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Midterm

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To do a later question in a problem, you can always assume a previous question even if you have not answered it.

Problem 1 :
Let T be a theory with infinite models in a language $\mathcal{L}$ with one sort. Assume that:

- T eliminates quantifiers;
- For all $A \subseteq M \vDash T$, $\mathrm{acl}(A) = A$.
- For all $\mathcal{L}$-formula $\varphi(x,y)$, there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that for all $M \vDash T$ and $a \in M^y$, if $|\varphi(M,a)| \geqslant k$, then $\varphi(M,a)$ is infinite. We say that T eliminates $\exists^\infty$.

Let $M \vDash T$ and let $<$ be a total order on M. We say that $<$ is generic if for all M-definable infinite $X \subseteq M$ and $a < b \in M \cup \{+\infty, -\infty\}$, $X \cap (a,b) \neq \varnothing$, where (a,b) is the open interval between a and b.

1. Show that if T is strongly minimal and only has infinite models, then it eliminates $\exists^\infty$.

   Solution: Assume T does not eliminate $\exists^\infty$. So for some formula $\varphi(x,y)$, for all $i \in \mathbb{Z}_{\geqslant 0}$, there exists $M_i \vDash T$ and $a_i \in M_i^y$ such that $i \leqslant |\varphi(M_i,a_i)| < \infty$. Let us first assume that $|x| = 1$[1]. Note that since M_i is infinite, $\neg\varphi(M_i,a_i)$ is infinite.

   Let $\mathfrak{U}$ be a non-principal ultrafilter on $\mathbb{Z}_{\geqslant 0}$, $M := \prod_{i\to\mathfrak{U}} M_i \vDash T$ and $a := [(a_i)_i]_{\mathfrak{U}} \in M^y$. Then for all $i \in \mathbb{Z}_{\geqslant 0}$, since $M_j \vDash \exists^{\geqslant i}x\ \varphi(x,a_j) \wedge \exists^{\geqslant i}x\ \neg\varphi(x,a_j)$ for all $j \geqslant i$, we have $|\varphi(M,a)| \geqslant i$ and $|\neg\varphi(M,a)| \geqslant i$, for all i, i.e. $\varphi(M,a)$ is infinite and coinfinite, contradicting strong minimality of T.

   We now proceed by induction on $|x|$. We just proved that we cannot have $|x| = 1$. If $x = (s,t)$ where $|t| = 1$, then, by induction, there exists k such that for all $M \vDash T$, $a \in M^y$ and $b \in M^t$, if $|\varphi(M,b,a)| \geqslant k$, then it is infinite and if $|\exists s\varphi(s,M,a)| \geqslant k$, it is infinite. If $\varphi(M,a)$ is finite, the projection to M^t is finite so size at most k. Moreover, for all b in the projection, $\varphi(M,b,a)$ is finite of size at most k so $\varphi(M,a)$ is size at most k^2. But that contradicts our initial hypothesis regarding φ.

2. Let $M \vDash T$ be infinite and $<$ be a total order on M. Show that there exists $N \succcurlyeq M$ with a generic order extending $<$.

   Solution: First, let us show that we can build $N \succcurlyeq M$ and an order on N extending that of M, such that any infinite M-definable subset of some N^x has a non empty intersection with any interval whose bounds are in $M \cup \{-\infty, +\infty\}$. Let $N \succcurlyeq M$ be $(|M| + |\mathcal{L}|)^+$-saturated. It follows that any infinite N-definable set has cardinality at least $\kappa^+ := (|M| + |\mathcal{L}|)^+$. Let $\{(a_i, b_i, X_i) : i \in \kappa\}$ be an enumeration of all triples $a < b \in M \cup \{-\infty, +\infty\}$and infinite M-definable set $X \subseteq N$. We construct a total order $<_i$ extending $<$ and all $<_j$ for $j < i$, on some $A_i \subseteq N$ with $|A_i| < \kappa$, by induction

[1] I was sneaky, but no one noticed... The definition I gave for elimination of $\exists^\infty$ is, *a priori*, more general than the definition usually considered since I allowed arbitrary finite tuples for x. But, as we will see, both definitions are equivalent

on i. Assume that $<_j$ is built for all $j < i$. Since X_i is large enough, we can find some$c_i \in X_i \smallsetminus \bigcup_{j<i} A_j$. Then $<_i$ is the order on $\{c_i\} \cup \bigcup_{j<i} A_j$ extending all the $<_j$ and such that c_i is just above a_i. Let $<_\kappa = \bigcup_{i<\kappa} <_i$. We extend $<_\kappa$ to a total order on N by choosing an order on $N \smallsetminus (\bigcup_{i<\kappa} A_i)$ and setting all these elements above the elements of $\bigcup_{i<\kappa} A_i)$.

