Imaginaries and definable types in valued differential fields

Silvain Rideau

Université Paris-Sud, École Normale Supérieure

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Valued fields

Let (K, v) be a valued field. We will denote by:

- $\mathcal{O} = \{x \in K \mid \mathrm{v}(x) \geq 0\}$ its valuation ring;
- $\mathfrak{M} = \{x \in K \mid \mathrm{v}(x) > 0\}$ its maximal ideal;
- $k = \mathcal{O} / \mathfrak{M}$ its residue field.

Definition (Hahn series field)

Let k be a field and Γ be an ordered abelian group. The field $k((t^{\Gamma}))$ consists of the formal power series $\sum_{\gamma \in \Gamma} c_{\gamma} t^{\gamma}$ with coefficients in k whose support $\{\gamma \mid c_{\gamma} \neq 0\}$ is well ordered.

Differential fields

Definition (Differential field)

A derivation on a field K is a group endohomomorphism ∂ of $(K,+)$ such that for all $x, y \in K$:

$$\partial(xy) = \partial(x)y + x\partial(y).$$

Example

- The field of meromorphic functions on some open subset of $\mathbb{C}$ with the usual derivation.
- The field of germs at $+\infty$ of infinitely differentiable real functions with the usual derivation.
- If (k, ∂) is a differential field, $k((t^\Gamma))$ can be made naturally into a differential field by setting $\partial(\sum_\gamma c_\gamma t^\gamma) = \sum_\gamma \partial(c_\gamma) t^\gamma$.

Valued differential fields

We want to study fields equipped with both a valuation and a derivation. Three classes of such fields have been studied:

- Fields where there is no interaction between the valuation and the derivation (Michaux, Guzy-Point).
- Hardy fields, field of transseries and more generally H-fields (Aschenbrenner-van den Dries-van der Hoeven).
- Valued fields with a contractive derivation, i.e. a derivation ∂ such that:

$$\forall x, y \in K \; \mathrm{v}(\partial(x)) \geq \mathrm{v}(x).$$

Some model theory

We work in the language $\mathcal{L}_{\partial,\mathrm{div}} := \{\mathbf{K}; 0, 1, +, -, \cdot, \partial, \mathrm{div}\}$ where $x \,\mathrm{div}\, y$ is interpreted as $\mathrm{v}(x) \leq \mathrm{v}(y)$.

Theorem (Scanlon, 2000)

The theory of equicharacteristic zero valued fields with a contractive derivation has a model completion $\mathrm{VDF}_{\mathcal{EC}}$ which is complete and eliminates quantifiers.

The theory $\mathrm{VDF}_{\mathcal{EC}}$ is the theory of valued fields with a contractive derivation such that:

- The field is ∂-Henselian;
- The value group of the constant field is equal to the value group of the whole field;
- The residue field is differentially closed;
- The value group is divisible.

Example

If (k, ∂) is differentially closed and Γ is divisible, then $k((t^{\Gamma})) \models \mathrm{VDF}_{\mathcal{EC}}$.

Imaginaries

An imaginary is an equivalent class of an $\varnothing$-definable equivalence relation.

Example

- Let $(X_y)_{y\in Y}$ be an $\varnothing$-definable family of sets. Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$. The set $Y/\equiv$ is a moduli space for the family $(X_y)_{y\in Y}$. We say that $\ulcorner X_y \urcorner := y/\equiv$ is the canonical parameter of X_y.
- Let $p(x)$ be a definable type. Then $\{\ulcorner \mathrm{d}_p x \phi(x;y) \urcorner \mid \phi(x;y) \in \mathcal{L}\}$ is called the canonical basis of p.
- Let G be a definable group and $H \trianglelefteq G$ be a subgroup. The group G/H is interpretable but *a priori* not definable.

Definition

A theory T eliminates imaginaries if for all $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f defined on D such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$

Shelah's eq construction

Definition

Let T be a theory. For all $\varnothing$-definable equivalence relation $E \subseteq \prod_i S_i$, let S_E be a new sort and $f_E : \prod S_i \to S_E$ be a new function symbol. Let

$$\mathcal{L}^{\mathrm{eq}} := \mathcal{L} \cup \{S_E, f_E \mid E \text{ is an } \varnothing\text{-definable equivalence relation}\}$$

and

$$T^{\mathrm{eq}} := T \cup \{f_E \text{ is onto and } \forall x, y\, (f_E(x) = f_E(y) \leftrightarrow xEy)\}.$$

Remark

- Let $M \models T$, then M can naturally be enriched into a model of T^{eq} that we denote M^{eq}.
- We will denote by $\mathcal{R}$ the set of $\mathcal{L}$-sorts. They are called the real sorts.
- The theory T^{eq} eliminates imaginaries.

