Midterm

Model theory of valued fields

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You can always use a previous question when answering a later question, even if you did not prove it.

- Let $\mathfrak{L}$ be the language with three sorts $\mathbf{K}$ (with the ring language), $\mathbf{RV}$ (with the ring language) and $\mathbf{\Gamma}$ (with the ordered group language), a function $\mathrm{rv} : \mathbf{K} \to \mathbf{RV}$ and a function $\mathrm{v} : \mathbf{RV} \to \mathbf{\Gamma}$.
- Any valued field (K, v) can be made into an $\mathfrak{L}$-structure by interpreting K as the ring K, $\mathbf{\Gamma}$ as vK as an ordered monoid — with $-$ as the inverse on $vK^\times$ and $-\infty = \infty$ — and $\mathbf{RV}$ as $K/1 + \mathfrak{m}$ with its multiplicative structure, $0 = \mathrm{rv}(0)$, $+$ is interpreted as the trace of addition when it is well-defined and 0 otherwise, and $-\mathrm{rv}(x) = \mathrm{rv}(-x)$.
- Let T denote the $\mathfrak{L}$-structure of algebraically closed non trivially valued fields.

Problem 1. Let $M \vDash T$.

1. Show that for every $\xi, \upsilon, \zeta \in \mathbf{RV}(M)$, $\zeta \cdot (\xi + \upsilon) = (\zeta \cdot \xi) + (\zeta \cdot \upsilon)$.
2. Show that for every $(\xi_i)_{i<n} \in \mathbf{RV}(M)$, $\sum_i \xi_{i<n} := \xi_0 + \sum_{0<i<n} \xi_i \in \bigoplus_i \xi_i$ and $0 \in \bigoplus_i \xi_i$ if and only if, for some permutation σ of n, $\sum_i \xi_{\sigma(i)} = 0$.
3. Let $\alpha \in \mathbf{RV}(M)$ and $P \in \mathbf{K}(M)[x]$ such that $0 \in \mathrm{rv}(P)(\alpha)$. Show that there exists $a \in \mathbf{K}(M)$ such that $\mathrm{rv}(a) = \alpha$ and $P(a) = 0$.
4. Let $A \leqslant M$, $\alpha \in \mathbf{RV}(A)$, $P \in \mathbf{K}(A)[x]$ minimal such that $0 \in \mathrm{rv}(P)(\alpha)$, $a \in \mathbf{K}(M)$ such that $\mathrm{rv}(a) = \alpha$ and $P(a) = 0$ and C be the structure generated by Aa. Show that $\mathbf{RV}(C) = \mathbf{RV}(A)$.
5. Show that T eliminates quantifiers.
   [Hint : You can consider a maximal $\mathfrak{L}$-embedding $f : C \leqslant M \to N$. Start by showing that $\mathbf{RV}(C) = \mathrm{rv}(\mathbf{K}(C))$.]

Solution.

1. If $\zeta = 0$, we have $\zeta \cdot (\xi + \upsilon) = 0 = (\zeta \cdot \xi) + (\zeta \cdot \upsilon)$. So let us assume that $\zeta \neq 0$. We have $\zeta \cdot (\xi \oplus \upsilon) = (\zeta \cdot \xi) \oplus (\zeta \cdot \upsilon)$. Indeed, if x, y, z are such that $\mathrm{rv}(x) = \xi$, $\mathrm{rv}(y) = \upsilon$ and $\mathrm{rv}(z) = \zeta$, then $\mathrm{rv}(z)\mathrm{rv}(x + y) = \mathrm{rv}(zx + zy) \in (\zeta \cdot \xi) \oplus (\zeta \cdot \upsilon)$. Conversely, if $\zeta \neq 0$ and x, y, z, z' are such that $\mathrm{rv}(x) = \xi$, $\mathrm{rv}(y) = \upsilon$ and $\mathrm{rv}(z) = \mathrm{rv}(z') = \zeta$, we have $\mathrm{rv}(zx + z'y) = \mathrm{rv}(z)\mathrm{rv}(x + z^{-1}z'y) \in \zeta \cdot (\xi \oplus \zeta^{-1}\zeta\upsilon) = \zeta \cdot (\xi \oplus \upsilon)$.
   It follows that, $(\zeta \cdot \xi) \oplus (\zeta \cdot \upsilon)$ is well-defined if and only if $\zeta \cdot (\xi \oplus \upsilon)$ is, equivalently $\xi \oplus \upsilon$ is. If they are all well defined, then both $\zeta \cdot (\xi + \upsilon)$ and $(\zeta \cdot \xi) + (\zeta \cdot \upsilon)$ are elements of the