We now build an elementary chain $(N_i)_{i\in\omega}$, by induction on i, with $N_0 = M$ and N_{i+1} built from N_i as above. Then $N = \bigcup_i N_i \succcurlyeq M$ and for any $a < b \in N \cup \{-\infty, +\infty\}$ and infinite N-definable $X \subseteq N$, there exists an i such that $a < b \in N_i \cup \{-\infty, +\infty\}$ and X is N_i-definable. But then, since $N_{i+1} \succcurlyeq N$, $X \cap N_{i+1}$ is an infinite N_i-definable set and $(a, b) \cap X \cap N_{i+1} \neq \varnothing$ by construction. So N has the required properties.

3. Let $\mathcal{L}_<$ be $\mathcal{L}$ with a new binary symbol $<$. Show that there exists an $\mathcal{L}_<$-theory $T_<$ such that models of $T_<$ are exactly the models of T where $<$ is generic (with respect to the $\mathcal{L}$-structure of M).

   Solution: For every $\mathcal{L}$-formula $\varphi(x, y)$, where $|x| = 1$, let k_φ be the bound given by elimination of $\exists^\infty$ and Ψ_φ be the sentence $\forall y(\exists^{\geqslant k_\varphi} x\ \varphi(x, y) \to (\forall a \forall b\ a < b \to (\exists x\ a < x \wedge x < b \wedge \varphi(x, y)) \wedge (\forall a\ (\exists x\ x < a \wedge \varphi(x, y)) \wedge (\exists x\ x > a \wedge \varphi(x, y))))$. Then $T_< = T \cup \{\Psi_\varphi : \varphi(x, y)\ \mathcal{L}\text{-formula}\}$ has the required properties. Indeed, $M \vDash T_<$ then $M \vDash T$ and for every infinite $X = \varphi(M, a)$, where $a \in M^y$, then $|X| \geqslant k_\varphi$ and hence, by Ψ_φ, X is dense in $(M, <)$. Conversely, if $M \vDash T$ has a generic order $<$, then for any formula $\varphi(x, y)$ and $a \in M^y$, if $|\varphi(M, a)| \geqslant k_\varphi$, then $\varphi(M, a)$ is infinite and hence, by genericity, intersects every open interval.

4. Show that $T_<$ eliminates quantifiers.

   Solution: Let $M, N \vDash T_<$, $A \subseteq M$, $f : A \to N$ be a partial embedding and assume N is $|A|^+$-saturated. Pick any $c \in M$. If $c \in \mathrm{acl}(A) = A$, then f is already defined at a. Otherwise, for any $\mathcal{L}$-formula $\varphi(x, a)$, where $a \in A^y$, if $M \vDash \varphi(c, a)$, then $\varphi(M, a)$ is infinite. By quantifier elimination in T, it follows that $\varphi(N, f(a))$ is also infinite. So it intersects any open interval. Then $\pi(x) = \{\varphi(x, f(a)) : M \vDash \varphi(c, a),\ a \in A^y \text{ and } \varphi \text{ is an } \mathcal{L}\text{-formula}\} \cup \{x > f(a) : a \in A \text{ and } c < a\} \cup \{x > f(a) : a \in A \text{ and } c > a\}$ is finitely satisfiable. Here, we are using the fact that the intersection of $\mathcal{L}(A)$-definable sets containing c is still a $\mathcal{L}(A)$-definable set containing c, that f respects the order and that the non-empty intersection of open intervals is an open interval. Let d realize π in N, then f can be extended by sending c to d.

5. Show that $T_<$ is complete.

   Solution: Since the interpretation of any constant is in $\mathrm{acl}(\varnothing) = \varnothing$, it follows that $\mathcal{L}$ does not have any constant. So $T_<$ is a theory that eliminates quantifiers in a language without constants, so it is complete.

