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Solutions to the review problems

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Problem 1 (Model theory) **:**

1. Let us first prove that b) implies a). Let us assume that $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$ and let φ, a_i and m be as in a). We have $\mathcal{M} \vDash \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$ and hence $\mathcal{A} \vDash \exists x_0\, \varphi(x_0, a_1, \ldots, a_n)$. In particular, there exists $a_0 \in A$ such that $\mathcal{A} \vDash \varphi(a_0, a_1, \ldots, a_n)$. But because $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$, we also have $\mathcal{M} \vDash \varphi(a_0, a_1, \ldots, a_n)$.

   Let us now prove a) implies b). Let $\mathcal{A} \leqslant \mathcal{M}$ verify a). We prove by induction on φ that for all $a_1, \ldots a_n \in A$, $\mathcal{A} \vDash (a_1, \ldots, a_n)$ if and only if $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$. If φ is atomic, this is an immediate consequence of the fact that $\mathcal{A}$ is a substructure of $\mathcal{M}$. If $\varphi = \neg\psi$, then $\mathcal{A} \vDash \varphi(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \nvDash \varphi(a_1, \ldots, a_n)$ (we are using the induction), if and only if $\mathcal{M} \nvDash \varphi(a_1, \ldots, a_n)$, if and only if $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$. If $\varphi = \varphi_1 \wedge \varphi_2$, then $\mathcal{A} \vDash \varphi(a_1, \ldots, a_n)$ if and only if $\mathcal{A} \vDash \varphi_1(a_1, \ldots, a_n)$ and $\mathcal{A} \vDash \varphi_2(a_1, \ldots, a_n)$, if and only if $\mathcal{M} \nvDash \varphi_1(a_1, \ldots, a_n)$ and $\mathcal{M} \vDash \varphi_2(a_1, \ldots, a_n)$ (we are using the induction), if and only if $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$.

   Let us now assume that $\varphi = \exists x_0\, \psi(x_0, x_1, \ldots, x_n)$. We have $\mathcal{A} \vDash \varphi(a_1, \ldots, a_n)$ if and only if there exists $a_0 \in A$ such that $\mathcal{A} \vDash \psi(a_0, a_1, \ldots, a_n)$, if and only if there exists $a_0 \in A$ such that $\mathcal{M} \vDash \psi(a_0, a_1, \ldots, a_n)$ (we are using the induction), if and only if there exists $m \in M$ such that $\mathcal{M} \vDash \psi(m, a_1, \ldots, a_n)$ (we are using hypothesis a)), if and only if $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$.

   This concludes the proof as all other connectives and quantifiers can be expressed using these three.

2. Recall that, in $\mathcal{L}(M)$, we denote the constant associated to $a \in M$ by $\underline{a}$.

   Let us prove that a) implies b). Let f be as in a) and let $\mathcal{N}^\star$ be the enrichment of $\mathcal{N}$ to $\mathcal{L}(M)$ such that $\underline{a}^{\mathcal{N}^\star} = f(a)$. Let $\varphi(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}^{\mathrm{el}}(\mathcal{M})$ (where φ is an $\mathcal{L}$-formula). By definition $\mathcal{M}^\star \vDash \varphi(\underline{a}_1, \ldots, \underline{a}_n)$ and hence $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$. Because f is an isosmorphism on its image, $f(\mathcal{M}) \vDash \varphi(f(a_1), \ldots, f(a_n))$ and because $f(\mathcal{M})$ is an elemtary substructure of $\mathcal{N}$, we also have $\mathcal{N} \vDash \varphi(f(a_1), \ldots, f(a_n))$ and hence $\mathcal{N} \vDash \varphi(\underline{a}_1, \ldots, \underline{a}_n)$.