same singleton and are thus equal. If none of them are well defined, then $\zeta \cdot (\xi + \upsilon) = 0 = (\zeta \cdot \xi) + (\zeta \cdot \upsilon)$.

2. We proceed by induction on n. For the second statement we also prove that we can choose σ such that $\xi_{\sigma(0)}$ has minimal valuation. The case $n = 2$ follows from the definition. Let us assume that the statement holds for $n - 1$. We have $\xi_0 + \sum_{0<i<n} \xi_i \in \xi_0 \oplus \sum_{0<i<n} \xi_i \in \xi_0 \oplus \bigoplus_{0<i<n} \xi_i = \bigoplus_{i<n} \xi_i$. In particular, if for some permutation σ, $\sum_{i<n} \xi_{\sigma i} = 0$, then $0 \in \bigoplus_{i<n} \xi_{\sigma i} = \bigoplus_{i<n} \xi_i$.
   Let us now assume that $0 \in \bigoplus_{i<n} \xi_i = \xi_0 \oplus \bigoplus_{0<i<n} \xi_i$. If $\bigoplus_{0<i<n} \xi_i$ is well defined, we have $0 \in \xi_0 \oplus \sum_{0<i<n} \xi_i$. So $\sum_{i<n} \xi_i = 0$ and $\mathrm{v}(\xi_0) = \mathrm{v}(\bigoplus_{0<i<n} \xi_i)$ is minimal. If $0 \in \bigoplus_{0<i<n} \xi_i$, let σ be such that $\sum_{0<i<n} \xi_{\sigma i} = 0$ and $\mathrm{v}(\xi_{\sigma(1)})$ is minimal. If $\mathrm{v}(\xi_0) \leqslant \mathrm{v}(\xi_{\sigma(1)})$, $\mathrm{v}(\sum_i \xi_i) = \mathrm{v}(\xi_0)$ is an element of $\bigoplus_i \xi_i$ whose valuation is the minimal valuation of the ξ_i, contradicting that $0 \in \bigoplus_i \xi_i$. It follows that $\mathrm{v}(\xi_0) > \mathrm{v}(\xi_{\sigma(1)}) = \mathrm{v}(\sum_{1<i<n} \xi_{\sigma(i)})$ and hence, $\xi_{\sigma(1)} + (\xi_0 + \sum_{1<i<n} \xi_{\sigma(i)}) = \sum_{1<i<n} \xi_{\sigma(i)} = 0$.
3. Let $P = c\prod_i (x - e_i)$. Then $0 \in \mathrm{rv}(P)(\alpha) \subseteq \mathrm{rv}(c) \cdot \bigoplus_I \prod_{i \in I} \mathrm{rv}(-e_i)\alpha^{|I|} = \mathrm{rv}(c)\prod_i \mathrm{rv}(\alpha \oplus \mathrm{rv}(-e_i))$. It follows that for some i, $0 \in \alpha \oplus \mathrm{rv}(-e_i)$, *i.e.* $\alpha = \mathrm{rv}(e_i)$.
