Transferring Imaginaries

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What is your name?

An imaginary is an equivalent class of an $\varnothing$-definable equivalence relation.

Example

- Let $(X_y)_{y \in Y}$ be an $\varnothing$-definable family of sets. Define $y_1 \equiv y_2$ whenever $X_{y_1} = X_{y_2}$. The set $Y/{\equiv}$ is a "moduli space" for the family $(X_y)_{y \in Y}$.
- Let G be a definable group and $H \trianglelefteq G$ be a definable subgroup. The group G/H is interpretable but *a priori* not definable.

Definition

A theory T eliminates imaginaries if for all $\varnothing$-definable equivalence relation $E \subseteq D^2$, there exists an $\varnothing$-definable function f defined on D such that for all $x, y \in D$:

$$xEy \iff f(x) = f(y).$$

What is your quest?

Definition

- A type p over M is said to be definable (over A) if for all formula $\phi(x;y)$ there is a formula $\theta(y)$ such that

$$\phi(x;a) \in p \text{ if and only if } M \vDash \theta(a).$$

  We will often write $\mathrm{d}_p\, x\, \phi(x;y) = \theta(y)$.
- A theory is said to be stable if every type over every model of T is definable.

Proposition (Shelah, 1978)

Let $A \subseteq M \vDash T$ stable, $p \in \mathcal{S}(A)$ and $p_1, p_2 \in \mathcal{S}(M)$ be two distinct extensions of p to M definable over A. Then there exists an $\mathcal{L}(A)$-definable finite equivalence relation E and a_1, $a_2 \in M$ such that:

- a_1 and a_2 are not E-equivalent;
- $p_i(x) \vdash xEa_i$.

What is your quest?

- If T is stable and eliminates imaginaries, $A = \mathrm{acl}(A) \subseteq M \models T$, then types over A have a unique definable extension to M.

Assume T eliminates imaginaries.

- If X if definable, then X has a smallest (definably closed) set of definition. We denote it $\ulcorner X \urcorner$.
- If p is a definable type, then p has a smallest (definably closed) set of definition. It is called the canonical basis of p.

- Proving elimination of imaginaries in specific structures can have (more or less direct) applications.

What is the airspeed velocity of an unladen swallow?

- The theory of algebraically closed fields eliminates imaginaries in the language of rings.
- The theory of differentially closed fields of characteristic zero eliminates imaginaries in the language of differential rings.
- O-minimal groups eliminate imaginaries.
  For example, any O-minimal enrichment of $(\mathbb{R}, 0, 1, +, -, \cdot)$.
- Infinite sets do not eliminate imaginaries:
  - The quotient of M^n by the action of $\mathfrak{S}_n$ is not represented.
- $\mathbb{Q}_p$ does not eliminate imaginaries in the ring language :
  - $\mathbb{Z}$ can be interpreted as $\mathbb{Q}_p^\star / \mathbb{Z}_p^\star$;
  - All infinite definable subsets of $\mathbb{Q}_p^n$ have cardinality continuum.
- Henselian valued fields do not eliminate imaginaries in the language of valued rings.

Shelah's eq construction

Definition

Let T be a theory. For all $\varnothing$-definable equivalence relation $E \subseteq \prod_i S_i$, let S_E be a new sort and $f_E : \prod S_i \to S_E$ be a new function symbol. Let

$$\mathcal{L}^{eq} := \mathcal{L} \cup \{S_E, f_E \mid E \text{ is an } \varnothing\text{-definable equivalence relation}\}$$

and

$$T^{eq} := T \cup \{f_E \text{ is onto and } \forall x, y\, (f_E(x) = f_E(y) \leftrightarrow xEy)\}.$$

Remark

- Let $M \vDash T$, then M can naturally be enriched into a model of T^{eq} that we denote M^{eq}.
- We will denote by $\mathcal{R}$ the set of $\mathcal{L}$-sorts. They are called the real sorts.
- The theory T^{eq} eliminates imaginaries.
- We will denote by dcl^{eq} (acl^{eq}) the definable (algebraic) closure in T^{eq}.