   Let now $\mathcal{N}^\star$ be as in b). We define $f : \mathcal{M} \to \mathcal{N}$ by $f(a) = \underline{a}^{\mathcal{N}}$. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and $a_1, \ldots, a_n \in M$. We have $\mathcal{N} \vDash \varphi(f(a_1), \ldots, f(a_n))$ if and only if, $\mathcal{N} \vDash \varphi(\underline{a}_1, \ldots, \underline{a}_n)$ (by definition of f), if and only if $\varphi(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}^{\mathrm{el}}(\mathcal{M})$ (one implication is by hypothesis b), the other by hypothesis b) applied to $\neg\varphi$), if and only if $\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)$ if and only if $f(\mathcal{M}) \vDash \varphi(f(a_1), \ldots, f(a_n))$. We have just proved that $f(\mathcal{M})$ is an elementary substructure of $\mathcal{N}$.

3. By compactness, it suffices to show that every finite $T_0 \subseteq \mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \mathcal{D}^{\mathrm{el}}(\mathcal{N})$ is consistent. Assume one of them is not. We have $T_0 \subseteq \mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \{\varphi_i(\underline{a}_1, \ldots, \underline{a}_n) : 0 < i < k\}$ where $a_j \in N$ and $\varphi_i(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}^{\mathrm{el}}(\mathcal{N})$. Let $\psi = \bigwedge_{0<i<k} \varphi_i$, then $\psi(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}^{\mathrm{el}}(\mathcal{N})$ and $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \{\psi(\underline{a}_1, \ldots, \underline{a}_n)\}$ is inconsistent and thus $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \vdash \neg\psi(\underline{a}_1, \ldots, \underline{a}_n)$. Because the constants $\underline{a}_i$ do not appear in $\mathcal{D}^{\mathrm{el}}(\mathcal{M})$ (that is why we had to be careful to choose distinct new constants in $\mathcal{L}(M)$ and $\mathcal{L}(N)$), we

have $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \vdash \forall x_1 \ldots \forall x_n \neg\psi(x_1, \ldots, x_n)$ and hence $\mathcal{M} \models \forall x_1 \ldots \forall x_n \neg\psi(x_1, \ldots, x_n)$. But $\mathcal{M} \equiv \mathcal{N}$ and $\forall x_1 \ldots \forall x_n \neg\psi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-sentence, so $\mathcal{N} \models \forall x_1 \ldots \forall x_n \neg\psi(x_1, \ldots, x_n)$, in particular $\mathcal{M} \models \neg\psi(a_1, \ldots, a_n)$, a contradiction with that fact that $\psi(\underline{a}_1, \ldots, \underline{a}_n) \in \mathcal{D}^{\mathrm{el}}(\mathcal{N})$.

Therefore, the theory $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \mathcal{D}^{\mathrm{el}}(\mathcal{N})$ is consistent.

4. Let us first prove that b) implies a). Let $\mathcal{O}$ be as in b) and φ be an $\mathcal{L}$-sentence. We have $\mathcal{M} \models \varphi$ if and only if $\mathcal{O} \models \varphi$ if and only if $\mathcal{N} \models \varphi$ and hence $\mathcal{M} \equiv \mathcal{N}$.

   Let us now prove that a) implies b). Let us assume a). By the previous question, $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \mathcal{D}^{\mathrm{el}}(\mathcal{N})$ is consistent. Let $\mathcal{O} \models \mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \mathcal{D}^{\mathrm{el}}(\mathcal{N})$. By question 2, there exists elementary embeddings $f : \mathcal{M} \to \mathcal{O}$ and $g : \mathcal{N} \to \mathcal{O}$.

Problem 2 (λ-calculus) **:**

1. Let t be a normal λ-term such that $\vdash t : A \to A$. The last rule to be applied in the derivation of $\vdash t : A \to A$ cannot be (Ax) as the context is empty. Let us assume it is $(\to_E)$ and hence t is an application. Because t is normal, there exists a variable $x \in V$ and normal terms $t_1, \ldots t_n$ such that $t = (\ldots((x)t_1)\ldots)t_n$. But we saw in class that if $\Gamma \vdash t$, $\mathrm{fvar}(t) \subseteq \mathrm{fvar}(\Gamma)$. But here $\Gamma = \varnothing$ and hence t cannot contain a free variable. So the last rule cannot be $(\to_E)$ and it has to be $(\to_I)$.

