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Solutions to homework 4

Problem 1 :
Let $\mathcal{L}_n$ be the language with one sort and n binary predicates $(E_i)_{0\leqslant i<n}$.

1. Give an $\mathcal{L}_n$-theory T_n such that in models of T_n, for all i, the E_i are equivalence relations, E_{i+1} is finer than E_i (i.e. every E_{i+1}-class is included in an E_i-class), E_0 has infinitely many classes, every E_i class is covered by infinitely many E_{i+1}-classes and the classes of E_{n-1} are infinite.

   Solution: We define:

$$\begin{aligned} T_n \quad := \quad & \{\forall x\, xE_ix : 0 \leqslant i < n\} \\ & \cup\{\forall x\forall y\, xE_iy \to yE_ix : 0 \leqslant i < n\} \\ & \cup\{\forall x\forall y\forall z\,(xE_iy \wedge yE_iz) \to xE_iz : 0 \leqslant i < n\} \\ & \cup\{\forall x\forall y\, xE_{i+1}y \to xE_iy : 0 \leqslant i < n-1\} \\ & \cup\{\forall x\exists y_1 \ldots \exists y_k \bigwedge_j xE_iy_j \wedge \bigwedge_{j_1\neq j_2} \neg y_{j_1}E_{i+1}y_{j_2} : 0 \leqslant i < n-1 \text{ and } k > 0\} \\ & \cup\{\exists y_1 \ldots \exists y_k \bigwedge_{j_1\neq j_2} \neg y_{j_1}E_0y_{j_2} : k > 0\} \\ & \cup\{\forall x\exists y_1 \ldots \exists y_k \bigwedge_j xE_{n-1}y_j \wedge \bigwedge_{j_1\neq j_2} \neg y_{j_1} = y_{j_2} : k > 0\} \end{aligned}$$

2. How many countable models does T_n have, up to isomorphism?

   Solution: There is only one countable model of T_n up to isomorphism. Indeed, let M, $N \vDash T_n$ be coutable. Since E_0 has infinitely many classes, there must be $\aleph_0$ many of them. Let us fix a bijection f_0 between M/E_0 and N/E_0. Fix an E_0 class x in M. Then both x and $f_0(x)$ are convered by infinitely many, hence countably many, E_1-classe. So one can fix a bijection $f_{1,x}$ between x/E_1 and $f(x)/E_1$. Let $f_1 := \bigcup_{x\in M/E_0} f_{1,x}$. Then f_1 is a bijection between M/E_1 and N/E_1. If $\pi_{1,0}$ is the projection $S/E_1 \to S/E_0$ where S is the unique sort of $\mathcal{L}_n$, then $\pi_{1,0} \circ f_1 = f_0 \circ \pi_{1,0}$. Similarly, we build by induction maps f_i for all $i \leqslant n$ where E_n denotes the equality. The map f_n is a bijection between M and N such that for all $i < n$, $\pi_{n,i}\circ f_n = f_i\circ\pi_{n,i}$, so f_n is an $\mathcal{L}_n$-isomorphism.

3. Let M, $N \vDash T_n$, $A \subseteq M$ finite, $f : A \to N$ be a partial embedding and $a \in M$. Show that f can be extended to a.

   Solution: Let E_{-1} denote the trivial equivalence relation and E_n denote the equality. If A is empty, pick any point in $c \in N$, then sending a to c is a partial embedding. Otherwise, let $\hat{a}^i$ denote the E_i-class of a and i_0 be maximal such that there exists $b \in A \cap \hat{a}^{i_0}$. Let $b_0 \in A$ be such that $aE_{i_0}b_0$. If $i_0 = n$ then $a = b_0 \in A$ and we do not have to extend f. Otherwise, pick $c \in N$ such that $cE_{i_0}f(b_0)$ and $\neg cE_{i_0+1}f(b)$ for any $b \in A$. We can always find such a c since every E_{i_0} is covered by infinitely many E_{i_0+1}-classes. Note that i_0 is also maximal such that there exists $d \in f(A) \cap \hat{c}^{i_0}$.