Imaginaries in fields

Theorem (Poizat, 1983)

The theory of algebraically closed fields in $\mathcal{L}_{\mathrm{rg}} := \{\mathbf{K}; 0, 1, +, -, \cdot\}$ and the theory of differentially closed fields in $\mathcal{L}_{\partial} := \mathcal{L}_{\mathrm{rg}} \cup \{\partial\}$ both eliminate imaginaries.

One cannot hope for such a theorem to hold for algebraically closed valued fields in $\mathcal{L}_{\mathrm{div}} := \mathcal{L}_{\mathrm{rg}} \cup \{\mathrm{div}\}$. Indeed,

- $K = \mathbb{C}((t^{\mathbb{Q}})) \vDash \mathrm{ACVF}$;
- $\mathbb{Q} = K^{\star} / \mathcal{O}^{\star}$ is both interpretable and countable;
- All definable set $X \subseteq K^n$ are either finite or have cardinality continuum.

Imaginaries in valued fields

Let (K, v) be a valued field, we define:

- $\mathbf{S}_n := \mathrm{GL}_n(K)/\mathrm{GL}_n(\mathcal{O})$.
  It is the moduli space of rank n free $\mathcal{O}$-submodules of K^n.
- $\mathbf{T}_n := \mathrm{GL}_n(K)/\mathrm{GL}_{n,n}(\mathcal{O})$ where $\mathrm{GL}_{n,n}(\mathcal{O})$ consists of the matrices $M \in \mathrm{GL}_n(\mathcal{O})$ whose reduct modulo $\mathfrak{M}$ has only zeroes on the last column but for a 1 in the last entry.
  It is the moduli space of $\bigcup_{s \in \mathbf{S}_n} s/\mathfrak{M}s = \{a + \mathfrak{M}s \mid s \in \mathbf{S}_n \text{ and } a \in s\}$.

Let $\mathcal{L}_{\mathcal{G}} := \{\mathbf{K}, (\mathbf{S}_n)_{n \in \mathbb{N}_{>0}}, (\mathbf{T}_n)_{n \in \mathbb{N}_{>0}}; \mathcal{L}_{\mathrm{div}}, \sigma_n : \mathbf{K}^{n^2} \to \mathbf{S}_n, \tau_n : \mathbf{K}^{n^2} \to \mathbf{T}_n\}$.

Theorem (Haskell-Hrushovski-Macpherson, 2006)

The $\mathcal{L}_{\mathcal{G}}$-theory of algebraically closed valued fields eliminates imaginaries.

Question

What about $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$?

Imaginaries and definable types

Proposition (Hrushovski, 2014)

Let T be a theory such that:

1. For all $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$ and all $\mathcal{L}^{\mathrm{eq}}(A)$-definable type p, then p is in fact $\mathcal{L}(\mathcal{R}(A))$-definable.
2. For all set X definable with parameters there exist an $\mathcal{L}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-definable type p which is consistant with X.
3. Finite sets have real canonical parameters.

Then T eliminates imaginaries.

Remark

It suffices to prove hypothesis 1 in dimension 1.

An aside: the invariant extension property

Definition

We say that T has the invariant extension property if for all $M \vDash T$ and $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, every type over A has a global A-invariant extension.

Proposition

The following are equivalent:

(i) The theory T has the invariant extension property.

(ii) For all set X definable with parameters there exists an $\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner)$-invariant type p which is consistant with X.

Remark

If T is NIP then the above are also equivalent to:

(iii) Forking equals dividing and Lascar strong type, Kim-Pillay strong type and strong type coincide.

Computing the canonical basis of types in DCF_0

- Let $p(x)$ be an $\mathcal{L}_\partial$-type over $M \models \mathrm{DCF}_0$ and let $\nabla_\omega(p)$ denote the $\mathcal{L}_{\mathrm{rg}}$-type of $(\partial^n(x))_{n\in\mathbb{N}})$ over M.
- By quantifier elimination, the map ∇_ω is injective. So we can identify $\mathcal{S}_x^{\mathcal{L}_\partial}(M)$ with a subset of $\mathcal{S}_{x_\omega}^{\mathcal{L}_{\mathrm{rg}}}(M)$.
- Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$ and assume p is $\mathcal{L}_\partial^{\mathrm{eq}}(A)$-definable. By elimination of imaginaries in ACF, the canonical basis of $\nabla_\omega(p)$ is contained in $\mathbf{K}(A)$. In particular, p is $\mathcal{L}_\partial(\mathbf{K}(A))$-definable.