   It follows that $t = \forall x\, u$ and that $x : A \vdash u : A$ holds. Because A is not of the form $B \to C$, the last applied rule to prove $x : A \vdash u : A$ cannot be $(\to_I)$. Be cause u is normal and does not begin with a λ, it is of the form $u = (\ldots((y)t_1)\ldots)t_n$ for some $y \in V$ and $t_i \in \Lambda$. So the n previous rules applied have to be $(\to_E)$ and there are types $A_1, \ldots A_n$ such that $x : A \vdash y : A_1 \to (\ldots \to (A_n \to A)\ldots)$ holds. But the only applicable rule would be (Ax) which only applies if $y = x$, $n = 0$ and $u = x$. So $t = \lambda x\, x$.

2. Let t be a normal λ-term such that $\vdash t : A \to A$. By similar considerations as above, $t = \lambda x\, u$ and $x : A \vdash u : A \to A$ holds. Let us assume that $u = (\ldots((y)t_1)\ldots)t_n$ for some $y \in V$ and $t_i \in \Lambda$. As above, there are types $A_1, \ldots A_n$ such that $x : A \vdash y : A_1 \to (\ldots \to (A_n \to (A \to A))\ldots)$ holds. But that typing statement cannot hold because $A \neq A_1 \to (\ldots \to (A_n \to (A \to A))\ldots)$ for any choice of n and A_i.

   It follows that the last rule applied is $(\to_I)$, that $u = \lambda y\, v$ and that $\{x : A, y : A\} \vdash v : A$ holds. The last applied rule cannot be $(\to_I)$ because A is not of the form $B \to C$. So $v = (\ldots((z)t_1)\ldots)t_n$ for some $z \in V$ and $t_i \in \Lambda$. As above, there must exist types A_i such that $\{x : A, y : A\} \vdash z : A_1 \to (\ldots \to (A_n \to (A \to A))\ldots)$ holds. The only applicable rule is (Ax) and hence $z = x$ or y and $n = 0$. We have proved that $t = \lambda x \lambda y\, x$ or $t = \lambda x \lambda y\, y$.

Problem 3 (Boolean algebras) **:**

1. Let us first prove that a) implies b). Let $X \subseteq A$ whose lower upper bound is 1. Then because f is complete, the lower upper bound of $f(X)$ is $f(1) = 1$.

   Let us now prove that b) implies a). Let $X \subseteq A$ whose upper bound in A is a. Let $Y = X \cup \{a^{\mathbf{c}}\}$. Any upper bound c of Y is an upper bound of X so $a \leqslant c$. Moroever, because $a^{\mathbf{c}} \in Y$, $a^{\mathbf{c}} \leqslant c$ and hence $c = 1$. By hypothesis a), the lower upper bound of $f(Y) = f(X) \cap \{f(a)^{\mathbf{c}}\}$ is 1.

For all all $x \in X$, $x \leqslant a$ and hence $f(x) \leqslant f(a)$. It follows that $f(a)$ is an upper bound of X. Moreover, let $c \in B$ be any upper bound of $f(X)$, then $c \cup f(a)^{\mathbf{c}}$ is an upper bound of $f(Y)$ and hence $c \cup f(a)^{\mathbf{c}} = 1$. Applying De Morgan's law, we get that $c^{\mathbf{c}} \cap f(a) = 0$ and hence $(1+c) \cdot f(a) = f(a) + c \cdot f(a) = 0$, i.e. $c \cdot f(a) = f(a)$ and $f(a) \leqslant c$. So $f(a)$ is an upper bound that is smaller that any other upper bound. It is the lower upper bound of $f(X)$.