Shelah's eq construction

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$$T^{\text{eq}} := T \cup \{f_E \text{ is onto and } \forall x, y\, (f_E(x) = f_E(y) \leftrightarrow xEy)\}.$$

Proposition

A theory T (with two constants) eliminates imaginaries if and only if for all $M \models T$ and $e \in M^{\text{eq}}$, there exists a tuple $a \in M$ such that

$$e \in \text{dcl}^{\text{eq}}(a) \text{ and } a \in \text{dcl}^{\text{eq}}(e).$$

Weak elimination

Definition

A theory T weakly eliminates imaginaries if for all $M \vDash T$ and $e \in M^{\mathrm{eq}}$, there exists a tuple $a \in M$ such that

$$e \in \mathrm{dcl}^{\mathrm{eq}}(a) \text{ and } a \in \mathrm{acl}^{\mathrm{eq}}(e).$$

Proposition

A theory T eliminates imaginaries if and only if:

1. T weakly eliminates imaginaries.
2. For all $M \vDash T$, the quotient of M^n by the action of $\mathfrak{S}_n$ is represented.

Example

- Infinite sets weakly eliminate imaginaries.
- Any strongly minimal theory weakly eliminates imaginaries.

Encoding functions

- A finite valued function $X \to Y$ is a subset of $X \times Y$ such that for all $x \in X$, the set Y_x is finite.

Proposition

The following are equivalent:

1. T weakly eliminates imaginaries
2. Every set definable in models of T has a smallest (algebraically closed) set of definition.
3. Every finite valued function $M \to M$ definable in $M \vDash T$ has a smallest (algebraically closed) set of definition.

Covering functions

- Let T be an $\mathcal{L}$-theory and $T' \supseteq T_\forall$ be an $\mathcal{L}'$-theory. Let $M' \vDash T'$ and $M \vDash T$ containing M'.
- Assume that every finite valued function f definable in M' is covered by a finite valued function g defined in M.
- One would like to deduce elimination of imaginaries in T' from elimination of imaginaries in T.
- There are a number of problems:
  - No control the domain of f.
  - g is not canonical (unless it can somehow be taken minimal).
  - The smallest set of definition of g might contain points from $M \setminus M'$.
  - Unclear how to recover f from g.

Covering functions

In the case of the field $(\mathbb{R}, 0, 1, +, -, \cdot)$:

- Take any finite valued function f definable in $\mathbb{R}$. Let g be the Zariski closure of f. Then g is a finite valued function definable in $\mathbb{C}$.
- Let $A \subseteq \mathbb{C}$ be the the smallest set of definition of g.
- The smallest set of definition of $g \cap \mathbb{R}$ is $A \cap \mathbb{R}$.
- f can be recovered from $g \cap \mathbb{R}$ using the order and the fact that every definable $X \subseteq \mathbb{R}$ has a smallest subset of definition.

Proposition (Hrushovski-Martin-R., 2014)

Let T' be a theory of fields such that, for all $M \vDash T'$ and $A \subseteq M$:

1. $\mathrm{dcl}(A) = \mathrm{acl}(A) \subseteq \overline{A}^{\mathrm{alg}}$;
2. Every definable $X \subseteq M$ has a smallest subset of definition.

Then T eliminates imaginaries.

Remark

Hypothesis 1 holds in $\mathbb{Q}_p$ but not hypothesis 2 (in the language of rings).

Covering functions

Proposition (Hrushovski-Martin-R., 2014)

Let T be an $\mathcal{L}$-theory that eliminates quantifiers and imaginaries and $T' \supseteq T_\forall$ an $\mathcal{L}'$-theory. Assume that, for all $M' \models T'$, $M \models T$ containing M' and $A \subseteq M'$:

1. $\mathrm{dcl}_{\mathcal{L}'}(A) = \mathrm{acl}_{\mathcal{L}'}(A) \subseteq \mathrm{acl}_{\mathcal{L}}(A)$;
2. Every definable $X \subseteq M'$ has a smallest subset of definition;
3. For all $e \in \mathrm{dcl}_M(M')$, there exists $e' \in M'$ such that for all $\sigma \in \mathrm{Aut}(M)$ stabilising M' globally,

$$\sigma(e) = e \text{ if and only if } \sigma(e') = e';$$

4. Assume $A = \mathrm{acl}_{\mathcal{L}'}(A)$ and let $p \in \mathcal{S}_1^{\mathcal{L}'}(A)$. Then there exists $\widetilde{p} \in \mathcal{S}_1^{\mathcal{L}}(M)$ definable over A such that $p \cup \widetilde{p}|_{M'}$ is consistent.

Then T' eliminates imaginaries.