Computing the canonical basis of definable types in $\mathrm{VDF}_{\mathcal{EC}}$

- Let $p(x)$ be an $\mathcal{L}_{\partial,\mathrm{div}}$-type over $M \models \mathrm{VDF}_{\mathcal{EC}}$ and let $\nabla_\omega(p)$ denote the $\mathcal{L}_{\mathrm{div}}$-type of $(\partial^n(x))_{n\in\mathbb{N}})$ over M.
- By quantifier elimination, the map ∇_ω is injective. So we can identify $\mathcal{S}_x^{\mathcal{L}_{\partial,\mathrm{div}}}(M)$ with a subset of $\mathcal{S}_{x_\omega}^{\mathcal{L}_{\mathrm{div}}}(M)$.
- One issue: if $p(x)$ is $\mathcal{L}_{\partial,\mathrm{div}}(M)$-definable, then $\nabla_\omega(p)$ might not be $\mathcal{L}_{\mathrm{div}}(M)$-definable as its definition scheme is given by $\mathcal{L}_{\partial,\mathrm{div}}$-formulas.
- Let $\phi(x_\omega; y)$ be an $\mathcal{L}_{\mathrm{div}}$-formula then and $a \models \nabla_\omega(p)$ we have:

$$\underbrace{\phi(a;M)}_{\text{externaly } \mathcal{L}_{\mathrm{div}}\text{-definable}} = \underbrace{\mathrm{d}_p x\phi(x,\partial(x),\ldots,\partial^n(x);M)}_{\mathcal{L}_{\partial,\mathrm{div}}\text{-definable}}$$

Question

Let X be a set that is both externaly $\mathcal{L}_{\mathrm{div}}$-definable and $\mathcal{L}_{\partial,\mathrm{div}}$-definable (with parameters). Is it automatically $\mathcal{L}_{\mathrm{div}}$-definable (with parameters)?

Definable types in enrichments of NIP theories

Definition (Uniform stable embeddedness)

Let M be some structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formula $\phi(x; y)$ there exists a formula $\psi(x; z)$ such that for all tuple $c \in M$,

$$\phi(A; c) = \psi(A; a)$$

for some tuple $a \in A$.

Proposition (Simon-R.)

Let T be an NIP be an $\mathcal{L}$-theory and $\widetilde{T}$ be a complete enrichment of T in a language $\widetilde{\mathcal{L}}$. Assume that there exits $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.
Let X be a set that is both externaly $\mathcal{L}$-definable and $\widetilde{\mathcal{L}}$-definable, then X is $\mathcal{L}$-definable.

In particular, any $\mathcal{L}$-type which is $\widetilde{\mathcal{L}}$-definable is in fact $\mathcal{L}$-definable.

Externally definable sets in NIP theories

Proposition (Simon-R.)

Let T be an NIP $\mathcal{L}$-theory, $U(x)$ be a new predicate and $\phi(x;t) \in \mathcal{L}$. There exists $\psi(x;s) \in \mathcal{L}$ and $\theta \in \mathcal{L}_U$ a sentence such that for all $M \models T$ and $U \subseteq M^{|x|}$ we have:

$$U \text{ is externally } \phi\text{-definable} \Rightarrow M_U \models \theta_U \Rightarrow U \text{ is externally } \psi\text{-definable.}$$

- It follows that (a uniform version of) the previous proposition's conclusion is a first order statement.
- Hence it suffices to find one model of T where it holds (uniformly enough); for example, a model where all externally $\mathcal{L}$-definable sets are $\mathcal{L}$-definable.

Computing the canonical basis of types in $\mathrm{VDF}_{\mathcal{EC}}$ (II)

- Let (k, ∂) be differentially closed. Then $k((t^{\mathbb{R}}))$ is uniformly stably embedded as a valued field in every elementary extension and it can be made into a model of $\mathrm{VDF}_{\mathcal{EC}}$.
- It follows that if $p(x)$ is an $\mathcal{L}_{\partial,\mathrm{div}}$-type over $M \models \mathrm{VDF}_{\mathcal{EC}}$ which is $\mathcal{L}^{\mathrm{eq}}_{\partial,\mathrm{div}}(A)$-definable for some $A = \mathrm{dcl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$, then $\nabla_\omega(p)$ is $\mathcal{L}_{\mathrm{div}}(M)$-definable and hence its canonical basis is included in $\mathcal{G}(A)$ and so is the canonical basis of p itself.

Theorem

The theory $\mathrm{VDF}^{\mathcal{G}}_{\mathcal{EC}}$ eliminates imaginaries and has the invariant extension property.

Thanks!