2. Let us first prove that a) implies b). Let I be a complete ideal and $f : A \to A/I$ be the canonical projection. Then $f^{-1}(0) = I$ and let us show that f is complete. Let $X \subseteq A$ whose lower upper is 1 and let $f(c)$ be an upper bound of $f(X)$. For all $x \in X$, $f(x) \leqslant f(c)$ and thus $f(x \cap c^{\mathbf{c}}) = f(x) \cap f(c)^{\mathbf{c}} = 0$, i.e. $x \cap c^{\mathbf{c}} \in I$. Moreover, Let b be some upper bound of $\{x \cap c^{\mathbf{c}} : x \in X\}$ (we want to show that $\circ c \leqslant b$). For all $x \in X$, we have $b \cup c \geqslant (x \cap c^{\mathbf{c}}) \cup (x \cap c) = x \cap (x^{\mathbf{c}} \cup c) = x$. So $b \cup c$ is an upper bound of X and hence $b \cup c = 1$. It follows immediately that $b \geqslant c^{\mathbf{c}}$. Moreover $c^{\mathbf{c}}$ is clearly an upper bound of $\{x \cap c^{\mathbf{c}} : x \in X\}$, so it is the lower upper bound.

   But we proved earlier that $\{x \cap c^{\mathbf{c}} : x \in X\} \subseteq I$ and hence, as I is complete, $c^{\mathbf{c}} \in I$, i.e. $f(c)^{\mathbf{c}} = 0$ and $f(c) = 1$.

   Let us now prove that b) implies a). Let f be as in b) and let $X \subseteq I$ whose lower upper bound is $a \in A$. By completeness of f, $f(a)$ is the lower upper bound of $f(X) = \{0\}$ and $f(a) = 0$, i.e. $a \in I$.

3. Let us first prove it is closed under addition. Let $a, b \in \bigcap_j I_j$. We have $a + b \in I_j$ for all j and hence $a + b \in \bigcap_j I_j$. Let now $a \in A$ and $b \in \bigcap_j I_j$. For all j, we have $a \cdot b \in I_j$ and hence $a \cdot b \in \bigcap_j I_j$. Moreover $1 \notin I_j$ for all j so $1 \notin \bigcap_j I_j$ and $0 \in I_j$ for all j so $0 \in \bigcap_j I_j$. Finally, let $X \subseteq \bigcap_j I_j$, then the lower upper bound of X is in each of the I_j and hence in $\bigcap_j I_j$.

4. Let $J = \{I \subseteq A : I \text{ is a complete ideal and } X \subseteq I\}$ be non empty and $I_0 = \bigcap_{I \in J} I$. By the previous question I_0 is a complete ideal. It clearly contains X and it is contained in any complete ideal that contains X, so it is the smallest element of J.

5. First of all N is an ideal. If a and $c \in N$ then for all $b \in I$, $(a+b) \cdot c = a \cdot c + b \cdot c = 0 + 0 = 0$. If $a \in A$ and $c \in N$, then $a \cdot c \cdot b = a \cdot 0 = 0$. Moreover $0 \cdot b = 0$ for all $b \in I$ so $0 \in N$ and if $1 \in N$, then for all $b \in I$, $b = b \cap 1 = 0$, contradicting the fact that $I \neq 0$.

   Let us now prove that N is complete. Let $X \subseteq N$ whose lower upper bound in A is a. Let $b \in I$, we have to show that $a \cap b = 0$. For all $x \in X$, $x \cap b = 0$ and hence $x \leqslant b^{\mathbf{c}}$. It follows that $b^{\mathbf{c}}$ is an upper bound of X and hence $a \leqslant b$, i.e. $a \cap b = 0$.

   Let $b \in N \cap I$. We have $b = b \cap b = 0$. There remains to show that there is no (proper[1]) complete ideal containing $I \cup N$. Let us first show that le lower uppper bound of $I \cup N$ is 1. Let $a \in A$ be an upper bound of $I \cup N$, then for all $b \in I$, $b \leqslant a$ and hence $b \cap a^{\mathbf{c}} = 0$ so $a^{\mathbf{c}} \in N$ and hence $a^{\mathbf{c}} \leqslant a$, i.e. $1 + a = (1 + a)a = a + a = 0$ and $a = 1$.

   If there existed a complete ideal J containing $I \cup N$, it would contain 1, a contradiction.

[1]Remeber that we assumed all ideal s to be proper.