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Homework 4

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Problem 1 :

1. a) In N_1 this formula is interpreted as: for all x and $y \in \mathbb{N}$, $x + (y + 1) = (x + y) + 1$, which indeed holds.
   b) In N_2, it is interpreted the same way, except that x and y are now in $\mathbb{Z}$, but it is also true.
   c) In N_3, it is interpreted as, for x and $y \in \mathbb{Z}$, $x \cdot (-y) < -(x \cdot y)$ which does not hold.
2. a) In N_1, this formulas negates the symmetry of equality. Therefore it cannot hold.
   b) Similarly in N_2.
   c) This formula holds in N_3 because for all x, $x < x + 1$ holds but $x + 1 < x$ does not hold.
3. a) This formula holds in N_1 because $+1$ is a function and so obviously if $x = y$, then $x + 1 = y + 1$.
   b) Similarly in N_2.
   c) This formula does not hold in N_3 because $-$ reverses inequalities.
4. a) This formula holds in N_1. Indeed, $x = 0$ is not of the form $y + 1$ for any y.
   b) This formula does not hold in N_2 as $y \mapsto y + 1$ is onto in $\mathbb{Z}$.
   c) This formula does not hold in N_3. For any choice of x there is a $z < x$ and hence $-(-z) < x$.
5. a) This formula does not hold in N_1. Indeed no x exists such that $x = x + 1$.
   b) Similarly in N_2
   c) This formula does not hold in N_3. Indeed the only x such that $x = -x$ is 0 and in that case for all y $x \cdot y = 0$ which is not strictly smaller than 0.

Problem 2 :

1. Let $\varphi = \forall x \exists y\, y < x$. This formula expresses that there does not exist a minimal element in the structure. Therefore, it holds in $(\mathbb{Z}, <)$ but not in $(\mathbb{N}, <)$.
2. Let $\varphi = \exists x \exists y\, (x < y \wedge \forall z\, \neg(x < z \wedge z < y))$. This formula holds in $(\mathbb{Z}, <)$. Take, for example $x = 0$ and $y = 1$ (or in fact any two successive integer). In $(\mathbb{Q}, <)$ that does not hold because there is a rational strictly between any two rational.
3. Let $\psi(x)$ be the formula $\exists y\, y + y = x$. Then $\mathbb{Z} \vDash \psi(a)$ if and only if $a = 2k$ for some $k \in \mathbb{Z}$, i.e. a is even. So $\mathbb{Z} \vDash \neg\psi(a)$ if and only if a is odd.