6. Let $M \vDash T_<$ and $A \subseteq M$. Show that $\mathrm{acl}(A) = A$ (here, the algebraic closure is understood in M as an $\mathcal{L}_<$-structure).

   Solution: Pick any $b \in M \smallsetminus A$, then c is not algebraic over A in M as an $\mathcal{L}$-structure, so any $\mathcal{L}$-formula $\varphi(x, a)$, with $a \in A^y$ and $M \vDash \varphi(c, a)$, is such that $\varphi(M, a)$ is infinite. Assume M is $|A|^+$-saturated, let $B \subseteq M$ be any countable set and let $\pi(x) = \{\varphi(x, a) : M \vDash \varphi(c, a),\ a \in A^y \text{ and } \varphi \text{ is an } \mathcal{L}\text{-formula}\} \cup \{x > a : a \in A \text{ and } c < a\} \cup \{x > a : a \in A \text{ and } c > a\} \cup \{x \neq b : b \in B\}$. As in the previous question, π is finitely satisfiable so it is satisfied in M. It follows from quantifier elimination in $T_<$, that any realisation of π has the same type as c over A and hence $\mathrm{tp}(c/A)$ has infinitely many realizations in M, so $a \notin \mathrm{acl}(A)$.

7. Assume $\mathcal{L}$ is countable. Show that $T_<$ is ω-categorical if and only if T is ω-categorical.

   Solution: Let us first assume that T is ω-categorical and let $M \models T_<$ be countable. Pick any finite $A \subseteq M$ and $p \in \mathcal{S}_x^M(A)$ where $|x| = 1$. Let $a \in A$ be the maximal element of $\{c \in A : "c < x" \in p\}$ — if this set is empty, let $a = -\infty$ —, $b \in A$ be the minimal element of $\{c \in A : "x < c" \in p\}$ — if this set is empty, let $b = +\infty$ — and let $\varphi(x)$ be an $\mathcal{L}(A)$-formula isolating $p|_{\mathcal{L}}$ — this formula exists because T is ω-categorical and hence so is $\mathcal{D}^{\mathrm{el}}(A)$, by counting types. If $\varphi(M)$ is finite, then any realization of p is in $\mathrm{acl}(A) = A$ and hence p contains a formula of the form $x = c$; it is then obviously realized by c in M . If $\varphi(M)$ is infinite, then, by density, we can find $c \in \varphi(M) \cap (a, b)$. Then, $c \models p|_{\mathcal{L}}$ and for all $d \in A$, $d < c$ if and only if $"d < x" \in p$. By quantifier elimination, it follows that $c \models p$. So every countable model of $T_<$ is saturated and $T_<$ is ω-categorical.

   If T $<$ is ω-categorical, let $M \models T_<$ be countable and x be a finite tuples of variables. Then M realizes only finitely many $\mathcal{L}_<$-types in variables x, so $M|_{\mathcal{L}} \models T$ realizes only finitely many $\mathcal{L}$-types in variables x. It follows that T is ω-categorical.

Problem 2 :
Let T be a theory, $\kappa > |\mathcal{L}|$, $M \models T$ be κ-saturated, and $X \subseteq M^x$ be $\varnothing$-definable. Consider the following statements.

(i) Any M-definable set $Y \subseteq X^n$ is X-definable.

(ii) For all $a \in M^z$, there exists $C \subseteq X$ such that $|C| < \kappa$ and for all $b \in M^z$, $\mathrm{tp}^M(a/C) = \mathrm{tp}^M(b/C)$ implies $\mathrm{tp}^M(a/X) = \mathrm{tp}^M(b/X)$.

(iii) For all $a, b \in M^z$, $\mathrm{tp}^{M^{\mathrm{eq}}}(a/C) = \mathrm{tp}^{M^{\mathrm{eq}}}(b/C)$ implies $\mathrm{tp}^M(a/X) = \mathrm{tp}^M(b/X)$, where $C = \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X)$.

