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# Final exam

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Remember that you can always assume a previous question (even if you have not proved it) to prove a later one.
Here are the typing rules for simply typed $\lambda$-calculus:

$$(\mathrm{Ax})\ \frac{}{\Gamma \cup \{x : A\} \vdash x : A}$$

$$(\to_I)\ \frac{\Gamma \cup \{x : A\} \vdash t : B}{\Gamma \vdash \lambda x\, t : A \to B} \qquad (\to_E)\ \frac{\Gamma_1 \vdash t : A \to B \qquad \Gamma_2 \vdash u : A}{\Gamma_1 \cup \Gamma_2 \vdash (t)u : B}$$

**Problem 1** (Atomless Boolean algebras) **:**
We say that a Boolean algebra $A$ is atomless if there are no atoms in $A$. In all of this problem we will assume that $A$ is a countable atomless Boolean algebra.

1. Show that for all $a \in A$, if $a \neq 0$, then there exists $c_1, c_2 \in A \smallsetminus \{0\}$ such that $c_1 \cup c_2 = a$ and $c_1 \cap c_2 = 0$.
2. Let $E$ be the set of words on $\{0, 1\}$ and let $A = \{a_k : k \in \mathbb{N}\}$. Show that there exists elements $\varepsilon_w \in A$ for all $w \in E$ such that:
   - for all $w \in E$, $\varepsilon_w \neq 0$;
   - $\varepsilon_\varnothing = 1$, where $\varnothing$ denotes the empty word;
   - for all $w \in E$, $\varepsilon_{w0} \cap \varepsilon_{w1} = 0$;
   - for all $w \in E$, $\varepsilon_{w0} \cup \varepsilon_{w1} = \varepsilon_w$;
   - for all $w \in E$ such that $|w| = k$, if $\varepsilon_w \cap a_k \neq 0$ and $\varepsilon_w \cap a_k^{\mathbf{c}} \neq 0$, then $\varepsilon_{w1} = \varepsilon_w \cap a_k$ and $\varepsilon_{w0} = \varepsilon_w \cap a_k^{\mathbf{c}}$.
3. Let $f \in \{0, 1\}^{\mathbb{N}}$. We denote by $w_k(f)$ the word $f(0) \cdots f(k)$. Show that if $a \cap \varepsilon_{w_k(f)} = 0$ then for all $n \geqslant k$, $a \cap \varepsilon_{w_n(f)} = 0$ and if $a \cap \varepsilon_{w_k(f)} \neq 0$ then for all $n \leqslant k$, $a \cap \varepsilon_{w_n(f)} \neq 0$.
4. Let $f \in \{0, 1\}^{\mathbb{N}}$ and $a \in A$. Show that one and only one of the following statements hold:
   - for all $n \in \mathbb{N}$, $\varepsilon_{w_n(f)} \cap a \neq 0$;
   - for all $n \in \mathbb{N}$, $\varepsilon_{w_n(f)} \cap a^{\mathbf{c}} \neq 0$.
5. Let $f \in \{0, 1\}^{\mathbb{N}}$. We define $U_f = \{a \in A : \forall n \in \mathbb{N}, \varepsilon_{w_n(f)} \cap a \neq 0\}$. Show that $U_f$ is an ultrafilter on $A$.
6. Let $U$ be an ultrafilter on $A$, show that there exist words $(w_n)_{n \in \mathbb{N}}$ such that:
   - $|w| = n$;
   - $w_n$ is a prefix of $w_{n+1}$ (i.e. the first $n$ letters of $w_{n+1}$);
   - $\varepsilon_{w_n} \in U$.
7. Let $h : \{0, 1\}^{\mathbb{N}} \to \mathcal{S}(A)$ be defined by $h(f) = U_f$. Show that $h$ is a bijection.

**Problem 2** (Model theory of $\mathbb{Z}$) **:**
Let $\mathcal{Z} = \{\mathbb{Z}, 0, 1, +, -, \cdot, <\}$ where the symbols are interpreted with their standard interpretation. Let $\overline{n} = 1 + \cdots + 1$ $n$-times if $n \in \mathbb{N}_{>0}$, $\overline{0} = 0$ and $\overline{n} = -(\overline{-n})$ otherwise.

1. Show that there exists $\mathcal{M} \equiv \mathcal{Z}$ and $a \in \mathcal{M}$ such that $\overline{n}^{\mathcal{M}}$ divides $a$ for all $n \in \mathbb{Z} \smallsetminus \{0\}$ (Recall that, $a$ divides $b$ if there exists $c$ such that $b = a \cdot c$).

   Hint: Add a new constant to the language.

2. Let $\mathcal{M} \equiv \mathcal{Z}$ and $\varphi(x)$ be a formula. Show that if $\mathcal{M} \models \varphi(\overline{n})$ for all $n \in \mathbb{Z}$, then $\mathcal{M} \models \forall x\, \varphi(x)$.

3. Let $\mathcal{M} \equiv \mathcal{Z}$ not be isomorphic to $\mathcal{Z}$. Show that $\{\overline{n}^{\mathcal{M}} : n \in \mathbb{Z}\} \subseteq M$ is not definable.

**Problem 3** (Axiomatizability of equivalence relations) **:**
Let $\mathcal{L} = \{E\}$. Which of the following classes of $\mathcal{L}$-structures are axiomatizable/finitely axiomatizable (for each example if they are axiomatizable/finitely axiomatizable give a theory that does so and if they are not give a proof of that fact that they are not; if the class is axiomatizable but not finitely so, you should give an infinite theory axiomatizing it and a proof that no finite theory works).

1. The class of all $\mathcal{L}$-structures where $E$ is an equivalence relation;

2. The class of all $\mathcal{L}$-structures where $E$ is an equivalence relation with finitely many classes;

3. The class of all $\mathcal{L}$-structures where $E$ is an equivalence relation whose classes are finite;

4. The class of all $\mathcal{L}$-structures where $E$ is an equivalence relation with two infinite classes;

5. The class of all $\mathcal{L}$-structures where $E$ is an equivalence relation with exactly one class of size $n$ for all $n \in \mathbb{N}_{>0}$ (and possibly some infinite classes).

**Problem 4** ($\lambda$-calculus) **:**

1. Let $A \in W$ be a type variable. Which are the normal $\lambda$-terms $t$ such that $\vdash t : (A \to A) \to A$?

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3. Let $t \in \Lambda$, $A \in T$ and $\Gamma$ be a context. Assume that $\Gamma \vdash t : A$. Show that for all $u \in \mathrm{sub}(t)$, there exists a context $\Gamma'$ and $A' \in T$ such that $\Gamma' \vdash u : A'$ holds.

4. Let $Y = \lambda f\, (\lambda x\, (f)(x)x)\lambda x\, (f)(x)x$ and $t \in \Lambda$. Show that $(Y)t$ is $\beta$-equivalent to $(t)(Y)t$

5. Is there a type $A \in T$ and a context $\Gamma$ such that $\Gamma \vdash Y : A$ holds.