Exam

Model theory of valued fields

April 6 2021

You can always use a previous question when answering a later question, even if you did not prove it.

Problem 1. Let k be a field. We define $k(t^{\mathbb{Q}}) := \bigcup_{n\in\mathbb{Z}_{>0}} k((t^{1/n}))$. Let v be the (unique) valuation on $k(t^{\mathbb{Q}})$ extending the t-adic valuation.

1. Show that $k(t^{\mathbb{Q}})$ is henselian.
2. Assume k is characteristic zero. Show that $k(t^{\mathbb{Q}})$ is algebraically closed if and only if k is.

Problem 2. Let $\mathfrak{L}$ be the language with two sorts B and C, each with the (additive) group language, a predicate $A \subseteq B$ and a map $\rho : B \to C$. Let T be the theory starting that $A \leqslant B$ and C are torsion free divisible abelian groups, ρ is a surjective group morphism with kernel A. Let $M, N \vDash T$, $M_0 \leqslant M$, $f : M_0 \to N$ be an embedding.

1. Let $g : A(M) \to A(N)$ be a group embedding extending $f|_A$ and $h : C(M) \to C(N)$ be a group embedding extending $f|_C$. Show that f,g and h have a common extension to an embedding $A(M) \cup B(M_0) \cup C(M) \to N$.
2. Assuming that $A(M) \cup C(M) \leqslant M_0$, show that f extends to an embedding $M \to N$.
3. Show that T resplendently eliminates B-quantifiers: for every $A \cup C$-enrichment $\mathfrak{L}'$, any $\mathfrak{L}'$-formula is equivalent modulo T to an $\mathfrak{L}'$-formula whose quantifiers range over either C or A.
4. Show that $M \equiv N$ if and only if $A(M) \equiv A(N)$ and $C(M) \equiv C(N)$.

Problem 3.

1. Show that there exists unique $V_n \in \mathbb{Z}[x_0, \ldots, x_n]$ such that $w_{p^n}(V(x)) = \mathbb{1}_{n\neq 0} p w_{p^{n-1}}(x)$ and give an explicit formula for V_i.
2. Show that, for every ring R, $V : W(R) \to W(R) := x \mapsto (V_i(x))_{i\geqslant 0}$ is an additive group morphism.