Covering functions

Proposition

Let T_i be an $\mathcal{L}_i$-theory that eliminates quantifiers and imaginaries and $T' \supseteq \bigcup_i T_{i,\forall}$ an $\mathcal{L}'$-theory. Assume that, for all $M' \models T'$, $M_i \models T_i$ containing M' and $A \subseteq M'$:

1. $\mathrm{dcl}_{\mathcal{L}'}(A) = \mathrm{acl}_{\mathcal{L}'}(A) \subseteq \mathrm{acl}_{\mathcal{L}_i}(A)$;
2. Every definable $X \subseteq M'$ has a smallest subset of definition;
3. For all $e \in \mathrm{dcl}_{M_i}(M')$, there exists $e' \in M'$ such that for all $\sigma \in \mathrm{Aut}(M_i)$ stabilising M' globally,
$$\sigma(e) = e \text{ if and only if } \sigma(e') = e';$$
4. Assume $A = \mathrm{acl}_{\mathcal{L}'}(A)$ and let $p \in \mathcal{S}_1^{\mathcal{L}'}(A)$. Then there exists $\widetilde{p}_i \in \mathcal{S}_1^{\mathcal{L}_i}(M_i)$ definable over A such that $p \cup \bigcup_i \widetilde{p}_i|_{M'}$ is consistent.

Then T' weakly eliminates imaginaries.

Some results

- All the imaginaries in $\mathbb{R}$ come from ACF (and hence they can be eliminated).
- All the imaginaries in real closed valued fields come from ACVF (whose imaginaries were described by Haskell, Hrushovski and Macpherson).
- All the imaginaries in $\mathbb{Q}_p$ come from ACVF.
- All the imaginaries in $\prod_p \mathbb{Q}_p / \mathfrak{U}$ come from ACVF.

Adding new functions

- If T is an $\mathcal{L}$-theory, we may want to form T_σ the $\mathcal{L} \cup \{\sigma\}$-theory of models of T with an automorphism.
- We will mainly be interested in T_A, the model companion of T_σ, if it exists (and from now on, we will assume it exists).

Proposition (Chatzidakis-Pillay, 1998)

Assume T is strongly minimal, then T_A weakly eliminates imaginaries.

Proposition (Hrushovski, 2012)

Let T be a stable theory that eliminates imaginaries. Assume that T has 3-uniqueness, then T_A eliminates imaginaries.

Adding new functions

- Let T be some $\mathcal{L}$-theory, f be new function symbol and $T' \supseteq T$ be an $\mathcal{L} \cup \{f\}$-theory.
- Let $M \models T'$. We define:

$$\begin{array}{rccc} \nabla_\omega : & \mathcal{S}_x^{\mathcal{L}'}(M) & \to & \mathcal{S}_{x_\omega}^{\mathcal{L}}(M) \\ & \mathrm{tp}_{\mathcal{L}'}(a/M) & \mapsto & \mathrm{tp}_{\mathcal{L}}(f_\omega(a)/M) \end{array}$$

  where $f_\omega(a) = (f^n(a))_{n\in\mathbb{N}}$.
- We assume that ∇_ω is injective (this is a form of quantifier elimination).
- That does not, in general, hold in T_A.
- It does hold in differentially closed fields of characteristic zero and separably closed fields of finite imperfection degree (and their valued equivalents).

Imaginaries and definable types

Proposition (Hrushovski, 2014)

Let T be a theory such that:

1. For every definable set X there exist an $\mathcal{L}^{\mathrm{eq}}(\mathrm{acl}^{\mathrm{eq}}(\ulcorner X \urcorner))$-definable type p which is consistent with X.
2. Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}} \models T^{\mathrm{eq}}$. If $p \in \mathcal{S}(M)$ is $\mathcal{L}^{\mathrm{eq}}(A)$-definable, then p is $\mathcal{L}(\mathcal{R}(A))$-definable.

Then T weakly eliminates imaginaries.

Remark

It suffices to prove hypothesis 1 in dimension 1.

Prolongations and canonical basis

In the case of differentially closed fields $(K, 0, 1, +, -, \cdot, \delta)$:

- Hypothesis I is true because DCF_0 is stable.
- Let $M \vDash \mathrm{DCF}_0$ and $p \in \mathcal{S}^{\mathcal{L}_\partial}(M)$.
- Let $A = \mathrm{acl}^{\mathrm{eq}}(A) \subseteq M^{\mathrm{eq}}$ and assume p is $\mathcal{L}_\partial^{\mathrm{eq}}(A)$-definable. By elimination of imaginaries in ACF, the canonical basis of $\nabla_\omega(p)$ is contained in $\mathbf{K}(A)$. In particular, p is $\mathcal{L}_\partial(\mathbf{K}(A))$-definable.

