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Review problems

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Problem 1 (Model theory) **:**
Let $\mathcal{M}$ be a substructure of $\mathcal{N}$. We say that $\mathcal{M}$ is an elementary substructure if for every formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots a_n \in M$,

$$\mathcal{M} \vDash \varphi(a_1, \ldots, a_n) \text{ if and only if } \mathcal{N} \vDash \varphi(a_1, \ldots, a_n)$$

1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{A} \leqslant \mathcal{M}$ be a substructure. Show that the following are equivalent:
   a) For every $\mathcal{L}$-formula $\varphi(x_0, x_1, \ldots, x_n)$, $a_1, \ldots a_n \in A$ and $m \in M$, if $\mathcal{M} \vDash \varphi(m, a_1, \ldots a_n)$, then there exists $a \in A$ such that $\mathcal{M} \vDash \varphi(a, a_1, \ldots a_n)$.
   b) $\mathcal{A}$ is an elementary substructure of $\mathcal{M}$.

   Hint: for a) $\Rightarrow$ b), proceed by induction on formulas.

2. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Show that the following are equivalent:
   a) There exists an embedding $f : \mathcal{M} \to \mathcal{N}$ such that $f(M)$ is an elementary substructure of $\mathcal{N}$ (We say that f is an elementary embedding).
   b) The structure $\mathcal{N}$ can be enriched into an $\mathcal{L}(M)$-structure $\mathcal{N}^\star$ such that $\mathcal{N} \vDash \mathcal{D}^{\mathrm{el}}(\mathcal{M})$.

3. Let $\mathcal{M}$ and $\mathcal{N}$ be elementarily equivalent $\mathcal{L}$-structures. Show that $\mathcal{D}^{\mathrm{el}}(\mathcal{M}) \cup \mathcal{D}^{\mathrm{el}}(\mathcal{N})$ is consistent (one has to be careful to chose the new constants in $\mathcal{L}(M)$ and $\mathcal{L}(N)$ so that they are discinct, i.e. $\mathcal{L}(M) \cap \mathcal{L}(N) = \mathcal{L}$)

4. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structure. Show that the following are equivalent:
   a) $\mathcal{M} \equiv \mathcal{N}$.
   b) There exists an $\mathcal{L}$-structure $\mathcal{O}$ and elementary embedding $f : \mathcal{M} \to \mathcal{O}$ and $g : \mathcal{N} \to \mathcal{O}$.

Problem 2 (λ-calculus) **:**
Let $A \in W$ be a type variable.

1. Show that the only normal λ-terms such that $\vdash t : A \to A$ is $\lambda x\, x$.

2. Show that the only two normal λ-terms such that $\vdash t : A \to (A \to A)$ are $\lambda x \lambda y\, x$ and $\lambda x \lambda y\, y$.

Problem 3 (Boolean algebras) **:**
Let A be a Boolean algebra. An ideal $I \subseteq A$ is said to be complete it for all $X \subseteq I$ such that X has a lower upper bound $a \in A$, then $a \in I$. A homomorphism of Boolean algebras $f : A \to B$ is said to be complete if for all $X \subseteq A$, such that X has a lower upper bound $a \in A$, then $f(a)$ is the lower upper bound of $f(X)$.

1. Let $f : A \to B$ be a homomorphism of Boolean algebras. Show that the following are equivalent:

a) f is complete;

b) For all $X \subseteq A$ whose lower upper bound is 1, then 1 is also the lower upper bound of $f(X)$.

2. Let $I \subseteq A$. Show that the following are equivalent:

   a) I is a complete ideal;

   b) There exists a complete homomorphism of Boolean algebras $f : A \to B$ such that $f^{-1}(0) = I$.

3. Let $(I_j)_{j\in J}$ be a family of complete ideals, show that $\bigcap_j I_j$ is a complete ideal.

4. Let $X \subseteq A$ be some set, show that if $\{I \subseteq A : I \text{ is a complete ideal and } X \subseteq I\}$ is non empty it has a smallest element, usually called the complete ideal generated by X.

5. Let $I \neq \{0\}$ be a complete ideal and define $N = \{a \in A : a \cap b = 0 \text{ for all } b \in I\}$. Show that N is a complete ideal, that $N \cap I = \{0\}$ and that no proper complete ideal contains both I and N.

6. Let $B = \{I \subseteq A : I \text{ complete ideal}\} \cup \{A\}$ (i.e. the set of proper and non proper ideals of A). Show that $(B, \subseteq)$ is a distributive and complemented lattice such that every set in B has an upper lower bound and a lower upper bound (we say that B is complete).

7. Let $I_a = \{c \in A : a \leqslant c\}$. Show that $f : a \mapsto I_a$ is a injective homomorphism of Boolean algebras $A \to B$.

8. Show that every element in B is the lower upper bound of $f(X)$ for some $X \subseteq A$.

We say that B is the completion of A.