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Homework 11

Problem 1 :
Let M be an $\mathcal{L}$-structure, $\mathcal{L}_0 \subseteq \mathcal{L}$. We denote by MR (respectively MR_0) the Morley rank computed in models of M (respectively $M|_{\mathcal{L}_0}$). Let φ be an $\mathcal{L}_0(M)$-formula.

1. Show that $\mathrm{MR}_0(\varphi) \leqslant \mathrm{MR}(\varphi)$.
2. Assume that $\mathrm{Th}(M)$ is strongly minimal. Show that $\mathrm{MR}_0(\varphi) = \mathrm{MR}(\varphi)$.

Problem 2 :
Let $M_0 \preccurlyeq M$ be some structures and $f : D \to E$ be M_0-definable in M. Assume that M_0 is $\aleph_0$-saturated and that M is $|M_0|^+$-saturated.

1. Assume that for all $e \in E$, $\mathrm{MR}(f^{-1}(e)) \leqslant 0$, show that $\mathrm{MR}(D) \leqslant \mathrm{MR}(E)$.
2. Pick $e \in E$ and assume that $\mathrm{MR}(D) > \gamma + \mathrm{MR}(f^{-1}(e))$ for some ordinal γ. Show that there exists an M_0-definable $D' \subseteq D$ such that $\mathrm{MR}(D') > \gamma$ and $f^{-1}(e) \cap D'$ is finite or empty.

   Hint: Proceed by induction on $\mathrm{MR}(f^{-1}(e))$.
3. Assume that for all $e \in E$, $\mathrm{MR}(f^{-1}(e)) \leqslant \alpha$ for some ordinal $\alpha > 0$, that $\mathrm{MR}(E) < \infty$ and that $\mathrm{MD}(E) = 1$. Show that $\mathrm{MR}(D) \leqslant \alpha(\mathrm{MR}(E) + 1)$.

   Hint: Proceed by induction on $\mathrm{MR}(E)$.
4. Assume that for all $e \in E$, $\mathrm{MR}(f^{-1}(e)) \leqslant \alpha$ for some ordinal $\alpha > 0$ and that $\mathrm{MR}(E) < \infty$. Show that $\mathrm{MR}(D) \leqslant \alpha(\mathrm{MR}(E) + 1)$.

If ordinal arithmetics is a mystery to you, just assume that all the ordinal involved are integers.

Problem 3 :
Let $A \subseteq M \vDash T$ totally transcendental and $p \in S_x(A)$. Let I be some total order and $(a_i)_{i\in I}$ be a sequence of elements from M^x. We say that $(a_i)_{i\in I}$ is a Morley sequence of p if there exists $q \in S_x(M)$ non forking extension of p such that, for all $i \in I$, $a_i \vDash q|_{A\cup\{a_j : j<i\}}$.

1. Assume p is stationary and let $(a_i)_{i\in I}$ be a Morley sequence of p. Show that $(a_i)_{i\in I}$ is $\mathcal{L}(A)$-indiscernible (i.e. it is an indiscernible sequence in M viewed as an $\mathcal{L}(A)$-structure).
2. Let $J \subseteq I$ be an initial segment of I without a greatest element and $(a_i)_{i\in I}$ be an $\mathcal{L}(A)$-indiscernible sequence. Pick any $i \in I \smallsetminus J$, tuple a from $(a_j)_{j<i}$ and $\mathcal{L}(A)$-formula $\varphi(x, y)$. Assume that $M \vDash \varphi(a_i, a)$. Show that there exists a tuple c from $(a_j)_{j\in J}$ such that $\mathrm{MR}(\varphi(x, c)) = \mathrm{MR}(\varphi(x, a))$ and $M \vDash \varphi(a_i, c)$.
3. Let J and $(a_i)_{i\in I}$ be as above and let $p = \mathrm{tp}(a_i/A \cup \{a_j : j \in J\})$ for any $i \in I \smallsetminus J$. Show that $(a_i)_{i\in I\smallsetminus J}$ is a Morley sequence of p.

Problem 4 :
Let M be $\aleph_1$-saturated, $\varphi(x,y)$ be some formula and $a \in M^y$. We say that $\varphi(M,a)$ is weakly normal if for all $(a_i)_{i\in\omega} \in M^y$ such that $\text{tp}(a_i) = \text{tp}(a)$ and if $i \neq j$, $\varphi(M,a_i) \neq \varphi(M,a_j)$, then $\bigcap_{i\in\omega} \varphi(M,a_i) = \varnothing$.

1. Show that if $\varphi(M,a)$ is not weakly normal, then, for all cardinal κ, there exists $N \succcurlyeq M$ and $(a_i)_{i\in\kappa} \in N^y$, such that for all $i \in \kappa$, $\text{tp}(a_i) = \text{tp}(a)$, for all $i \neq j$, $\varphi(N,a_i) \neq \varphi(N,a_j)$ and $\bigcap_{i\in\kappa} \varphi(N,a_i) \neq \varnothing$.
2. Show that $\varphi(M,a)$ is weakly normal if and only if, for all $b \in \varphi(M,a)$, $\ulcorner\varphi(M,a)\urcorner \subseteq \text{acl}^{\text{eq}}(b)$.