4. For every $Q = \sum_i c_i x^i \in \mathbf{K}(A)[x]$ with $\deg(Q) < \deg(P)$, by minimality, $0 \notin \mathrm{rv}(Q)(\alpha)$ and hence $\mathrm{rv}(Q)(\alpha) = \{\mathrm{rv}(Q(a))\} = \sum_i \mathrm{rv}(c_i)\alpha^i \in \mathbf{RV}(A)$, by question 2. It follows that $\mathrm{rv}(\mathbf{K}(C)) = \mathrm{rv}(\mathbf{K}(A)[a]) \subseteq \mathbf{RV}(A)$ ahd hence $\mathbf{RV}(C) = \mathbf{RV}(A)$.
5. Let us consider a maximal embedding $f : C \leqslant M \to N$. As shown in class, it suffices to show that $C = M$. Let $\alpha \in \mathbf{RV}(C)$ and $P \in \mathbf{K}(C)[x]$ minimal such that $0 \in \mathrm{rv}(P)(\alpha)$. By question 3, there exists $a \in M$ such that $P(a) = 0$ and $\mathrm{rv}(a) = \alpha$. By question 2, $f_\star P$ is minimal such that $0 \in \mathrm{rv}(f_\star P)(f(\alpha))$. By question 3, we also find $b \in N$ such that $f_\star P(b) = 0$ and $\mathrm{rv}(b) = f(\alpha)$. By minimality, P is the minimal polynomial of a over $\mathbf{K}(A)$ and $f_\star P$ is the minimal polynomial of b over $\mathbf{K}(f(C))$. So $f|_{\mathbf{K}}$ extends to a ring morphism $g|_{\mathbf{K}}$ sending a to b. Let $g = g|_{\mathbf{K}} \cup f$. To show that it is an $\mathfrak{L}$-morphism, it suffices to check that it commutes with rv. But for every $Q = \sum_i c_i x^i \in \mathbf{K}(C)[x]$ of degree smaller than P, $g(\mathrm{rv}(Q(a))) = f(\sum_i \mathrm{rv}(c_i)\alpha^i) = \sum_i \mathrm{rv}(f(c_i))\mathrm{rv}(b)^i = \mathrm{rv}(f_\star Q(b)) = \mathrm{rv}(f(Q(a)))$. By maximality of f, we have $a \in C$ and hence $\mathbf{RV}(C) = \mathrm{rv}(\mathbf{K}(C))$.
   Now $f|_{\mathbf{K} \cup \mathbf{\Gamma}}$ naturally induces an $\mathfrak{L}_{\mathbf{k},\mathbf{\Gamma}}$-morphism on the $\mathfrak{L}_{\mathbf{k},\mathbf{\Gamma}}$-structure generated by $\mathbf{K}(C) \cup \mathbf{\Gamma}(C)$ into the $\mathfrak{L}_{\mathbf{k},\mathbf{\Gamma}}$-structure of N. By quantifier elimination, f extends to an elementary $g : M \to N^\star$, where $h : N \to N^\star$ is elementary. Then the map induced by g on $\mathbf{K}(M) \cup \mathbf{\Gamma}(M) \cup \mathbf{RV}(M)$ is an $\mathfrak{L}$-morphism and, since $\mathbf{RV}(C) = \mathrm{rv}(\mathbf{K}(C))$, it extends f. So $C = M$. $\square$

