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Homework 8

Due November 8th

The questions indicated as (Harder) are optional and will not be taken in account in the grade.

Problem 1 (nilpotent elements and radical ideals) **:**
Let R be a unitary commutative ring. An element $x \in R$ is said to be nilpotent if there exists $n \in \mathbb{Z}_{>0}$ such that $x^n = 0$.

1. What are the nilpotent elements in $\mathbb{Z}/36\mathbb{Z}$?
2. Show that $\{x \in R : x \text{ nilpotent}\}$ is an ideal. It is called the nilradical of R.
3. Assume that x is nilpotent, show that $1 - x$ is a unit.
4. Assume that x is nilpotent, show that for all $u \in R^\star$, $u + x$ is a unit.
5. (Harder) Let $S \subseteq R \smallsetminus \{0\}$ be closed under multiplication. Show that there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $\mathfrak{p} \cap S = \varnothing$.
6. Let $x \in R$ not be nilpotent. Show that there exists a prime ideal $\mathfrak{p} \subseteq R$ such that $x \notin \mathfrak{p}$.

   Hint: Use the previous question!
7. Let N be the nilradical of R, show that
$$N = \bigcap_{\mathfrak{p} \subseteq R \text{ prime}} \mathfrak{p}.$$
8. Let $I \subseteq R$ be an ideal. We define $\sqrt{I} := \{x \in R : x^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$. Show that $\sqrt{I} \subseteq R$ is an ideal.
9. Let $f : R \to S$ be a unitary ring homomorphism, $\mathfrak{p} \subseteq S$ be a prime ideal. Show that $f^{-1}(\mathfrak{p}) \subseteq R$ is a prime ideal.
10. Let $I \subseteq R$ be an ideal and $\pi : R \to R/I$ be the canonical projection. Let $N_I \subseteq R/I$ be its nilradical. Show that $\sqrt{I} = \pi^{-1}(N_I)$.
11. Let $I \subseteq R$ be an ideal. Show that
$$\sqrt{I} = \bigcap_{\substack{I \subseteq \mathfrak{p} \subseteq R \\ \mathfrak{p} \text{ prime}}} \mathfrak{p}.$$