   Define $g : A\cup\{a\} \to N$ extending f by sending a to c. Since $A\cup\{a\}$ is a substructure of M and f is an partial embedding, we only have to check that for all $b \in A$ and all $i \leqslant n$, aE_ib if and only if $cE_if(b)$. Assume aE_ib for some i and b. By maximality of i_0 we have $i \leqslant i_0$ and therefore $bE_iaE_ib_0$, so $f(b)E_if(b_0)E_ic$. The converse is symmetric.

4. Show that any f as in the previous question is a partial elementary embedding from M into N.

   Solution: We show, by induction on $\varphi(x)$, that for any $g \subseteq f$ with finite domain and $a \in \mathrm{dom}(g)^x$, $M \vDash \varphi(a)$ if and only if $N \vDash \varphi(g(a))$. If φ is atomic, this is immediate because g is a partial embedding. The induction goes through easily for Boolean combinations so the only case we are left to prove is when $\varphi = \exists y\, \psi(y, x)$.

   If $M \vDash \exists y\, \psi(y, a)$, then we can find $c \in M$ such that $M \vDash \psi(c, a)$. By the previous question we can find h extendending g, such that h has a finite domain and h is defined at c. By induction we have $N \vDash \psi(h(c), h(a)$, in particular, $N \vDash \exists y\, \psi(y, h(a))$. Note that M and N play symmetric roles, so the converse is also true.

   If we apply what we just proved to f itself, we see that f is elementary.

5. Show that f is also elementary even when A is not finite.

   Solution: Let $\varphi(x)$ be an $\mathcal{L}$-formula and $a \in A^x$. By the previous question $f|_a$ is elementary and thus $M \vDash \varphi(a)$ if and only if $N \vDash \varphi(f(a))$.

6. Let $M \vDash T_n$, $A \subseteq M$ non empty and $p \in \mathcal{S}_x^M(A)$ where $|x| = 1$. To simplify notations, let E_n be the equality and E_{-1} be the trivial equivalence relation with just one equivalence class. Show that one (and only one) of the following holds:

   - there exists $a \in A$ such that p is the unique type containing $x = a$;
   - there exists $i \in \{-1, 0, \dots, n\}$ and $a \in A$ such that p is the unique type containing $xE_i a$ and, for all $c \in A$, $\neg xE_{i+1}c$.

   Solution: Let i_0 be maximal such that there is a formula of the form $xE_{i_0}a$ in p. If $i = n$, then p contains $x = a$ for some $a \in A$. If $q \in \mathcal{S}_x^M(A)$ also contains $x = a$, then the only possible realisation of q in any elementary extension of M is a. Since q is realized in some elementary extension of M, then a must be a realization of q. Similarly, a realizes p. So $q = \mathrm{tp}(a/A) = p$.

   If $i_0 < n$, let $a \in A$ be such that p contains $xE_{i_0}a$. By maximality of i_0, p contains $\neg yE_{i+1}c$ for all $c \in A$. Let $q \in \mathcal{S}_x^M(A)$ have the same property. Changing M for an elementary extension, we may assume that M is $|A|^+$-saturated. Let d (resp. e) be a realisation of p (resp. q) in M. Then the map $f : A \cup \{d\} \to A \cup \{e\}$ is a partial embedding (cf. Question 3). By Question 5, f is a partial elementary embedding and so $p = \mathrm{tp}(d/A) = \mathrm{tp}(e/A) = q$.

7. Show that T_n is κ-stable for all $\kappa \geqslant \aleph_0$.

   Solution: Let $A \subseteq M \vDash T$ have cardinality $\kappa \geqslant \aleph_0$. In the previous question, we have described all types over A. There are $|\kappa|$ of the first kind and at most $n|A|$ of the second. It follows that $|\mathcal{S}_x^M(A)| = \kappa$.

