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Solutions to homework 9

Problem 1 :
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages, T_1 an $\mathcal{L}_1$-theory and T_2 an $\mathcal{L}_2$-theory. Let $\mathcal{L} := \mathcal{L}_1 \cap \mathcal{L}_2$. Let $T = \{\varphi\ \mathcal{L}\text{-sentence} : T_1 \vDash \varphi\}$. Let us assume that both T_1 and T_2 are satisfiable.

1. Let $M \vDash T$, show that there exists $A \vDash T_1$ such that $M \preccurlyeq A|_{\mathcal{L}}$.

   Solution: Let $\Sigma := \mathcal{D}^{\mathrm{el}}_{\mathcal{L}}(M) \cup T_1$. We have to show that Σ is satisfiable. Let us assume that $\Sigma_0 \subseteq \Sigma$ is finite and not satisfiable. We have finitely many $\mathcal{L}$-formulas $\varphi_i(x)$ and tuples $m_i \in M^{x_i}$ such that $\varphi_i(m_i) \in \mathcal{D}^{\mathrm{el}}_{\mathcal{L}}(M)$ and $\Sigma_0 \subseteq \{\varphi_i(M) : i \leqslant k\} \cup T_1$. Let $\theta(x) := \bigwedge_i \varphi_i(x_i)$. Since Σ_0 is not satisfiable, it follows that $T_1 \vDash \neg\theta(m)$. Since the constants m do not appear in T_1, it follows that $T_1 \vDash \forall x\, \neg\theta(x)$. In particular $\forall x\, \theta(x) \in T$ and $M \vDash \forall x\, \theta(x)$. But this contradicts the fact that $M \vDash \theta(m)$. We conclude by compactness.

2. Let $\mathcal{L}'$ be any language containing $\mathcal{L}$, A be an $\mathcal{L}'$-structure and M be an $\mathcal{L}$-structure such that $A|_{\mathcal{L}} \preccurlyeq M$. Show that there exists an $\mathcal{L}'$-structure B such that $A \preccurlyeq B$ and $M \preccurlyeq B|_{\mathcal{L}}$.

   Solution: Let $\Sigma := \mathcal{D}^{\mathrm{el}}(A) \cup \mathcal{D}^{\mathrm{el}}_{\mathcal{L}}(M)$. Here the constants for A are identified with those for $A \subseteq M$. We have to show that Σ is satisfiable. Let us assume that $\Sigma_0 \subseteq \Sigma$ is finite and not satisfiable. There exists finitely many $\mathcal{L}(A)$-formulas $\varphi_i(x_i)$ and tuples $m_i \in M \smallsetminus A$ such that $\varphi_i(m_i) \in \mathcal{D}^{\mathrm{el}}_{\mathcal{L}}(M)$ and $\Sigma_0 \subseteq \mathcal{D}^{\mathrm{el}}(A) \cup \{\varphi_i(m_i)\}$. Let $\theta(x) := \bigwedge_i \varphi_i(x_i)$. Since Σ_0 is not satisfiable, it follows that $\mathcal{D}^{\mathrm{el}}(A) \vDash \neg\theta(m)$. Since the constants m do not appear in $\mathcal{L}'(A)$, $\mathcal{D}^{\mathrm{el}}(A) \vDash \forall x\, \neg\theta(x)$. In particular, $A \vDash \forall x\, \neg\theta(x)$. Since $A|_{\mathcal{L}} \preccurlyeq M$, we also have $M \vDash \forall x\, \neg\theta(x)$, contradicting the fact that $M \vDash \theta(m)$. We conclude by compactness.

3. Assume that $T \cup T_2$ is satisfiable. Show that $T_1 \cup T_2$ is satisfiable.

   Solution: For all $i \in \omega$, we build, by induction, $A_i \vDash T_2$, $B_i \vDash T_1$ such that $A_i|_{\mathcal{L}} \preccurlyeq B_i|_{\mathcal{L}} \preccurlyeq A_{i+1}|_{\mathcal{L}}$, $A_i \preccurlyeq A_{i+1}$ and $B_i \preccurlyeq B_{i+1}$. Let A_0 be any model of $T \cup T_2$. By Question 1, we get B_0. The other induction steps follow from Question 2.