1. Show that (i) implies (iii).

   Solution: Let $a \in M^z$ and $\varphi(t, z)$ be some formula, where t is a tuple of n variables sorted like x. The a-definable set $Y = \varphi(M, a) \cap X^n$ is also X-definable by (i). So $\ulcorner Y \urcorner \subseteq \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X) = C$ and hence Y is C-definable (in M^{eq}). It follows that the formula $\forall t\ (t \in Y \leftrightarrow \varphi(z, t) \wedge t \in X^n)$ is a formula in $\mathrm{tp}^{M^{\mathrm{eq}}}(a/C)$, so if $\mathrm{tp}^{M^{\mathrm{eq}}}(b/C) = \mathrm{tp}^{M^{\mathrm{eq}}}(a/C)$, $\varphi(b, M) \cap X^n = Y$ and for all $m \in X^n$, $M \models \varphi(a, m)$ if and only if $M \models \varphi(b, m)$. Since this holds for every formula φ, $tp^M(a/X) = tp^M(b/X)$.

2. Assume that (i) does not hold. Show that there exists $a \in M^x$ and a formula $\varphi(t, y)$ such that, for any $C \subseteq X$ with $|C| < \kappa$, there exists $b_1, b_2 \in X^n$ with $\mathrm{tp}(b_1/C) = \mathrm{tp}(b_2/C)$, $M \models \varphi(a, b_1)$ and $M \models \neg\varphi(a, b_2)$.

   Solution: If (i) does not hold, there exists $\varphi(t, z)$ and $a \in M^z$, such that $\varphi(M, a) \cap X^n$ is not X-definable. Pick any $C \subseteq X$ with $|C| < \kappa$ and $\psi(t)$ an $\mathcal{L}^{\mathrm{eq}}(C)$-formula. If for all $b_1, b_2 \in X^n$, $M \models (\psi(b_1) \leftrightarrow \psi(b2)) \rightarrow (\varphi(b_1, a) \rightarrow \varphi(b_2, a))$, then, since $\varphi(M, a) \cap X^n$ is neither X^n nor $\varnothing$ that are both X-definable, $\psi(M) = \varphi(M, a) \cap X^n$ or $\neg\psi(M) = \varphi(M, a) \cap X^n$, a contradiction. It follows that there exists $b_1, b_2 \in X^n$ such that $M \models (\psi(b_1) \leftrightarrow \psi(b_2)) \wedge \varphi(b_1, a) \wedge \neg\varphi(a, b_2)$. By compactness, there exists $b_1, b_2 \in X^n$ such that $M \models \varphi(b_1, a) \wedge \neg\varphi(a, b_2)$ and for all $\mathcal{L}(C)$-formula $\psi(t)$, $M \models \psi(b_1) \leftrightarrow \psi(b_2)$ — i.e. $\mathrm{tp}(b_1/C) = \mathrm{tp}(b_2/C)$.

3. Show that (i), (ii) and (iii) are equivalent.

   Solution: We have shown that (i) implies (iii) in question 2.1. If (i) fails, let a and $\varphi(t,z)$ be as in 2.2. and for any choice of $C \subseteq X^n$, let b_1, b_2 be as in 2.2. By κ-saturation, we find $a' \in M^z$ such that $\mathrm{tp}(ab_1/C) = \mathrm{tp}(a'b_2/C)$. But then $tp(a_1/C) = \mathrm{tp}(a_2/C)$ and $M \vDash \neg\varphi(a,b_2) \wedge \varphi(a',b_2)$ — i.e. $\mathrm{tp}(a/X)\neg tp(a'/X)$. So (ii) fails.

   There remains to prove that (iii) implies (ii). Let us assume (iii) holds. Pick any $a \in M^z$ and let $C = \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X)$. We can find $D \subseteq M$ with $|D| = |C|$ and $C \subseteq \mathrm{dcl}^{\mathrm{eq}}(D)$. Indeed, any $c \in C$ lives in a sort $S_{\varphi,y}$ for some formula $\varphi(x,y)$ and, by surjectivity, $c = f_{\varphi,y}(d)$ for some $d \in M^y$. If $\mathrm{tp}(a/D) = \mathrm{tp}(b/D)$, then $\mathrm{tp}^{M^{\mathrm{eq}}}(a/C) = \mathrm{tp}^{M^{\mathrm{eq}}}(b/C)$, and hence, by (ii) $\mathrm{tp}(a/X) = \mathrm{tp}(b/X)$.

