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Homework 8

Due November 12th

Problem 1 :
Let $(A, 0, 1, +, \cdot)$ be a boolean algebra. For all x and $y \in A$, define $x \leqslant' y$ to hold if $y \leqslant x$ holds.

1. Prove that $(A, \leqslant')$ is a distributive, complemented lattice.
2. Prove that $x \mapsto x^{\mathbf{c}}$ is a Boolean algebra isomorphism between $(A, \leqslant)$ and $(A, \leqslant')$.
3. Let $(A, 0', 1', +', \cdot')$ be the Boolean algebra structure on A induced by $\leqslant'$. Describe $0'$, $1'$, $+'$ and $\cdot'$ in terms of 0, 1, $+$ and $\cdot$.

Problem 2 :
A Boolean algebra A is said to be complete if every set $X \subseteq A$ has a lower upper bound (which we denote by $\bigcup_{x \in X} x$).

1. Let E be some set. Show that $\mathcal{P}(E)$ is a complete Boolean algebra (for the usual boolean algebra structure).
2. Let $f : A \to B$ be an isomorphism of Boolean algebras. Show that A is complete if and only if B is complete.
3. Show that a Boolean algebra A is complete if and only if every set $X \subseteq A$ has an upper lower bound (which we denote by $\bigcap_{x \in X} x$) and show that $\bigcap_{x \in X} x = (\bigcup_{x \in X} x^{\mathbf{c}})^{\mathbf{c}}$.
4. Let $a \in A$ be an atom and $X \subseteq A$. Show that if $a \leqslant \bigcup_{x \in X} x$ then $a \leqslant x$ for some $x \in X$.
5. Show that every atomic complete Boolean algebra is isomorphic to the Boolean algebra $\mathcal{P}(\mathcal{A})$ of its set of atoms $\mathcal{A}$.