   Let $M := \bigcup_i A_i = \bigcup_i B_i$. Then M can be made into both an $\mathcal{L}_1$-structure and an $\mathcal{L}_2$-structure. Note that the $\mathcal{L}$-structure induced by these two chains coincide and hence M can be made into an $\mathcal{L}_1 \cup \mathcal{L}_2$-structure. We have $M \succcurlyeq A_0 \vDash T_2$ and $M \succcurlyeq B_0 \vDash T_1$, so $M \vDash T_1 \cup T_2$.

4. Let φ be an $\mathcal{L}_1$-sentence and ψ be an $\mathcal{L}_2$-sentence. Assume that $\varphi \vDash \psi$ (i.e. any $\mathcal{L}_1 \cup \mathcal{L}_2$-structure which is a model of φ is also a model of ψ). Show that there exists an $\mathcal{L}$-sentence θ such that $\varphi \vDash \theta$ and $\theta \vDash \psi$.

   Solution: We have that $\{\varphi\} \cup \{\neg\psi\}$ is not a satisfiable $\mathcal{L}_1 \cup \mathcal{L}_2$-theory. Let $T = \{\theta\ \mathcal{L}\text{-theory} : \varphi \vDash \theta\}$. By the previous question, $T \cup \{\neg\psi\}$ is not satisfiable. By compactness, there exists finitely many $\theta_i \in T$ such that $\{\theta_i : i \leqslant k\} \cup \{\neg\psi\}$ is not satisfiable. Let $\theta := \bigwedge_i \theta_i \in T$. Then $\theta \vDash \psi$ and, by definition of T, $\varphi \vDash \theta$.

Problem 2 :
Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two languages, T an $\mathcal{L}$-theory and $\varphi(x)$ an $\mathcal{L}$-formula whoses variables are in $\mathcal{L}_0$-sorts. Assume that for all M, $N \models T$. If $M|_{\mathcal{L}_0} = N|_{\mathcal{L}_0}$ then $\varphi(M) = \varphi(N)$. Let $\mathcal{L}'$ be a copy of $\mathcal{L}$ such that $\mathcal{L} \cap \mathcal{L}' = \mathcal{L}_0$. When ψ is an $\mathcal{L}$ formula, let ψ' denote the $\mathcal{L}'$-formula obtained by changing the $\mathcal{L}$-symbols of ψ into the corresponding $\mathcal{L}'$-symbols. Let $T' := \{\psi' : \psi \in T\}$.

1. Show that $T \cup T' \models \forall x\, \varphi(x) \to \varphi'(x)$.

   Solution: Let $M \models T \cup T'$. Let N be $M|_{\mathcal{L}'}$ considered as an $\mathcal{L}$-structure. Then $N|_{\mathcal{L}_0} = M|_{\mathcal{L}_0}$. By our hypothesis, $\varphi'(M) = \varphi(N) = \varphi(M)$ and hence $M \models \forall x\, \varphi(x) \to \varphi'(x)$.

2. Show that there exists an $\mathcal{L}$-sentence θ such that in every $\mathcal{L} \cup \mathcal{L}'$-structure M, we have $M \models \forall x\, (\theta \wedge \varphi(x)) \to (\theta' \to \varphi'(x))$.

   Solution: By compactness, we can find $\theta \in T$ such that $\theta \wedge \theta' \models \forall x\, \varphi(x) \to \varphi'(x)$. Let M be an $\mathcal{L} \cup \mathcal{L}'$-structure and pick $m \in M^x$. Assume that $M \models \theta \cup \varphi(m)$ and $M \models \theta'$, then we have $M \models \forall x\, \varphi(x) \to \varphi'(x)$ and hence $M \models \varphi'(m)$.

3. Show that there exists an $\mathcal{L}_0$-formula $\psi(x)$ such that $T \models \forall x\, \varphi(x) \leftrightarrow \psi(x)$.

   Hint: Use the last question of the previous problem.

