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Solutions to the midterm

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Problem 1 :

1. Let G be acting on a set X, show that $x \sim y$ defined by $\exists g \in G\, g \star x = y$ is an equivalence relation on X.

   Solution: Pick any $x, y, z \in X$. Since $x = 1 \star x$, we have $x \sim x$ and $\sim$ is reflexive. Now assume $x \sim y$. So there exists a $g \in G$ such that $y = g \star x$. Then $x = g^{-1} \star (g \star x) = g^{-1} \star y$ so $y \sim x$. It follows that $\sim$ is symmetric. Finally, assume that $x \sim y$ and $y \sim z$, then there exists $g, h \in G$ such that $y = g \star x$ and $z = h \star y$. Then $w = h \star (g \star x) = (hg) \star x$ and $x \sim z$. So $\sim$ is transitive. Since $\sim$ is reflexive, symmetric and transitive, it is an equivalence relation.

2. State Cauchy's theorem.

   Solution: Let G be a finite group and p be a prime dividing the order of G, then there exists $x \in G$ such that $|x| = p$.

Problem 2 :

Let G be a group and $H \leqslant G$ be a subgroup. For all g, $l \in G$, we define $g \star lH = glH$.

1. Show that $\star$ is a group action of G on G/H.

   Solution: Pick $g, h, l \in G$, we have $g \star (h \star lH) = g \star (hlH) = ghlH = gh \star lH$ and $1 \star lH = (1 \cdot l)H = lH$. So $\star$ is a group action.

   To be complete, we should also check that the function $\star$ is well defined: pick $g, l, k \in G$ such that $lH = kH$, i.e. $l = kh$ for some $h \in H$. Then $gkH = glhH = glH$ since $h \in H$, so $\star$ is well defined.

2. Let $K = \{g \in G : \forall l \in G,\ g \star lH = lH\}$. Show that $K \leqslant H$ and $K \trianglelefteq G$.

   Solution: K is the kernel of the permutation representation associated to the action $\star$. So $K \trianglelefteq G$. Now, pick any $g \in K$, then for all $l \in G$, $g \star lH = lH$. In particular, $g \star H = gH = H$ so $g \in H$. It follows that $K \leqslant H$.

3. Let $n = [G : H]$ and $k = [H : K]$. Show that k divides $(n-1)!$.

   Solution: We have $[G : K] = [G : H][H : K] = nk$. Also, we have a group homomorphism $\theta : G \to S_{G/H}$, the permutation representation. By the first isomorphism theorem $G/K \cong \theta(G) \leqslant S_{G/H}$. By Lagrange, it follows that $nk = |G/K| = |\theta(G)|$ divides $|S_{G/H}| = n!$ and hence k divides $(n-1)!$.

4. Assume that $|G|$ is finite and n is the smallest prime dividing $|G|$. Show that $k = 1$. Conclude that $H \trianglelefteq G$.

   Solution: By Lagrange $nk = [G : K]$ divides $|G|$ and hence so does k. Let p be a prime dividing k, then $p \geqslant n$ by hypothesis. By the previous question, k divides $(n-1)!$. But all the prime factors of $(n-1)!$ are strictly smaller than n, a contradiction. It follows that k is not a multiple of any prime and hence $k = 1$. Since $k = [H : K] = 1$, we have $H = K \trianglelefteq G$.

Problem 3 :
Let G be a group and $K_i \trianglelefteq G$ for $i = 1, 2$ be such that $[G : K_i] < \infty$.

1. Show that $[K_1 : K_1 \cap K_2] < \infty$. Conclude that $[G : K_1 \cap K_2] < \infty$.

   Solution: Since $K_2 \trianglelefteq G$, $K_1K_2 \leqslant G$. Moreover, by the second isomorphism theorem $K_1/(K_1 \cap K_2) \cong K_1K_2/K_2 \leqslant G/K_2$. By Lagrange, it follows that $[K_1 : K_1 \cap K_2]$ divides $|G/K_2|$, in particular, it is finite. Since $[G : K_1 \cap K_2] = [G : K_1][K_1 : K_1 \cap K_2]$, and both indices on the right are finite, $[G : K_1 \cap K_2]$ is finite.