $\mathrm{bd}(T) \geq \lambda$ if there is an inp-pattern of depth λ with $|x| = 1$.
- A theory is NTP_2 if and only if $\mathrm{bd}(T) < \infty$.
- $\mathrm{bd}(\mathrm{ACVF}) = \mathrm{bd}(\mathrm{RCF}) = \mathrm{bd}(p\mathrm{CF}) = 1$

Burden in $\mathrm{SP}T\mathrm{C}_{\mathfrak{d}}^{\mathrm{perf}}$

Theorem

$$\mathrm{bd}(\mathrm{SP}T\mathrm{C}_{\mathfrak{d}}^{\mathrm{perf}}) \leq \sum_i \mathrm{bd}(T_i).$$

- $T_{i<n} = \mathrm{ACVF}$,

$$\mathrm{bd}(\mathrm{PAC}_{\mathfrak{d}}^{\mathrm{perf}} + n \text{ independent valuations}) = n.$$