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Homework 6

Problem 1 :
Let M be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $a \in M$ is algebraic over A if there exists an $\mathcal{L}(A)$-formula φ such that $M \models \varphi(a)$ and $|\varphi(M)|$ is finite. We define $\mathrm{acl}(A) = \{a \in M : a \text{ is algebraic over } A\}$.

1. Show that $|A| \leqslant |\mathrm{acl}(A)| \leqslant |A| + |\mathcal{L}|$.
2. Show that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.
3. Let $a \in \mathrm{acl}(A)$, show that $\mathrm{tp}(a/A) \in \mathcal{S}^M(A)$ is isolated.
4. Show that $a \in \mathrm{acl}(A)$ if and only if for all $N \succcurlyeq M$, $\mathrm{tp}(a/A)$ only has finitely many realisations in N.

   Hint: Use compactness to prove that $a \notin \mathrm{acl}(A)$ implies that there is some $N \succcurlyeq M$ in which $\mathrm{tp}(a/A)$ has infinitely many realisations.
5. Let $\kappa = |\mathcal{L}| + |A|$. Assume that M is κ^+-saturated. Show that:
$$\mathrm{acl}(A) = \bigcap_{A \subseteq N \preccurlyeq M} N.$$
6. Let $\mathrm{Aut}(M/A)$ be the set of $\mathcal{L}$-automorphisms of M that fix A pointwise. We say that $a \in M$ has a finite orbit under the action of $\mathrm{Aut}(M/A)$ if the set $\{\sigma(a) : \sigma \in \mathrm{Aut}(M/A)\}$ is finite.

   Assume that M is strongly κ^+-homogeneous and κ^+-saturated. Show that:
$$\mathrm{acl}(A) = \{a \in M : a \text{ has a finite orbit under the action of } \mathrm{Aut}(M/A)\}.$$
7. Let $\mathcal{R} := (\mathbb{R}, +, \cdot)$. Show that the only automorphism of $\mathcal{R}$ is the identity, but that $\mathrm{acl}(\mathbb{Q})$ is a strict subset of $\mathbb{R}$.

Problem 2 :
Let $\mathcal{L}$ be a countable language and T be a complete $\mathcal{L}$-theory.

1. Pick $p \in \mathcal{S}_x(T)$ and c a new tuple of constants. Let $T_p := T \cup p(c)$. Show that if T_c is $\aleph_0$-categorical, then so is T.

   Hint: Show that the restriction map $\mathcal{S}_y(T_c) \to \mathcal{S}_y(T)$ is unto.
2. Assume that T is not $\aleph_0$-categorical, show that it has a least three countable models up to isomorphism.
3. Let $\mathcal{L}_C = \mathcal{L}_< \cup \{c_i : i \in \omega\}$ and $\mathrm{DLO}_C := \mathrm{DLO} \cup \{c_i < c_j : i < j\}$. Show that DLO_C is complete and has three countable models up to isomorphism.