   Solution: Let c be a new tuple of constants sorted as x. By the previous question, we have that $\theta \wedge \varphi(c) \models \theta' \to \varphi'(c)$. By Question 2.4, there exists an $\mathcal{L}_0(c)$-formula χ such that $\theta \wedge \varphi(c) \models \chi$ and $\chi \models \theta' \to \varphi'(c)$. Let $\psi(x)$ be an $\mathcal{L}_0$-formula such that $\chi = \psi(c)$. We have that $T \models \forall x\, \varphi(x) \to \psi(x)$. Also $T' \models \forall x\, \psi(x) \to \varphi'(x)$. By definition of T', it follows that $T \models \forall x\, \psi(x) \to \varphi(x)$ and thus, $T \models \forall x\, \psi(x) \leftrightarrow \varphi(x)$.

Problem 3 :
Let M be an $\mathcal{L}$-structure, $A \subseteq B \subseteq M$ and $\mathfrak{U}$ be a non principal ultrafilter on A. We define

$$\mathrm{Av}(\mathfrak{U}/B) := \{\varphi(x)\ \mathcal{L}(B)\text{-formula} : \{a \in A : M \models \varphi(a)\} \in \mathfrak{U}\}.$$

1. Show that $\mathrm{Av}(\mathfrak{U}/B)$ is a complete $\mathcal{L}(B)$-type.

   Solution: Let us first prove that $\mathrm{Av}(\mathfrak{U}/B)$ is finitely satisfiable. Let $phi_i(x)$ be finitely many formulas in $\mathrm{Av}(\mathfrak{U}/B)$. Let $A_i := \{a \in A : M \models \varphi_i(a)\} \in \mathfrak{U}$. Since $\mathfrak{U}$ is a filter, $\bigcap_i A_i$ is in $\mathfrak{U}$ and is therefore non empty. Moreover, since $\mathfrak{U}$ is an ultrafilter, if $\{a \in A : M \models \varphi(a)\} \notin \mathfrak{U}$, then its complement $\{a \in A : M \models \neg\varphi(a)\}$ is in $\mathfrak{U}$ and $\neg\varphi \mathrm{Av}(\mathfrak{U}/B)$. So $\mathrm{Av}(\mathfrak{U}/B)$ is a complete type.

2. Assume M is $|A|^+$-saturated. For all $i \in \mathbb{Z}_{\geqslant 0}$, pick by induction $b_{i+1} \models \mathrm{Av}(\mathfrak{U}/A \cup \{b_j : j < i\})$. Show that $(b_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ is a sequence which is indiscernible over A.

   Solution: Let us first prove that if c_1 and c_2 are tuples in B with the same type over A and $\varphi(x, y)$ be an $\mathcal{L}$-formula, then $\varphi(x, c_1) \in \mathrm{Av}(\mathfrak{U}/B)$ if and only if $\varphi(x, c_2) \in \mathrm{Av}(\mathfrak{U}/B)$. Indeed, since c_1 and c_2 have the same type over A, $\{a \in A : \varphi(a, c_1)\} = \{a \in A : \varphi(a, c_2)\}$. Also note that if $B \subseteq C$, then $\mathrm{Av}(\mathfrak{U}/B) \subseteq \mathrm{Av}(\mathfrak{U}/C)$.

   We now prove by induction on n that for all $i_1 < \ldots < i_n$, $b_{i_1}, \ldots, b_{i_n}$ has the same type over A as $b_1, \ldots, b_n$. If $b_{i_1}, \ldots, b_{i_n}$ has the same type over A as $b_1, \ldots, b_n$, then, by our first remark, $b_{i_1}, \ldots, b_{i_n}, b_{i_{n+1}}$ has the same type over A as $b_1, \ldots, b_n, b_{i_{n+1}}$. By our second remark, $b_{i_{n+1}}$ has the same type as b_{n+1} over $Ab_1 \ldots b_n$. It follows that $b_{i_1}, \ldots, b_{i_n}, b_{i_{n+1}}$ has the same type over A as $b_1, \ldots, b_n, b_{n+1}$.