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Homework 9

Problem 1 :
Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two languages, T_1 an $\mathcal{L}_1$-theory and T_2 an $\mathcal{L}_2$-theory. Let $\mathcal{L} := \mathcal{L}_1 \cap \mathcal{L}_2$. Let $T = \{\varphi\ \mathcal{L}\text{-sentence} : T_1 \vDash \varphi\}$. Let us assume that both T_1 and T_2 are satisfiable.

1. Let $M \vDash T$, show that there exists $A \vDash T_1$ such that $M \preccurlyeq A|_{\mathcal{L}}$.
2. Let $\mathcal{L}'$ be any language containing $\mathcal{L}$, A be an $\mathcal{L}'$-structure and M be an $\mathcal{L}$-structure such that $A|_{\mathcal{L}} \preccurlyeq M$. Show that there exists an $\mathcal{L}'$-structure B such that $A \preccurlyeq B$ and $M \preccurlyeq B|_{\mathcal{L}}$.
3. Assume that $T \cup T_2$ is satisfiable. Show that $T_1 \cup T_2$ is satisfiable.
4. Let φ be an $\mathcal{L}_1$-sentence and ψ be an $\mathcal{L}_2$-sentence. Assume that $\varphi \vDash \psi$ (i.e. any $\mathcal{L}_1 \cup \mathcal{L}_2$-structure which is a model of φ is also a model of ψ). Show that there exists an $\mathcal{L}$-sentence θ such that $\varphi \vDash \theta$ and $\theta \vDash \psi$.

Problem 2 :
Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two languages, T an $\mathcal{L}$-theory and $\varphi(x)$ an $\mathcal{L}$-formula whoses variables are in $\mathcal{L}_0$-sorts. Assume that for all $M, N \vDash T$. If $M|_{\mathcal{L}_0} = N|_{\mathcal{L}_0}$ then $\varphi(M) = \varphi(N)$. Let $\mathcal{L}'$ be a copy of $\mathcal{L}$ such that $\mathcal{L} \cap \mathcal{L}' = \mathcal{L}_0$. When ψ is an $\mathcal{L}$ formula, let ψ' denote the $\mathcal{L}'$-formula obtained by changing the $\mathcal{L}$-symbols of ψ into the corresponding $\mathcal{L}'$-symbols. Let $T' := \{\psi' : \psi \in T\}$.

1. Show that $T \cup T' \vDash \forall x\, \varphi(x) \to \varphi'(x)$.
2. Show that there exists an $\mathcal{L}$-sentence θ such that in every $\mathcal{L} \cup \mathcal{L}'$-structure M, we have $M \vDash \forall x\, (\theta \wedge \varphi(x)) \to (\theta' \to \varphi'(x))$.
3. Show that there exists an $\mathcal{L}_0$-formula $\psi(x)$ such that $T \vDash \forall x\, \varphi(x) \leftrightarrow \psi(x)$.

   Hint: Use the last question of the previous problem.

Problem 3 :
Let M be an $\mathcal{L}$-structure, $A \subseteq B \subseteq M$ and $\mathfrak{U}$ be a non principal ultrafilter on A. We define

$$\mathrm{Av}(\mathfrak{U}/B) := \{\varphi(x)\ \mathcal{L}(B)\text{-formula} : \{a \in A : M \vDash \varphi(a)\} \in \mathfrak{U}\}.$$

1. Show that $\mathrm{Av}(\mathfrak{U}/B)$ is a complete $\mathcal{L}(B)$-type.
2. Assume M is $|A|^+$-saturated. For all $i \in \mathbb{Z}_{\geqslant 0}$, pick by induction $b_{i+1} \vDash \mathrm{Av}(\mathfrak{U}/A \cup \{b_j : j < i\})$. Show that $(b_i)_{i \in \mathbb{Z}_{\geqslant 0}}$ is an indiscernible sequence.