8. Show that T_n has a saturated model of cardinality κ for all $\kappa \geqslant \aleph_0$.

   Solution: Let $M = \kappa^n$ and E_i be defined by "the first $i+1$ coordinates are equal". Then $M \vDash T_n$. Let $A \subseteq M$ be such that $|A| < \kappa$. Pick any $p \in \mathcal{S}_x^M(A)$. If p contains $x = a$, then, as seen above, it is realized by a. If not there is $a \in A$ and $i_0 < n$ such that p contains $xE_{i_0}a$ and $\neg xE_{i_0+1}c$ for all $c \in A$. The E_{i_0}-class of a is covered by κ many E_{i_0+1}-classe and, since $|A| < \kappa$, not all of them contain a point from A. So we can find $d \in \hat{a}^{i_0} \smallsetminus (\bigcup_{c\in A} \hat{c}^{i_0+1})$. By uniqueness of p, we must have $M \vDash p(d)$.

9. Let $\mathcal{L}_\infty := \bigcup_n \mathcal{L}_n$ and $T_\infty := \bigcup_n T_n$. Show that T_∞ is a satisfiable $\mathcal{L}_\infty$-theory.

   Solution: Since each T_n is satisfiable, T_∞ is finitely consistent and hence, by compactness consistent. One can also construct a model of T_∞ by taking $M = \kappa^\omega$ for any cardinal κ and define E_n to hold if "the first $n+1$ coordinates are equal".

10. Let $M \models T_\infty$ and $p \in \mathcal{S}_x^M(M)$ where $|x| = 1$. Let E_{-1} be the trivial equivalence relation with just one equivalence class. Show that one (and only one) of the following holds:

    - there exists $a \in M$ such that p is the unique type containing $x = a$;
    - there exists $a \in M$ such that p is the unique type containing $\neg x = a$ and $xE_i a$ for all $i \in \mathbb{Z}_{\geqslant 0}$;
    - there exists $i \in \mathbb{Z}_{\geqslant 0}$ and $a \in M$ such that p is the unique type containing $xE_i a$ and, for all $c \in M$, $\neg y E_{i+1} c$;
    - there is no $a \in M$ such that p contains $xE_i a$, for all $i \in \mathbb{Z}_{\geqslant 0}$, and there exists $a_i \in M$ such that for all $i \in \mathbb{Z}_{\geqslant 0}$, p is the unique type containing $xE_i a_i$ and $\neg x E_{i+1} a_i$, for all $i \in \mathbb{Z}_{\geqslant 0}$.

    Solution: Note that these four cases are clearly mutually exclusive.

    If p contains $x = a$ for some $a \in M$, then, as seen before p is the unique such type. Let us now assume that p contains $x \neq a$ for all $a \in M$. Let $I = \{i \in \mathbb{Z}_{\geqslant 0} : \exists a \in M\, p \text{ contains } xE_i a\}$. Let us first assume that $I \subset \mathbb{Z}_{\geqslant 0}$. Let i_0 be its maximal element and $a \in M$ be such that p contains $xE_{i_0} a$. Then p is of the third kind. Note that for all $n > i_0$, $p|_{\mathcal{L}_n}$ contains $xE_{i_0} a$ and $\neg x E_{i_0+1} c$ for all $c \in M$. If $q \in\in \mathcal{S}_x^M(M)$ id another type containing $xE_{i_0} a$ and $\neg x E_{i_0+1} c$ for all $c \in M$, then, by Question 6, for all $n > i_0$, $p|_{\mathcal{L}_n} = q|_{\mathcal{L}_n}$, so $p = q$.

    Now let us assume that $I = \mathbb{Z}_{\geqslant 0}$. If there is an $a \in M$ such that $xE_i a$ for all i, then we are in the second case. Note that $p|_{\mathcal{L}_n}$ contains $xE_{n-1} a$ and $x \neq c$ for all $c \in M$, so, by Question 6, p is unique.