4. Assume (i). Let $N \succcurlyeq M$, $a \in N^z$ and $X(N) := \psi(N)$ for any formula ψ such that $X = \psi(M)$. Show that $\mathrm{tp}(a/X)$ is realized in M if and only if $\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X(N)) \subseteq M^{\mathrm{eq}}$.

   Solution: First, let us show that $X(N)$ has (i) in N. Fix a formula $\varphi(t,z)$ where t is a tuple of n variables sorted like x and let $\Sigma(z) = \{\forall s\ s \in X^m \to (\exists tt \in X^n \wedge \neg(\varphi(t,z) \leftrightarrow \chi(t,s))) : \chi\ \mathcal{L}\text{-formula}\}$. By (i), Σ is not satisfiable in M and hence, by κ-saturation, it is not finitely satisfiable. So there exists χ_i for $i \leqslant n$ such that for all $a \in M^z$, $\varphi(M,a) \cap \psi(M)^n = \chi_i(M,b)$ for some $b \in X^m$. This is a first order statement so it also holds in N and hence (i) holds of $X(N)$ in N.

   If $\mathrm{tp}(a/X)$ is realized by some $b \in M^z$, then for every $c \in \mathrm{dcl}^{\mathrm{eq}}(b) \cap \mathrm{dcl}^{\mathrm{eq}}(X)$, there exists $\mathcal{L}^{\mathrm{eq}}$-definable maps f and g and $d \in X^n$ such that $f(b) = g(d)$, but this formula is in $\mathrm{tp}(b/X)$, so it also holds of a and $\mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X(N)) = \mathrm{dcl}^{\mathrm{eq}}(b) \cap \mathrm{dcl}^{\mathrm{eq}}(X) \subseteq M^{\mathrm{eq}}$. Conversely, if $C := \mathrm{dcl}^{\mathrm{eq}}(a) \cap \mathrm{dcl}^{\mathrm{eq}}(X(N)) \subseteq M^{\mathrm{eq}}$, let $b \in M$ realize $\mathrm{tp}(a/C)$. By (iii) we have $tp(b/X) = tp(a/X)$ which is therefore realized in M.

5. Assume (i) and M is saturated (and $|M| > |\mathcal{L}|$). Let $a, b \in M^z$ be such that $\mathrm{tp}(a/X) = \mathrm{tp}(b/X)$. Show that there exists $\sigma \in \mathrm{Aut}(M/X) = \{\sigma \in \mathrm{Aut}(M) : \sigma|_X = \mathrm{id}\}$ such that $\sigma(a) = b$.

   Solution: Let us show that the set I of partial elementary embedding from M to M whose domain is of the form $A \cup X$ where $|A| < |M|$ and whose restriction to X is the identity, has the back and forth. It is obviously non-empty as it contains the identity on X. Now, pick any $f \in I$, with domain $A \cup X$ and $c \in M$. let $p := tp(c/A \cup X)$ and let $q := f_\star p \in \mathcal{S}_x(f(A) \cup X)$. Note that $\mathrm{dcl}^{\mathrm{eq}}(f(A) \cup X)$ is the image, by (the unique extension of) f (to $dcleq(A \cup X)$), of $\mathrm{dcl}^{\mathrm{eq}}(A \cup X)$. Hence, for any $d \vDash q$ in $N \succcurlyeq M$, $\mathrm{dcl}^{\mathrm{eq}}(f(A) \cup d) \cap \mathrm{dcl}^{\mathrm{eq}}(f(A) \cup X(N)) \subseteq \mathrm{dcl}^{\mathrm{eq}}(f(A) \cup X) \subseteq M^{\mathrm{eq}}$. Applying the previous question in M as a model of $\mathcal{D}^{\mathrm{el}}(f(A))$, we find $d \vDash q$ in M and we can extend f. The other direction is symmetric.

   Since $\mathrm{tp}(a/X) = \mathrm{tp}(b/X)$, the map fixing X and sending a to b is an element of I. Let $\{m_\alpha : \alpha \in |M|\} = M$. Using back and forth, we build, by induction, a coherent system of partial elementary embeddings f_α and g_α extending f, such that m_α is in the domain of f_α and in the image of g_α. The union of all these partial elementary embeddings is an isomorphism of M, fixing X pointwise and sending a to b.