Silvain Rideau
1091 Evans
silvain.rideau@berkeley.edu
www.normalesup.org/~srideau/en/teaching

Homework 5

Due October 9th

Problem 1 :
Let m, $n \in \mathbb{Z}_{>0}$ be such that $\gcd(m,n) = 1$. Let G be a group of order mn, $H \leqslant G$ such that $|H| = n$ and $K \trianglelefteq G$.

1. Show that there exists m_0 dividing m such that $|HK| = m_0 n$.
2. Show that there exists n_0 dividing n such that $|K| = m_0 n_0$.
3. Show that $H \cap K$ is maximal among subgroups of K whose order divides n.
4. Show that $\gcd(|HK/K|, |G/HK|) = 1$.
5. Assume that $|G| = p^\alpha r$ where $\gcd(p,r) = 1$, $|K| = p^\beta s$ where $\gcd(p,s) = 1$ and $|H| = p^\alpha$. Show that $|H \cap K| = p^\beta$ and $|HK/K| = p^{\alpha-\beta}$.

Problem 2 :
Let G be a group, $N \trianglelefteq G$ and $H \leqslant G$. Assume $H \cap N = \{1\}$.

1. Show that the map $f : N \times H \to NH$ defined by $f((n,h)) = n \cdot h$ is a bijection.
2. Show that f is a group isomorphism if and only if $H \leqslant \mathrm{C}_G(N)$. Here $N \times H$ is considered as a group with the usual coordinatewise group law.
3. Show that there exists a group homomorphism $\theta : H \to \mathrm{Aut}(N)$ such that for all $n \in N$ and $h \in H$, $h \cdot n = [\theta(h)](n) \cdot h$.
4. Let us define the operation on $N \times H$: $(n_1, h_1) \star (n_2, h_2) = (n_1 \cdot [\theta(h_1)](n_2), h_1 \cdot h_2)$. Show that $(N \times H, \star)$ is a group and that it is isomorphic to $(NH, \cdot)$.