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Homework 2

Problem 1 :
Let C be a set of finite $\mathcal{L}$-structures. Let $T = \{\varphi : \varphi$ is a sentence and for all $M \in C$, $M \vDash \varphi\}$.

1. Give a necessary and sufficient condition for T to have an infinite model.
2. Assume that T has infinite models, give a theory T' such that the models of T' are exactly the infinite models of T.
3. Show that $T' \vDash \varphi$ if and only if there exists some $n \in \mathbb{N}$ such that for all $M \in C$ of cardinality greater than n, $M \vDash \varphi$.

Problem 2 :
A sentence φ is said to be universal if it is of the form $\forall x_1 \ldots \forall x_n\, \psi(x_1, \ldots, x_n)$ where ψ is quantifier free.

1. Let φ be universal, M an $\mathcal{L}$-structure, and $N \leqslant M$. Show that if $M \vDash \varphi$ then $N \vDash \varphi$.
2. Let T be an $\mathcal{L}$-theory and $\overline{c}$ a tuple of new constants (i.e. that do not appear in $\mathcal{L}$). Let $\varphi(\overline{x})$ be an $\mathcal{L}$-formula, such that $\overline{x}$ is sorted as $\overline{c}$. Show that if $T \vDash \varphi(\overline{c})$ then $T \vDash \forall \overline{x}\, \varphi(\overline{x})$.
3. Let $M \vDash T_\forall$. Show that the $\mathcal{L}(M)$-theory $\Delta_M(M) \cup T$ is consistent.
4. Let T and T' be two theories, show that the following are equivalent:
   a) $T_\forall \subseteq T'_\forall$;
   b) Every model of T' can be embedded in a model of T.
5. Show that T is stable under substructure (i.e. if $N \vDash T$ and $f : M \to N$ is an embedding, then $M \vDash T$) if and only if T is equivalent to $T_\forall$.
6. Let T be the $\mathcal{L}_{\mathrm{rg}}$-theory of algebraically closed fields (where $\{\mathbf{K}; 0 : \mathbf{K}, 1 : \mathbf{K}, - : \mathbf{K} \to \mathbf{K}, + : \mathbf{K}^2 \to \mathbf{K}, \cdot : \mathbf{K}^2 \to \mathbf{K}\}$). Which are the models of $T_\forall$.
7. Let $\mathcal{L} = \{X; <: X^2\}$ and T, T' be two $\mathcal{L}$-theories containing the theory of infinite total orders. Show that $T_\forall = T'_\forall$.