    Finally, there remains the case where for all $a \in M$, there is a maximal i such that $xE_i a$ is in p. Pick some $i \in \mathbb{Z}_{\geqslant 0}$. Then there is an $a \in M$ such that $xE_i a$ is in p. If p contains $\neg x E_{i+1} a$, take $a_i = a$. Otherwise, because $M \models T_\infty$, we can find $a_i \in M$ such that $a_i E_i a$ and $\neg a_i E_{i+1} a$. Then p contains $xE_i a_i$ and $\neg x E_{i+1} a_i$. Note that there is no $a \in M$ such that $aE_i a_i$ for all $i \in \mathbb{Z}_{\geqslant 0}$. Indeed, if such an a existed then p would contain $xE_i a$ for all $i \in \mathbb{Z}_{\geqslant 0}$. Let $q \in\in \mathcal{S}_x^M(M)$ also contain $xE_i a_i$ and $\neg x E_{i+1} a_i$, for all $i \in \mathbb{Z}_{\geqslant 0}$. For all $a \in M$, there is an i such that $aE_i a_i$ does not hold, it follows that q contains $\neg x E_i a$ and hence $x \neq a$. So both $p|_{\mathcal{L}_n}$ and $q|_{\mathcal{L}_n}$ contains $xE_{n-1} a_{n-1}$ and $x \neq a$ for all $a \in M$. By Question 6, $p|_{\mathcal{L}_n} = q|_{\mathcal{L}_n}$ and hence $p = q$.

11. Let $M \models T_\infty$, $A \subseteq M$ be infinite and x a tuple of variables. Show that $|\mathcal{S}_x^M(A)| \leqslant |A|^{\aleph_0}$ and that this bound is sharp.

    Solution: Let us first assume that $|x| = 1$. By downwards Löwenheim-Skolem, we can find $M_0 \preccurlyeq M$ containing A and such that $|M_0| = |A|$. There is a surjection $\mathcal{S}^M(M_0) \to \mathcal{S}^M(A)$. If $|\mathcal{S}_x^M(M_0)| \leqslant |M_0|^{\aleph_0}$, then $|\mathcal{S}_x^M(A)| \leqslant |M_0|^{\aleph_0} = |A|^{\aleph_0}$. So we may assume that $A = M$.

    The previous question gives us a complete description of $\mathcal{S}_x^M(M)$. There are $|M|$ types of the first kind, $|M|$ types of the second kind, $|M|\aleph_0 = |M|$ types of the third kind and at most $|M|^{\aleph_0}$ types of the fourth kind (they only depend on the choice of the a_i). It follows that $|\mathcal{S}_x^M(M)| \leqslant |M|^{\aleph_0}$.

Now assume $|xy| = n + 1$ where $|y| = 1$. To any $p \in \mathcal{S}^M_{xy}(A)$, we can associate $q \in \mathcal{S}^M_x(A)$ its restriction to the first n variables and $r \in \mathcal{S}^{M^\star}_y(Aa)$ where a realizes q in some $M^\star \succcurlyeq M$. Moreover, knowing q are r completely determines p and $|Aa| = |A|$. By induction and the dimension 1 case, $|\mathcal{S}^M_{xy}(A)| \leqslant (|A|^{\aleph_0})^2 = |A|^{\aleph_0}$.

Let us now show that the bound is sharp. Let $M := \kappa^\omega$ and let E_i be defined by "the first $i+1$ coordinates are equal". Let $A := \{\varepsilon \in \kappa^\omega : \exists i_0 \forall i > i_0\, \varepsilon(i) = 0\}$. We have $|A| = \kappa$. Pick any ε, $\eta \in M$. There exists an i such that $\varepsilon(i) \neq \eta(i)$. Let $\nu \in A$ be defined by $\nu(j) = \varepsilon(j)$ for all $j \leqslant i$ and $\nu(j) = 0$ for all $j > i$. Then $\varepsilon E_i \nu$ and $\neg \eta E_i \nu$. It follows that $\mathrm{tp}(\varepsilon/A) \neq \mathrm{tp}(\eta/A)$ and hence, if $|x| = 1$, $|\mathcal{S}^M_x(A)| \geqslant \kappa^{\aleph_0} = |A|^{\aleph_0}$. As above, an easy induction shows that this also holds if $|x| > 1$.