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Homework 9

Due November 19th

Problem 1 :
Let A be a finite Boolean algebra. Let $\mathcal{A}$ denote the set of its atoms.

1. Show that $\bigcap_{a\in\mathcal{A}} a^{\mathbf{c}} = 0$.
2. Show that every ultrafilter on A is principal.
3. Show that if F is a principal ultrafilter, then $\{F\}$ is open in $\mathcal{S}(A)$.
4. Show that every subset of $X \subseteq \mathcal{S}(A)$ is open (and hence closed).
5. Give a new proof that A is isomorphic to $\mathcal{P}(\mathcal{A})$.

Problem 2 :
Let E be an infinite set. We denote by $\mathcal{F}(E) \subseteq \mathcal{P}(E)$ the Boolean subalgebra of finite and cofinite subsets of E.

1. Show that the only non principal ultrafilter on $\mathcal{F}(E)$ is the Fréchet filter $\mathfrak{F} = \{X \subseteq E : X^{\mathbf{c}} \text{ is finite}\}$.
2. Recall that if A is a Boolean algebra and $a \in A$, we denote by $U_a = \{b \in A : a \leqslant b\}$ the filter generated by a. Show that $\mathcal{S}(\mathcal{F}(E)) = \{U_{\{e\}} : e \in E\} \cup \{\mathfrak{F}\}$.
3. Let $X \in \mathcal{F}(E)$. Recall that $V_X = \{F \in \mathcal{S}(\mathcal{F}(E)) : X \in F\}$. Show that the following are equivalent:
   a) X is cofinite;
   b) V_X is cofinite;
   c) V_X is not finite;
   d) $\mathfrak{F} \in V_X$.
4. Show that if $V \subseteq \mathcal{S}(\mathcal{F}(E))$ is open if and only if one of the following happens:
   - V does not contain $\mathfrak{F}$;
   - V is cofinite and contains $\mathfrak{F}$.