Midterm 1

September 22nd

- To do a later question in a problem, you can always assume a previous question even if you have not answered it.
- I am aware that this is long. I don't expect you to do everything.
- There are 2 class material questions (in Problem 1) and 2 independent problems. You don't have to do them in any particular order.
- Using a pen and writing clearly makes it easier for me to grade your midterm.

Problem 1 :
The following questions are about material covered in class. When asked to prove something you are only allowed to use results that we proved before that particular result.

1. Define the center of a group and prove it is a subgroup.

2. Prove that every infinite cyclic group is isomorphic to $\mathbb{Z}$.

Problem 2 :

1. Compute the disjoint cycle decomposition and the order of the product $(0,1,2) \cdot (2,3) \cdot (0,1,2,3)$.

2. Let $\tau = (0,1) \in S_n$, for $n \geqslant 2$. Show that $\sigma \in \mathrm{C}_{S_n}(\tau)$ if and only if $\sigma(\{0,1\}) = \{0,1\}$.

Problem 3 :

Let G be a group and p be some prime number.

1. Let $a,\ b \in G$ be such that $|a| = |b| = p$. Show that either $\langle a \rangle = \langle b \rangle$ or $\langle a \rangle \cap \langle b \rangle = \{1\}$.

2. Assume that $a \notin \langle b \rangle$, show that $\{a^i b^j : i, j \in \mathbb{Z}\}$ has at least p^2 elements.

3. Assume that G is Abelian, $|G| = p^2$ and that every element in $G \smallsetminus \{1\}$ is order p. Show that $G \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.