Problem 2. Let $M \models T$ and $A \leqslant M$.

1. Let $a, b \in \mathbf{K}(M)$ with $\mathrm{v}(\mathrm{rv}(a)) = \mathrm{v}(\mathrm{rv}(b)) \notin \mathbb{Q} \cdot \mathrm{v}(\mathbf{RV}(A))$. Show that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.

Let $f : \mathbf{\Gamma} \to \mathbf{RV}$ be $\mathcal{L}(A)$-definable.

2. Show that $\mathrm{v}(f(\mathbf{\Gamma})) \subseteq \mathbb{Q} \cdot \mathbf{\Gamma}(A)$.
3. Show that $\mathrm{v} \circ f$ has finite image.
4. Show that f has finite image.

Solution.

1. Let $A \leqslant C \leqslant M$ be maximal such that $\mathrm{rv}(\mathbf{K}(C)) \subseteq \mathbf{RV}(A)$. For every $\alpha \in \mathbf{RV}(A)$, let

$P \in \mathbf{K}(C)[x]$ be minimal such that $0 \in \mathrm{rv}(P)(\alpha)$. By 1.3, we find $c \in \mathbf{K}(M)$ such that $P(c) = 0$ and $\mathrm{rv}(c) = \alpha$. By 1.4, the $\mathbf{RV}$ part of the structure generated by Cc is $\mathbf{RV}(A)$. By maximality of C, $\mathrm{rv}(\mathbf{K}(C)) = \mathbf{RV}(A)$. In particular, $\mathrm{v}(\mathbf{RV}(A)) = \mathrm{v}(\mathbf{K}(C))$.
By the characterisation of ramified 1-types in ACVF, we have $\mathrm{tp}_{\mathfrak{L}_{\mathbf{k},\mathbf{\Gamma}}}(a/\mathbf{K}(C) \cup \mathbf{\Gamma}(C)) = \mathrm{tp}_{\mathfrak{L}_{\mathbf{k},\mathbf{\Gamma}}}(b/\mathbf{K}(C) \cup \mathbf{\Gamma}(C))$. It follows that $\mathrm{tp}_{\mathfrak{L}}(a/C) = \mathrm{tp}_{\mathfrak{L}}(b/C)$ and hence $\mathrm{tp}_{\mathfrak{L}}(a/A) = \mathrm{tp}_{\mathfrak{L}}(b/A)$.

2. Fix some $\gamma \in \mathbf{\Gamma}(M)$ and let $\delta = \mathrm{v}(f(\gamma))$. If $\delta \notin \mathbb{Q} \cdot \mathrm{v}(\Gamma(A))$, then by question 1, for any $\alpha \in \mathbf{RV}(M)$ with $\mathrm{v}(\alpha) = \delta$, we have $\mathrm{tp}(\alpha/A\gamma\delta) = \mathrm{tp}(f(\gamma)/A\gamma\delta)$ and hence $\alpha = f(\gamma)$, contradicting that $\mathrm{v}^{-1}(\delta)$ is infinite.
3. If $\mathrm{v} \circ f(\mathbf{\Gamma})$ is infinite, then, by compactness, we find $h : M \to M^\star$ elementary and $\gamma \in \mathbf{\Gamma}(M^\star)$ such that $\mathrm{v}(f(\gamma)) \notin \mathbb{Q} \cdot \mathrm{v}(\Gamma(A)$, contradicting question 2.
4. Fix $a \in \mathbf{K}(M)$ and $X = (\mathrm{v} \circ f)^{-1}(\mathrm{v}(a))$. It suffices to show that $f(X)$ is finite. Then the function $g := \mathrm{rv}(a)^{-1} \cdot f\big|_X : X \subseteq \mathbf{\Gamma} \to \mathbf{k}$ is definable and since $\mathbf{k}$ and $\mathbf{\Gamma}$ are orthogonal, the graph of g is of the form $\bigcup_{i<n} X_i \times \{c_i\}$ and $g(X) = \{c_i : i < n\}$ is finite. It follows that $f(X) = \{\mathrm{rv}(a) \cdot c_i : i < n\}$ is also finite. □

Problem 3. Let (K, v) be a valued field $\gamma \in vK^\times_{\geqslant 0}$ and $(\xi_i)_{i<n} \in \mathrm{RV}_\gamma = K/1 + \gamma\mathfrak{m}$. Show that $\{\sum_{i<n} x_i : \mathrm{rv}_\gamma(x_i) = \xi_i\}$ is an open ball and give its radius.

Solution. Let $\delta := \min_i \mathrm{v}(\xi_i)$ and let us show that $\{\sum_{i<n} x_i : \mathrm{rv}_\gamma(x_i) = \xi_i\} = \mathring{\mathrm{B}}(\sum_i x_i, \gamma + \delta)$, where $\mathrm{rv}_\gamma(x_i) = \xi_i$. Let y_i such that $\mathrm{rv}_\gamma(y_i) = \xi_i = \mathrm{rv}_\gamma(x_i)$, then $\mathrm{v}(\sum_i x_i - \sum_i y_i) = \mathrm{v}(\sum_i x_i - y_i) \geqslant \min_i \mathrm{v}(x_i - y_i) > \min_i \mathrm{v}(x_i) + \gamma \geqslant \delta + \gamma$. Conversely, let c be such that $\mathrm{v}(c) > \gamma + \delta$, i_0 be such that $\mathrm{v}(\xi_{i_0})$ is minimal. Let $y_{i_0} = x_{i_0} + c$ and $y_i = x_i$ otherwise. Then $\mathrm{rv}_\gamma(y_i) = \xi_i$ and $\sum_i y_i = \sum_i x_i + c$. □