Prolongations and canonical basis

If T is not stable, the previous strategy has a serious flaw:

- If p is $\mathcal{L}'(M)$-definable, there is no reason for $\nabla_\omega(p)$ to be $\mathcal{L}(M)$-definable.
- Let $\phi(x_\omega; y)$ be an $\mathcal{L}$-formula then

$$\phi(x_\omega; a) \in \nabla_\omega(p) \text{ if and only if } M \vDash \mathrm{d}_p\, x\, \phi(f_\omega(x); a).$$

- Let $a \vDash \nabla_\omega(p)$ we have:

$$\underbrace{\phi(a; M)}_{\text{externally } \mathcal{L}\text{-definable}} \quad = \quad \underbrace{\mathrm{d}_p\, x\, \phi(f_\omega(x); M)}_{\mathcal{L}'\text{-definable}}.$$

and we wish this set to be $\mathcal{L}$-definable.

NIP theories

Definition

Let $\phi(x;y)$ be a formula and M a structure, we say that ϕ has the independence property in M if there exists $(a_n)_{n\in\mathbb{N}}$ and $(b_X)_{X\subseteq\mathbb{N}}$ such that:

$$M \models \phi(a_n; b_X) \text{ if an only if } n \in X$$

We say that the theory T is NIP (not the independence property) if no formula has the independence property in any model of T.

Example

- All stable theories are NIP.
- All O-minimal theories are NIP.
- ACVF is NIP.

Definable types in enrichments of NIP theories

Definition (Stable embeddedness)

Let M be some structure and $A \subseteq M$. We say that A is stably embedded in M if for all formula $\phi(x;y)$ and all $c \in M$, there exists a formula $\psi(x;z)$ such that

$$\phi(A;c) = \psi(A;a)$$

for some tuple $a \in A$.

Proposition (Simon-R., 2015)

Let T be an NIP be an $\mathcal{L}$-theory and $\widetilde{T}$ be a complete enrichment of T in a language $\widetilde{\mathcal{L}}$. Assume that there exits $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.
Let X be a set that is both externally $\mathcal{L}$-definable and $\widetilde{\mathcal{L}}$-definable, then X is $\mathcal{L}$-definable.

In particular, any $\mathcal{L}$-type which is $\widetilde{\mathcal{L}}$-definable is in fact $\mathcal{L}$-definable.

Definable types in enrichments of NIP theories

Definition (Uniform stable embeddedness)

Let M be some structure and $A \subseteq M$. We say that A is uniformly stably embedded in M if for all formula $\phi(x;y)$, there exists a formula $\psi(x;z)$ such that for all tuple $c \in M$,

$$\phi(A;c) = \psi(A;a)$$

for some tuple $a \in A$.

Proposition (Simon-R., 2015)

Let T be an NIP be an $\mathcal{L}$-theory and $\widetilde{T}$ be a complete enrichment of T in a language $\widetilde{\mathcal{L}}$. Assume that there exits $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.
Let X be a set that is both externally $\mathcal{L}$-definable and $\widetilde{\mathcal{L}}$-definable, then X is $\mathcal{L}$-definable.

In particular, any $\mathcal{L}$-type which is $\widetilde{\mathcal{L}}$-definable is in fact $\mathcal{L}$-definable.

Prolongations and canonical basis II

Proposition

Let T be some $\mathcal{L}$-theory that eliminates imaginaries, f be new function symbol and $T' \supseteq T$ be a complete $\mathcal{L} \cup \{f\}$-theory. Assume that:

1. ∇_ω is injective.
2. For every $\mathcal{L}'$-definable set X there exist an $\mathcal{L}^{eq}(\mathrm{acl}^{eq}(\ulcorner X \urcorner))$-definable $\mathcal{L}$-type p which is consistent with X.
3. There exits $M \models \widetilde{T}$ such that $M|_{\mathcal{L}}$ is uniformly stably embedded in every elementary extension.

Then T' eliminates imaginaries.

Some results II

- All the imaginaries in DCF_0 come from ACF (and hence they can be eliminated).
- All the imaginaries from separably closed fields (be it with λ-functions or Hasse derivations) come from ACF.
- All the imaginaries in Scanlon's theory of differential valued fields come from ACVF.
- All the imaginaries from separably closed valued fields come from ACVF.

Thanks!